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L^p THEORY OF DIFFERENTIAL FORMS ON MANIFOLDS

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ABSTRACT. In this paper, we establish a Hodge-type decomposition for the L^p space of differential forms on closed (i.e., compact, oriented, smooth) Riemannian manifolds. Critical to the proof of this result is establishing an L^p estimate which contains, as a special case, the L^2 result referred to by Morrey as Gaffney's inequality. This inequality helps us show the equivalence of the usual definition of Sobolev space with a more geometric formulation which we provide in the case of differential forms on manifolds. We also prove the L^p boundedness of Green's operator which we use in developing the L^p theory of the Hodge decomposition. For the calculus of variations, we rigorously verify that the spaces of exact and coexact forms are closed in the L^p norm. For nonlinear analysis, we demonstrate the existence and uniqueness of a solution to the A-harmonic equation.

1. INTRODUCTION

This paper contributes primarily to the development of the L^p theory of differential forms on manifolds. The reader should be aware that for the duration of this paper, manifold will refer only to those which are Riemannian, compact, oriented, C^∞ smooth and without boundary. For $p = 2$, the L^p theory is well understood and the L^2-Hodge decomposition can be found in **[M]**. However, in the case $p \neq 2$, the L^p theory has yet to be fully developed. Recent applications of the L^p theory of differential forms on $\mathbb{R}^n$ to both quasiconformal mappings and nonlinear elasticity continue to motivate interest in this subject. Specifically, in the case of quasiconformal mappings, see **[IM]** and **[I]**, and in the case of nonlinear elasticity see **[RRT]** and **[IL]**. We expose many of the techniques used for $p = 2$, add critical new techniques for $p \neq 2$ and provide a general framework for developing the L^p theory of forms on manifolds. Also, we carry out this program for the restricted class of manifolds mentioned above as well as provide applications to both the calculus of variations and the study of A-harmonic equations.

Let $\bigwedge^l M$ denote the lth exterior power of the cotangent bundle. Also, let $C^\infty(\bigwedge^l M)$ denote the space of smooth l-forms on M (i.e., sections of $\bigwedge^l M$). The familiar Hodge decomposition for $C^\infty(\bigwedge^l M)$ says that $\omega = h + \Delta\beta$ where $dh = d^*h = 0$, d is exterior differentiation, d^* is coexterior differentiation and

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$\Delta \equiv dd^* + d^*d$ is the Laplace-Beltrami operator. Actually, the decomposition is even more descriptive (§6, [M] or [W]), but this will serve us here. We express this decomposition as

$$C^\infty\left(\bigwedge^l M\right) = \mathscr{H} \oplus \Delta C^\infty\left(\bigwedge^l M\right) \tag{1.1}$$

Let $L^p(\wedge^l M)$ denote the space of measurable l-forms on M satisfying $\int_M |\omega|^p < \infty$. Perhaps the first complication in replacing the left side of (1.1) with $L^p(\bigwedge^l M)$ is the fact that the meaning of d^* and d of an L^p form is unclear. This leads to the introduction of the Sobolev spaces $\mathscr{W}^{1,p}(\bigwedge^l M)$. There is a classical definition available (see [M]). Using this definition and Gaffney's inequality for L^2, it is possible to introduce a potential operator

$$\Omega : L^2\left(\bigwedge^l M\right) \to \mathscr{W}^{2,2}\left(\bigwedge^l M\right) \tag{1.2}$$

which yields the decomposition

$$L^2\left(\bigwedge^l M\right) = \mathscr{H} \oplus \Delta\Omega L^2\left(\bigwedge^l M\right) \tag{1.3}$$

In fact, the result is even better. Namely, we have the following identity which uniquely determines the potential.

$$\omega = h + \Delta\Omega(\omega) \tag{1.4}$$

for $\omega \in L^2(\bigwedge^l M)$.

In §5, we define an L^p analogue to Ω. In keeping with some other standard references (e.g., [W]), we refer to this operator as Green's operator and denote it by G. Of course, before G can be effectively exploited, its L^p theory must be developed. This leads us to a more geometric definition of Sobolev space (see §3). Namely,

$$\mathscr{W}^{1,p}\left(\bigwedge^l M\right) \equiv \left\{\omega \in \mathscr{W}\left(\bigwedge^l M\right) : \omega, d\omega, d^*\omega \in L^p\right\} \tag{1.5}$$

where $\mathscr{W}(\bigwedge^l M)$ is the space of l-forms which have generalized partials (again, see §3). In order to make use of this definition, we require that it be equivalent to the usual one. It turns out that showing that the usual Sobolev space is imbedded in ours presents little difficulty but the reverse is quite challenging. A key step is showing that for any smooth l-form with compact support in $\mathbb{R}^n$, we have the Gaffney type inequality

$$\|\nabla\omega\|_p^p \le C \int_{\mathbb{R}^n} (|d\omega|^p + |d^*\omega|^p) \qquad \text{(see §4)} \tag{1.6}$$

where $C = C(n, p)$ and $1 < p < \infty$. Using this Euclidean result, we establish a local version of (1.6) for an arbitrary manifold (see [M] for the case $p = 2$) which gives equivalence of (1.5) with the usual Sobolev space. We indulge ourselves a bit by commenting that both (1.6) and the rest of our techniques are valid for a much wider class of manifolds than those treated here. Unfortunately, manifolds which are noncompact or with boundary require a study of growth conditions for the metric tensor. Such concerns would distract us from the more concise presentation of techniques we intend to give. Consequently, those results will appear separately (see [Sc] for some further discussion).

Using Ω, we then give a definition for Green's operator and establish fundamental results about its L^p theory leading fairly quickly to the L^p-Hodge decomposition (see §6).

Finally, using Hodge's decomposition, we are able to rigorously establish the closedness of the spaces of exact and coexact forms in the L^p norm. Of course, such information is essential to the calculus of variations and in the case of differential forms on manifolds, it constitutes a nontrivial part of this calculus. Further, we exploit the L^p-Hodge decomposition in defining a nonlinear operator from the exact L^p forms to the exact L^q forms. Appealing again to this decomposition as well as to Browder's theory, we show that there exists a unique (modulo closed forms) solution to the A-harmonic equation.

2. Notation and preliminary results

Unfortunately, the notational complexities of the local expressions of the exterior and coexterior derivatives often obscure very elegant facts concerning these operators. We take some time here to expose, as cleanly as possible, one such fact which will be of essential importance (specifically in §4).

Fix $1 \le l \le n$. For all $I \equiv \{1 \le i_1 < \dots < i_l \le n\}$, $J \equiv \{1 \le j_1 < \dots < j_l \le n\}$ and all $1 \le i, j \le n$, there are polynomials a_{ij}^{IJ}, b_i^{IJ} and c^{IJ}, so that for any l-form ω, represented in any system, we have

$$\begin{aligned} |d^*\omega|^2 + |d\omega|^2 = &\sum_{i,j,I,J} a_{ij}^{IJ}(g)\frac{\partial \omega_I}{\partial x^i}\frac{\partial \omega_J}{\partial x^j} \\ &+ \sum_{i,I,J} b_i^{IJ}(g, \nabla g)\frac{\partial \omega_I}{\partial x^i}\omega_J \\ &+ \sum_{IJ} c^{IJ}(g, \nabla g)\omega_I\omega_J . \end{aligned} \tag{2.1}$$

Perhaps some explanation is required. The notation $a_{ij}^{IJ}(g)$ means that the polynomial a_{ij}^{IJ} has exactly enough variables to accommodate all the components of g and that a_{ij}^{IJ} is being evaluated pointwise at the components of $g(x)$. Similarly, b_i^{IJ} and c^{IJ} have exactly enough variables to accommodate all the components of g as well as all the partials of these components. For later use, when the metric tensor is fixed, we will usually write

$$|d^*\omega|^2 + |d\omega|^2 = \sum a_{ij}^{IJ}\frac{\partial \omega_I}{\partial x^i}\frac{\partial \omega_J}{\partial x^j} + \sum b_i^{IJ}\frac{\partial \omega_I}{\partial x^i}\omega_J + \sum c^{IJ}\omega_I\omega_J . \tag{2.2}$$

An easily overlooked fact is that these polynomials, a_{ij}^{IJ}, b_i^{IJ} and c^{IJ} have absolutely nothing to do with coordinate systems. They are being evaluated at points depending on the representation of the metric tensor and consequently the values of $a_{ij}^{IJ}(g)$, $b_i^{IJ}(g, \nabla g)$ and $c^{IJ}(g, \nabla g)$ at a given point of the manifold depend on the coordinate system. The explicit forms of these polynomials are not given here since they are quite complicated and play no role in forthcoming analysis. Another fact that will be useful is that when the metric tensor is locally represented with constant coefficients then $a_{ij}^{IJ}(g) = a_{ij}^{IJ}$ are constant over the domain of the system and $b_i^{IJ}(g, 0) = c^{IJ}(g, 0) = 0$. This simplifies (2.2) giving

$$|d^*\omega|^2 + |d\omega|^2 = \sum a_{ij}^{IJ} \frac{\partial \omega_I}{\partial x^i} \frac{\partial \omega_J}{\partial x^j}. \tag{2.3}$$

In particular, when $g_{ij} = \delta_{ij}$, the constants are exactly those occurring for Euclidean space.

