

A CONSTRUCTIVE PROOF OF THE POINCARÉ-BIRKHOFF THEOREM

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ABSTRACT. In this paper, with the use of the homotopy method, a constructive proof of the Poincaré-Birkhoff theorem is given. This approach provides a global method for finding fixed points of area-preserving maps and periodic solutions of Duffing equations.

1. INTRODUCTION

As is well known, the Poincaré-Birkhoff theorem, proposed by Poincaré [38] in 1912 and proven by Birkhoff [7] in 1913, has been generalized by many authors, for example, Birkhoff [8-10], Jacobowitz [29], Ding [20], [21], Conley and Zehnder [12], [13] (about Arnold's conjecture [6]), Franks [24], [25] and Flucher [22]. Today, these notable results have become powerful tools in studying the dynamical systems and periodic solutions for some significant equations, for example Newton equations and Duffing equations. The related works can be found in [14, 16-20, 23, 25, 28, 29, 35] (and [15, 32, 34, 36] utilizing Moser's twist theorem [37]), and the references therein.

As an important aspect for applications, it is also significant to have a way of finding fixed points of area-preserving maps, such as Poincaré maps of Newton equations and Duffing equations. Generally, such a map of an annulus which twists the boundary curves in opposite directions has at least two fixed points, and the sum of indices of fixed points on such an annulus equals zero (see [25]). These result in some difficulties in utilizing the classical numerical methods, for example, the Newton method and the continuation method. The main reasons are the local convergence of the former and the strong regularity of the latter, along the manifold of solutions.

The main aim of this paper is to present a global method of finding fixed points of area-preserving maps and periodic solutions of Duffing equations. More precisely, with use of the continuous homotopy method, we shall establish constructively the following Poincaré-Birkhoff theorem given by Jacobowitz [29] and Ding [21].

Let A denote an annular region in $R^2 \setminus \{0\}$, whose inner boundary Γ_1 and outer boundary Γ_2 are two disjoint closed simple curves. By D_i we denote

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the bounded open region bounded by Γ_i , $i = 1, 2$. Hence, $A = \bar{D}_2 \setminus D_1$, and $0 \in D_1 \subset D_2$.

Theorem. Let $T: A \rightarrow T(A) \subset \mathbb{R}^2 \setminus \{0\}$ be an area-preserving homeomorphism. Suppose:

(i) T has the polar coordinates

$$r^* = f(r, \theta), \quad \theta^* = \theta + g(r, \theta)$$

such that $g(r, \theta) > 0$ on Γ_1 and $g(r, \theta) < 0$ on Γ_2 where f and g are C^2 continuous and 2π -periodic in θ .

(ii) There exists a continuous area-preserving map $T_1: \bar{D}_2 \rightarrow \mathbb{R}^2$ such that $T_1|_A = T$, and $0 \in T_1(D_1)$.

Then T has at least two fixed points in A .

Now let us make some comments.

(a) In our result, the inner boundary curve Γ_1 is not necessarily star shaped relative to the origin.

(b) The continuous homotopy method which we use has been proposed originally by Keller [30], Kellogg, Li and Yorke [31] and Smale [39]. This method has played an important role in various problems of finding fixed points or zeros of maps; see, for example, [1-3, 11, 26, 27, 33, 40]. There are three distinct, but interrelated, aspects of the homotopy method: (1) construction of the right homotopy map, (2) theoretical proof of global convergence for this homotopy map, and (3) tracking the zero curve of this homotopy map. The first aspect is to link the problem considered to a simpler one by considering a single parameter problem. The second aspect is a key to that method, because it guarantees the third aspect, that is, following paths to find the desired solutions. There have been some typical algorithms in the third aspect; see, for example, [2, 3, 27].

Our constructive proof means for almost every point near the set $g(r, \theta) = 0$, there exists a C^1 path passing to that point such that two ends of this path are two distinct fixed points of the area-preserving map considered. Hence, by utilizing the usual path-following methods, one can find numerically two fixed points. In particular, when fixed points of the map are isolated, an end of that C^1 path leads to a fixed point with positive index and another end leads to one with negative index. Because the zeros of $g(r, \theta) = 0$ are easier to determine than the fixed points of T (for example, using the Newton method), we provide an effective global method of finding fixed points of area-preserving maps and periodic solutions of the Duffing equations. To our best knowledge, such global methods seem not to have been applied to this problem. Of course, the aspect of path-following can be carried out by utilizing the typical methods. This makes us focus our attention on the theoretical proof of the global convergence for the given homotopy (Newton-type), that is the constructive proof.

This paper is organized as follows. Section 2 is the main part, which exhibits such a "constructive proof" of the Poincaré-Birkhoff theorem. There it will be seen that for the singular case, T has infinite fixed points in the set $g(r, \theta) = 0$, and finding fixed points is generally simpler; but for the nonsingular case, the problem becomes more complicated. In Section 3, we outline a general framework of applications to the Duffing equations. As stated above, we use the path-following algorithm given by Allgower and Georg [3]. Nevertheless, we

also provide an experimental example, which shows the efficiency of the reduced algorithm.

It should be pointed out that recently Alpern and Prasad [4, 5] have established a very interesting combinational proof of the famous Conley-Zehnder-Franks theorem. This approach is different from ours. In particular, our approach reduces finding fixed points of area-preserving maps into following solutions of suitable ordinary differential equations with initial values. Consequently, it is convenient to implement to applications.

2. CONSTRUCTIVE PROOF

In this section, we give a constructive proof of the Poincaré-Birkhoff theorem. From this proof we can obtain a global method of finding two fixed points for area-preserving maps. The following lemmas are vital to our discussion.

