SOME NEW HOMOGENEOUS EINSTEIN METRICS ON SYMMETRIC SPACES

MEGAN M. KERR

ABSTRACT. We classify homogeneous Einstein metrics on compact irreducible symmetric spaces. In particular, we consider symmetric spaces with $\text{rank}(M) > 1$, not isometric to a compact Lie group. Whenever there exists a closed proper subgroup G of $\text{Isom}(M)$ acting transitively on M we find all G -homogeneous (non-symmetric) Einstein metrics on M .

1. INTRODUCTION

In this paper we look at compact symmetric spaces presented homogeneously, i.e. as $M = G/H$, where $G = \text{Isom}_0(M)$ is simple, and we consider the cases where there exists a closed subgroup $G' \subset G$ which acts transitively on M . Denote by H' the isotropy subgroup in G' , then $M = G'/H'$. Since G' is smaller than G , we expect more G' -invariant metrics on M than G -invariant metrics, and thus we can hope for non-symmetric G' -invariant Einstein metrics on our symmetric space M . We find the following.

Lemma 1.1. *Let M be a compact irreducible symmetric space of rank > 1 , M not isometric to a compact Lie group with biinvariant metric. Let $G = \text{Isom}_0(M)$, and $M = G/H$. Then there exists a subgroup $G' \subset G$ acting transitively on $M \Leftrightarrow$*

1. $G = \text{SO}(2n)$, $H = \text{U}(n)$, $G' = \text{SO}(2n-1)$, $H' = \text{U}(n-1)$ ($n \geq 4$);
2. $G = \text{SU}(2n)$, $H = \text{Sp}(n)$, $G' = \text{SU}(2n-1)$, $H' = \text{Sp}(n-1)$ ($n \geq 3$);
3. $G = \text{SO}(7)$, $H = \text{SO}(2)\text{SO}(5)$, $G' = \text{G}_2$, $H' = \text{U}(2)$ ($\text{U}(2) \subset \text{SU}(3)$);
4. $G = \text{SO}(8)$, $H = \text{SO}(3)\text{SO}(5)$, $G' = \text{Spin}(7)$, $H' = \text{SO}(4)$ ($\text{SO}(4) \subset \text{G}_2$).

Theorem 1.2. *Among the compact irreducible symmetric spaces of rank > 1 , not isometric to a Lie group with a biinvariant metric, only $G_2^+(\mathbb{R}^7)$, $G_3^+(\mathbb{R}^8)$, and $\text{SO}(2n)/\text{U}(n)$, for $n \geq 4$, carry non-symmetric homogeneous Einstein metrics. The Grassmannians $G_2^+(\mathbb{R}^7)$ and $G_3^+(\mathbb{R}^8)$ each carry two and $\text{SO}(2n)/\text{U}(n)$ carries one; the only homogeneous Einstein metric on $\text{SU}(2n)/\text{Sp}(n)$ is the symmetric metric.*

Let $\mathcal{M}_{G'}$ denote the space of G' -invariant metrics of volume one. Our results are summarized in the following table.

G/H	G'/H'	$\dim \mathcal{M}_{G'}$	no. Einstein
$\text{SO}(2n)/\text{U}(n)$	$\text{SO}(2n-1)/\text{U}(n-1)$	1	2
$\text{SU}(2n)/\text{Sp}(n)$	$\text{SU}(2n-1)/\text{Sp}(n-1)$	2	1
$\text{SO}(7)/\text{SO}(2)\text{SO}(5)$	$\text{G}_2/\text{U}(2)$	2	3
$\text{SO}(8)/\text{SO}(3)\text{SO}(5)$	$\text{Spin}(7)/\text{SO}(4)$	2	3

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The analogous results for symmetric spaces of rank 1 were studied in [Z] and for compact Lie groups with biinvariant metrics in [DA-Z].