For the formulation of Sobolev space in §3, it will be useful to define

$$[\omega]_p \equiv |\omega|^p + |d\omega|^p + |d^*\omega|^p \tag{2.4}$$

for differential forms ω and $1 \leq p < \infty$. Indeed, we make immediate use of this notation by citing the following pointwise inequality:

$$[f\omega]_p \leq C[f]_p[\omega]_p \tag{2.5}$$

where f is a function, ω is a differential form, $C = C(p)$ and $1 \leq p \leq \infty$. Another use of (2.4) is given by observing that locally, say within open U compactly contained in a regular coordinate system, we have

$$[\omega]_p \leq C(U, p)|\nabla\omega|^p \tag{2.6}$$

where $|\nabla\omega|^p \equiv (\sum |\frac{\partial \omega_I}{\partial x^k}|^2)^{\frac{p}{2}}$, $\omega = \sum \omega_I dx^I$, $\{x^1, \dots, x^n\}$ are local coordinates and regular coordinate system is defined in §3. Recall that, classically, the $(1, p)$-Sobolev space of l-forms is given (see **[M]**) as

$$\mathscr{W}^{1,p}\left(\bigwedge^l M\right) \equiv \{\omega \text{ with generalized gradient } : \omega \text{ and } |\nabla\omega| \in L^p\} \tag{2.7}$$

and so (2.6) expresses that d and d^* are continuous linear operators of $\mathscr{W}^{1,p}$ to L^p. As we shall see, (2.6) also gives that the classical Sobolev space (i.e., (2.7)) is imbedded in the one given by a more geometric definition which we provide in §3.

The familiar integration by parts formula,

$$(du, v) \equiv \int_M \langle du, v\rangle = \int_M \langle u, d^*v\rangle \equiv (u, d^*v) \tag{2.8}$$

for C^∞-smooth forms expresses a duality relationship between d and d^* that is of critical importance. Once we show equivalence of (2.7) with (1.5), the Meyer and Serrin result asserting density of the smooth forms in classical Sobolev space will allow us to argue that (2.8) holds for $u \in \mathscr{W}^{1,p}(\bigwedge^l M)$ and $v \in \mathscr{W}^{1,q}(\bigwedge^l M)$ with p and q are Hölder conjugate, $1 < p, q < \infty$.

Finally, we mention one fact from abstract measure theory which will be useful in §4. Suppose (X, μ) is a measure space and $\mathscr{A} \equiv \{A_i : i \in \mathbb{N}\}$ is a cover of X by measurable sets. Denoting the multiplicity of $\mathscr{A}$ by

$$K \equiv \sup\{\sum_{A \in \mathscr{A}} \chi_A(x) : x \in X\}$$

we have

$$\sum_{i=1}^{\infty} \int_{A_i} f d\mu \leq K \int_X f d\mu. \tag{2.9}$$

3. A formulation of Sobolev space

We take a moment to introduce the so-called **classical** or **usual Sobolev spaces.** Given an l-form which is locally integrable ($\omega \in L^1_{\text{loc}}(\bigwedge^l M)$), we say that it has a **generalized gradient** in case, for each coordinate system, the pullbacks of the coordinate functions of ω have generalized gradient in the familiar sense (see [S]). We set

$$\mathscr{W}\left(\bigwedge^l M\right) \equiv \left\{\omega \in L^1_{\text{loc}}\left(\bigwedge^l M\right) : \omega \text{ has generalized gradient}\right\}$$

Of course, for the manifolds we consider here, $L^1_{\text{loc}} = L^1$. If we now choose an atlas for M, say $\mathscr{A}$, then for $(U, \phi \equiv (x^1, \dots, x^n)) \in \mathscr{A}$, we can define the **local gradient modulus** by

$$|\nabla_U \omega(x)|^2 \equiv \sum_{I,k} |\frac{\partial \omega_I}{\partial x^k}(x)|^2$$

and the **global gradient modulus** by

$$|\nabla \omega(x)|^2 \equiv \sum_{U \in \mathscr{A}} |\nabla_U \omega(x)|^2 . \tag{3.1}$$

Notice that this definition of modulus depends on the atlas chosen and that the gradient itself was not defined. If we choose $\mathscr{A}$ so that it is a locally finite cover of M, then we may define (classical) Sobolev space as

$$\mathscr{W}^{1,p}_{\mathscr{A}}\left(\bigwedge^l M\right) \equiv \{\omega \in \left(\bigwedge^l M\right) : \omega,\ |\nabla\omega| \in L^p\}$$

with norm $\|\omega\|_p + \|\nabla\omega\|_p$. Similarly, $\mathscr{W}^{k,p}_{\mathscr{A}}$ is constructed for $k = 2, 3, \dots$.

Simple examples demonstrate that it is possible to choose, in perfectly reasonable ways, two atlases which yield Sobolev spaces that are not equivalent as normed linear spaces. It is important then to specify some class of atlases, call them **regular**, all of which yield equivalent Sobolev spaces. When referring to a coordinate system (U, ϕ) as **regular**, we shall mean that there is another system (V, ψ) with $\overline{U}$ compact, $\overline{U} \subset V$ and $\psi|_U = \phi$. A **regular atlas** is simply a locally finite cover by such systems. From here on, classical Sobolev space refers to one constructed as above using a regular atlas. This is all fairly familiar and once again, [M] is a fine reference. Further, it is also well known that many of the results concerning Sobolev space in $\mathbb{R}^n$ are transferred and that perhaps cheif among these is the Sobolev Imbedding Theorem.

Unfortunately, this definition is unsatisfying from a geometric perspective. We would like to define these spaces without reference to coordinate systems. We propose the following definition.

Definition 3.2. The $(1, p)$*-Sobolev space of differential forms* on M is given by

$$\mathscr{W}^{1,p}\left(\bigwedge^l M\right) \equiv \left\{\omega \in \mathscr{W}\left(\bigwedge^l M\right) \cap L^p\left(\bigwedge^l M\right) : d\omega \in L^p\left(\bigwedge^{l+1} M\right) \right.$$
$$\left. \text{and } d^*\omega \in L^p\left(\bigwedge^{l-1} M\right)\right\}$$

with norm

$$\|\omega\|^p_{1,p} \equiv \int_M [\omega]_p .$$

The fact that classical Sobolev space is imbedded in the one given by this new formulation is a nearly immediate consequence of the pointwise estimate (2.6). However, the reverse is far from clear. We shall find a regular atlas yielding classical Sobolev space for which the imbedding is reversible. Since the manifolds of concern here are compact, this issue will be essentially a local one. The next section is dedicated to establishing the local bounds.

4. Gaffney's inequality

Given a point $y \in M$ and $p = 2$, we may choose a regular coordinate system, $(U, \phi \equiv (x^1, \ldots, x^n))$, containing y and a constant, say $C > 0$, so that

$$\int_U |\nabla_U \omega|^p \equiv \int_U (\sum_{I,i} |\frac{\partial \omega_I}{\partial x^i}|^2)^{\frac{p}{2}} \leq C \int_U (|\omega|^p + |d\omega|^p + |d^*\omega|^p) \tag{4.1}$$

for any $\omega \in C_0^\infty(\bigwedge^l U)$. See for example [**M**, p. 292]. This result is a cornerstone in the variational method leading to the decomposition theorem and it is our goal in this section to establish this for $p \neq 2$. We point out here that (4.1) is quite simple for $p = 2$ while for $p \neq 2$, it is far from clear even for $M \equiv \mathbb{R}^n$.

In the proof of upcoming Proposition 4.3, we will make use of two identities for the Riesz transforms in $\mathbb{R}^n$. For appropriate definitions, L^p theory and other basic results, see [**S**].

Lemma 4.2. *Let R_i denote the ith Riesz transform. Then*

(1) $\int_{\mathbb{R}^n} f(x)(R_i g)(x)dx = \int_{\mathbb{R}^n} g(x)(R_i f)(x)dx$

$(f \in L^p,\ g \in L^q,\ p + q = pq$ *and* $1 < p, q < \infty)$.

(2) $R_j(\sum_{i=1}^n R_i \frac{\partial f}{\partial x^i}) = -\frac{\partial f}{\partial x^j}$ *for* $f \in \mathscr{W}^{1,p}(\mathbb{R}^n)$.

Proposition 4.3 (Gaffney type inequality for L^p; Euclidean case).

$$\|\nabla \omega\|_p^p \leq C D_p(\omega) \qquad \textit{for } \omega \in C_0^\infty\left(\bigwedge^l \mathbb{R}^n\right)$$

where $D_p(\omega) \equiv \int_{\mathbb{R}^n} |d\omega|^p + |d^*\omega|^p$ *and* $C = C(n, p)$.