Lemma 1 (Sard's theorem). *If $\phi: U \subset R^m \rightarrow R^n$ is a C^k map on the open set U with $k > \max\{0, m - n\}$, then the set of singular values of ϕ has n -dimensional Lebesgue measure zero. Consequently, the set of regular values of ϕ is dense in R^n .*

Lemma 2 (the parametrized Sard theorem [11]). *Let $V \subset R^n$, $U \subset R^m$ be open sets, and $\phi: V \times U \rightarrow R^k$ a C^r map, where $r > \max\{0, m - k\}$. If $0 \in R^k$ is a regular value of ϕ , then for almost all $a \in V$, 0 is a regular value of $\phi_a = \phi(a, \cdot)$.*

Lemma 3 (Garcia and Zangwill [26]). *Let $V \subset R^n$ be an open set and let $\phi: \bar{V} \times [0, 1] \rightarrow R^n$ be a C^1 function and 0 a regular value on V . Then $H^{-1} = \{(x, t) \in V \times [0, 1]; \phi(x, t) = 0\}$ is a finite number of disjoint continuously differentiable paths. Any path is either a loop in $\bar{V} \times [0, 1]$ or starts from a boundary point of $\bar{V} \times [0, 1]$ and ends at another boundary point of $\bar{V} \times [0, 1]$.*

Lemma 4 (Garcia and Zangwill [26]). *Let $\phi: \bar{V} \times [0, 1] \rightarrow R^n$ be a C^1 homotopy, and 0 a regular value on V , where $V \subset R^n$ is an open bounded set. Then each solution $x(s)$ of the initial value problem*

$$\frac{dx_i}{ds} = (-1)^{i+1} \det \phi'_i, \quad x_i(0) = x_{i0} \quad (i = 1, \dots, n+1)$$

determines a C^1 path in $\phi^{-1}(0)$, where s is a parameter and

$$\phi'_i = (\phi_{x_1}, \dots, \phi_{x_{i-1}}, \phi_{x_{i+1}}, \dots, \phi_{x_{n+1}}).$$

Lemma 5. *Under the assumptions of the Theorem, there exists an area-preserving map $T_2: \bar{D}_2 \rightarrow R^2$ such that $T_2|_A = T$ and $T_2(0) = 0$.*

Proof. By the assumption (ii), we have $P = T_1^{-1}(0) \in D_1$. Hence there exists a ball $B_\delta(P)$ with center P and the radius δ such that $\bar{B}_\delta(P) \subset D_1$.

Let $P_1 \in \partial B \cap \overline{OP}$, where \overline{OP} denotes the line segment with end points 0 and P . Set $O_1 = \frac{1}{2}[P + P_1]$. Choose a polar coordinate system (ρ, ϕ) in R^2 which has O_1 as its pole. With this coordinate system, define $S: R^2 \rightarrow R^2$ by

$$\rho^* = \rho, \quad \phi^* = \phi + h(\rho),$$

where $h(\rho)$ is a C^∞ function satisfying $h(\rho) = 0$, for $\rho \geq \delta/2$; $h(\rho) = \pi$, for $0 < \rho \leq \delta/4$. Obviously, S is an area-preserving diffeomorphism, and

$$S(0) = P, \quad S|_A = \text{id (identity)}.$$

Define $T_2 = T_1 \circ S$. Then $T_2: \bar{D}_2 \rightarrow R^2$ is an area-preserving homeomorphism, because it is a composition of such maps. Moreover,

$$T_2|_A = T_1 \circ S|_A = T_1|_A = T, \quad T_2(0) = T_1 \circ S(0) = T_1(P) = 0,$$

which completes the proof of the lemma.

For convenience, we also define a map $\Pi: R^2 \setminus \{0\} \rightarrow (R_+ \setminus \{0\}) \times [0, 2\pi]$ by

$$\Pi(x, y) = \begin{cases} \left(\sqrt{x^2 + y^2}, \arccos \frac{x}{\sqrt{x^2 + y^2}} \right), & y \geq 0, \\ \left(\sqrt{x^2 + y^2}, \pi + \arccos \frac{x}{\sqrt{x^2 + y^2}} \right), & y < 0. \end{cases}$$

Proof of the theorem. Let

$$\begin{aligned} F(p) &= T(p) - p = \Pi^{-1}(\tilde{f}(\Pi(p)), g(\Pi(p))), \\ J &= \{p \in A: g(\Pi(p)) = 0\}, \end{aligned}$$

where $\tilde{f} = f(r, \theta) - r$. By Lemma 5, without loss of generality, we may assume that the map T_1 satisfies $T_1(0) = 0$. Hence, $T_1: \bar{D}_2 \setminus \{0\} \rightarrow R^2 \setminus \{0\}$. With the polar coordinate system, we can write T_1 in the following form:

$$r^* = f_1(r, \theta), \quad \theta^* = \theta + g_1(r, \theta), \quad \text{in } \bar{D}_2 \setminus \{0\},$$

where f_1 and g_1 are continuous on $\bar{D}_2 \setminus \{0\}$. Since $T_1|_A = T$, we have

$$(1) \quad f_1(r, \theta) = f(r, \theta), \quad g_1(r, \theta) = g(r, \theta), \quad \text{on } A.$$

From the property of the polar coordinate system it follows that f_1 is 2π -periodic in θ , and for some integer k , $g_1(r, \theta) = g(r, \theta) + 2k\pi$, on $\bar{D}_2 \setminus \{0\}$. Since $g_1(r, \theta)$ is continuous and $\bar{D}_2 \setminus \{0\}$ is connected, k is independent of (r, θ) . From (1) and the periodicity of $g(r, \theta)$, we obtain $k = 0$, which implies that $g_1(r, \theta)$ is also 2π -periodic in θ .

Define

$$\tilde{T}(r, \theta) = (f_1(r, \theta), \theta + g_1(r, \theta)) \quad \text{on } (R_+ \setminus \{0\}) \times [0, 2\pi]$$

and set

$$F(r, \theta) = (F_1(r, \theta), F_2(r, \theta)) \equiv T_1(r, \theta) - (r, \theta), \quad \text{on } (R_+ \setminus \{0\}) \times [0, 2\pi].$$

We shall complete the proof by the following two steps.

1°. *Singular case.* For all $(r, \theta) \in \Pi A$, the Jacobian $F'(r, \theta)$ is singular. Take a simply connected region $\Omega \subset D_2$ such that $\bar{D}_1 \subset \Omega$ and $\partial\Omega \subset J$. Indeed, by (i) there exists a bounded and connected branch $\Omega_1 \subset R^2 \setminus J$ such that

$$0 \in \Omega_1 \quad \text{and} \quad \partial\Omega_1 \subset J.$$

Thus D_1 is a region. Let K denote the set of all closed simple curves ν lying in Ω_1 . By $D(\nu)$ we denote the bounded region bounded by ν . Set

$$\Omega = \bigcup \{D(\nu): \nu \text{ lies in } \Omega_1\}.$$

Then Ω is the desired simply connected region.

