UNIFORM HARMONIC APPROXIMATION OF BOUNDED FUNCTIONS

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ABSTRACT. Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω . We characterize those pairs (Ω, E) which have the following property: every function which is bounded and continuous on E and harmonic on E^0 can be uniformly approximated by functions harmonic on Ω . Several related results concerning both harmonic and superharmonic approximation are also established.

1. INTRODUCTION AND RESULTS

Let Ω be an open set in $\mathbb{C}$, let Ω^* denote its Alexandroff (one point) compactification, and let $\text{Hol}(\Omega)$ denote the collection of holomorphic functions on Ω . Further, let E be a relatively closed subset of Ω and let $C(E)$ denote the collection of continuous complex-valued functions on E .

Theorem A. *The following are equivalent:*

- (a) *for each f in $C(E) \cap \text{Hol}(E^0)$ and each positive number ε there exists g in $\text{Hol}(\Omega)$ such that $|g - f| < \varepsilon$ on E ;*
- (b) *for each bounded function f in $C(E) \cap \text{Hol}(E^0)$ and each positive number ε there exists g in $\text{Hol}(\Omega)$ such that $|g - f| < \varepsilon$ on E ;*
- (c) *$\Omega^* \setminus E$ is connected and locally connected.*

The equivalence of (a) and (c) in Theorem A is the celebrated theorem of Arakelyan (see [1] or [2]). The addition of condition (b) here is due to Stray [16, Theorem 1]. Stray [17, p. 359] subsequently asked whether conditions (a) and (b) of Theorem A remain equivalent when “holomorphic” is replaced by “harmonic” throughout.

The harmonic analogue of Arakelyan’s theorem was recently established by the author in [8]: see Theorem B below. One of the results of the present paper (Theorem 2 below) identifies which sets E have the harmonic analogue of property (b) in Theorem A and, in so doing, gives a negative answer to Stray’s question in all dimensions. When $n = 2$ it simplifies to a result of Nersesyan [15]. However, the general characterization is more delicate, and new arguments are required to deal with higher dimensions.

From now on Ω denotes an open set in Euclidean space $\mathbb{R}^n$ ($n \geq 2$) and E is a relatively closed subset of Ω . If $A \subseteq \mathbb{R}^n$, then $C(A)$ denotes the collection of real-valued continuous functions on A , and $\mathcal{H}(A)$ (resp. $\mathcal{S}(A)$) is the class of functions which are harmonic (resp. superharmonic) on some open set which contains A . We

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say that A is Ω -bounded if $\overline{A}$ is a compact subset of Ω , and denote by $\widehat{E}$ the union of E with the Ω -bounded (connected) components of $\Omega \setminus E$. The harmonic analogue of Arakelyan's theorem is as follows. The reader is referred to [8, Theorem 5] for this result, or to [9] for a general exposition of the results in this area. An account of thin sets and the fine topology may be found in Doob [6, 1.XI] or Helms [13, Chapter 10].

Theorem B. *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω . The following are equivalent:*

- (a) *for each u in $C(E) \cap \mathcal{H}(E^0)$ and each positive number ε there exists ν in $\mathcal{H}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (b) (i) $\Omega \setminus \widehat{E}$ and $\Omega \setminus E^0$ *are thin at the same points of E , and*
(ii) *for each compact subset K of Ω there is a compact subset L of Ω which contains every Ω -bounded component of $\Omega \setminus (E \cup K)$ whose closure intersects K .*

It will become clear from the results below that, if we require the function u in (a) above to be bounded, then condition (b) can be significantly relaxed; that is, approximation is possible on a larger class of sets E .

Let $\Omega^* = \Omega \cup \{\mathcal{A}\}$, where $\mathcal{A}$ denotes the Alexandroff point for Ω . If ω is a connected open subset of Ω , then $\mathcal{A}$ is said to be *accessible from ω* if there is a continuous function $f : [0, +\infty) \rightarrow \omega$ such that $f(t) \rightarrow \mathcal{A}$ as $t \rightarrow +\infty$. We denote by $\widetilde{E}$ (or, sometimes, by $(E)^\sim$) the union of E with the components of $\Omega \setminus E$ from which $\mathcal{A}$ is not accessible. Clearly $\widehat{E} \subseteq \widetilde{E}$, but this inclusion may be strict: see Example 1 below. Our main results on approximation of bounded functions are as follows.

Theorem 1. *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω . The following are equivalent:*

- (a) *for each bounded function u in $\mathcal{H}(E)$ and each positive number ε there exists ν in $\mathcal{H}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (b) *for each bounded function u in $\mathcal{S}(E)$ and each positive number ε there exists ν in $\mathcal{S}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (c) (i) $\Omega \setminus \widetilde{E}$ and $\Omega \setminus E$ *are thin at the same points of E , and*
(ii) *for each compact subset K of Ω there is a compact subset L of Ω such that $\Omega \setminus \widetilde{E}$ and $\Omega \setminus (\widetilde{E} \cup K)^\sim$ are thin at the same points of $E \setminus L$.*

Theorem 2. *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω . The following are equivalent:*

- (a) *for each bounded function u in $C(E) \cap \mathcal{H}(E^0)$ and each positive number ε there exists ν in $\mathcal{H}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (b) *for each bounded function u in $C(E) \cap \mathcal{S}(E^0)$ and each positive number ε there exists ν in $C(\Omega) \cap \mathcal{S}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (c) (i) $\Omega \setminus \widetilde{E}$ and $\Omega \setminus E^0$ *are thin at the same points of E , and*
(ii) *for each compact subset K of Ω there is a compact subset L of Ω such that $\Omega \setminus \widetilde{E}$ and $\Omega \setminus (\widetilde{E} \cup K)^\sim$ are thin at the same points of $E \setminus L$.*

The proof of Theorem 1 (resp. Theorem 2) actually yields a little more: if (c) holds, then any function in $\mathcal{H}(E)$ (resp. in $C(E) \cap \mathcal{H}(E^0)$) which is bounded on $\partial E \cap \Omega$ can be uniformly approximated on E by functions in $\mathcal{H}(\Omega)$.

When $n = 2$ Theorems 1 and 2 simplify to the following slight reformulation of a result of Nersisyan [15].

Corollary 1. *Let Ω be an open set in $\mathbb{R}^2$ and E be a relatively closed subset of Ω . The following are equivalent:*

- (a) *for each bounded function u in $C(E) \cap \mathcal{H}(E^0)$ and each positive number ε there exists ν in $\mathcal{H}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (b) *for each bounded function u in $\mathcal{H}(E)$ and each positive number ε there exists ν in $\mathcal{H}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (c) (i) $\partial\tilde{E} = \partial E$, and
 (ii) *for each compact subset K of Ω there is a compact subset L of Ω such that $\partial\tilde{E} \setminus L = \partial((\tilde{E} \cup K)^\sim) \setminus L$.*

The thinness conditions in Theorems 1 and 2 are new, but similar in spirit to those in [8]. That paper contains some illustrative examples which are relevant also in the present context. Below we illustrate condition (c) of Corollary 1 by some further examples.