Proof. For $\omega = \sum \omega_I dx^I$, set

$$F \equiv |\nabla \omega|^{p-2} \nabla \omega \text{ and } \phi \equiv \sum_{j=1}^n R_j F^j.$$

F^j in this case is the differential form $|\nabla \omega|^{p-2} \partial \omega / \partial x^j$, where $\partial \omega / \partial x^j$ is the l-form with Ith $(I = \{1 \leq i_1 < \ldots < i_l \leq n\})$ component $\partial \omega_I / \partial x^j$. Then $R_j F^j$ means the differential form with Ith component $R_j(|\nabla \omega|^{p-2} \partial \omega_I / \partial x^j)$. Let u satisfy

$$\nabla u = R\phi \equiv (R_1 \phi, \ldots, R_n \phi). \tag{4.4}$$

See for example [**S**, §V] to confirm that such a solution u exists. Notice that this means

$$\frac{\partial u_I}{\partial x^k} = R_k \phi_I = R_k(\sum_{j=1}^n R_j F^j)_I = R_k(\sum_{j=1}^n R_j(|\nabla \omega|^{p-2} \frac{\partial \omega_I}{\partial x^j})).$$

Recalling the notation from (2.8) and using basic relationships between d, d^*, Δ, ∇ and div, we observe

$$\begin{aligned}
(d\omega, du) + (d^*\omega, d^*u) &= \int_{\mathbb{R}^n} \langle (dd^* + d^*d)\omega, u\rangle = \int_{\mathbb{R}^n} \langle \Delta\omega, u\rangle \\
&= \int_{\mathbb{R}^n} \langle \sum (\Delta\omega_I) dx^I, u\rangle = \int_{\mathbb{R}^n} \langle \sum_I \operatorname{div}\nabla\omega_I dx^I, u\rangle \\
&= \int_{\mathbb{R}^n} \langle \operatorname{div}(\nabla\omega), u\rangle = -\int_{\mathbb{R}^n} \langle \nabla\omega, \nabla u\rangle \\
&= -\int_{\mathbb{R}^n} \sum_{k=1}^{n} \langle (\nabla\omega)^k, R_k\phi\rangle \qquad \text{(by 4.4)} \\
&= -\int_{\mathbb{R}^n} \sum_{k=1}^{n} \langle (\nabla\omega)^k, \sum_{j=1}^{n} R_k R_j F^j\rangle \\
&= -\int_{\mathbb{R}^n} \sum_{k=1}^{n} \sum_{j=1}^{n} \langle (\nabla\omega)^k, R_k R_j F^j\rangle
\end{aligned}$$

Now applying Lemma 4.2(1) to this last expression gives

$$= -\int_{\mathbb{R}^n} \sum_{k=1}^{n} \sum_{j=1}^{n} \langle R_k R_j (\nabla\omega)^k, F^j\rangle = -\int_{\mathbb{R}^n} \sum_{j=1}^{n} \langle \sum_{k=1}^{n} R_k R_j (\nabla\omega)^k, F^j\rangle$$

Next, applying Lemma 4.2(2) gives

$$\begin{aligned}
&= \int_{\mathbb{R}^n} \sum_{j=1}^{n} \langle (\nabla\omega)^j, F^j\rangle = \int_{\mathbb{R}^n} \langle \nabla\omega, F\rangle \\
&= \int_{\mathbb{R}^n} \langle \nabla\omega, |\nabla\omega|^{p-2}\nabla\omega\rangle \\
&= \int_{\mathbb{R}^n} |\nabla\omega|^p \\
&= \|\nabla\omega\|_p^p .
\end{aligned}$$

So we have

$$\begin{aligned}
\|\nabla\omega\|_p^p &= \left| \int_{\mathbb{R}^n} \langle d\omega, du\rangle + \langle d^*\omega, d^*u\rangle \right| \\
&\le \int_{\mathbb{R}^n} |\langle d\omega, du\rangle| + |\langle d^*\omega, d^*u\rangle| \\
&\le \|d\omega\|_p \|du\|_q + \|d^*\omega\|_p \|d^*u\|_q \\
&\le (\|d^*\omega\|_p^p + \|d\omega\|_p^p)^{\frac{1}{p}} (\|d^*u\|_q^q + \|du\|_q^q)^{\frac{1}{q}} \\
&= D_p(\omega)^{\frac{1}{p}} D_q(u)^{\frac{1}{q}} .
\end{aligned}$$

This last inequality follows from the numerical fact that for $a_1, a_2, b_1, b_2 \ge 0$, we have $a_1 b_1 + a_2 b_2 \le (a_1^p + a_2^p)^{\frac{1}{p}} (b_1^q + b_2^q)^{\frac{1}{q}}$ when p, q are Hölder conjugate.

We now make use of (2.6) to continue with

$$
\begin{aligned}
\|\nabla\omega\|_p^p &\le C(D_p\omega)^{\frac{1}{p}}\|\nabla u\|_q \\
&\le C(D_p\omega)^{\frac{1}{p}}\|R\|_q\|\phi\|_q \qquad \text{(boundedness of } R) \\
&\le C(D_p\omega)^{\frac{1}{p}}\|F\|_q \\
&= C(D_p\omega)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^n}(|\nabla\omega|^{p-2}|\nabla\omega|)^q\Big)^{\frac{1}{q}} \\
&= C(D_p\omega)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^n}|\nabla\omega|^{q(p-1)}\Big)^{\frac{1}{q}} \\
&= C(D_p\omega)^{\frac{1}{p}}\|\nabla\omega\|_p^{p-1}
\end{aligned}
$$

For notational simplicity, we are employing the convention of generically denoting constants by C, despite the fact that the constant may be larger from one inequality to the next. Dividing both sides by $\|\nabla\omega\|_p^{p-1}$ gives

$$\|\nabla\omega\|_p \le C(n, p)(D_p\omega)^{\frac{1}{p}}$$

as desired. $\square$

Remark. Notice that Proposition 4.3 can be immediately strengthened to $\mathscr{W}_0^{1,p}(\bigwedge^l\mathbb{R}^n)$. Indeed, choosing $\omega_n \in C_0^\infty(\bigwedge^l\mathbb{R}^n)$ so that $\omega_n \to \omega$ in $\mathscr{W}^{1,p}$ and observing that $D_p(\omega_n) \to D_p(\omega)$ gives the desired strengthening.

Proposition 4.5 (Gaffney type inequality). *Given $y \in M$, there is a regular system, $(U, \phi \equiv (x^1, \dots, x^n))$, containing y and a constant, $C = C_U(n, p)$ (depending on U, dimension and p), satisfying*

$$\int_U |\nabla_U\omega|^p \equiv \int_U \Big(\sum_{I,i}\Big|\frac{\partial\omega_I}{\partial x^i}\Big|^2\Big)^{\frac{p}{2}} \le C\|\omega\|_{1,p}^p$$

for any $\omega \in \mathscr{W}^{1,p}(\bigwedge^l M)$ with $\operatorname{spt}(\omega) \subset U$.

Proof. Initially choose a system $(V, \phi \equiv (x^1, \dots, x^n))$ so that $g_{ij}(y) = \delta_{ij}$ and $\phi(V) = B$ where B is a Euclidean ball centered at $\phi(y) = 0$ and g_{ij} are the components of the metric tensor w.r.t. ϕ. We observe that given $\epsilon > 0$ there is $C(\epsilon)$ and $0 < \rho = \rho(\epsilon) \le 1$ giving

$$
\begin{aligned}
\int_U \Big|\sum(a_{ij}^{IJ} - a_{ij}^{IJ}(\delta))\frac{\partial\omega_I}{\partial x^i}\frac{\partial\omega_J}{\partial x^j} + \sum b_i^{IJ}\frac{\partial\omega_I}{\partial x^i}\omega_J + \sum c^{IJ}\omega_I\omega_J\Big|^{\frac{p}{2}} \\
\le \epsilon\int_U |\nabla_U\omega|^p + C(\epsilon)\|\omega\|_p^p
\end{aligned}
\tag{4.6}
$$

where $U \equiv \phi^{-1}(\rho B)$, $\omega \in \mathscr{W}^{1,p}(\bigwedge^l M)$ with $\operatorname{spt}(\omega) \subset U$, the special polynomials a_{ij}^{IJ}, b_i^{IJ} and c^{IJ} were given in §2 and δ denotes the tensor with constant components δ_{ij}. To see this estimate, consider

$$\begin{aligned}
&\int_U \left|\sum (a_{ij}^{IJ} - a_{ij}^{IJ}(\delta))\frac{\partial \omega_I}{\partial x^i}\frac{\partial \omega_J}{\partial x^j} + \sum b_i^{IJ}\frac{\partial \omega_I}{\partial x^i}\omega_J + \sum c^{IJ}\omega_I\omega_J\right|^{\frac{p}{2}} \\
&\le \int_U \left(\sum |a_{ij}^{IJ} - a_{ij}^{IJ}(\delta)|\left|\frac{\partial \omega_I}{\partial x^i}\frac{\partial \omega_J}{\partial x^j}\right| + \sum \left|b_i^{IJ}\frac{\partial \omega_I}{\partial x^i}\omega_J\right| + \sum |c^{IJ}\omega_I\omega_J|\right)^{\frac{p}{2}} \\
&\le \int_U \Big[\sum \|a_{ij}^{IJ} - a_{ij}^{IJ}(\delta)\|_{U,\infty}\left(\left|\eta\frac{\partial \omega_I}{\partial x^i}\right|^2 + \left|\eta^{-1}\frac{\partial \omega_J}{\partial x^j}\right|^2\right) \\
&\quad + \sum \|b_i^{IJ}\|_{U,\infty}\left(\left|\frac{\partial \omega_I}{\partial x^i}\right|^2 + |\omega_J|^2\right) \\
&\quad + \sum \|c^{IJ}\|_{U,\infty}(|\eta\omega_I|^2 + |\eta^{-1}\omega_J|^2)\Big]^{\frac{p}{2}} \\
&\le K\Big[\int_U |\nabla_U\omega|^p \sum |a_{ij}^{IJ} - a_{ij}^{IJ}(\delta)| + \eta^{\frac{p}{2}}\int_U |\nabla_U\omega|^p + (\eta^{\frac{p}{2}} + \eta^{\frac{-p}{2}})\int_U |\omega|^p\Big].
\end{aligned}$$

Recalling that $U \equiv \phi^{-1}(\rho B)$, we see that the continuity of $a_{ij}^{IJ}(g)$ and continuity of g along with $g(y) = \delta$ provide for the existence of a small $\rho > 0$ for which

$$\|a_{ij}^{IJ} - a_{ij}^{IJ}(\delta)\|_{U,\infty} \equiv \|a_{ij}^{IJ}(g) - a_{ij}^{IJ}(\delta)\|_{U,\infty} < \frac{\epsilon}{2K}.$$

Also, we may choose a small $\eta > 0$ so that $\eta^p < \frac{\epsilon}{2K}$. Denoting $C(\epsilon) \equiv K(\eta^p - \eta^{-p})$, we have (4.6).