We claim:

$$F_1(r, \theta) \equiv 0, \quad \text{on } \partial\Omega.$$

By (i), there exists $(r_0, \theta_0) \in \Pi A$ such that $\text{rank } F'(r_0, \theta_0) = 1$. Hence by Lemma 1, almost every $r \in \text{range}(F_1)$ is a regular value of F_1 . First, we prove

$$(2) \quad F_1(r, \theta) \equiv c \text{ (a constant), on } \partial\Omega.$$

If not, then by the continuity of F_1 , $F_1(\Pi\partial\Omega)$ is an interval in R . Let S_1 denote the set of all regular values in $F_1(\Pi\partial\Omega)$. Since $\partial\Omega$ is bounded and closed, and F_1 is C^1 , S_1 is open in R , and consequently, $S_2 = F_1^{-1}(\Pi S_1)$ is open in the relative topology of $\Pi\partial\Omega$. By the inverse function theorem, for each $(r_0, \theta_0) \in S_2$, there exists a positive number $\eta = \eta(r_0, \theta_0)$ such that on $[\theta_0 - \eta, \theta_0 + \eta]$, the equation

$$(3) \quad F_1(r, \theta) = F_1(r_0, \theta_0)$$

has a unique solution $(r(\theta), \theta)$ satisfying

$$r(\theta_0) = r_0, \quad |r(\theta) - r_0| \leq \eta,$$

if $F_{1r}(r_0, \theta_0) \neq 0$; or on $[r_0 - \eta, r_0 + \eta]$, equation (3) has a unique solution $(r, \theta(r))$ satisfying

$$\theta(r_0) = \theta_0, \quad |\theta(r) - \theta_0| \leq \eta,$$

if $F_{1\theta}(r_0, \theta_0) \neq 0$. We can choose a small η such that $F_{1r} \neq 0$, on $I_\eta = [r_0 - \eta, r_0 + \eta] \times [\theta_0 - \eta, \theta_0 + \eta]$; or $F_{1\theta} \neq 0$ on I_η . For definiteness, let us assume it is the former case. Since F' is singular, there exists a continuous function $\lambda(r, \theta)$ defined on I_η such that

$$(4) \quad (F_{2r}, F_{2\theta}) = \lambda(r, \theta)(F_{1r}, F_{1\theta}) \text{ on } I_\eta.$$

From (4) and

$$0 \equiv \frac{dF_1(r(\theta), \theta)}{d\theta} = F_{1r} \frac{dr}{d\theta} + F_{1\theta}$$

it follows that

$$\frac{dF_2(r(\theta), \theta)}{d\theta} = F_{2r} \frac{dr}{d\theta} + F_{2\theta} = \lambda(r(\theta), \theta) \frac{dF_1(r(\theta), \theta)}{d\theta} = 0, \text{ on } I_\eta.$$

Hence on $I_\eta \cap \partial\Omega$,

$$(5) \quad (F_1(r, \theta), F_2(r, \theta)) \equiv (F_1(r_0, \theta_0), F_2(r_0, \theta_0)) = (F_1(r_0, \theta_0), 0),$$

which implies

$$(r(\theta), \theta) \in \partial\Omega.$$

Set $I(\eta(r, \theta)) = (r - \eta(r, \theta), r + \eta(r, \theta)) \times (\theta - \eta(r, \theta), \theta + \eta(r, \theta))$. Since the set of all members in $\{I(\eta(r, \theta)): (r, \theta) \in S_2\}$ disjoint from each other is countable, by (5) S_1 is also countable, which contradicts the openness of S_1 . Therefore (2) holds.

Now we prove that $c = 0$. If $c \neq 0$, then from $F_1(r, \theta) \equiv c$ on $\partial\Omega$, we have $f(r, \theta) = r + c$, on $\Pi\partial\Omega$. Note

$$g(r, \theta) \equiv 0, \text{ on } \Pi\partial\Omega.$$

Therefore,

$$\Pi\hat{T}(\Omega) = \{(r^*, \theta^*): r^* = r + c, \theta^* = \theta, (r, \theta) \in \Pi\Omega\} = G(\Pi\Omega),$$

where $G(r, \theta) = (r + c, \theta)$. Hence

$$\Pi\tilde{T}(\Omega) = G(\Pi\Omega).$$

Given any $\Delta S = \{(r, \theta): 0 < r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\} \subset \Pi\Omega$, we have

$$G(\Delta S) \subset \Pi T_1(\Omega),$$

$$\begin{aligned} \text{meas}[G(\Delta S)] &= \frac{\pi}{360}(\theta_2 - \theta_1)[(r_2 + c)^2 - (r_1 + c)^2] \\ &= \frac{\pi}{360}(\theta_2 - \theta_1)(r_2 - r_1)(r_2 + r_1 + 2c) \\ &= \frac{2\pi c}{360}(\theta_2 - \theta_1)(r_2 - r_1) + \text{meas}(\Delta S). \end{aligned}$$

Thus, when $c \neq 0$, we obtain

$$\text{meas}[T_1(\Omega)] = \text{meas}[\Pi T_1(\Omega)] = \text{meas}[G(\Pi\Omega)] \neq \text{meas}(\Pi\Omega) = \text{meas}(\Omega),$$

a contradiction. This proves that $c = 0$, and consequently, every point in $\partial\Omega$ is a fixed point of T .

2° Regular case. For some $a_0 \in \Pi A$, $F'(a_0)$ is nonsingular. We claim that for some $(r_0, \theta_0) \in \Pi A$,

$$f(r_0, \theta_0) - r_0 \neq 0.$$

If not, then

$$f(r, \theta) - r \equiv 0, \text{ on } \Pi A.$$

Choose any $\bar{\theta} \in g(\Pi A)$ and consider the equation

$$f(r, \theta) - r \equiv 0, \quad g(r, \theta) - \bar{\theta} \equiv 0.$$

Denote $J(\bar{\theta}) = \{\Pi^{-1}(r, \theta) \in A: g(r, \theta) - \bar{\theta} = 0\}$. Hence

$$\begin{aligned} (f_r - 1)dr + f_\theta d\theta &\equiv 0, \text{ on } \Pi A, \\ g_r dr + g_\theta d\theta &\equiv 0, \text{ on } \Pi J(\bar{\theta}). \end{aligned}$$

Therefore,

$$\det \begin{pmatrix} f_r - 1 & f_\theta \\ g_r & g_\theta \end{pmatrix} = 0, \text{ on } \Pi J(\bar{\theta}),$$

which leads to a contradiction, because $\bar{\theta}$ is arbitrary.