Example 1. Let $\Omega = \mathbb{R}^2$, and for each m in $\mathbb{N}$ let W_m denote the open rectangle $((m+1)^{-1}, m^{-1}) \times (0, m)$. We define

$$E_1 = \partial \left(\left(\bigcup_{m=1}^{\infty} W_m \right) \cup (0, 1)^2 \right), \quad E_2 = \partial \left(\left(\bigcup_{m=1}^{\infty} W_{2m} \right) \cup (0, 1)^2 \right),$$

and $E_3 = E_2 \cup \{(1/2, 1/2)\}$. Then $\mathbb{R}^2 \setminus E_k$ ($k = 1, 2, 3$) consists of two components, and the point at infinity is not accessible from the component which contains the point $(1/4, 1/4)$. Clearly $\partial\tilde{E}_2 = \partial E_2$, but $\partial\tilde{E}_1 \neq \partial E_1$ and $\partial\tilde{E}_3 \neq \partial E_3$.

Example 2. Let $\Omega = \mathbb{R}^2$, let W_m be as in Example 1, and let

$$F_1 = \partial \left(\left(\bigcup_{m=1}^{\infty} W_m \right) \cup (-\infty, 1)^2 \right), \quad F_2 = \partial \left(\left(\bigcup_{m=1}^{\infty} W_{2m} \right) \cup (-\infty, 1)^2 \right),$$

$$F_3 = F_2 \cup \left\{ \left(\frac{4m+1}{4m(2m+1)}, m \right) \in \mathbb{R}^2 : m \in \mathbb{N} \right\}.$$

Then condition (c)(ii) of Corollary 1 is satisfied when $E = F_2$. However, it is not satisfied when $E = F_1$ or $E = F_3$, as can be seen by taking K to be $[0, 1] \times \{1\}$.

It is now easy to see that condition (c) of Corollary 1 is satisfied when $E = E_2$ and when $E = F_2$ in $\mathbb{R}^2$. However, this is not true of condition (b) in Theorem B. This provides the promised negative answer to the question of Stray recorded above when $n = 2$, and a simple modification of this example gives a counterexample also in higher dimensions, in the light of Theorems B and 2 and of Lemmas 1 and 2 below.

The proofs of Theorems 1 and 2 rely in part on results and techniques of Gauthier, Goldstein and Ow [11], [12], Labrèche [14] and the author [8]. In particular, they implicitly rely on fusion results for harmonic and superharmonic functions. Theorem 1 is proved in §§2–4. Theorem 2 and Corollary 1 are then derived in §5. In §6 we indicate refinements of Theorems 1 and 2 in which better than uniform approximation of bounded functions is achieved.

2. PROOF OF THEOREM 1 (c) $\Rightarrow$ (b)

2.1. In this section we collect together some preliminary observations associated with condition (c) of Theorem 1.

Lemma 1. *Suppose that K and L are compact subsets of Ω such that $K \subseteq L$ and such that $\Omega \setminus \tilde{E}$ and $\Omega \setminus (\tilde{E} \cup K)^\sim$ are thin at the same points of $E \setminus L$. Let W denote the union of the components of $\Omega \setminus (\tilde{E} \cup K)$ from which $\mathcal{A}$ is not accessible. If $W \neq \emptyset$, then:*

- (i) *each point of $(\partial W \cap \Omega) \setminus L$ is regular for the Dirichlet problem on W , and*
- (ii) *$(\partial W \cap \Omega) \setminus L \subseteq \partial((\tilde{E} \cup K)^\sim)$.*

To prove (i), suppose that z is a point of $(\partial W \cap \Omega) \setminus L$ which is irregular for the Dirichlet problem on W . Then $\Omega \setminus W$, and hence also $\Omega \setminus (\tilde{E} \cup K)^\sim$, is thin at z . However, $\Omega \setminus \tilde{E}$ contains W , and so is nonthin at z . Clearly

$$\partial W \cap \Omega \subseteq (\partial \tilde{E} \cap \Omega) \cup \partial K \subseteq E \cup \partial K,$$

so $z \in E \setminus L$. This contradicts our hypothesis, and thus (i) must hold.

To prove (ii), let $A = (\partial W \cap \Omega) \setminus (L \cup \partial((\tilde{E} \cup K)^\sim))$. Then

$$A \subseteq (\partial(\tilde{E} \cup K) \cap \Omega) \setminus \partial((\tilde{E} \cup K)^\sim) \subseteq ((\tilde{E} \cup K)^\sim)^0,$$

so $\Omega \setminus (\tilde{E} \cup K)^\sim$ is certainly thin at each point of A . Also,

$$A \subseteq (\partial(\tilde{E} \cup K) \cap \Omega) \setminus L \subseteq E \setminus L,$$

so $\Omega \setminus \tilde{E}$ is thin at each point of A , by hypothesis. Hence W is also thin at each point of A . Thus A is a relatively open subset of ∂W which has zero harmonic measure for each component of W (see [6, 1.XI,13]). Each point of A is therefore irregular for the Dirichlet problem on W . Since $A \subseteq (\partial W \cap \Omega) \setminus L$, it follows from (i) that $A = \emptyset$. Hence (ii) holds, and the lemma is proved.

Lemma 2. *Suppose that $\Omega \setminus \tilde{E}$ and $\Omega \setminus E$ are thin at the same points of E and let $W = \tilde{E} \setminus E$. If $W \neq \emptyset$, then:*

- (i) *each point of $\partial W \cap \Omega$ is regular for the Dirichlet problem on W , and*
- (ii) *$\partial W \cap \Omega \subseteq \partial \tilde{E}$.*

The proof of Lemma 2 is similar to that of Lemma 1, so we omit the details.

Let $\partial^\infty A$ denote the boundary of a set A in $\mathbb{R}^n \cup \{\infty\}$. If W is as in Lemma 2, then each point y of $\partial^\infty W \cap \partial^\infty \Omega$ is regular for the Dirichlet problem on W . To see this, we note that, if y were an irregular boundary point of W , then $\mathbb{R}^n \setminus W$ would be thin at y , and it follows that almost all rays emanating from y lie initially in $W \cup \{y\}$. This is impossible, in view of the definition of W .

Theorem C. *Let ω be a connected open subset of Ω from which $\mathcal{A}$ is not accessible. If s is a subharmonic function on ω and*

$$\limsup_{x \rightarrow y} s(x) \leq a \quad (y \in \partial \omega \cap \Omega),$$

then $s \leq a$ on ω .

Theorem C is implied by a result of Fuglede [7, §4]; an elementary proof may be found in [5].

We will use H_f^U to denote the Perron-Wiener-Brelot solution (when it is defined) of the Dirichlet problem on an open set U with boundary function f defined on $\partial^\infty U$.

2.2. In the next two lemmas we show that, if condition (c)(i) of Theorem 1 holds, then bounded functions in $\mathcal{S}(E)$ can be uniformly approximated on E by functions in $\mathcal{S}(\tilde{E})$.