Remark 1.3. The following result should be true for all compact irreducible symmetric spaces M , up to diffeomorphism: If G is a compact connected Lie group acting transitively and effectively on M , then G is conjugate to a subgroup of $\text{Isom}_0(M)$. This would imply that Theorem 1.2 classifies *all* homogeneous Einstein metrics on compact irreducible symmetric spaces of rank > 1 , not isometric to a Lie group. Such a result is well known for rank 1 symmetric spaces, but does not seem to be known for all symmetric spaces of rank > 1 . Partial results can be found in [O2], [O3], [S], [T].

For example, in [O3, Thm. 1] Oniščik showed that if M is diffeomorphic to $G_{2k}(\mathbb{R}^n)$ for n even, $n > 5$, $1 < k < \frac{n-2}{2}$, or for n odd, $2 < k < \frac{n-3}{2}$, and if a compact connected Lie group G acts transitively and effectively on M , then G is conjugate to $\text{SO}(n)$ with the standard action. Tsukada proved in [T] that if M is diffeomorphic to $G_{2k+1}(\mathbb{R}^{2n})$, and G is compact, connected, and simple, then if G acts transitively and effectively, the action of G is conjugate to the standard action.

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2. PRELIMINARIES

A Riemannian manifold (M, g) is Einstein if $\text{Ric}(X, Y) = \lambda g(X, Y)$ for some constant λ , for all vector fields X, Y .

We say a Riemannian manifold (M, g) is a symmetric space if for all $p \in M$ there exists an isometry $\sigma_p : M \rightarrow M$ such that $\sigma_p(p) = p$ and $(d\sigma_p)_p = -\text{Id}$. Symmetric spaces make up a class of manifolds which includes spheres, projective spaces, and Grassmannians; their geometry is well understood. In fact, every symmetric space is homogeneous.

A manifold M is defined to be G -homogeneous if we have a Riemannian metric g and a closed subgroup $G \subset \text{Isom}(M, g)$ such that for any p and $q \in M$, there exists a $g \in G$ with $g(p) = q$. We write $H_p = \{g \in G \mid g(p) = p\}$, called the isotropy subgroup corresponding to p . Notice H_p is compact, since $H_p \subset \text{O}(T_p M)$. Via the map $g \mapsto g(p)$ we identify the two manifolds G/H_p and M . Any two isotropy subgroups H_p and H_q are conjugate: if $q = g(p)$, then $g^{-1}H_q g = H_p$, hence we will usually suppress the point p .

Given a homogeneous manifold G/H , where G is compact and H is closed, what metrics can we put on G/H so that G acts by isometries?

Just as a left-invariant metric on a Lie group is determined by any inner product on its Lie algebra, a G -invariant metric on G/H is determined by an inner product on $\mathfrak{g}/\mathfrak{h} \cong T_{[H]}(G/H)$, with the additional requirement that the inner product be $\text{Ad}(H)$ -invariant. We identify the quotient $\mathfrak{g}/\mathfrak{h}$ with an $\text{Ad}(H)$ -invariant complement $\mathfrak{p}$ to $\mathfrak{h}$ in $\mathfrak{g}$; compactness of H guarantees such a $\mathfrak{p}$ exists. If $\mathfrak{g}$ is semisimple then the Killing form is $\text{Ad}(H)$ -invariant and we can use it to define $\mathfrak{p} = \mathfrak{h}^\perp$. We want to consider all $\text{Ad}(H)$ -invariant inner products on $\mathfrak{p}$.

A homogeneous space $M = G/H$ is said to be isotropy irreducible if the isotropy action, denoted $\chi : H \rightarrow \text{GL}(T_p M)$, or equivalently $\text{Ad} : H \rightarrow \text{GL}(\mathfrak{p})$, is an irreducible representation of H . When this is the case, the G -invariant metric on G/H is unique, up to scaling, and it is Einstein. When G/H is a symmetric space

with G simple and $G = \text{Isom}_0(G/H)$, then G/H is an irreducible symmetric space. (In fact, the only irreducible symmetric space with G not simple is $(K \times K)/\Delta K$, for K a compact simple Lie group, and $(K \times K)/\Delta K$ is isometric to K with a biinvariant metric.)