By Lemma 1 and the above claim, almost every point $(\bar{r}, \bar{\theta}) \in (F_1(\Pi A), (\sigma_1, \sigma_2))$ satisfying $\bar{r} \neq 0$ is a regular value of (F_1, F_2) , where $\sigma_1 = \max_{p \in \Gamma_2} g(r(p), \theta(p)) < \min_{p \in \Gamma_1} g(r(p), \theta(p)) = \sigma_2$. For such a regular value $(\bar{r}, \bar{\theta})$, set

$$T_{\bar{\theta}}: r^* = f_1(r, \theta), \quad \theta^* = \theta + g_1(r, \theta) - \bar{\theta}, \quad \bar{\theta} \in (\sigma_1, \sigma_2).$$

Obviously, $T_{\bar{\theta}}$ is an area-preserving homeomorphism defined on $\Pi(D_2 \setminus \{0\})$. Set

$$F_{\bar{\theta}}(r, \theta) = (f_1(r, \theta) - r, g_1(r, \theta) - \bar{\theta});$$

and define a homotopy map $H: \Pi\bar{A} \times \Pi\bar{A} \times [-\lambda(\bar{r}), 1] \rightarrow R^2$ by

$$H_{\bar{\theta}}(p_0, p, \lambda) = F_{\bar{\theta}}(p) - (1 - \lambda)F_{\bar{\theta}}(p_0),$$

where $p = (r, \theta)$, $F_{\bar{\theta}}(p_0) = (\bar{r}, 0)$, $p_0 \in F_{\bar{\theta}}^{-1}(\bar{r}, 0)$, and $\lambda(\bar{r})$ is a positive constant such that

$$(6) \quad \frac{1}{2}|\bar{r}|\lambda(\bar{r}) \geq \max\{r: r = \sqrt{x^2 + y^2}, (x, y) \in A \cup T(A)\}.$$

Notice that if for some $p_0 = (r, \theta) \in \Pi A$, $F'(r, \theta)$ is nonsingular, then

$$(7) \quad \frac{\partial H}{\partial(r, \theta)} = -(1 - \lambda)F'(r, \theta)$$

is also nonsingular for $\lambda < 1$. Set $V = \{P = (r, \theta) \in F^{-1}(F(\Pi A)): (\bar{r}, \bar{\theta}) \text{ is a regular value of } F \text{ on } \Pi A \text{ and } \bar{r} \neq 0, \bar{\theta} \in (\sigma_1, \sigma_2)\}$. By Lemma 2, for almost all $p_0 \in V$, 0 is a regular value of $H_{p_0} \equiv H(p_0, \cdot)$, for $\lambda \in [-\lambda(\bar{r}), 1]$. Denote by V_1 all such regular values.

Choose any $(\bar{r}, \bar{\theta})$ such that $F^{-1}(\bar{r}, \bar{\theta}) \subset V_1$. Let $x_1 = r$, $x_2 = \theta$, $x_3 = \lambda$ and $P = (x_1, x_2)$. Then by Lemmas 3, 4, for each $p_0 = (r_0, \theta_0) \in F^{-1}(\bar{r}, \bar{\theta})$, the solution $(P(s, p_0), \lambda(s, p_0))$ of the initial value problem

$$\begin{aligned} \frac{dx_i}{ds} &= (-1)^{i+1} \det H'_i, \quad i = 1, 2, 3, \\ (P(0), \lambda(0)) &= (p_0, 0) \end{aligned}$$

determines a C^1 path $(P(s, p_0), \lambda(s, p_0))$ in $H_{p_0}^{-1}(0)$, where

$$\begin{aligned} H'_1 &= (H'_{x_2}, H'_{x_3}) = \det \begin{pmatrix} f_{\theta} & \bar{r} \\ g_{\theta} & 0 \end{pmatrix} = -\bar{r}g_{\theta}, \\ H'_2 &= (H'_{x_1}, H'_{x_3}) = \det \begin{pmatrix} f_r - 1 & \bar{r} \\ g_r & 0 \end{pmatrix} = -\bar{r}g_r, \\ H'_3 &= (H'_{x_1}, H'_{x_2}) = \det \begin{pmatrix} f_r - 1 & f_{\theta} \\ g_r & g_{\theta} \end{pmatrix} (= F'(p)); \end{aligned}$$

and

$$\begin{aligned} H^{-1}(0) &= \{p = (r, \theta) \in \Pi A: H(p_0, p, \lambda) = 0, \\ (8) \quad &\lambda \in [-\lambda(\bar{r}), 1], p_0 \in F^{-1}(\bar{r}, \bar{\theta})\} \\ &= \{P(s, p_0) \in \Pi A: \lambda(s, p_0) \in [-\lambda(\bar{r}), 1], p_0 \in F^{-1}(\bar{r}, \bar{\theta})\}. \end{aligned}$$

Since $\bar{\theta} \in (\sigma_1, \sigma_2)$ and

$$H(p_0, P(s, p_0), \lambda(s, p_0)) = 0,$$

we have

$$\begin{aligned} \tilde{g}_1(r(s, p_0), \theta(s, p_0)) &\equiv g_1(r(s, p_0), \theta(s, p_0)) - \bar{\theta} \\ &= (1 - \lambda(s, p_0)) \cdot 0 = 0, \text{ for } -\lambda(\bar{r}) \leq \lambda(s, p_0) \leq 1. \end{aligned}$$

Hence,

$$(9) \quad J[\bar{\theta}] \equiv \{(r, \theta): (r, \theta) = (r(s, p_0), \theta(s, p_0))\} \subset \Pi A.$$

By Lemma 3, $(P(s, p_0), \lambda(s, p_0))$ is either a loop in $\Pi \bar{A} \times [-\lambda(\bar{r}), 1]$ or starts from boundary points of $\Pi \bar{A} \times [-\lambda(\bar{r}), 1]$ and ends at boundary points of $\Pi \bar{A} \times [-\lambda(\bar{r}), 1]$. We shall prove that the former is impossible. Let $(P(s, p_0), \lambda(s, p_0))$ be a loop. By $\Omega(p_0)$ we denote the bounded region bounded by $P(s, p_0)$. We claim

$0 \in \Omega(p_0)$ and the orbit $\{P(s, p_0)\}$ is star shaped about the origin 0.