Now that U is in hand, observe that there is a constant $C > 0$ so that

$$\int_U \left|\sum a_{ij}^{IJ}(\delta)\frac{\partial \omega_I}{\partial x^i}\frac{\partial \omega_J}{\partial x^j}\right|^{\frac{p}{2}} \ge C\int_U |\nabla_U\omega|^p. \tag{4.7}$$

To see this estimate, recall from (2.3) that for Euclidean forms we have

$$\sum a_{ij}^{IJ}(\delta)(\partial_i\omega_I)(\partial_j\omega_J) = |d\omega|^2 + |d^*\omega|^2$$

where ∂_i denotes the Euclidean partial with respect to the ith standard basis vector. This gives

$$\int_{\rho B} \left|\sum a_{ij}^{IJ}(\delta)(\partial_i\omega_I)(\partial_j\omega_J)\right|^{\frac{p}{2}} = \int_{\rho B}(|d\omega|^2 + |d^*\omega|^2)^{\frac{p}{2}}. \tag{4.8}$$

Now observe that

$$\begin{aligned}
\int_U \left|\sum a_{ij}^{IJ}(\delta)\frac{\partial \omega_I}{\partial x^i}\frac{\partial \omega_J}{\partial x^j}\right|^{\frac{p}{2}} &\ge C_1\int_{\rho B}\left|\sum a_{ij}^{IJ}(\delta)\left(\frac{\partial \omega_I}{\partial x^i}\right)_\phi\left(\frac{\partial \omega_J}{\partial x^j}\right)_\phi\right|^{\frac{p}{2}} \\
&= C_1\int_{\rho B}\left|\sum a_{ij}^{IJ}(\delta)(\partial_i\omega_\phi)_I(\partial_j\omega_\phi)_J\right|^{\frac{p}{2}} \\
&= C_1\int_{\rho B}(|d\omega_\phi|^2 + |d^*\omega_\phi|^2)^{\frac{p}{2}} \qquad \text{(by 4.8)} \\
&\ge C_2\int_{\rho B}|\nabla\omega_\phi|^p \qquad \text{(by Proposition 4.3)} \\
&\ge C\int_U |\nabla_U\omega|^p.
\end{aligned}$$

Notice that when C_1 and C were introduced, there was a dependence on the metric tensor hence on the location of y and when C_2 was introduced, Proposition 4.3 was used which gives a dependence on dimension and p. Also observe that we are using the notation ω_ϕ to indicate the 'pullback' of the form ω to Euclidean space via the chart ϕ (i.e., ω_ϕ is the Euclidean form with components $(\omega_\phi)_I \equiv (\omega_I)_\phi \equiv \omega_I \circ \phi^{-1}$).

Before giving the final analysis, consider the following L^p estimate.

$$\begin{aligned}
\int_U |\sum a_{ij}^{IJ}(\delta)\frac{\partial \omega_I}{\partial x^i}\frac{\partial \omega_J}{\partial x^j}|^{\frac{p}{2}} &\le C\int_{\rho B}|\sum a_{ij}^{IJ}(\delta)(\frac{\partial \omega_I}{\partial x^i})_\phi(\frac{\partial \omega_J}{\partial x^j})_\phi|^{\frac{p}{2}}\\
&\le C\sum |a_{ij}^{IJ}(\delta)|\int_{\rho B}|(\partial_i\omega_\phi)_I(\partial_j\omega_\phi)_J|^{\frac{p}{2}}\\
&\le C\sum(\int_{\rho B}|(\partial_i\omega_\phi)_I|^p)^{\frac{1}{2}}(\int_{\rho B}|(\partial_j\omega_\phi)_J|^p)^{\frac{1}{2}}\\
&\le CD_p(\omega_\phi) \qquad \text{(by Remark after 4.3)}\\
&\le C\|\omega\|_{1,p} < \infty .
\end{aligned}$$

This means $|\sum a_{ij}^{IJ}(\delta)\frac{\partial\omega_I}{\partial x^i}\frac{\partial\omega_J}{\partial x^j}|^{\frac{1}{2}}$ is an L^p function. Thus we may add and subtract its L^p norm without fear. We are now in position to make the final string of estimates. First though, keep in mind the following numerical fact.

$$|a+b|^r \ge -|b|^r + 2^{1-r}|a|^r \tag{4.9}$$

for arbitrary real numbers a and b but with $r > 0$. For completeness, notice that (4.9) follows by $|a|^r = |a+b-b|^r \le (|a+b|+|b|)^r \le 2^{r-1}(|a+b|^r + |b|^r)$ for $r \ge 1$ and for $0 < r \le 1$ we can replace 2^{r-1} with 1. Finally, let us estimate.

$$\begin{aligned}
\|\omega\|_{1,p}^p - \|\omega\|_p^p &= \int_U |d\omega|^p + |d^*\omega|^p \ge C_1\int_U(|d\omega|^2 + |d^*\omega|^2)^{\frac{p}{2}}\\
&\qquad \text{(where } C_1 = 2^{1-\frac{p}{2}} \text{ for } p \ge 2 \text{ and } C_1 = 1 \text{ for } 1 \le p \le 2)\\
&= C_1\int_U|\sum a_{ij}^{IJ}\frac{\partial\omega_I}{\partial x^i}\frac{\partial\omega_J}{\partial x^j} + \sum b_i^{IJ}\frac{\partial\omega_I}{\partial x^i}\omega_J + \sum c^{IJ}\omega_I\omega_J|^{\frac{p}{2}} \quad (2.2)\\
&= C_1\int_U|\sum(a_{ij}^{IJ} - a_{ij}^{IJ}(\delta))\frac{\partial\omega_I}{\partial x^i}\frac{\partial\omega_J}{\partial x^j} + \sum b_i^{IJ}\frac{\partial\omega_I}{\partial x^i}\omega_J\\
&\qquad + \sum c^{IJ}\omega_I\omega_J + \sum a_{ij}^{IJ}(\delta)\frac{\partial\omega_I}{\partial x^i}\frac{\partial\omega_J}{\partial x^j}|^{\frac{p}{2}}\\
&\ge C_2\int_U|\sum a_{ij}^{IJ}(\delta)\frac{\partial\omega_I}{\partial x^i}\frac{\partial\omega_J}{\partial x^j}|^{\frac{p}{2}}\\
&\quad - C_1\int_U|\sum(a_{ij}^{IJ} - a_{ij}^{IJ}(\delta))\frac{\partial\omega_I}{\partial x^i}\frac{\partial\omega_J}{\partial x^j}\\
&\qquad + \sum b_i^{IJ}\frac{\partial\omega_I}{\partial x^i}\omega_J + \sum c^{IJ}\omega_I\omega_J|^{\frac{p}{2}} \quad \text{(by 4.9)}\\
&\ge C_3\int_U|\nabla_U\omega|^p - \frac{C_3}{2}\int_U|\nabla_U\omega|^p - C_4\|\omega\|_p^p \qquad \text{(by 4.6 and 4.7)}
\end{aligned}$$

This gives

$$\|\omega\|_{1,p}^p + (C_4 - 1)\|\omega\|_p^p \ge \frac{C_3}{2}\int_U|\nabla_U\omega|^p .$$

By factoring a large enough number from the left-hand side, the result follows. □

Proposition 4.10. *There is a regular atlas for M, say* $\mathscr{A} \equiv \{(U_i, \phi_i)\}_{i=1}^N$, *satisfying*

$$\sum_{i=1}^N \int_{U_i} |\nabla_i \omega|^p \le C\|\omega\|_{1,p}^p$$

for any $\omega \in \mathscr{W}^{1,p}(\bigwedge^l M)$ *and some* $C = C(\mathscr{A}, n, p)$.

Proof. Using the compactness of M and Proposition 4.5, we may select finitely many systems $\{(V_i, \phi_i)\}_{i=1}^N$ and $C \equiv C(\mathscr{A}, n, p) > 0$ so that

$$\int_{V_i} |\nabla_i \omega|^p \le C\|\omega\|_{1,p}^p \qquad (\nabla_i \text{ w.r.t. } \phi_i) \tag{4.11}$$

for $i = 1, 2, \ldots, N$ and any $\omega \in \mathscr{W}^{1,p}(\bigwedge^l M)$ with $\operatorname{spt}(\omega) \subset V_i$. Also, while choosing the V_i, we can choose open $U_i \subset V_i$ and partition of unity $\{\zeta_i\}_{i=1}^N$ satisfying: $\bigcup U_i = M$, $\operatorname{spt}(\zeta_i) \subset V_i$ and $\zeta_i(x) = 1$ when $x \in U_i$. Now,

$$\begin{aligned}\int_{U_i} |\nabla_i \omega|^p = \int_{U_i} |\nabla_i(\zeta_i\omega)|^p &\le \int_{V_i} |\nabla_i(\zeta_i\omega)|^p \\ &\le C\int_{V_i} [\zeta_i\omega]_p \qquad \text{(by (4.11))} \\ &\le C\int_{V_i} [\omega]_p \qquad \text{(by (2.2))}.\end{aligned}$$

Thus

$$\sum_{i=1}^N \int_{U_i} |\nabla_i\omega|^p \le C\sum_{i=1}^N \int_{V_i} [\omega]_p \le C\int_M [\omega]_p \qquad \text{(by (2.9))}. \quad \square$$

An immediate consequence of Proposition 4.10 and the comments following Definition 3.2 is

Corollary 4.12. *Regular atlases yield classical Sobolev space* (§3) *equivalent to the geometric one* (*Definition* 3.2).