Lemma 3. *Suppose that $\Omega \setminus \tilde{E}$ and $\Omega \setminus E$ are thin at the same points of E and let $W = \tilde{E} \setminus E$. Further, let F be a relatively closed subset of Ω such that $F \subseteq W$ and F is nonthin at each of its points. Then, for each positive number ε , there is a nonnegative continuous superharmonic function w , on an open set ω which contains $\tilde{E}$, such that $w \in \mathcal{H}(\omega \setminus F)$ and $w = 1$ on F and $w \leq \varepsilon$ on E .*

In proving Lemma 3 we may assume that $F \neq \emptyset$, for otherwise we can choose w to be the zero function. Let $\Omega_0 = \Omega \setminus F$. In this proof $\hat{R}_s^A$ will denote the regularized reduced function (balayage) of a nonnegative superharmonic function s on Ω_0 relative to a subset A of Ω_0 .

Let $\varepsilon \in (0, 1)$ and let (K_m) be a sequence of compact subsets of Ω such that $K_m \subset K_{m+1}^0$ for each m and $\bigcup_m K_m = \Omega$. Further, let

$$L(m, k) = \{x \in \Omega \setminus K_m^0 : \text{dist}(x, \tilde{E}) \geq k^{-1}\} \quad (m, k \in \mathbb{N}),$$

where $\text{dist}(x, A)$ denotes the Euclidean distance from a point x to a set A . Then

$$\hat{R}_1^{L(m, k)}(x) \uparrow \hat{R}_1^{\Omega \setminus (\tilde{E} \cup K_m^0)}(x) \quad (x \in \Omega_0; m \in \mathbb{N}; k \rightarrow \infty).$$

We temporarily fix m in $\mathbb{N}$. Let $s_m(x)$ denote the above limit and let $D = (\partial W \cap \Omega) \setminus K_m$. Then $s_m = 1$ at points of D where $\Omega \setminus \tilde{E}$ is nonthin. Let S denote the complementary set of points of D . Since $\Omega \setminus \tilde{E}$ is thin at points of S , so also is $\Omega_0 \setminus E$, in view of the hypothesis and the fact that $\partial W \cap \Omega \subseteq E$. Hence the smaller set $W \setminus F$ is thin at points of S , and so S carries zero harmonic measure for $W \setminus F$. Let χ_A denote the characteristic function valued 1 on a set A and 0 elsewhere in $\mathbb{R}^n \cup \{\infty\}$. Then

$$s_m(x) \geq H_{\chi_{D \setminus S}}^{W \setminus F}(x) = H_{\chi_D}^{W \setminus F}(x) \quad (x \in W \setminus F).$$

Let

$$V_m = \{x \in W \setminus (K_{m+1} \cup F) : s_m(x) < 1 - \varepsilon/2\} \cup (F \setminus K_{m+1}).$$

Then V_m is an open set such that $\overline{V_m} \cap \Omega \subset W$ in view of Lemma 2(i). Clearly $s_m \geq 1 - \varepsilon/2$ on $W \setminus (K_{m+1} \cup V_m)$.

Similarly we define

$$L(0, k) = \{x \in \Omega : \text{dist}(x, \tilde{E}) \geq k^{-1}\} \quad (k \in \mathbb{N}),$$

$s_0 = \hat{R}_1^{\Omega \setminus \tilde{E}}$, and

$$V_0 = \{x \in W \setminus F : s_0(x) < 1 - \varepsilon/2\} \cup F.$$

Let

$$V = \bigcup_{m=0}^{\infty} V_m.$$

Then V is an open set which contains F and which satisfies $\bar{V} \cap \Omega \subseteq W$. It follows from Dini's theorem that there exists k_0 in $\mathbb{N}$ such that

$$(1) \quad \hat{R}_1^{L(0, k_0)}(x) \geq 1 - \varepsilon \quad (x \in \partial V \cap K_3).$$

Similarly, for each m in $\mathbb{N}$, we can choose k_m in $\mathbb{N}$ such that

$$(2) \quad \hat{R}_1^{L(m, k_m)}(x) \geq 1 - \varepsilon \quad (x \in \partial V \cap (K_{m+3} \setminus K_{m+2}^0); m \in \mathbb{N}).$$

Now let

$$L = \bigcup_{m=0}^{\infty} L(m, k_m),$$

let $\omega = \Omega \setminus L$ and

$$w(x) = \begin{cases} H_{\chi_{\partial F \cap \Omega}}^{\Omega \setminus (F \cup L)}(x) & (x \in \Omega \setminus (F \cup L)), \\ 1 & (x \in F). \end{cases}$$

Thus ω is an open set which contains $\tilde{E}$, and w is a nonnegative continuous superharmonic function on ω . (It is continuous at points of $\partial F \cap \Omega$ by the nonthinness of F at such points.) Clearly $w \in \mathcal{H}(\omega \setminus F)$. Since $w \leq 1 - \hat{R}_1^L$ on $\Omega \setminus (F \cup L)$, it follows from (1) and (2) that $w \leq \varepsilon$ on $\partial V \cap \Omega$. Hence, by the maximum principle, we see that $w \leq \varepsilon$ on $\omega \setminus V$ and so on all of E .

This completes the proof of Lemma 3.

Lemma 4. *Suppose that $\Omega \setminus \tilde{E}$ and $\Omega \setminus E$ are thin at the same points of E , let u be a bounded function in $\mathcal{S}(E)$ and let $\varepsilon > 0$. Then there exists a bounded function ν in $\mathcal{S}(\tilde{E})$ such that $|\nu - u| < \varepsilon$ on E . Further, if u is continuous, then it can be arranged that ν is also continuous.*

To see this, let Ω, E, u and ε be as in the first sentence of the lemma. (We may assume that $\varepsilon < 1$.) Then there is an open set V such that $E \subseteq V$ and $u \in \mathcal{S}(V)$, and a positive number a such that $|u| < a$ on V . If $\tilde{E} \subseteq V$, then there is nothing to prove. Otherwise, let $W = \tilde{E} \setminus E$ and let F be a relatively closed subset of Ω such that $F \subseteq W$ and $W \subseteq V \cup F^0$, and such that F is nonthin at each of its points.

It follows from Lemma 3 that there is a nonnegative continuous superharmonic function w , on an open set ω which contains $\tilde{E}$, such that $w \in \mathcal{H}(\omega \setminus F)$ and $w = 1$ on F and $w < \varepsilon/(2a + 1)$ on E . We now define

$$\nu_1(x) = \begin{cases} u(x) & (x \in V \setminus W), \\ \min\{u(x), a + 1 - (2a + 1)w(x)\} & (x \in W \setminus F), \\ -a & (x \in F). \end{cases}$$

Since

$$a + 1 - (2a + 1)w(x) > a + 1 - \varepsilon > a > u(x) \quad (x \in \partial W \cap \Omega)$$

and $w \in \mathcal{H}(\omega \setminus F)$, the function ν_1 is superharmonic on $V \setminus F$. Since

$$a + 1 - (2a + 1)w(x) = -a < u(x) \quad (x \in \partial F \cap \Omega),$$

the function ν_1 is equal to $a + 1 - (2a + 1)w(x)$ on an open set which contains F . Hence the function ν defined by $\nu = \nu_1 + (2a + 1)w$ is bounded and superharmonic on $\omega \cap (V \cup W)$. Further, $u \leq \nu < u + \varepsilon$ on E . Finally, if u is continuous, then the above construction clearly results in ν also being continuous.