First, $\{P(s, p_0)\}$ is star shaped about the origin 0. Set $P(s) = P(s, p_0) = (r(s), \theta(s))$, $\lambda(s) = \lambda(s, p_0)$. If not, then there would exist $s_0 \in R$ such that $\theta(s_0)$ is minimal and $d\theta/ds|_{s=s_0} = 0$. Since $d\theta/ds = g_r \bar{r}$, we see $g_r(r(s_0), \theta(s_0)) = 0$. Notice that the line $\theta = \theta(s_0)$ is a tangent of $(r(s), \theta(s))$ at $(r(s_0), \theta(s_0))$. By the area-preserving property of $T_{\bar{\theta}}$, we have

$$\det T'_{\bar{\theta}}(r, \theta) = \det \begin{pmatrix} f_r & f_{\theta} \\ g_r & 1 + g_{\theta} \end{pmatrix} = r.$$

Hence for any $(r, \theta) \in \Pi A$,

$$(10) \quad \Delta r^* \Delta \theta^* = \det \begin{pmatrix} f_r & f_{\theta} \\ g_r & 1 + g_{\theta} \end{pmatrix} \Big|_{(r+\xi\Delta r, \theta+\xi\Delta\theta)} \Delta r \Delta \theta = (r + \xi\Delta r) \Delta r \Delta \theta,$$

whenever $|\Delta r|, |\Delta \theta|$ are sufficiently small. Since for $s_1, s_2 \in R$, along the curve $\{P(s)\}$, we get

$$(11) \quad \Delta r^* \Delta \theta^* = \Delta r^* \Delta \theta = f_r(r + \xi\Delta r, \theta + \xi\Delta\theta) \Delta r \Delta \theta.$$

From

$$(12) \quad \begin{aligned} r^*(s) &= f(r(s), \theta(s)) = (1 - \lambda(s))\bar{r} + r(s), \\ \theta^*(s) &= g(r(s), \theta(s)) - \bar{\theta} = 0, \\ \text{rank}(H_r, H_{\theta}, H_{\lambda})|_{(r(s), \theta(s), \lambda(s)), \lambda(s) < 1} &= 2, \end{aligned}$$

it follows that

$$(13) \quad \begin{aligned} g_{\theta}(r(s_0), \theta(s_0)) &\neq 0, \quad \frac{d\theta^*(s)}{ds} \Big|_{s=s_0} = \frac{d\theta(s)}{ds} \Big|_{s=s_0} = 0, \\ \frac{dr^*(s)}{ds} \Big|_{s=s_0} &= f_r \frac{dr(s)}{ds} \Big|_{s=s_0}. \end{aligned}$$

Because $P(s)$ is a loop, we can choose such an s_0 to possess the following property:

(P) For every $\epsilon > 0$, $\text{meas}\{\theta(s) : s \in [s_0 - \epsilon, s_0 + \epsilon]\} > 0$.

Then using (10), (11), and (13) yields

$$0 \neq r(s_0) = f_r(r(s_0), \theta(s_0)) [1 + g_{\theta}(r(s_0), \theta(s_0))] = f_r(r(s_0), \theta(s_0)).$$

Then

$$g_{\theta}(r(s_0), \theta(s_0)) = 0.$$

Consequently,

$$\text{rank}(H_r, H_{\theta}, H_{\lambda})|_{(r(s_0), \theta(s_0), \lambda(s_0))} = 1,$$

a contradiction.

It is clear that $0 \in \Omega(p_0)$. By the claim and (12), we have

$$(14) \quad T_{\bar{\theta}}[\Omega(p_0)] \subset \Omega(p_0), \text{ if } \bar{r} < 0;$$

or

$$(15) \quad T_{\bar{\theta}}[\Omega(p_0)] \supset \Omega(p_0), \text{ if } \bar{r} > 0.$$

Notice that $\lambda(s) \neq 1$. Therefore from (14) and (15) it follows that

$$\text{meas}[T_{\bar{\theta}}(\Omega(p_0))] \neq \text{meas}(\Omega(p_0)),$$

a contradiction. This shows that $\{(P(s), \lambda(s))\}$ is not a loop.

By (6),(9) and (12), we get

$$(16) \quad \lambda(s) \geq -\frac{1}{2}\lambda(\bar{r}), \text{ for } s \in R.$$

Hence $\Gamma(p_0) = \{(P(s), \lambda(s)) : \lambda(s) \leq 1\}$ starts from the points in $\Pi A \times \{1\}$ and ends at the points in $\Pi A \times \{1\}$. Set

$$S_{\pm} = \{q \in \Pi A : P(s, p_0) \text{ leads to } q, \text{ as } \lambda(s) \rightarrow 1, s \in R_{\pm}\}.$$

We claim

$$(17) \quad S_+ \cap S_- = \emptyset.$$

If not, then $S_+ \cap S_- \neq \emptyset$. By the above arguments, for each $s \in R$ with $\lambda(s) < 1$, $\theta(s)$ is not an extreme value. Otherwise, for some $s_0 \in R$, $\theta(s_0)$ is an extreme value. Then for any interval I containing s_0 , the property (P) does not hold. Therefore

$$\theta(s) \equiv \text{a constant, for } \lambda(s) < 1.$$

Since $\text{rank}(H_r, H_\theta, H_\lambda)|_{(p(s), \lambda(s)), \lambda(s) < 1} = 2$, we have

$$\frac{dr(s)}{ds} \neq 0, \text{ for } \lambda(s) < 1.$$

This shows that $S_+ \cap S_- = \emptyset$, a contradiction. Hence when $\lambda(s) < 1$,

$$(18) \quad \frac{d\theta(s)}{ds} \geq 0 \quad (\text{or } \frac{d\theta(s)}{ds} \leq 0).$$

From (17) and (18) it follows that there exist $\theta_1, \theta_2 \in R$ with $|\theta_1 - \theta_2| = 2\pi$ such that

$$\theta(s) \rightarrow \begin{cases} \theta_1, & \text{as } \lambda(s) \rightarrow 1, s \in R_+, \\ \theta_2, & \text{as } \lambda(s) \rightarrow 1, s \in R_-. \end{cases}$$

Then $\Gamma(p_0)$ bounds a bounded region $\Omega(p_0)$. Notice that for each $\theta \in (\min\{\theta_1, \theta_2\}, \max\{\theta_1, \theta_2\})$, $\{(r, \theta) : r \geq 0\} \cap \Gamma(p_0)$ contains only one point. Thus by (12), $\text{meas}(T_\theta(\Omega(p_0))) \neq \text{meas}(\Omega(p_0))$. This contradiction shows that (17) is true.