5. Harmonic fields and Green's operator

Definition 5.1. We define and denote the *harmonic l-fields* by

$$\mathscr{H}\left(\bigwedge^l M\right) \equiv \left\{\omega \in \mathscr{W}\left(\bigwedge^l M\right) : d\omega = d^*\omega = 0, \right.$$
$$\left. \omega \in L^p \text{ for some } 1 < p < \infty \right\}.$$

Proposition 5.2. $\mathscr{H}(\bigwedge^l M) \subset C^\infty(\bigwedge^l M)$.

Proof. Let $\omega \in \mathscr{H}(\bigwedge^l M)$. Thus there is $1 < p < \infty$ so that $\omega \in L^p$. Also $d^*\omega = d\omega = 0$ gives $\omega \in \mathscr{W}^{1,p}$. If $p > n$ then $\omega \in C(\bigwedge^l M)$ and hence $\omega \in L^2$. If $p \le n$ then choose $r < p$ so that for some positive integer, say k, we have $\frac{nr}{n-rk} > n$. Now $\omega \in L^r$ and $d^*\omega = d\omega = 0$ imply $\omega \in \mathscr{W}^{1,r}$. Since Corollary 4.12 gives that $\|\cdot\|_{1,p}$ is equivalent to the classical Sobolev norm, we may apply the Sobolev imbedding theorem to get $\omega \in L^{1,\frac{nr}{n-r}}$. Of course we still have $d^*\omega = d\omega = 0$ so that $\omega \in \mathscr{W}^{1,\frac{nr}{n-r}}$. We repeat this process k times to get $\omega \in \mathscr{W}^{1,\frac{nr}{n-rk}}$ so that ω has a continuous representative and hence $\omega \in L^2$. In [**M**, Chapter 7], it is shown that the L^2 harmonic fields are C^∞ and we have just observed that $\mathscr{H} \subset L^2$ $\square$.

This regularity result reveals that even though we have expanded the space of forms from C^∞ to L^p, we haven't introduced any new harmonic fields. Consequently, it is classically known that $\mathscr{H}(\bigwedge^l M)$ constitutes a finite dimensional real vector space.

In analogy with the classical definition of the Dirichlet integral, we define the **L^p-Dirichlet integral** by

$$D_p(\omega) \equiv \int_M |d\omega|^p + |d^*\omega|^p \tag{5.3}$$

and use the 'perp' notation to denote the orthogonal complement of $\mathscr{H}$ in L^1, as

$$\mathscr{H}^\perp \equiv \{\omega \in L^1 : (\omega, h) = 0 \text{ for all } h \in \mathscr{H}\} \tag{5.4}$$

Employing only minor modifications of the reasoning in [**M**, Chapter 7], we see that

$$0 < \eta \equiv \inf\{D_p(\omega) : \|\omega\|_p = 1, \omega \in \mathscr{H}^\perp \cap \mathscr{W}^{1,p}\}$$

as well as

$$D_p(\omega) \le \|\omega\|_{1,p}^p \le \frac{1+\eta}{\eta} D_p(\omega) \tag{5.5}$$

for $\omega \in H^\perp \cap \mathscr{W}^{1,p}$. This means that $D_p^{\frac{1}{p}}$ is a norm equivalent to $\|\cdot\|_{1,p}$ on $H^\perp \cap \mathscr{W}^{1,p}$. But of course the L^p norm is equivalent to $\|\cdot\|_{1,p}$ on $\mathscr{H} \subset \mathscr{W}^{1,p}$ and $(H^\perp \cap \mathscr{W}^{1,p}) \oplus \mathscr{H} = \mathscr{W}^{1,p}$. These facts motivate the following observations and definition.

Lemma 5.6. *For $\omega \in L^1(\bigwedge^l M)$ there is unique $H(\omega) \in \mathscr{H}$ such that*

$$(\omega - H(\omega), h) = 0 \qquad \text{for all } h \in \mathscr{H}. \tag{5.7}$$

Proof. Let $\{e^1, \dots, e^N\}$ be an orthonormal basis for $\mathscr{H}$ (in L^2 of course). Set

$$H(\omega) \equiv \sum (\omega, e^k) e^k$$

and write $h = \sum h_k e^k$, then

$$(\omega - H(\omega), h) = (\omega, h) - (H(\omega), h) = \sum(\omega, e^k)h_k - \sum(\omega, e^k)h_k = 0.$$

Now if $h' = \sum h'_k e^k$ also satisfies

$$(\omega - h', h) = 0 \qquad \text{for all } h \in \mathscr{H}$$

then

$$\begin{aligned} 0 &= (\omega - H(\omega), h' - H(\omega)) - (\omega - h', h' - H(\omega)) \\ &= (h' - H(\omega), h' - H(\omega)) = \|h' - H(\omega)\|_2^2 = 0. \end{aligned}$$

Thus, $h' = H(\omega)$. $\square$

Definition 5.8. Given $\omega \in L^1$, we set $H(\omega)$ to be the unique element of $\mathscr{H}$ guaranteed by Lemma 5.6 and refer to $H(\omega)$ as either the harmonic projection or sometimes the harmonic part of ω.

Proposition 5.9 (Harmonic Projection).

(1) *$H : L^p \to \mathscr{H}$ is a bounded linear projection (regardless of the norm on $\mathscr{H}$).*

(2) *$L^p = (L^p \cap \mathscr{H}^\perp) \oplus \mathscr{H}$ for $1 \le p \le \infty$.*

(3) *For $1 < p < \infty$ we have that $\|H(\omega)\|_p + D_p^{\overset{\perp}{p}}(\omega)$ is a norm equivalent to $\|\omega\|_{1,p}$ on $\mathscr{W}^{1,p} = (\mathscr{W}^{1,p} \cap \mathscr{H}^\perp) \oplus \mathscr{H}$.*

Proof. As discussed after Proposition 5.2, $\mathscr{H}$ is finite dimensional as a real vector space. This gives part (1). Part (2) follows since $\omega = (\omega - H(\omega)) + H(\omega)$ and if $\tau \in \mathscr{H}^\perp \cap \mathscr{H}$ then $0 = (\tau, \tau) = \|\tau\|_2^2$ implies $\tau = 0$. For part (3), we noted in (5.5) that $D_p^{\overset{\perp}{p}}$ and $\|\cdot\|_{1,p}$ are equivalent norms on $\mathscr{H}^\perp \cap \mathscr{W}^{1,p}$. The definition of $\|\cdot\|_{1,p}$ and $\mathscr{H}$ give that $\|\cdot\|_{1,p} = \|\cdot\|_p$ on $\mathscr{H}$. Part (3) is then a consequence of the inverse function theorem. □

In [**W**, Chapter 6], **Green's operator** is defined as

$$G : C^\infty\left(\bigwedge^l M\right) \to \mathscr{H}^\perp \cap C^\infty\left(\bigwedge^l M\right)$$

by letting $G(\omega)$ be the unique element of $\mathscr{H}^\perp \cap C^\infty(\wedge^l M)$ satisfying Poisson's equation

$$\Delta G(\omega) = \omega - H(\omega). \tag{5.10}$$

That there is such a unique operator is part of the Hodge theory and can be found in any standard reference such as [**W**]. We would like to define Green's operator more generally for L^p. This work will be broken into the cases of $2 \le p < \infty$ and $1 < p < 2$ and greatly facilitated by the following information which can be found in [**M**, Chapter 7]. We are given an operator $\Omega : L^2 \cap \mathscr{H}^\perp \to \mathscr{W}^{1,2} \cap \mathscr{H}^\perp$ which is rather extensively developed there. Further, for $p \ge 2$, Morrey gives

$$d\Omega(\omega),\ d^*\Omega(\omega) \text{ and } \Omega(\omega) \in \mathscr{W}^{1,p} \cap \mathscr{H}^\perp \tag{5.11}$$

and the estimate

(5.12)
$$\|dd^*\Omega(\omega)\|_2 + \|d^*d\Omega(\omega)\|_2 + \|d\Omega(\omega)\|_2 + \|d^*\Omega(\omega)\|_2 + \|\Omega(\omega)\|_2 \le C\|\omega\|_2.$$

He also shows that $\Omega(\omega)$ is the unique form in $\mathscr{W}^{1,p} \cap \mathscr{H}^\perp$ satisfying

$$\Delta\Omega(\omega) = \omega \qquad \text{for } \omega \in \mathscr{H}^\perp \cap L^p \qquad (p \ge 2). \tag{5.13}$$

With these facts in hand, we see that we can make the following

Definition 5.14. For $p \ge 2$, we define Green's operator

$$G : L^p\left(\bigwedge^l M\right) \to \mathscr{W}^{1,p} \cap \mathscr{H}^\perp$$

by $G(\omega) \equiv \Omega(\omega - H(\omega))$, where H is the harmonic projection (see Definition 5.8).

Remark. Notice that $G(\omega) \equiv \Omega(\omega - H(\omega))$ so that by (5.13) we have $\Delta G(\omega) = \omega - H(\omega)$. By the uniqueness result for Poisson's equation, we find that Definition 5.14 extends the one given above.

Proposition 5.15. *For $p \geq 2$, there is $C = C(p)$ so that*

$$\|dd^*\Omega(\omega)\|_p + \|d^*d\Omega(\omega)\|_p + \|d\Omega(\omega)\|_p + \|d^*\Omega(\omega)\|_p + \|\Omega(\omega)\|_p \leq C\|\omega\|_p$$

for all $\omega \in L^p$.