In 1962 A.L. Oniščik classified all simple compact Lie algebras $\mathfrak{g}$ with subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$, such that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$. In terms of transitive group actions, let G be the simply connected compact Lie group corresponding to $\mathfrak{g}$ and let G' , G'' be subgroups corresponding to $\mathfrak{g}'$, $\mathfrak{g}''$, respectively, then $G/G' = G''/(G' \cap G'')$ and $G/G'' = G'/(G' \cap G'')$. When G/G' or G/G'' is a symmetric space, Oniščik's list tells us when a subgroup of G still acts transitively. Here are the symmetric spaces on his list [O1]: (See the appendix of this paper for the non-symmetric homogeneous spaces on his list.)

$$\begin{array}{lll}
\text{SO}(2n)/\text{SO}(2n-1) = \text{U}(n)/\text{U}(n-1) & & = S^{2n-1} \\
\text{SO}(2n)/\text{SO}(2n-1) = \text{SU}(n)/\text{SU}(n-1) & & = S^{2n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) = \text{Sp}(n)/\text{Sp}(n-1) & & = S^{4n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) = \text{Sp}(n)\text{U}(1)/\text{Sp}(n-1)\text{U}(1) & & = S^{4n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) = \text{Sp}(n)\text{Sp}(1)/\text{Sp}(n-1)\text{Sp}(1) & & = S^{4n-1} \\
\text{SO}(7)/\text{SO}(6) & = \text{G}_2/\text{SU}(3) & = S^6 \\
\text{SO}(8)/\text{SO}(7) & = \text{Spin}(7)/\text{G}_2 & = S^7 \\
\text{SO}(16)/\text{SO}(15) & = \text{Spin}(9)/\text{Spin}(7) & = S^{15} \\
\text{SU}(2n)/\text{U}(2n-1) & = \text{Sp}(n)/\text{Sp}(n-1)\text{U}(1) & = \mathbb{C}P^{2n} \\
\text{SO}(2n)/\text{U}(n) & = \text{SO}(2n-1)/\text{U}(n-1) & = \text{spec. orth. cx. str. on } \mathbb{R}^{2n} \\
\text{SU}(2n)/\text{Sp}(n) & = \text{SU}(2n-1)/\text{Sp}(n-1) & = \text{spec. orth. quat. str. on } \mathbb{C}^{2n} \\
\text{SO}(7)/\text{SO}(2)\text{SO}(5) & = \text{G}_2/\text{U}(2) & = G_2^+(\mathbb{R}^7) \\
\text{SO}(8)/\text{SO}(3)\text{SO}(5) & = \text{Spin}(7)/\text{SO}(4) & = G_3^+(\mathbb{R}^8).
\end{array}$$

Each of these symmetric spaces in the left-hand presentation is an irreducible symmetric space. Up to scaling, each has exactly one Einstein metric, the symmetric metric, homogeneous with respect to the left-hand presentation. However, with respect to the right-hand presentation, only the sixth and seventh symmetric spaces are isotropy irreducible.

The first nine examples are discussed in [Z]. In this paper we consider the last four examples in the table. Of these, the first is originally described in [W-Z]. On $\text{SU}(2n)/\text{Sp}(n)$, the only homogeneous Einstein metric is the original one. However, the last two spaces each carry two new Einstein metrics, homogeneous with respect to the right-hand presentation.