Using (12), we know that every point $p \in S_+ \cup S_-$ is a fixed point of T_θ . By (17), T_θ has at least two fixed points in A . In particular, when S_+ and S_- are isolated sets, letting $\{P_+\} = S_+$ and $\{P_-\} = S_-$, we have $\text{index}(T_\theta, P_{\pm}) = \pm 1$. Indeed, choose $\epsilon > 0$ such that $B_\epsilon(P_+) \cap B_\epsilon(P_-) = \emptyset$. By Sard's theorem, there exist two sequences of regular values, $\{\lambda(s_k)\}_{s_k > 0}$ and $\{\lambda(t_k)\}_{t_k < 0}$ such that

$$\lambda(s_k) \rightarrow 1, \lambda(t_k) \rightarrow 1, k \rightarrow \infty, P(s_k) \in B_\epsilon(P_+), P(t_k) \in B_\epsilon(P_-).$$

Hence, for sufficiently large k ,

$$(19) \quad \begin{aligned} \deg(\text{id} - T_\theta, B_\epsilon(P_+), 0) &= \deg(\text{id} - T_\theta, B_\epsilon(P_+), r_k) \\ &= \sum_{p \in (\text{id} - T_\theta)^{-1}(r_k)} \text{sgn det}(T'_\theta(p)), \end{aligned}$$

where $r_k = -(1 - \lambda(s_k)) \binom{r}{0}$. Notice

$$P(s_k) \rightarrow P_+, \lambda(s_k) < 1, \lambda(s_k) \rightarrow 1, s_k > 0.$$

Therefore, from (19) it follows that

$$\text{index}(T_{\bar{\theta}}, P_+) = 1.$$

Similarly,

$$\text{index}(T_{\bar{\theta}}, P_-) = -1.$$

Now we prove that T has at least two fixed points in A . In the same way as in choosing $(\bar{r}, \bar{\theta})$, choose a sequence $\{(\bar{r}_k, \bar{\theta}_k)\}$ such that $\bar{\theta}_k \in (\sigma_1, \sigma_2)$, $\bar{\theta}_k \rightarrow 0$, $k \rightarrow \infty$, and without loss of generality, let $\bar{r}_k \rightarrow \bar{r}_0 \neq 0$. Set

$$H_k(p_k, p, \lambda) = F_{\bar{\theta}_k}(p) - (1 - \lambda)F_{\bar{\theta}_k}(p_k),$$

where $p_k \in F_{\bar{\theta}_k}^{-1}(\bar{r}_k, 0)$. Then the Cauchy problem

$$(20) \quad \begin{aligned} \frac{dx_i}{ds} &= (-1)^{i+1} \det(H_k)'_i, & i &= 1, 2, 3, \\ (p_k(0), \lambda_k(0)) &= (p_k, 0) \end{aligned}$$

determines a C^1 path $(P_k(s), \lambda_k(s))$ in $(H_k)_{p_k}^{-1}(0)$. Applying the Arzela-Ascoli theorem and passing to a subsequence if necessary, we may assume $P_k \rightarrow P^*$, and for each compact interval I with $\lambda_k(s) \leq 1$,

$$(21) \quad P_k(s) \rightarrow P(s), \quad \lambda_k(s) \rightarrow \lambda(s) \text{ uniformly on } I.$$

Set

$$S_{k\pm} = \{q \in \Pi A : P_k(s) \text{ leads to } q, \text{ as } \lambda_k(s) \rightarrow 1, s \in R_{\pm}\}$$

and

$$D_{\pm} = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} S_{k\pm}.$$

Obviously, every point $p \in D_+ \cup D_-$ is a fixed point of T in A . We claim that $D_+ \cup D_-$ contains at least two points.

Indeed, if not, then $D_+ \cup D_- = \{p\}$. Since $\bar{r}_k \rightarrow \bar{r}_0 \neq 0$, we have

$$\lim_{k \rightarrow \infty} \sup \{|r_k(s) - r_k(s_1)| + |\theta_k(s) - \theta_k(s_1) \pmod{2\pi}| : \lambda_k(s), \lambda_k(s_1) < 1\} > 0.$$

Notice if $D_+ \cup D_- = \{p\}$, then

$$(22) \quad \begin{aligned} \theta'_k(s) &\neq 0, \text{ for } \lambda_k(s) < 1, \\ \lim_{k \rightarrow \infty} \text{meas}\{\theta_k(s) : \lambda_k(s) \leq 1\} &\geq 2\pi. \end{aligned}$$

First, we have

$$\lim_{k \rightarrow \infty} \text{meas}\{\theta_k(s) : \lambda_k(s) \leq 1\} \neq 0.$$

If not, without loss of generality, we assume

$$\lim_{k \rightarrow \infty} \text{meas}\{\theta_k(s) : \lambda_k(s) \leq 1\} = 0.$$

Hence $\lim_{k \rightarrow \infty} \text{meas}\{r_k(s) : \lambda_k(s) \leq 1\} > 0$. Since when $\lambda_k(s) < 1$, $\theta_k(s)$ is monotonic, and if $\theta'_k(s) = 0$, then $\theta_k(s) \equiv \text{a constant}$, and $r'_k(s) \neq 0$. We have

$$D_+ \cup D_- = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \{r_k(s) : \lambda_k(s) \leq 1\}.$$

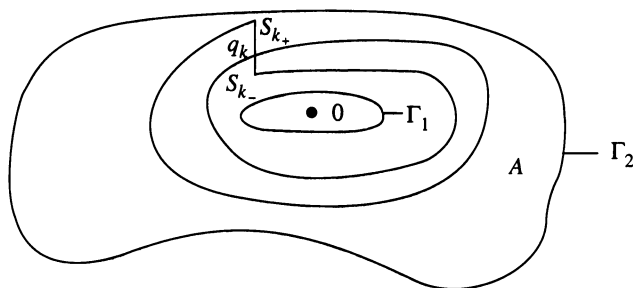


FIGURE 1

This shows

$$\text{meas}(D_+ \cup D_-) = \lim_{k \rightarrow \infty} \text{meas}\{r_k(s) : \lambda_k(s) < 1\} > 0,$$

a contradiction.