Proof. We apply the closed graph theorem. Let $\|\omega_n\|_p + \|G(\omega_n) - v\|_p \to 0$ as $n \to \infty$. Since L^p is imbedded in L^2, we see that $\|\omega_n\|_2 + \|G(\omega_n) - v\|_2 \to 0$ as $n \to \infty$. But now

$$\begin{aligned} \|G(\omega_n)\|_2 &= \|G(\omega_n - H(\omega_n))\|_2 \\ &\leq C\|\omega_n - H(\omega_n)\|_2 \qquad \text{(by 5.12)} \\ &\leq C(\|\omega_n\|_2 + \|H(\omega_n)\|_2) \qquad \text{(by triangle inequality)} \\ &\leq C\|\omega_n\|_2 \qquad \text{(by 5.9)} \\ &\to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Thus $v = 0$ and so the closed graph theorem says that G is bounded. We repeat this argument for d^*dG, dd^*G, dG and d^*G to finish the proof. □

We will observe that for $p < 2$, we have Proposition 5.15 as well. In preparation for the next result, we recall that for smooth forms, Green's operator commutes with anything the Laplacian does (e.g., d^* and d, see [W]) and is selfadjoint. In particular, when $\eta, \omega \in C^\infty(\bigwedge^l M)$, we have

$$\text{(5.16)} \qquad \begin{cases} (G(\eta), \omega) = (\eta, G(\omega)), \\ dG(\eta) = Gd(\eta), \\ d^*G(\omega) = Gd^*(\omega). \end{cases}$$

Proposition 5.17. *For $1 < p < 2$, there is $C = C(p)$ so that*

$$\|dd^*\Omega(\omega)\|_p + \|d^*d\Omega(\omega)\|_p + \|d\Omega(\omega)\|_p + \|d^*\Omega(\omega)\|_p + \|\Omega(\omega)\|_p \leq C\|\omega\|_p$$

for all $\omega \in C^\infty$.

Proof. Set $\eta_n \equiv G(\omega)(|G(\omega)|^2 + \frac{1}{n})^{\frac{p-2}{2}}$ and observe that $\eta_n \in C^\infty$. Also notice that by the Lebesgue Dominated Convergence Theorem (LDCT),

$$\text{(5.18)} \qquad \|\eta_n\|_q^q \to \|G(\omega)\|_p^p \qquad \text{as } n \to \infty.$$

Next, observe that $|(G(\omega), \eta_n)|$ increases to $\|G(\omega)\|_p^p$ (again by the LDCT). Thus, given $\epsilon > 0$, we may select a large positive integer, say N, so that for $n > N$ we have $\|G(\omega)\|_p^p \leq |(G(\omega), \eta_n)| + \epsilon$. Now we have

$$\begin{aligned} \|G(\omega)\|_p^p &\leq |(G(\omega), \eta_n)| + \epsilon \\ &= |(\omega, G(\eta_n))| + \epsilon \qquad \text{(by 5.16)} \\ &\leq \|\omega\|_p\|G(\eta_n)\|_q + \epsilon \qquad \text{(by Hölder)} \\ &\leq C\|\omega\|_p\|\eta_n\|_q + \epsilon \qquad \text{(by 5.15)} \\ &\to C\|\omega\|_p\|G(\omega)\|_q^{\frac{p}{q}} + \epsilon \qquad \text{(by 5.18)}. \end{aligned}$$

Thus, letting $\epsilon \to 0$, we see

$$\text{(5.19)} \qquad \|G(\omega)\|_p^p \leq C\|\omega\|_p\|G(\omega)\|_p^{\frac{p}{q}}.$$

Under the assumption that $\|G(\omega)\|_p > 0$, dividing 5.19 by $\|G(\omega)\|_p^{\frac{p}{q}}$ gives

(5.20) $$\|G(\omega)\|_p \leq C\|\omega\|_p .$$

Of course, when $\|G(\omega)\|_p = 0$, we see that (5.20) is immediate. Next, we set $\eta_n \equiv dG(\omega)(|dG(\omega)|^2 + \frac{1}{n})^{\frac{p-2}{2}}$. As above, $\eta_n \in C^\infty$ and by the LDCT,

(5.21) $$\|\eta_n\|_q^q \to \|dG(\omega)\|_p^p \qquad \text{as } n \to \infty .$$

Again we observe that $|(dG(\omega), \eta_n)|$ increases to $\|dG(\omega)\|_p^p$ by the LDCT. Thus, given $\epsilon > 0$, we may select a large positive integer, say N, so that for $n > N$ we have $\|dG(\omega)\|_p^p \leq |(dG(\omega), \eta_n)| + \epsilon$. Now we have

$$\begin{aligned}
\|dG(\omega)\|_p^p &\leq |(dG(\omega), \eta_n)| + \epsilon \\
&= |(\omega, G(d^*\eta_n))| + \epsilon \qquad \text{(by 2.8 and 5.16)} \\
&\leq \|\omega\|_p \|G(d^*\eta_n)\|_q + \epsilon \qquad \text{(by Hölder)} \\
&\leq C\|\omega\|_p \|\eta_n\|_q + \epsilon \qquad \text{(by 5.15 and 5.16)} \\
&\to C\|\omega\|_p \|dG(\omega)\|_q^{\frac{p}{q}} + \epsilon \qquad \text{(by 5.21)}.
\end{aligned}$$

Thus, as argued above we have the following analogue to (5.20)

(5.22) $$\|dG(\omega)\|_p \leq C\|\omega\|_p .$$

Finally, setting

$$\begin{aligned}
\eta_n &\equiv d^*G(\omega)(|d^*G(\omega)|^2 + \frac{1}{n})^{\frac{p-2}{2}}, \\
\eta_n &\equiv dd^*G(\omega)(|dd^*G(\omega)|^2 + \frac{1}{n})^{\frac{p-2}{2}}, \\
\eta_n &\equiv d^*dG(\omega)(|d^*dG(\omega)|^2 + \frac{1}{n})^{\frac{p-2}{2}}
\end{aligned}$$

in turn, we argue analogously to obtain

$$\begin{aligned}
\|d^*G(\omega)\| &\leq C\|\omega\|_p , \\
\|dd^*G(\omega)\| &\leq C\|\omega\|_p , \\
\|d^*dG(\omega)\| &\leq C\|\omega\|_p .
\end{aligned}$$

These inequalities, together with (5.20) and (5.22) give (5.17). □

Notice that §4 (e.g. Corollary 4.12) and Proposition 5.17 guarantee that G is a bounded linear operator of C^∞ (as a subspace of L^p) into $\mathscr{W}^{2,p} \cap \mathscr{H}^\perp$. This allows us to give

Definition 5.23. For $1 < p < 2$, we define Green's operator

$$G : L^p\left(\bigwedge^l M\right) \to \mathscr{W}^{2,p} \cap \mathscr{H}^\perp$$

to be the unique bounded linear extension guaranteed by the density of C^∞ in L^p and the boundedness of G into $\mathscr{W}^{2,p} \cap \mathscr{H}^\perp$.

Observe that for any $\omega \in L^p(\bigwedge^l M)$ and any $\eta \in L^q(\bigwedge^l M)$ with p, q Hölder conjugate indices, we have

(5.24) $$(G(\omega), \eta) = (\omega, G(\eta)).$$

The verification of (5.24) is a standard density argument using (5.16). Of course, selfadjointness is not the only useful property which is preserved by our extension of Green's operator to L^p. Indeed, we will use that G and Δ commute when operating on sufficiently smooth forms. This fact, together with (5.24), reveals that for $\omega \in L^p(\bigwedge^l M)$ and $\eta \in C^\infty(\bigwedge^l M)$, we have

$$(\Delta G(\omega), \eta) = (\omega, \Delta G(\eta)) \tag{5.25}$$

6. L^p-Hodge Decomposition

Proposition 6.1. *For $1 < p < \infty$ and $\omega \in L^p(\bigwedge^l M)$, we have $\Delta G(\omega) = \omega - H(\omega)$.*

Proof. Let $\omega_n \in C^\infty$ satisfy: $\|\omega - \omega_n\|_p \to 0$ as $n \to \infty$. According to the Remark following Definition 5.14,

$$\Delta G(\eta) = \eta - H(\eta) \tag{6.2}$$

when $\eta \in C^\infty$. Since H is L^p bounded, we see that $\Delta G(\omega_n) \to \omega - H(\omega)$ in L^p. Of course, this strong convergence implies weak convergence. We will now show that $\Delta G(\omega_n) \to \Delta G(\omega)$ weakly in L^p. By the uniqueness of weak limits we will then be done.