2.3. In this section we establish variants of Lemmas 3 and 4 which relate to condition (c)(ii) of Theorem 1. Suppose that condition (c)(ii) of Theorem 1 holds, let $y \in E$ (we dismiss the trivial case where $E = \emptyset$) and $K_0 = \{y\}$. Further, let $(K_m)_{m \geq 1}$ be a sequence of smoothly bounded compact subsets of Ω such that $K_m \subset K_{m+1}^0$ and $\hat{K}_m = K_m$ for each m in $\{0, 1, \dots\}$ and such that $\bigcup_m K_m = \Omega$. By deleting some members of $(K_m)_{m \geq 1}$ we can arrange, in view of condition (c)(ii), that $\Omega \setminus \tilde{E}$ and $\Omega \setminus (\tilde{E} \cup K_m)^\sim$ are thin at the same points of $E \setminus K_{m+1}^0$, for each m .

Lemma 5. *Suppose that condition (c)(ii) of Theorem 1 holds and let (K_m) be as described above. Further, let $l \in \mathbb{N}$, let*

$$(2) \quad U_l = (\tilde{E} \cup K_l)^\sim \setminus ((\tilde{E} \cup K_{l-1})^\sim \cup K_{l+2}),$$

and let F be a relatively closed subset of Ω such that $F \subseteq U_l$ and F is nonthin at each of its points. Then, for each positive number ε , there is a nonnegative continuous superharmonic function w , on an open set ω which contains $(\tilde{E} \cup K_{l-1})^\sim \cup (\bar{U}_l \cap \Omega)$, such that $w \in \mathcal{H}(\omega \setminus F)$ and $w = 1$ on F and $w \leq \varepsilon$ on $(\tilde{E} \cup K_{l-1})^\sim \cup (\partial U_l \cap \Omega)$.

In proving Lemma 5 we may assume that $F \neq \emptyset$, for otherwise we can choose w to be the zero function. Let $\Omega_0 = \Omega \setminus F$. We again use $\hat{R}_s^A$ to denote the regularized reduced function of a nonnegative superharmonic function s on Ω_0 relative to a subset A of Ω_0 .

Let $\varepsilon \in (0, 1)$ and

$$L(1, k) = \{x \in \Omega \setminus K_{l+1}^0 : \text{dist}(x, (\tilde{E} \cup K_{l-1})^\sim \cup \bar{U}_l) \geq k^{-1}\} \quad (k \in \mathbb{N}).$$

It follows from the construction of the sequence (K_m) that $\Omega \setminus ((\tilde{E} \cup K_{l-1})^\sim \cup \bar{U}_l)$ and $\Omega \setminus \tilde{E}$ are thin at the same points of $E \setminus K_{l+1}^0$. Hence, by the smoothness of ∂K_{l+2} , the sets $\Omega \setminus ((\tilde{E} \cup K_{l-1})^\sim \cup \bar{U}_l)$ and $\Omega \setminus \tilde{E}$ are thin at the same points of $\partial U_l \cap \Omega$. Let

$$s_1(x) = \hat{R}_1^{\Omega \setminus [(\tilde{E} \cup K_{l-1})^\sim \cup \bar{U}_l \cup K_{l+1}^0]}(x) \quad (x \in \Omega_0)$$

and

$$V_1 = \{x \in U_l \setminus F : s_1(x) < 1 - \varepsilon/2\} \cup F.$$

Similarly we define

$$L(m, k) = \{x \in \Omega \setminus K_{l+m}^0 : \text{dist}(x, (\tilde{E} \cup K_{l-1})^\sim \cup \bar{U}_l) \geq k^{-1}\} \quad (m \geq 2; k \in \mathbb{N}),$$

$$s_m(x) = \hat{R}_1^{\Omega \setminus [(\tilde{E} \cup K_{l-1})^\sim \cup \bar{U}_l \cup K_{l+m}^0]}(x) \quad (x \in \Omega_0),$$

and

$$V_m = \{x \in U_l \setminus (K_{l+m+1} \cup F) : s_m(x) < 1 - \varepsilon/2\} \cup (F \setminus K_{l+m+1}).$$

Let

$$V = \bigcup_{m=1}^{\infty} V_m.$$

We now argue as in the proof of Lemma 3 (we appeal to Lemma 1 in place of Lemma 2) to see that V is an open set which contains F and which satisfies $\overline{V} \cap \Omega \subseteq U_l$. Since

$$\widehat{R}_1^{L(1,k)}(x) \uparrow s_1(x) \quad (x \in \Omega_0; k \rightarrow \infty),$$

we can choose k_1 in $\mathbb{N}$ such that

$$(4) \quad \widehat{R}_1^{L(1,k_1)}(x) \geq 1 - \varepsilon \quad (x \in \partial V \cap K_{l+4}).$$

Similarly, if $m \geq 2$, we choose k_m in $\mathbb{N}$ such that

$$(5) \quad \widehat{R}_1^{L(m,k_m)}(x) \geq 1 - \varepsilon \quad (x \in \partial V \cap (K_{l+m+3} \setminus K_{l+m+2}^0)).$$

Now let

$$L = \bigcup_{m=1}^{\infty} L(m, k_m),$$

let $\omega = \Omega \setminus L$ and let

$$w(x) = \begin{cases} H_{\chi_{\partial F \cap \Omega}}^{\Omega \setminus (F \cup L)}(x) & (x \in \Omega \setminus (F \cup L)), \\ 1 & (x \in F). \end{cases}$$

Thus ω is an open set which contains $(\widetilde{E} \cup K_{l-1})^\sim \cup (\overline{U}_l \cap \Omega)$, and w is a nonnegative continuous superharmonic function on ω . Clearly $w \in \mathcal{H}(\omega \setminus F)$. Since $w \leq 1 - \widehat{R}_1^L$ on $\Omega \setminus (F \cup L)$, it follows from (4) and (5) that $w \leq \varepsilon$ on $\partial V \cap \Omega$. Hence, by the maximum principle, we see that $w \leq \varepsilon$ on $\omega \setminus V$ and so on $(\widetilde{E} \cup K_{l-1})^\sim \cup (\partial U_l \cap \Omega)$.

This completes the proof of Lemma 5.

Lemma 5 will now be used to prove the following.