Thus, $\theta_k(s)$ is strictly monotonic. Since $\text{meas}\{\theta_k(s) : \lambda_k < 1\} > 0$, by (P) and $D_+ \cup D_- = \{p\}$ we get (22), and $\theta'_k(s) \neq 0$. It follows that for each k , there exist $\theta_{k+}, \theta_{k-} \in R$ such that

$$(23) \quad \theta_k(s) \rightarrow \theta_{k\pm}, \text{ when } \lambda_k(s) \rightarrow 1, s \in R_{\pm}.$$

Thus,

$$(24) \quad |\theta_{k+} - \theta_{k-} \pmod{2\pi}| \rightarrow 0, |\theta_k(s) - \theta_0 \pmod{2\pi}| \rightarrow 0, \\ |r_k^*(s) - r_0| + |r_k(s) - r_0| \rightarrow 0, \text{ as } \lambda_k(s) \rightarrow 1, s \in R_{\pm}, k \rightarrow \infty,$$

where $(r_0, \theta_0) = p$. By (22) and Jordan's separation theorem, for each k , there exists a point $q_k = (r_k, \theta_k)$ lying in the line segment l_k determined by S_{k+} and S_{k-} , such that

$$\text{dist}(q_k, S_{k+}) = \inf\{\text{dist}(S_{k+}, q) : q \in (\{p_k(s)\} \cap l_k) \cup S_{k-}\}.$$

See Figure 1.

Then there exists $t_k \in R \cup \{-\infty\}$ such that $q_k \in \{p_k(t_k)\}$. Let N_k denote the line segment determined by $\{q_k\}$ and S_{k+} . Thus, the curve

$$\Gamma_k = \{P_k(s) : \lambda_k(s) \leq 1, s \in [t_k, \infty)\} \cup N_k$$

is a simple closed curve.

For any $\epsilon \in (0, \pi/8)$, let

$$\tilde{P}_k(s) = (\tilde{r}_k(s), \tilde{\theta}_k(s)) = P_k(s), \tilde{\lambda}_k(s) = \lambda_k(s), \text{ for } \theta_k(s) \notin (\theta_0 - \epsilon, \theta_0 + \epsilon).$$

By the Arzela-Ascoli theorem, without loss of generality, we may assume

$$\tilde{p}_k(s) \rightarrow p(s), \tilde{\lambda}_k(s) \rightarrow \lambda(s), \text{ uniformly with respect to } \theta_k(s) \notin (\theta_0 - \epsilon, \theta_0 + \epsilon).$$

This shows that $(P(s), \lambda(s))$ is a loop and satisfies

$$\lambda(s) \leq 1, T_1(P(s)) - P(s) = (1 - \lambda(s)) \begin{pmatrix} \bar{r}_0 \\ 0 \end{pmatrix}.$$

The set Ω is a bounded region bounded by $\{P(s)\}$. Then Ω is star shaped about the origin. Notice if $D_+ \cup D_- = \{p\}$, then $\lambda(s) \not\equiv 1$. Thus

$$\text{meas}(T_1(\Omega)) > \text{meas}(\Omega), \text{ for } \bar{r}_0 > 0,$$

$$\text{meas}(T_1(\Omega)) < \text{meas}(\Omega), \text{ for } \bar{r}_0 < 0,$$

which leads to a contradiction. This completes the proof of the theorem.

3. A NUMERICAL EXAMPLE TO THE DUFFING EQUATIONS

In the above section, we have established the global convergence of our homotopy. As an application, in this section, we shall give a general framework of finding periodic solutions for the Duffing equations and also shall provide a numerical example.

Consider the Duffing equations

$$(25) \quad x'' + F(t, x) = 0,$$

where $F: R \times R \rightarrow R$ is a continuous function which is twice continuously differentiable in the second variable and 2π -periodic in the first variable. Then (25) has the equivalent form

$$(26) \quad x' = y, \quad y' = -F(t, x).$$

Let $(x(t, x_0, y_0), y(t, x_0, y_0))$ denote the unique solution of (26) with the initial value $(x(0), y(0)) = (x_0, y_0)$. Then the Poincaré map $T: R^2 \rightarrow R^2$ defined by

$$T(x_0, y_0) = (x(2\pi, x_0, y_0), y(2\pi, x_0, y_0))$$

is an area-preserving homeomorphism (see [29]).

Set $x = r \cos \theta$, $y = r \sin \theta$. Then (26) is transformed into

$$(27) \quad \begin{aligned} r' &= r \sin \theta \cos \theta - \sin \theta F(t, r \cos \theta), \\ \theta' &= -\sin^2 \theta - \frac{1}{r} \cos \theta F(t, r \cos \theta). \end{aligned}$$

Denote by $(r(t, r_0, \theta_0), \theta(t, r_0, \theta_0))$ the unique solution of (27) with the initial value $(r(0), \theta(0)) = (r_0, \theta_0)$. Using the notation of the Theorem,

$$(28) \quad \begin{aligned} r_0^* &= r(2\pi, r_0, \theta_0) = f(r_0, \theta_0), \\ \theta_0^* &= \theta(2\pi, r_0, \theta_0) = g(r_0, \theta_0) + \theta_0 - 2k\pi, \quad k = 1, 2, \dots \end{aligned}$$

To compute the periodic solutions of (25), we first fix a variable $\theta_0 \in [0, 2\pi)$, and find a point r_0 near the set $\{r: g(r, \theta_0) = 0, r > 0\}$ by Newton's method. By the parametrized Sard theorem, the choice of such a point (r_0, θ_0) is of probability one. That zero is a regular value of the homotopy

$$(29) \quad H(r, \theta, \lambda) = \begin{pmatrix} f(r, \theta) - r \\ g(r, \theta) \end{pmatrix} - (1 - \lambda) \begin{pmatrix} f(r_0, \theta_0) - r_0 \\ 0 \end{pmatrix}.$$

When $\lambda = 0$, the solution of $H(r, \theta, \lambda) = 0$ is $(r_0, \theta_0, 0)$; and when $\lambda \rightarrow 1$, any limit point (r^*, θ^*) of the solutions of (29) satisfies $f(r^*, \theta^*) = r^*$, $g(r^*, \theta^*) = \theta^* + 2k\pi$, $k = 1, 2, \dots$

It is clear (29) is equivalent to the following initial value problem to the differential equation:

$$(30) \quad \frac{dH(r(s), \theta(s), \lambda(s))}{ds} = 0, \quad \lambda(0) = 0.$$

By the discussion in section 2, to find two solutions of (25), we only need to follow two different directions to $\lambda(s) = 1$ of the curve determined by (30).

Following the line of Allgower and Georg [3], we have the following algorithm.

Algorithm 1 (Euler-Newton method).