For any homogeneous space $M = G/H$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ on the Lie algebra level, we parametrize the space of G -invariant metrics on M by decomposing $\mathfrak{p}$ into its $\text{Ad}(H)$ -irreducible subspaces, $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_k$. If the $\mathfrak{p}_i$'s are pairwise inequivalent representations, a G -homogeneous metric is determined by an inner product on $\mathfrak{p}$ of the form $\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp \cdots \perp x_k Q|_{\mathfrak{p}_k}$, for Q an $\text{Ad}(H)$ -invariant inner product and $x_i > 0$ for all i . If $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are equivalent for some i

and j , then $\langle \mathfrak{p}_i, \mathfrak{p}_j \rangle$ does not necessarily vanish; however, in each of the examples in this paper we have $\mathfrak{p}_i \not\propto \mathfrak{p}_j$ for all $i \neq j$.

Assume G/H is compact, and let $S(g)$ denote the scalar curvature of g . Einstein metrics are the critical points of the total scalar curvature functional

$$T(g) = \int_M S(g) d\text{vol}_g$$

on the space $\mathcal{M}$ of Riemannian metrics of volume one [Ber], [H]. Let $\mathcal{M}_G$ denote the set of all G -invariant metrics of volume one on M . Notice that on $\mathcal{M}_G$, $T(g) \simeq S(g)$. Furthermore, critical points of $T|_{\mathcal{M}_G}$ are precisely G -invariant Einstein metrics of volume one [Bes, p.121].

If for our homogeneous space G/H , every homogeneous metric is diagonal, i.e., $\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp \cdots \perp x_k Q|_{\mathfrak{p}_k}$, with $x_i > 0$ for all i , then we use equation (1.3) for the scalar curvature given in [W-Z].

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i \ j} \frac{x_k}{x_i x_j}.$$

In their formula, for each i , $-B|_{\mathfrak{p}_i} = b_i Q|_{\mathfrak{p}_i}$, where B denotes the Killing form, and $d_i = \dim(\mathfrak{p}_i)$; the triple $\binom{k}{i \ j} = \sum Q([X_\alpha, X_\beta], X_\gamma)^2$, summed over $\{X_\alpha\}$, $\{X_\beta\}$, and $\{X_\gamma\}$: Q -orthonormal bases for $\mathfrak{p}_i$, $\mathfrak{p}_j$, and $\mathfrak{p}_k$, respectively. Notice $\binom{k}{i \ j}$ is symmetric in all three entries.

We are now ready to prove the theorem.

3. $\text{SO}(2n)/\text{U}(n)$

We begin with the symmetric space $\text{SO}(2n)/\text{U}(n)$. Consider the space of orthogonal complex structures on $\mathbb{R}^{2n}$, and let M_0 be the connected component containing J_0 , the complex structure represented by $\begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix}$ with respect to the standard basis on $\mathbb{R}^{2n}$. We will show that $M_0 \cong \text{SO}(2n)/\text{U}(n)$.

Let $J \in M_0$. Since J is an orthogonal complex structure $\|Jv\| = \|v\|$ and $J^2 = -\text{Id}$. We construct an orthonormal basis $\{v_\alpha\}$ for $\mathbb{R}^{2n}$ such that for $1 \leq i \leq n$, $Jv_i = v_{n+i}$ and $Jv_{n+i} = -v_i$. Let $v_1 = e_1$, and let $v_{n+1} = Jv_1$. We have $\langle v_1, Jv_1 \rangle = \langle Jv_1, J^2 v_1 \rangle = -\langle v_1, Jv_1 \rangle$, hence $\{v_1, v_{n+1}\}$ is an orthonormal basis for a J -invariant subspace. Let v_2 be any unit vector in $\text{span}\{v_1, v_{n+1}\}^\perp$, and let $v_{n+2} = Jv_2$.

Continue up to v_n, v_{2n} . With respect to the basis $\{v_\alpha\}$, $J = \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix}$. Let P be the change of basis transformation from the standard basis to $\{v_\alpha\}$, then $J = PJ_0P^{-1}$. The hypothesis that J be in the connected component containing J_0 corresponds exactly to the fact that P must be in $\text{SO}(2n)$. Via conjugation, $\text{SO}(2n)$ acts transitively on M_0 .