Lemma 6. *Suppose that condition (c)(ii) of Theorem 1 holds, let (K_m) be as described at the beginning of §2.3, let $l \in \mathbb{N}$ and let U_l be as in (3). If u is a bounded member of $\mathcal{S}((\widetilde{E} \cup K_{l-1})^\sim \cup K_{l+2})$, and if $\varepsilon > 0$, then there exists a bounded member ν of $\mathcal{S}((\widetilde{E} \cup K_{l-1})^\sim \cup (\overline{U}_l \cap \Omega))$ such that $|\nu - u| < \varepsilon$ on $(\widetilde{E} \cup K_{l-1})^\sim \cup (\partial U_l \cap \Omega)$. Further, if u is continuous, then it can be arranged that ν is also continuous.*

To prove this, let u be a bounded member of $\mathcal{S}((\widetilde{E} \cup K_{l-1})^\sim \cup K_{l+2})$ and let $\varepsilon \in (0, 1)$. Then there is an open set V such that $(\widetilde{E} \cup K_{l-1})^\sim \cup K_{l+2} \subseteq V$ and $u \in \mathcal{S}(V)$, and a positive number a such that $|u| < a$ on V . Next, let F be a relatively closed subset of Ω such that $F \subset U_l$ and F is nonthin at each of its points, and such that $U_l \subseteq V \cup F^0$. By Lemma 5 there is a nonnegative continuous superharmonic function w , on an open set ω which contains $(\widetilde{E} \cup K_{l-1})^\sim \cup (\overline{U}_l \cap \Omega)$, such that $w \in \mathcal{H}(\omega \setminus F)$ and $w = 1$ on F and $w < \varepsilon/(2a + 1)$ on $(\widetilde{E} \cup K_{l-1})^\sim \cup (\partial U_l \cap \Omega)$.

Let

$$\nu_1(x) = \begin{cases} u(x) & (x \in V \setminus U_l), \\ \min\{u(x), a + 1 - (2a + 1)w(x)\} & (x \in U_l \setminus F), \\ -a & (x \in F). \end{cases}$$

As in the proof of Lemma 4, the function ν_1 is superharmonic on $V \setminus F$, and the function ν defined by $\nu = \nu_1 + (2a + 1)w$ is bounded and superharmonic on $\omega \cap (V \cup U_l)$. Further,

$$u(x) \leq \nu(x) < u(x) + \varepsilon \quad (x \in (\widetilde{E} \cup K_{l-1})^\sim \cup (\partial U_l \cap \Omega)).$$

Since $(\tilde{E} \cup K_{l-1})^\sim \cup (\bar{U}_l \cap \Omega) \subset \omega \cap (V \cup U_l)$, this completes the proof of Lemma 6. (It is clear that, if u is continuous, then so also is ν .)

2.4. It will now be shown that (c) implies (b) in Theorem 1. Suppose that (c) holds, let u be a bounded function in $\mathcal{S}(E)$ and let $\varepsilon > 0$. In view of Lemma 4 there is a bounded function u_0 in $\mathcal{S}(\tilde{E})$ such that $|u_0 - u| < 2^{-1}\varepsilon$ on $\tilde{E}$. Let $\varphi_n : [0, +\infty) \rightarrow \mathbb{R} \cup \{+\infty\}$ be the function defined by $\varphi_2(t) = \log(1/t)$ or $\varphi_n(t) = t^{2-n}$ if $n \geq 3$. (We interpret $\varphi_n(0)$ as $+\infty$ in either case.) Also, let (K_l) be the sequence of compact subsets of Ω described at the beginning of §2.3, and let U_l be as in (3). Then u_0 is a bounded member of $\mathcal{S}((\tilde{E} \cup K_0)^\sim)$, since $K_0 = \{y\} \subseteq E$.

We proceed inductively as follows. Suppose that we have a bounded superharmonic function u_{l-1} on an open set ω_{l-1} which contains $(\tilde{E} \cup K_{l-1})^\sim$, where $l \in \mathbb{N}$. It follows (see [8, Lemma 2] or [9, Lemma 6.2]) that there exist ν_{l-1} in $\mathcal{S}(\mathbb{R}^n)$, points $x_1, \dots, x_m$ in $\mathbb{R}^n \setminus \omega_{l-1}$ and a nonnegative constant c such that

$$u_{l-1}(x) = \nu_{l-1}(x) - c \sum_{j=1}^m \varphi_n(|x - x_j|)$$

on some open set which contains the compact set $K_{l+3} \cap (\tilde{E} \cup K_{l-1})^\sim$. Thus u_{l-1} can be redefined off the set $(\tilde{E} \cup K_{l-1})^\sim$ so that it is superharmonic (and bounded above) on an open set that contains $(\tilde{E} \cup K_{l-1})^\sim \cup K_{l+2}$, apart from finitely many Newtonian (or logarithmic) singularities in $K_{l+2} \setminus (\tilde{E} \cup K_{l-1})^\sim$. Each of these singularities can be joined to $\mathcal{A}$ by a continuous path lying in $\Omega \setminus (\tilde{E} \cup K_{l-1})^\sim$. Hence, by a simple pole-pushing argument (see, for example, [9, §1.6]), there is a bounded member w_{l-1} of $\mathcal{S}((\tilde{E} \cup K_{l-1})^\sim \cup K_{l+2})$ such that $|w_{l-1} - u_{l-1}| < 2^{-l-3}\varepsilon$ on $\tilde{E} \cup K_{l-1}$. It follows from Lemma 6 that there is a bounded member s_{l-1} of $\mathcal{S}((\tilde{E} \cup K_{l-1})^\sim \cup (\bar{U}_l \cap \Omega))$ such that $|s_{l-1} - u_{l-1}| < 2^{-l-2}\varepsilon$ on $(\tilde{E} \cup K_{l-1})^\sim$. Finally, as before, we may redefine s_{l-1} off the set $(\tilde{E} \cup K_{l-1})^\sim \cup (\bar{U}_l \cap \Omega)$ so that it is superharmonic on an open set which contains $(\tilde{E} \cup K_l)^\sim$, apart from finitely many Newtonian (or logarithmic) singularities in $K_{l+2} \setminus (\tilde{E} \cup K_{l-1})^\sim$. Each of these singularities can be joined to $\mathcal{A}$ by a continuous path lying in $\Omega \setminus (\tilde{E} \cup K_{l-1})^\sim$, and so we can obtain a bounded member u_l of $\mathcal{S}((\tilde{E} \cup K_l)^\sim)$ such that $u_{l-1} \leq u_l < u_{l-1} + 2^{-l-1}\varepsilon$ on $(\tilde{E} \cup K_{l-1})^\sim$.

Since $\bigcup_l K_l = \Omega$, the sequence (u_l) , which is eventually defined and increasing on any given compact subset of Ω , converges on Ω to a function ν in $\mathcal{S}(\Omega)$ such that

$$|\nu - u| \leq \sum_{l=1}^{\infty} (u_l - u_{l-1}) + |u_0 - u| < \varepsilon \quad \text{on } E.$$

Thus condition (b) of Theorem 1 holds.

3. PROOF OF THEOREM 1 (c) $\Rightarrow$ (a)

3.1. We begin with harmonic analogues of Lemmas 4 and 6.

Lemma 7. *Suppose that $\Omega \setminus \tilde{E}$ and $\Omega \setminus E$ are thin at the same points of E , let u be a bounded function in $\mathcal{H}(E)$ and let $\varepsilon > 0$. Then there exists a bounded function ν in $\mathcal{H}(\tilde{E})$ such that $|\nu - u| < \varepsilon$ on E .*

To see this, let u be a bounded function in $\mathcal{H}(E)$ and let $\varepsilon > 0$. By Lemma 4 there exist bounded functions ν_1 and ν_2 in $\mathcal{S}(\tilde{E})$ such that $u < \nu_1 < u + \varepsilon$ and

$-u < \nu_2 < -u + \varepsilon$ on E . Thus $-\nu_2 < \nu_1$ on E , and hence, by Theorem C, on some open set ω which contains $\tilde{E}$. Let ν be the greatest harmonic minorant of ν_1 on ω . Then $-\nu_2 \leq \nu \leq \nu_1$ on ω . Hence ν is a bounded member of $\mathcal{H}(\tilde{E})$ and $|\nu - u| < \varepsilon$ on E .