Let $\eta_n \in C^\infty$ satisfy: $\|\eta_n - \eta\|_q \to 0$ as $n \to \infty$. Now

$$\begin{aligned}
|(\Delta G(\omega_n) - \Delta G(\omega), \eta)| &= |(\Delta G(\omega_n - \omega), \eta)| \\
&= |(\Delta G(\omega_n - \omega), \eta - \eta_k) + (\omega_n - \omega, \Delta G(\eta_k))| \quad \text{(by 5.25)} \\
&\le \|\Delta G(\omega_n - \omega)\|_p \|\eta - \eta_k\|_q + \|\omega_n - \omega\|_p \|\Delta G(\eta_k)\|_q \\
&\le \|\Delta G(\omega_n - \omega)\|_p \|\eta - \eta_k\|_q + \|\omega_n - \omega\|_p \|\eta_k - H(\eta_k)\|_q .
\end{aligned}$$

Now letting $k \to \infty$ gives

$$|(\Delta G(\omega_n) - \Delta G(\omega), \eta)| \le C\|\omega_n - \omega\|_p \to 0 \text{ as } n \to \infty .$$

As discussed, this settles the issue. □

Lemma 6.3. *If $\alpha \in \mathscr{W}^{1,p}(\bigwedge^{l-1} M)$, $\beta \in \mathscr{W}^{1,p}(\bigwedge^{l+1} M)$ and $h \in \mathscr{H}$ satisfy $0 = d\alpha + d^*\beta + h$, then $0 = d\alpha = d^*\beta = h$.*

Proof. Let $\phi \in C^\infty(\bigwedge^l M)$. According to the C^∞-Hodge Decomposition, there are $\eta \in C^\infty(\bigwedge^{l-1})$, $\omega \in C^\infty(\bigwedge^{l+1} M)$ and $\tau \in \mathscr{H}$ satisfying

$$\phi = d\eta + d^*\omega + \tau .$$

Notice that $(d^*\beta, d\eta) = (\beta, dd\eta) = (\beta, 0) = 0$ and $(h, d\eta) = (d^*h, \eta) = 0$ thanks to the duality between d and d^*. Linearity then gives

$$(d\alpha, d\eta) = (d\alpha + d^*\beta + h, d\eta) = (0, d\eta) = 0 . \tag{6.4}$$

Finally, we have

$$\begin{aligned}
(d\alpha, \phi) &= (d\alpha, d\eta) + (d\alpha, d^*\omega) + (d\alpha, \tau) \\
&= 0 + (\alpha, d^*d^*\omega) + (\alpha, d\tau) \qquad \text{(by 6.4)} \\
&= (\alpha, 0) + (\alpha, 0) \qquad (\text{since } d^*d^* = 0 \text{ and } \tau \in \mathscr{H}) \\
&= 0 .
\end{aligned}$$

Since $C^\infty(\bigwedge^l M)$ is dense in $L^q(\bigwedge^l M)$ for p, q Hölder conjugate and ϕ was arbitrary, we see that $d\alpha = 0$. Analogously, we see that $d^*\beta = 0$ and it follows that $h = 0$. □

Proposition 6.5 (The L^p Hodge Decomposition). *Let M be a compact, orientable, C^∞, Riemannian manifold without boundary and $1 < p < \infty$. We have*

(1) $L^p(\bigwedge^l M) = \Delta G(L^p) \oplus \mathscr{H} = dd^*G(L^p) \oplus d^*dG(L^p) \oplus \mathscr{H}$
$(1 \leq l \leq \dim(M))$.
(2) $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M) = dd^*G(L^p)$ *and* $d^*\mathscr{W}^{1,p}(\bigwedge^{l+1} M) = d^*dG(L^p)$.

Specifically, $\omega = \Delta G(\omega) + H(\omega)$ *for* $\omega \in L^p(\bigwedge^l M)$, *where G is Green's operator and H is harmonic projection.*

Proof. Proposition 6.1 says that $\Delta G(\omega) = \omega - H(\omega)$. Adding $H(\omega)$ to both sides of this equation and using the definition of Δ reveals that

$$\omega = dd^*G(\omega) + d^*dG(\omega) + H(\omega). \tag{6.6}$$

Since $dd^*G(L^p) \subset d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ and $d^*dG(L^p) \subset d^*\mathscr{W}^{1,p}(\bigwedge^{l+1} M)$, the uniqueness result (Lemma 6.3) and (6.6) give (1). For part (2), let $\omega \in \mathscr{W}^{1,p}(\bigwedge^{l-1} M)$. Part (1) gives

$$d\omega = d\alpha + d^*\beta + H(d\omega)$$

where $\alpha = d^*G(d\omega)$ and $\beta = dG(d\omega)$. Now Lemma 6.3 says $d^*\beta = H(d\omega) = 0$ so that $d\alpha = d\omega \in dd^*G(L^p)$ as desired. The equality $d^*\mathscr{W}^{1,p}(\bigwedge^{l+1} M) = d^*dG(L^p)$ is verified by analogous reasoning. □

Notice that we can make some geometric statements here. In particular, if $d\omega \in d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ and $d^*\eta \in d\mathscr{W}^{1,q}(\bigwedge^{l+1} M)$ for $1 < p, q < \infty$ and p, q Hölder conjugate, then $(d\omega, d^*\eta) = 0$. This expresses the fact that $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ and $d^*\mathscr{W}^{1,q}(\bigwedge^{l+1} M)$ are 'orthogonal' in some reasonable sense. It is provocative to reason that $0 = (d\omega, d^*\eta)$ by using the duality relationship between d and d^* to write

$$(d\omega, d^*\eta) = (dd\omega, \eta) = (0, \eta) = 0. \tag{6.7}$$

Unfortunately, $d\omega$ may only be in $L^p(\bigwedge^l M)$ at best. Thus, without using the general notion of distributions throughout this paper, applying d to $d\omega$ may not be possible. Fortunately though, the Meyers and Serrin density result gives a sequence, $\omega_n \in C^\infty(\bigwedge^{l-1} M)$, which approximates ω in $\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$. This means, in particular, that $d\omega_n$ approximates $d\omega$ in $L^p(\bigwedge^l M)$. Since (6.7) is valid for ω_n, we may write

$$\begin{aligned} |(d\omega, d^*\eta)| &= |(d\omega, d^*\eta) - (d\omega_n, d^*\eta)| \\ &= |(d\omega - d\omega_n, d^*\eta)| \\ &\leq \|d\omega - d\omega_n\|_p \|d^*\eta\|_q \to 0 \text{ as } n \to \infty. \end{aligned}$$

We summarize this reasoning in

Proposition 6.8. *For $\omega \in W^{1,p}(\bigwedge^{l-1} M)$, $\eta \in \mathscr{W}^{1,q}(\bigwedge^{l+1} M)$, $1 < p, q < \infty$ and p, q Hölder conjugate, we have $(d\omega, d^*\eta) = 0$ (i.e., $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ is 'orthogonal' to $d^*\mathscr{W}^{1,q}(\bigwedge^{l+1} M)$).*

There is some nice terminology here. We refer to the forms in $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ as **exact L^p forms** and those in $d^*\mathscr{W}^{1,p}(\wedge^{l+1} M)$ as **coexact L^p forms**. In order for $\omega \in C^\infty$ to be closed, it is equivalent to check that $(\omega, d^*\eta) = 0$ for all $\eta \in C^\infty$. Fortunately, this distributional understanding of closed is available for the L^p forms and Proposition 6.8 reveals that the **closed L^p forms** are exactly those in $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M) \oplus \mathscr{H}$. Similarly, the **coclosed L^p forms** are exactly those in $d^*\mathscr{W}^{1,p}(\bigwedge^{l+1} M) \oplus \mathscr{H}$.

7. Applications to nonlinear analysis and the calculus of variations

Proposition 7.1. *$d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ and $d^*\mathscr{W}^{1,p}(\bigwedge^{l+1} M)$ are closed subspaces of $L^p(\bigwedge^l M)$ for $1 < p < \infty$.*

Proof. Let $v \in \mathrm{Cl}_{L^p}(d\mathscr{W}^{1,p})$. This means there are $\omega_n \in C^\infty$ satisfying $\|d\omega_n - v\|_p \to 0$ as $n \to \infty$. The L^p-Hodge Decomposition says $\omega_n = \alpha_n + k_n$ where $dk_n = 0$ and $\alpha_n \in d^*C^\infty$. We have

$$\begin{aligned} \|\alpha_n\|_{1,p}^p &\le CD_p(\alpha_n) && (5.5) \\ &= C\|d\alpha_n\|_p^p && (d^*d^* = 0) \\ &= C\|d\omega_n\|_p^p && (dk_n = 0) \\ &\le CK && (K > 0, \text{ independent of } n). \end{aligned}$$

But since $\mathscr{W}^{1,p}$ is a reflexive Banach space (w.r.t. Sobolev norm), there is a (Sobolev) weakly convergent subsequence of α_n. For notational simplicity, denote the subsequence again by α_n and say $\alpha_n \to \alpha$ weakly in $\mathscr{W}^{1,p}$. Now $\|\cdot\|_p$ is (L^p) weakly lower semicontinuous and d is continuous from ($\mathscr{W}^{1,p}$, weak) to (L^p, weak). Consequently we have

$$\begin{aligned} \|d\alpha - v\|_p &\le \liminf \|d\alpha_n - v\|_p \\ &= \liminf \|d\omega_n - v\|_p = 0. \end{aligned}$$

Thus $d\alpha = v$ and $d\mathscr{W}^{1,p}$ is closed in L^p. Precisely analogous arguments give that $d^*\mathscr{W}^{1,p}$ is closed in L^p. □

Using this result, we may also give

Proposition 7.2. *Let $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ and $d\mathscr{W}^{1,q}(\bigwedge^{l-1} M)$ be regarded as subspaces of $L^p(\bigwedge^l M)$ and $L^q(\bigwedge^l M)$ respectively, where $1 < p, q < \infty$ and p, q are Hölder conjugate. The linear transformation*

$$\Phi : d\mathscr{W}^{1,q}(\bigwedge^{l-1} M) \to d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)^*$$

given by

$$\Phi(d\eta)(d\omega) \equiv (d\omega, d\eta)$$

is a Banach space isomorphism. Here, $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)^*$ denotes the dual of $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$.