The isotropy subgroup of J_0 is the set of all $P \in \text{SO}(2n)$ such that $PJ_0 = J_0P$. If we identify $\mathbb{R}^n \oplus \mathbb{R}^n \cong \mathbb{C}^n$ via $(u, v) \mapsto u + iv$ then J_0 is multiplication by i and hence $PJ_0 = J_0P$ implies $P \in \text{SO}(2n) \cap \text{GL}(n, \mathbb{C}) = \text{U}(n)$. Thus $M_0 \cong \text{SO}(2n)/\text{U}(n)$, where we have $\text{U}(n)$ embedded in $\text{SO}(2n)$ in the following way: $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. It is well known that $\text{SO}(2n)/\text{U}(n)$ is an irreducible symmetric space [W, p.287]. Let μ_n denote the standard complex n -dimensional representation of $\text{U}(n)$; the isotropy representation of $\text{U}(n)$ is $[\wedge^2 \mu_n]_{\mathbb{R}}$. Since $\wedge^2 \mu_n$ is unitary,

$[\wedge^2 \mu_n]_{\mathbb{R}}$ is the irreducible real representation whose complexification is isomorphic to the direct sum of $\wedge^2 \mu_n$ and its dual.

If we look at the low dimensional examples, for $n \leq 4$, we find $\mathrm{SO}(4)/\mathrm{U}(2) = S^2$, $\mathrm{SO}(6)/\mathrm{U}(3) = \mathbb{C}P^3$, and $\mathrm{SO}(8)/\mathrm{U}(4) = G_2^+(\mathbb{R}^8)$. For $n \geq 4$ the rank of the symmetric space is greater than one.

Notice that in our description above we let $v_1 = e_1$, so the subgroup $\mathrm{SO}(2n-1) \subset \mathrm{SO}(2n)$ fixing $\mathrm{span}\{e_1\}$ also acts transitively on M_0 . The isotropy subgroup of $\mathrm{SO}(2n-1)$ corresponding to J_0 is $\mathrm{U}(n-1) \subset \mathrm{SO}(2n-2)$, where $\mathrm{SO}(2n-2)$ is the subgroup fixing e_1 and e_{n+1} .

On the Lie algebra level we have, for $X, Y \in \mathfrak{gl}(n-1, \mathbb{R})$,

$$\mathfrak{u}(n-1) \cong \left\{ \begin{pmatrix} X & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & X \end{pmatrix} \middle| X = -X^t, Y = Y^t \right\} \subset \mathfrak{so}(2n-1);$$

$$\text{therefore, } \mathfrak{p} \cong \left\{ \begin{pmatrix} X & v & Y \\ -v^t & 0 & -w^t \\ Y & w & -X \end{pmatrix} \middle| X = -X^t, Y = -Y^t, v, w \in \mathbb{R}^{n-1} \right\}.$$

We find that $\mathfrak{p}$ decomposes into the sum of two irreducible representations. In fact $\mathrm{U}(n-1) \subset \mathrm{SO}(2n-2) \subset \mathrm{SO}(2n-1)$ gives rise to the following fibration:

$$\mathrm{SO}(2n-2)/\mathrm{U}(n-1) \rightarrow \mathrm{SO}(2n-1)/\mathrm{U}(n-1) \rightarrow \mathrm{SO}(2n-1)/\mathrm{SO}(2n-2) \cong S^{2n-2}.$$