Step 0: Given an initial point $\theta_0 \in [0, 2\pi)$, compute r_0 such that $g(r_0, \theta_0) = 0$ by Newton's method, and $\lambda(0) = 0$, an initial steplength $h_0 > 0$, $\tau \in (0, 1)$ and three small positive numbers $\epsilon_1, \epsilon_2, \epsilon_3 > 0$. Set $k := 0$.

Step 1: Compute a predictor direction η_k :

(a) Compute a unit tangent vector $\xi_k \in R^3$;

(b) Determine the direction η_k of predictor step:

If the sign of

$$\det \begin{pmatrix} DH(r_k, \theta_k, \lambda_k) \\ \xi_k^T \end{pmatrix}$$

does not change, then $\eta_k = \xi_k$;

If the sign of

$$\det \begin{pmatrix} DH(r_k, \theta_k, \lambda_k) \\ \xi_k^T \end{pmatrix}$$

changes, then $\eta_k = -\xi_k$;

Step 2: Compute the next iterative point (r_{k+1}, θ_{k+1}) :

(a) Compute a predictor point

$$(r_{k,j}, \theta_{k,j}, \lambda_{k,j}) = (r_k, \theta_k, \lambda_k) + \tau^j h_k \eta_k.$$

(b) Compute a corrector point

$$\begin{aligned} (r_{k+1,j}, \theta_{k+1,j}, \lambda_{k+1,j}) \\ = |(r_{k,j}, \theta_{k,j}, \lambda_{k,j})| - DH(r_k, \theta_k, \lambda_k)^+ H(r_{k,j}, \theta_{k,j}, \lambda_{k,j}), \\ j = 0, 1, 2, \dots \end{aligned}$$

until $\|H(r_{k+1,j}, \theta_{k+1,j}, \lambda_{k+1,j})\| \leq \epsilon_1$. Set $(r_{k+1}, \theta_{k+1}, \lambda_{k+1}) = (r_{k+1,j}, \theta_{k+1,j}, \lambda_{k+1,j})$.

Step 3: If $\lambda_{k+1} \in [1 - \epsilon_2, 1]$, then set $\lambda = 1$ and use Newton's method to compute the periodic solutions for the Duffing equations until the iterative point (r_k, θ_k) satisfying $\|H(r_k, \theta_k, 1)\| \leq \epsilon_3$; else choose a new steplength $h_{k+1} > 0$. $k := k + 1$, go to Step 1, where

$$DH(r, \theta, \lambda) = \frac{dH(r, \theta, \lambda)}{(dr, d\theta, d\lambda)}$$

and

$$DH(r, \theta, \lambda)^+ = DH(r, \theta, \lambda)^T (DH(r, \theta, \lambda) DH(r, \theta, \lambda)^T)^{-1}$$

is the Moore-Penrose inverse, and h_{k+1} is chosen as in [3].

Remark. By $f(r, \theta) = r(2\pi, r, \theta) - r$, $g(r, \theta) = \theta(2\pi, r, \theta) - \theta + 2k\pi$, $k = 1, 2, \dots$, we can use fourth order Runge-Kutta method to compute $f(r, \theta)$, $g(r, \theta)$ and the approximation of

$$\frac{df(r, \theta)}{dr}, \frac{df(r, \theta)}{d\theta}, \frac{dg(r, \theta)}{dr}, \frac{dg(r, \theta)}{d\theta}.$$

In the following, we give an example for the superlinear Duffing equation. For the general theory of such equations, refer to [28] and [29]. Set $g(r, \theta) = \theta(2\pi, r, \theta) - \theta + 2\pi$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.01$, $\epsilon_3 = 10^{-5}$. Numerical results are computed by double precision operations.

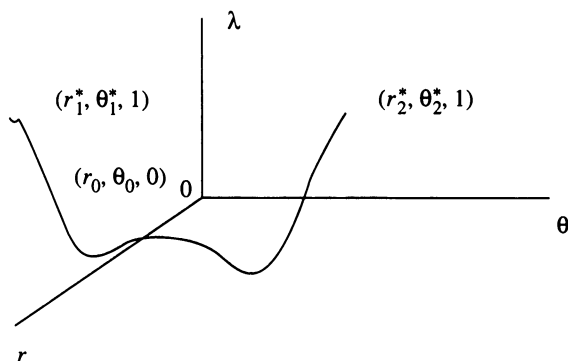


FIGURE 2. The homotopy path curve

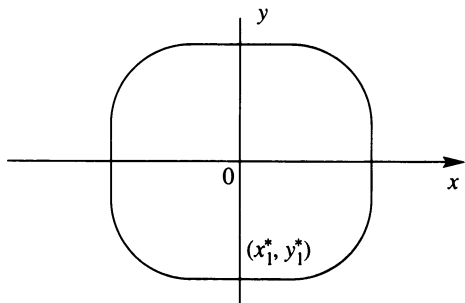


FIGURE 3

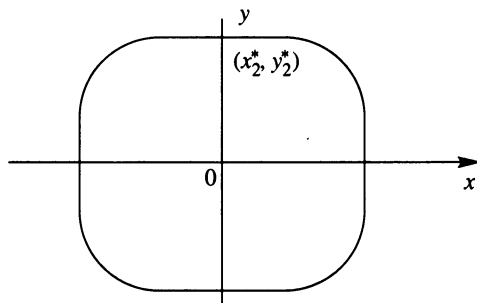


FIGURE 4

Example 1. Find at least two periodic solutions of $x'' + x^3 = (\sin t)/10$.

By (26) and (27), Example 1 can be transformed into

$$\begin{aligned} r' &= r \sin \theta \cos \theta - \sin \theta \left((r \cos \theta)^3 - \frac{\sin t}{10} \right), \\ \theta' &= -\sin^2 \theta - \frac{1}{r} \cos \theta \left((r \cos \theta)^3 - \frac{\sin t}{10} \right). \end{aligned}$$

By a simple computation, we obtain an initial value $(r_0, \theta_0) = (1.210101, -1.018535)$. Using homotopy (29), we can get two distinct solutions $(x_1^*, y_1^*) = (0, -0.955017)$ and $(x_2^*, y_2^*) = (0, 1.009132)$. In the following, Figure 2 is the homotopy path, which is generated by (29) and its initial point is (r_0, θ_0) ; Figures 3 and 4 are two periodic solutions of Example 1, and their initial points are (x_1^*, y_1^*) , (x_2^*, y_2^*) , respectively. Hence they are two distinct 2π -periodic solutions of the equation.

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