Proof. First of all, notice $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ and $d\mathscr{W}^{1,q}(\bigwedge^{l-1} M)$ are indeed reflexive Banach spaces since Proposition 7.1 gives that they are closed in L^p and L^q respectively. Linearity of Φ is obvious. For injectivity, suppose $(d\alpha, d\eta) = 0$ for all $\alpha \in \mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ and some $\eta \in \mathscr{W}^{1,q}(\bigwedge^{l-1} M)$. Let τ be an arbitrary element of $L^p(\bigwedge^l M)$. According to the L^p-Hodge Decomposition, we may write $\tau = d\alpha + d^*\beta + h$. This gives

$$\begin{aligned}(\tau, d\eta) &= (d\alpha + d^*\beta + h, d\eta)\\ &= (d\alpha, d\eta) + (d^*\beta, d\eta) + (h, d\eta)\\ &= 0 + (d^*\beta, d\eta) + (h, d\eta)\\ &= 0 + (h, d\eta) \qquad \text{(by Proposition 6.8)}\\ &= 0 \qquad (\text{since } d\eta \in \mathscr{H}^{\perp}).\end{aligned}$$

This is enough to conclude that the L^p form $d\eta$ is 0. It follows that Φ is injective. To get surjectivity, let $F \in d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)^*$. The Hahn-Banach Theorem says that there is $\tilde{F} \in L^p(\bigwedge^l M)^*$ with $\|F\| = \|\tilde{F}\|$ and $\tilde{F}|_{d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)} = F$. The Riesz representation theorem for L^p gives $\gamma \in L^q(\bigwedge^l M)$ satisfying $\tilde{F}(\omega) = (\omega, \gamma)$. It is then our job to show that when we restrict $\tilde{F}$ to exact L^p forms, there is an exact L^q form which can be used in place of γ. Again, by the L^p-Hodge Decomposition, we can write $\gamma = d\alpha + d^*\beta + h$. For any $d\omega \in d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$, this gives

$$\begin{aligned}F(d\omega) &= (d\omega, \gamma) = (d\omega, d\alpha) + (d\omega, d^*\beta) + (d\omega, h) = (d\omega, d\alpha)\\ &= \Phi(d\alpha)(d\omega).\end{aligned}$$

Thus, $\Phi(d\alpha) = F$ and Φ is surjective. If we verify that Φ is bounded then by the Inverse Function Theorem, we will be done. But this boundedness is clear since by Hölder's inequality we have

$$|(d\eta, d\omega)| \le \|d\eta\|_q \|d\omega\|_p \qquad (\text{i.e., } \|\Phi\| \le 1). \quad \square$$

We turn now to the study of (homogeneous) A-harmonic equations,

$$d^*A(du) = 0 \tag{7.4}$$

where A is a **monotone bundle mapping** from $\bigwedge^l M$ into itself. That is to say that A satisfies

(1) $|A(\zeta) - A(\xi)| \le K(|\zeta| + |\xi|)^{p-2}|\zeta - \xi|$,
(2) $\langle A(\zeta) - A(\xi) \mid \zeta - \xi\rangle \ge k(|\zeta| + |\xi|)^{p-2}|\zeta - \xi|^2$,
(3) $A(0) \in L^q(\bigwedge^l M)$,

where $1 < p, q < \infty$ are Hölder conjugate, ζ and ξ are arbitrary elements from the same fiber of $\bigwedge^l M$ and both K and k are independent of ζ and ξ. Of course, $A(du)(x) \equiv A(du(x))$ and $d^*A(du) = 0$ is meant in the distributional sense. Making (7.4) nonhomogeneous does not dramatically increase the complexity of the problem and will be dealt with more thoroughly in work to appear elsewhere. Also observe that conditions (1) and (3) give that

$A(\omega) \in L^q(\bigwedge^l M)$ for any $\omega \in L^p(\bigwedge^l M)$. Indeed,

$$\begin{aligned}\int_M |A(\omega)|^q &= \int_M |A(\omega) - A(0) + A(0)|^q \\ &\leq 2^q \|A(0)\|_q^q + 2^q \int_M |A(\omega) - A(0)|^q \\ &\leq 2^q \|A(0)\|_q^q + 2^q K \int_M (|\omega|^{p-2}|\omega|)^q \qquad \text{(by condition (1))} \\ &= 2^q \|A(0)\|_q^q + 2^q K \|\omega\|_p^p < \infty .\end{aligned}$$

This study is well motivated by the fact that for a large class of bundle maps

$$W : \bigwedge^l M \to \bigwedge^n M$$

there exist such A for which solving the A-harmonic equation (7.4) is equivalent to minimizing the associated functional

$$I[u] \equiv \int_M W(du) .$$

A familiar special case is the p-harmonic example where $W(du) = |du - \alpha|^p$ for a fixed L^p form α. In this case, the classical Euler-Lagrange equation yields $A(\zeta) = |\zeta - \alpha|^{p-2}(\zeta - \alpha)$.

According to the L^p-Hodge Decomposition (i.e., Proposition 6.5), we have

$$L^q(\bigwedge^l M) = d\mathscr{W}^{1,q}(\bigwedge^{l-1} M) \oplus d^*\mathscr{W}^{1,q}(\bigwedge^{l+1} M) \oplus \mathscr{H} .$$

For the purpose of effectively exploiting this decomposition, we define T to be the projection of L^q onto $d\mathscr{W}^{1,q}$. Precisely, if $\omega = d\alpha + d^*\beta + h$ where $\alpha \in \mathscr{W}^{1,q}(\bigwedge^{l-1} M)$, $\beta \in \mathscr{W}^{1,q}(\bigwedge^{l+1} M)$ and $h \in \mathscr{H}$, then

$$T(\omega) \equiv d\alpha . \tag{7.5}$$

We restrain ourselves from a thorough development of this operator and explore only those properties which will be immediately useful. The most apparent property of T is that it is a bounded linear projection of L^q with the coclosed forms as its kernel and the exact forms as its range. A slightly less obvious property of T is that when $TA(du) = 0$, then u is a solution of the A-harmonic equation (7.4). Indeed, for such u, $A(du)$ is coclosed giving $(A(du), d\omega) = 0$ for any $\omega \in \mathscr{W}^{1,p}(\bigwedge^{l-1} M)$, which is precisely what it means for u to solve $d^*A(du) = 0$ in the distributional sense. As we will now demonstrate, we can do even better. To be precise, we will show that

$$TA(d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)) = d\mathscr{W}^{1,q}(\bigwedge^{l-1} M). \tag{7.6}$$

Since Proposition 7.2 gives $(d\mathscr{W}^{1,p})^* = d\mathscr{W}^{1,q}$, it is natural to use Browder's theory to verify the surjection (7.6). Perhaps now it is more or less apparent why conditions (1) and (2) were imposed since, when we view TA as an operator from $d\mathscr{W}^{1,p}(\bigwedge^{l-1} M)$ to $d\mathscr{W}^{1,q}(\bigwedge^{l-1} M)$, they will yield that TA is monotone, continuous and coercive. Indeed, continuity follows from condition (1) and the boundedness of T. Next, Propositions 6.5 and 6.8 give that

$A(d\omega) = TA(d\omega) + \kappa$ where $(\kappa, d\eta) = 0$ for every $\eta \in \mathscr{W}^{1,p}(\bigwedge^{l-1} M)$. Thus, for all $\omega, \eta \in \mathscr{W}^{1,p}(\bigwedge^{l-1} M)$, we have

$$(TA(d\omega), d\eta) = (A(d\omega), d\eta). \tag{7.7}$$

We then argue that

$$\begin{aligned}(TA(d\omega) - TA(du) \mid d\omega - du) &= (A(d\omega) - A(du) \mid d\omega - du) \qquad \text{(by 7.7)}\\ &= \int_M \langle A(d\omega) - A(du) \mid d\omega - du\rangle \\ &\geq k \int_M (|d\omega| + |du|)^{p-2} |d\omega - du|^2 \geq 0\end{aligned}$$

which is exactly monotonicity for the operator TA. For coercivity, observe that

$$\begin{aligned}\frac{(TA(d\omega) \mid d\omega)}{\|d\omega\|_p} - \frac{(TA(0) \mid d\omega)}{\|d\omega\|_p} &= \frac{(A(d\omega) - A(0) \mid d\omega)}{\|d\omega\|_p} \qquad \text{(by 7.7)}\\ &= \frac{\int_M \langle A(d\omega) - A(0) \mid d\omega - 0\rangle}{\|d\omega\|_p} \\ &\geq k \frac{\int_M |d\omega|^p}{\|d\omega\|_p} \qquad \text{(by condition (2))}\\ &= k\|d\omega\|_p^{p-1}.\end{aligned}$$

Thus,

$$\frac{(TA(d\omega) \mid d\omega)}{\|d\omega\|_p} - \frac{(TA(0) \mid d\omega)}{\|d\omega\|_p} \to \infty$$

as $\|d\omega\|_p \to \infty$. Further,

$$\begin{aligned}\frac{|(TA(0) \mid d\omega)|}{\|d\omega\|_p} &= \frac{(A(0) \mid d\omega)}{\|d\omega\|_p} \qquad \text{(by 7.7)}\\ &\leq \|A(0)\|_q \qquad \text{(by Hölder's inequality)}.\end{aligned}$$

It follows that

$$\frac{(TA(d\omega) \mid d\omega)}{\|d\omega\|_p} \to \infty$$

as $\|d\omega\|_p \to \infty$. According to Browder's theory (see for example [Z]), TA is surjective. As discussed, this gives the existence of a solution to the A-harmonic equation (7.4). Finally, the estimates given above for monotonicity of TA reveal that the solution is unique up to a closed L^p form.

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