Both base and fibre are irreducible symmetric spaces. Let $\mathfrak{p}_1$ denote the $\mathrm{Ad} \mathrm{SO}(2n-2)$ -invariant complement to $\mathfrak{so}(2n-2)$ in $\mathfrak{so}(2n-1)$; in our fibration $\mathfrak{p}_1$ corresponds to the tangent space of the base. The representation of $\mathrm{U}(n-1)$ on $\mathfrak{p}_1$ is the restriction of the standard representation of $\mathrm{SO}(2n-2)$ on $\mathfrak{p}_1 \cong \mathbb{R}^{2n-2}$ to $\mathrm{U}(n-1)$, which is $[\mu_{n-1}]_{\mathbb{R}}$, again irreducible. Let $\mathfrak{p}_2$ denote the $\mathrm{Ad} \mathrm{U}(n-1)$ -invariant complement to $\mathfrak{u}(n-1)$ in $\mathfrak{so}(2n-2)$; $\mathfrak{p}_2$ corresponds to the tangent space of the fibre. This representation of $\mathrm{U}(n-1)$ on $\mathfrak{p}_2$ is the irreducible isotropy representation of the fibre symmetric space, $[\wedge^2 \mu_{n-1}]_{\mathbb{R}}$ [W, p.287]. We have $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

The dimensions of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are $2(n-1)$ and $(n-1)(n-2)$, respectively. We see that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are clearly inequivalent representations of $\mathrm{U}(n-1)$ for $n \neq 4$, and for $n = 4$, while μ_3 and $\wedge^2 \mu_3$ are equivalent representations on $\mathrm{SU}(3)$, they are inequivalent on the center of $\mathrm{U}(3)$. We apply Schur's lemma to know that $\langle \mathfrak{p}_1, \mathfrak{p}_2 \rangle$ and $\mathrm{Ric}(\mathfrak{p}_1, \mathfrak{p}_2)$ must vanish. Thus any $\mathrm{SO}(2n-1)$ -homogeneous metric on M_0 must be of the form $\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2}$, where $Q(X, Y) = -\frac{1}{2} \mathrm{tr}(XY)$ and $x_1, x_2 > 0$. We express the scalar curvature in terms of x_1 and x_2 using [W-Z, (1.3)]. The equation is

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i \ j} \frac{x_k}{x_i x_j}.$$

Since for $\mathfrak{so}(k)$, $-B(X, Y) = (k-2) \mathrm{tr}(XY)$, we have $b_1 = b_2 = 2(2n-3)$. From the fibration it follows that $[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{u}(n-1) \oplus \mathfrak{p}_2$, $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{u}(n-1)$, hence the only nonzero triple (up to rearrangements) is $\binom{2}{11}$.

Let E_{ij} denote the skew-symmetric matrix in $\mathfrak{so}(2n-1)$ with 1 in the ij^{th} entry and -1 in the ji^{th} entry, and zeros everywhere else.

$$\mathfrak{p}_1 = \mathrm{span}\{-E_{in}, E_{n,n+i} \mid 1 \leq i \leq n-1\},$$

$$\mathfrak{p}_2 = \mathrm{span}\{E_{ij} - E_{n+i,n+j}, E_{i,n+j} - E_{j,n+i} \mid 1 \leq i < j \leq n-1\}.$$

We find $\binom{2}{11} = 2(n-1)(n-2)$, and then we substitute into the scalar curvature equation to find S . Let $\tilde{S}$ be the equation for S with the boundary constraint that volume = 1:

$$S = \frac{2(n-1)(2n-3)}{x_1} + \frac{2(n-1)(n-2)^2}{x_2} - \frac{(n-1)(n-2)x_2}{2x_1^2},$$

$$\tilde{S} = S - \lambda(x_1^{2(n-1)}x_2^{(n-1)(n-2)} - 1).$$

We find the partial derivatives of $\tilde{S}$:

$$\frac{\partial \tilde{S}}{\partial x_1} = \frac{-2(n-1)(2n-3)}{x_1^2} + \frac{(n-1)(n-2)x_2}{x_1^3} - 2(n-1)\lambda x_1^{2n-3}x_2^{(n-1)(n-2)},$$

$$\frac{\partial \tilde{S}}{\partial x_2} = \frac{-2(n-1)(n-2)^2}{x_2^2} - \frac{(n-1)(n-2)}{2x_1^2}$$