Lemma 8. *Suppose that condition (c)(ii) of Theorem 1 holds, let (K_m) be as described at the beginning of §2.3, and let U_l be as in (3). If u is a bounded member of $\mathcal{H}((\tilde{E} \cup K_{l-1})^\sim \cup K_{l+2})$, where $l \in \mathbb{N}$, and if $\varepsilon > 0$, then there exists a bounded member ν of $\mathcal{H}((\tilde{E} \cup K_{l-1})^\sim \cup (\overline{U}_l \cap \Omega))$ such that $|\nu - u| < \varepsilon$ on $(\tilde{E} \cup K_{l-1})^\sim \cup (\partial U_l \cap \Omega)$.*

This can be deduced from Lemma 6 in the same way that Lemma 7 was deduced from Lemma 4.

3.2. It will now be shown that (c) implies (a) in Theorem 1. Suppose that (c) holds, let u be a bounded function in $\mathcal{H}(E)$ and let $\varepsilon > 0$. In view of Lemma 7 there is a bounded function u_0 in $\mathcal{H}(\tilde{E})$ such that $|u_0 - u| < 2^{-1}\varepsilon$ on $\tilde{E}$. Let (K_l) be the sequence of compact subsets of Ω described at the beginning of §2.3 and let U_l be as in (3). Then u_0 is a bounded member of $\mathcal{H}((\tilde{E} \cup K_0)^\sim)$, since $K_0 = \{y\} \subseteq E$.

We proceed inductively as follows. Suppose that $l \in \mathbb{N}$ and that there is a bounded harmonic function u_{l-1} on an open set ω_{l-1} that contains $(\tilde{E} \cup K_{l-1})^\sim$. It follows from a result of Gauthier, Goldstein and Ow (see [11] and [12], or [9, Theorem 3.5]) that there is a function ν_{l-1} , which is harmonic on Ω apart from isolated singularities in $\Omega \setminus (E \cup K_{l-1})^\sim$, and which satisfies $|\nu_{l-1} - u_{l-1}| < 2^{-l-4}\varepsilon$ on $(\tilde{E} \cup K_{l-1})^\sim$. Only finitely many of the singularities lie in the compact set K_{l+2} , and these singularities can be joined to $\mathcal{A}$ by a continuous path lying in $\Omega \setminus (\tilde{E} \cup K_{l-1})^\sim$. A pole-pushing argument now yields a bounded member w_{l-1} of $\mathcal{H}((\tilde{E} \cup K_{l-1})^\sim \cup K_{l+2})$ such that $|w_{l-1} - u_{l-1}| < 2^{-l-3}\varepsilon$ on $\tilde{E} \cup K_{l-1}$. It follows from Lemma 8 that there is a bounded member s_{l-1} of $\mathcal{H}((\tilde{E} \cup K_{l-1})^\sim \cup (\overline{U}_l \cap \Omega))$ such that $|s_{l-1} - u_{l-1}| < 2^{-l-2}\varepsilon$ on $(\tilde{E} \cup K_{l-1})^\sim$. A further approximation of s_{l-1} by a harmonic function on Ω with isolated singularities, followed by pole pushing, yields u_l in $\mathcal{H}((\tilde{E} \cup K_l)^\sim)$ such that $|u_l - u_{l-1}| < 2^{-l-1}\varepsilon$ on $(\tilde{E} \cup K_{l-1})^\sim$.

Since $\bigcup_l K_l = \Omega$, the sequence (u_l) converges locally uniformly on Ω to a function ν in $\mathcal{H}(\Omega)$ such that

$$|\nu - u| \leq \sum_{l=1}^{\infty} |u_l - u_{l-1}| + |u_0 - u| < \varepsilon \quad \text{on } E.$$

Thus condition (a) of Theorem 1 holds.

4. PROOF OF THEOREM 1 (a) $\Rightarrow$ (c) AND (b) $\Rightarrow$ (c)

4.1. The following lemma simultaneously shows that condition (c)(i) is implied by (a) and by (b) in Theorem 1.

Lemma 9. *Suppose that, for each bounded u in $\mathcal{H}(E)$ and each positive number ε , there exists ν in $\mathcal{S}(\tilde{E})$ such that $|\nu - u| < \varepsilon$. Then $\Omega \setminus \tilde{E}$ and $\Omega \setminus E$ are thin at the same points of E .*

To prove this we first note that, if Ω does not have a Green function (so $n = 2$) and $\tilde{E} = \Omega$, then $E = \Omega$. To see this, suppose otherwise, let $\varepsilon > 0$ and let $u(x) = \log(|x - y|/|x - z|)$, where y and z are distinct points of $\tilde{E} \setminus E$. Then u is a bounded member of $\mathcal{H}(E)$ and so, by hypothesis, there exists ν_ε in $\mathcal{S}(\Omega)$ such that

$|\nu_\varepsilon - u| < \varepsilon$ on E . Hence, by Theorem C, ν_ε is a lower bounded superharmonic function on Ω , and so is constant. Since ε can be arbitrarily small, this forces u to be constant on E and so leads to the contradictory conclusion that E is contained in a straight line or circle. We now discount the trivial case where $E = \Omega$. We define $\Omega_0 = \Omega$ if Ω has a Green function, and $\Omega_0 = \mathbb{R}^2 \setminus B$ otherwise, where B is a closed ball in $\Omega \setminus \tilde{E}$. Thus, in either case, Ω_0 has a Green function $G(\cdot, \cdot)$. All reduced functions below are with respect to superharmonic functions on Ω_0 .

We next claim that, if U is a component of $\tilde{E} \setminus E$, then each point of $\partial U \cap \Omega$ is regular for the Dirichlet problem on U . To see this, let $\varepsilon > 0$, let $y \in U$ and $u(x) = -G(x, y)$. Then there exists ν in $\mathcal{S}(\tilde{E})$ such that $|\nu - u| < \varepsilon$ on E and so, by Theorem C, $\nu - u + \varepsilon$ is a positive member of $\mathcal{S}(\tilde{E})$. Clearly $\nu - u + \varepsilon \geq G_U(y, \cdot)$, where $G_U(y, \cdot)$ is the Green function for U with pole y . Hence

$$\limsup_{x \rightarrow z} G_U(y, x) \leq 2\varepsilon \quad (z \in \partial U \cap \Omega).$$

The claim now follows in view of the arbitrary nature of ε .

Suppose that $\Omega \setminus E$ is nonthin at a point z of E , and let δ be a positive number such that $\overline{B(z, \delta)} \subset \Omega_0 \cap \Omega$, where $B(z, \delta)$ denotes the open ball of centre z and radius δ . Further, let $0 < \rho < \delta$, and let K be a compact subset of $B(z, \rho) \setminus E$ such that

$$(6) \quad u_\rho(z) \geq 1/2,$$

where $u_\rho = \hat{R}_1^K$.

Let $\varepsilon > 0$. Since $-u_\rho$ is a bounded member of $\mathcal{H}(E)$, we know by hypothesis that there exists ν_ε in $\mathcal{S}(\tilde{E})$ such that $|\nu_\varepsilon + u_\rho| < \varepsilon$ on E . By lower semicontinuity $\nu_\varepsilon + u_\rho > -\varepsilon$ on some open set which contains E , and hence contains $\partial \tilde{E} \cap \Omega$. We define V to be the union of $B(z, \delta)$ with $\tilde{E} \setminus E$. Then $\partial V \cap \Omega \subseteq E \cup \partial B(z, \delta)$. Also, $\partial^\infty V \cap \partial^\infty \Omega$ has zero harmonic measure for V . (For example, this can be seen from Theorem C if we define s to be the harmonic measure in V of the set $\partial^\infty V \cap \partial^\infty \Omega$, since all points of $\partial V \cap (\Omega \setminus \overline{B(z, \delta)})$ are regular.) Thus, if we define

$$V_m = \{x \in V : \text{dist}(x, \tilde{E}) < m^{-1}\} \quad (m \in \mathbb{N}),$$

then, for large values of m ,

$$-\hat{R}_{u_\rho}^{\Omega_0 \setminus V_m} - \nu_\varepsilon = H_{-u_\rho}^{V_m} - \nu_\varepsilon \leq H_{-u_\rho - \nu_\varepsilon}^{V_m} \leq \varepsilon \quad \text{on } E \cap B(z, \delta).$$

Hence

$$u_\rho - \hat{R}_{u_\rho}^{\Omega_0 \setminus V_m} < 2\varepsilon \quad \text{on } E \cap B(z, \delta).$$

If we let $m \rightarrow \infty$ and observe that ε can be arbitrarily small, then we see that

$$(7) \quad \hat{R}_{u_\rho}^{\Omega_0 \setminus (\tilde{E} \cap V)}(z) = u_\rho(z).$$

Let $w_\rho = R_1^{B(z, \rho)}$. Then $w_\rho \geq u_\rho$ on Ω_0 , so it follows from (6) and (7) that

$$\hat{R}_{w_\rho}^{\Omega_0 \setminus (\tilde{E} \cap B(z, \delta))}(z) \geq \hat{R}_{u_\rho}^{\Omega_0 \setminus (\tilde{E} \cap V)}(z) = u_\rho(z) \geq \frac{1}{2}.$$

Since ρ can be arbitrarily small, we conclude (see [6, 1.XI.3(a'')]) that $\Omega_0 \setminus (\tilde{E} \cap B(z, \delta))$, and hence $\Omega \setminus \tilde{E}$, is nonthin at z .

This establishes Lemma 9.

4.2. The following lemma simultaneously shows that condition (c)(ii) is implied by (a) and by (b) in Theorem 1.

Lemma 10. *Suppose that, for each bounded function u in $\mathcal{H}(E)$ and each positive number ε , there exists ν in $\mathcal{S}(\Omega)$ such that $|\nu - u| < \varepsilon$ on E . Then condition (c)(ii) of Theorem 1 holds.*

To prove this, we suppose that condition (c)(ii) fails to hold. Thus there is a compact subset K of Ω and a sequence (z_m) of distinct points of $E \setminus K$ such that (z_m) converges to a point of $\partial^\infty \Omega$, and such that

- (I) $\Omega \setminus \tilde{E}$ is nonthin at each point z_m ,
- (II) $\Omega \setminus (\tilde{E} \cup K)^\sim$ is thin at each point z_m .

If Ω does not have a Green function and $(\tilde{E} \cup K)^\sim = \Omega$, then $\tilde{E} = \Omega$ and so $E = \Omega$ (see §4.1). We now dismiss the trivial case where $E = \Omega$. We define $\Omega_0 = \Omega$ if Ω has a Green function, and $\Omega_0 = \mathbb{R}^2 \setminus B$ otherwise, where B is a closed ball in $\Omega \setminus (\tilde{E} \cup K)^\sim$. All reduced functions below are with respect to superharmonic functions on Ω_0 . Let $V = (\tilde{E} \cup K)^\sim \setminus (\tilde{E} \cup K)$. We recall that each point of $\partial^\infty V \cap \partial^\infty \Omega$ is regular for the Dirichlet problem on V . (See the remark preceding Theorem C.)

For each m in $\mathbb{N}$ we choose a positive number δ_m in $(0, m^{-1})$ such that the balls $\overline{B(z_m, \delta_m)}$ are pairwise disjoint and are contained in $(\Omega_0 \cap \Omega) \setminus K$. Further, the numbers δ_m should be small enough so that, if $W = V \cup (\bigcup_m B(z_m, \delta_m))$, then each point of $\partial^\infty W \cap \partial^\infty \Omega$ is regular for the Dirichlet problem on W . (This is possible in view of Wiener's criterion and the corresponding property of V .) Next we choose η_m in $(0, \delta_m)$ sufficiently small so that the functions w_m defined by

$$w_m = R_1^{B(z_m, \eta_m)}$$

satisfy $w_m < 2^{-m}$ on $\Omega_0 \setminus B(z_m, \delta_m)$. In view of property (II) above we can also arrange, by reducing the value of η_m , if necessary, that

$$(8) \quad \widehat{R}_{w_m}^{\Omega_0 \setminus [B(z_m, \delta_m) \cap (\tilde{E} \cup K)^\sim]}(z_m) < \frac{1}{4}$$

(see [6, 1.XI.3(a'')]).

In view of property (I) above, we can choose (for each m) a compact subset L_m of $B(z_m, \eta_m) \setminus \tilde{E}$ such that

$$(9) \quad u_m(z_m) > 3/4,$$

where

$$u_m = \widehat{R}_{w_m}^{L_m}.$$

Clearly $u_m \in \mathcal{H}(\Omega_0 \setminus L_m)$ for each m . Since $u_m \leq w_m < 2^{-m}$ on $\Omega_0 \setminus B(z_m, \delta_m)$, we see that the series $\sum_m u_m$ converges locally uniformly on Ω to a function u such that $0 \leq u < 2$ on Ω and $u \in \mathcal{H}(\tilde{E} \cup K)$. By hypothesis there exists ν in $\mathcal{S}(\Omega)$ such that $|\nu + u| < 1/6$ on E . By Theorem C and lower semicontinuity $\nu + u > -1/6$ on an open set U which contains $\tilde{E}$. Further, we may arrange (by truncation) that ν is bounded above on Ω , so there is a positive constant a such that $|\nu + u| < a$ on the compact set K . Let $f = \chi_K$. Then, in view of our choice of (δ_m) , we see that $H_f^W(x) \rightarrow 0$ as $x \rightarrow \mathcal{A}$.

We now choose m' large enough such that $H_f^W < (6a)^{-1}$ on $B(z_{m'}, \delta_{m'})$, and define $V' = V \cup B(z_{m'}, \delta_{m'})$ and

$$V_k = \{x \in V' : \text{dist}(x, (\tilde{E} \cup K)^\sim) < k^{-1}\} \quad (k \in \mathbb{N}).$$

We know (cf. §4.1) that $\partial^\infty V_k \cap \partial^\infty \Omega$ has zero harmonic measure for each component of V_k . Hence, for large values of k , we see that

$$\begin{aligned} -\widehat{R}_u^{\Omega_0 \setminus V_k}(x) - \nu(x) &\leq H_{-u-\nu}^{V_k}(x) \leq \frac{1}{6} + aH_f^{V_k}(x) \\ &\leq \frac{1}{6} + aH_f^W(x) < \frac{1}{3} \quad (x \in \tilde{E} \cap B(z_{m'}, \delta_{m'})), \end{aligned}$$

and so

$$u(x) - \widehat{R}_u^{\Omega_0 \setminus V_k}(x) < \frac{1}{2} \quad (x \in E \cap B(z_{m'}, \delta_{m'})).$$

Thus, if we let $k \rightarrow \infty$, we obtain

$$(10) \quad u(z_{m'}) - \widehat{R}_u^{\Omega_0 \setminus [V' \cap (\tilde{E} \cup K)^\sim]}(z_{m'}) \leq \frac{1}{2}.$$

However, since

$$\widehat{R}_u^A = \sum_m \widehat{R}_{u_m}^A$$

for any set A (see [6, 1.VI.3(f)]), we obtain

$$\begin{aligned} u(z_{m'}) - \widehat{R}_u^{\Omega_0 \setminus [V' \cap (\tilde{E} \cup K)^\sim]}(z_{m'}) &\geq u_{m'}(z_{m'}) - \widehat{R}_{u_{m'}}^{\Omega_0 \setminus [V' \cap (\tilde{E} \cup K)^\sim]}(z_{m'}) \\ &> \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \end{aligned}$$

in view of (8) and (9) and the fact that $u_{m'} \leq w_{m'}$. This contradicts (10), so condition (c)(ii) must hold.

This completes the proof of Theorem 1.

5. PROOF OF THEOREM 2 AND COROLLARY 1

5.1. Theorem 2 follows immediately from Theorem 1 and the following result due to Labrèche and Bensouda and Gauthier (see [14], [10], [4], or see [9, Theorems 3.17 and 6.11]). In this connection we observe that the proof that (c) implies (b) in Theorem 1 (see §2) yields a continuous approximating function ν if the original function u is continuous.

Theorem D. *Let Ω be an open set in $\mathbb{R}^n$ and E be a relatively closed subset of Ω . The following are equivalent:*

- (a) *for each u in $C(E) \cap \mathcal{H}(E^0)$ and each positive number ε there exists ν in $\mathcal{H}(E)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (b) *for each u in $C(E) \cap \mathcal{S}(E^0)$ and each positive number ε there exists a continuous function ν in $\mathcal{S}(E)$ such that $|\nu - u| < \varepsilon$ on E ;*
- (c) *$\Omega \setminus E$ and $\Omega \setminus E^0$ are thin at the same points of E .*

5.2. We will now prove Corollary 1. Let $n = 2$. Clearly (a) implies (b).

Suppose that (b) holds. It follows from Theorem 1 and Lemma 2 that $\partial E \cap \Omega = \partial \tilde{E} \cap \Omega$, and from the definition of $\tilde{E}$ that $\partial^\infty E \cap \partial^\infty \Omega = \partial^\infty \tilde{E} \cap \partial^\infty \Omega$. Hence $\partial E = \partial \tilde{E}$. Let K be a compact subset of Ω . Then, by Theorem 1, there is a compact subset L of Ω such that $\Omega \setminus \tilde{E}$ and $\Omega \setminus (\tilde{E} \cup K)^\sim$ are thin at the same points of $E \setminus L$. We may assume that $K \subseteq L$. It follows from Lemma 1 that $\partial \tilde{E} \cap (\Omega \setminus L) = \partial((\tilde{E} \cup K)^\sim) \cap (\Omega \setminus L)$. As before we deduce that $\partial \tilde{E} \setminus L = \partial((\tilde{E} \cup K)^\sim) \setminus L$. Thus condition (c) holds.

Finally, suppose that condition (c) holds, and let z be a point of E at which $\Omega \setminus \tilde{E}$ is thin. Then, since $n = 2$, there are arbitrarily small circles centered at z which are contained in $\tilde{E}$. Hence $z \in (\tilde{E})^0$. By condition (c)(i) it follows that $z \notin \partial E$, so $z \in E^0$ and $\Omega \setminus E^0$ is certainly thin at z . This shows that condition (c)(i) of Theorem 2 holds. Let K be a compact subset of Ω . Then, by condition (c)(ii) there is a compact subset L of Ω such that $\partial \tilde{E} \setminus L = \partial((\tilde{E} \cup K)^\sim) \setminus L$. Let z be a point of $E \setminus L$ at which $\Omega \setminus (\tilde{E} \cup K)^\sim$ is thin. Then, as in the previous paragraph, we see that $z \in ((\tilde{E} \cup K)^\sim)^0$. Thus $z \notin \partial \tilde{E}$ and so $z \in (\tilde{E})^0$. It follows that $\Omega \setminus \tilde{E}$ is also thin at z . Thus condition (c)(ii) of Theorem 2 holds. We can now apply Theorem 2 to see that condition (a) of Corollary 1 holds.

This completes the derivation of Corollary 1 from Theorems 1 and 2.

6. A REFINEMENT OF THEOREM 1

A further equivalent condition is added to Theorem 1 by the following result. It yields better-than-uniform approximation near points of $\partial^\infty E \cap \partial^\infty \Omega$ which are regular for the Dirichlet problem on Ω .

Theorem 3. *Let Ω be a connected open set with Green function $G_\Omega(\cdot, \cdot)$, and let $y \in \Omega$. If condition (c) of Theorem 1 holds, then for each bounded function u in $\mathcal{H}(E)$ and each positive number ε there exists ν in $\mathcal{H}(\Omega)$ such that*

$$|\nu(x) - u(x)| < \varepsilon \min\{1, G_\Omega(y, x)\} \quad (x \in E).$$

The proof of this result is essentially the same as the argument given in §§2, 3. For example, the right-hand side of (1) and (2) should now be replaced by

$$1 - \varepsilon \min\{1, G_\Omega(y, x)\},$$

so that the function w of Lemma 3 satisfies

$$w(x) \leq \varepsilon \min\{1, G_\Omega(y, x)\} \quad (x \in E).$$

This allows us to arrange that the function ν of Lemma 4 satisfies

$$|\nu(x) - u(x)| < \varepsilon \min\{1, G_\Omega(y, x)\} \quad (x \in E).$$

The appropriate pole-pushing estimates and approximation results can be found in [3] or [9, Chapter 3].

A similar refinement of Theorem 2 also holds.

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