$$- (n-1)(n-2)\lambda x_1^{2(n-1)}x_2^{(n-1)(n-2)-1}.$$

Setting both equations equal to zero is equivalent to the following equation:

$$\frac{n-1}{2}x_2 + 2(n-2)\frac{x_1^2}{x_2} = (2n-3)x_1.$$

We find that the solutions are $x_1 = \frac{1}{2}x_2$, and $x_1 = \frac{(n-1)}{2(n-2)}x_2$. The second solution is a (non-symmetric) $\text{SO}(2n-1)$ -invariant Einstein metric, discovered earlier in [W-Z, §3, Ex.6]. The first solution is the $\text{SO}(2n)$ -invariant symmetric metric, but this is not obvious until we see how to compare them.

Let $\tilde{\mathfrak{p}}$ denote the Q -orthogonal complement to $\mathfrak{u}(n)$ in $\mathfrak{so}(2n)$:

$$\tilde{\mathfrak{p}} = \text{span}\{E_{ij} - E_{n+i, n+j}, E_{i, n+j} - E_{j, n+i} \mid 1 \leq i < j \leq n\}.$$

We must project $\mathfrak{p}$ to $\tilde{\mathfrak{p}}$. We take a basis element of $\mathfrak{p}_1$: $-E_{in}$. Under the embedding of $\mathfrak{so}(2n-1)$ in $\mathfrak{so}(2n)$, $-E_{in} \mapsto -E_{i+1, n+1}$. Next we write $-E_{i+1, n+1}$ as the sum of an element in $\mathfrak{u}(n)$ and an element in $\tilde{\mathfrak{p}}$:

$$-E_{i+1, n+1} = -\frac{1}{2}(E_{i+1, n+1} + E_{1, n+i+1}) - \frac{1}{2}(E_{i+1, n+1} - E_{1, n+i+1}).$$

This shows that an element of norm $= \sqrt{x_1}$ is sent to an element of norm $= \frac{1}{\sqrt{2}}$. A basis element of $\mathfrak{p}_2$ is $\frac{1}{\sqrt{2}}(E_{ij} - E_{n+i, n+j})$, which the embedding sends to $\frac{1}{\sqrt{2}}(E_{i+1, j+1} - E_{n+i+1, n+j+1})$, already in $\tilde{\mathfrak{p}}$; hence an element of norm $= \sqrt{x_2}$ is sent to an element of norm $= 1$. The symmetric metric on $\text{SO}(2n)/\text{U}(n)$ is given by the restriction of Q to $\tilde{\mathfrak{p}}$. Hence it corresponds to $\frac{2}{1} = \frac{x_2}{x_1}$, i.e., $x_1 = \frac{1}{2}x_2$.

To see that these metrics are distinct, we can compare the scale-invariant product $(S)^{\frac{d}{2}}(V)^{\frac{1}{2}}$, where S is the scalar curvature, V is the volume, and d is the dimension of M . The first metric has $S = \frac{2n(n-1)^2}{x_2}$ and $V = x_1^{2(n-1)}x_2^{(n-1)(n-2)}$, so that

$$(S)^{\frac{n(n-1)}{2}}(V)^{\frac{1}{2}} = 2^{\frac{(n-1)(n-2)}{2}}(n(n-1)^2)^{\frac{n(n-1)}{2}}.$$

The second metric has $S = \frac{2n(n-2)(n^2-n-1)}{(n-1)x_2}$, and $V = (\frac{n-1}{2(n-2)})^{2(n-1)}x_2^{n(n-1)}$, hence

$$(S)^{\frac{n(n-1)}{2}}(V)^{\frac{1}{2}} = \left(\frac{2n(n-2)(n^2-n-1)}{n-1} \right)^{\frac{n(n-1)}{2}} \left(\frac{n-1}{2(n-2)} \right)^{n-1}.$$

4. $\mathrm{SU}(2n)/\mathrm{Sp}(n)$

Our next example is the symmetric space $M = \mathrm{SU}(2n)/\mathrm{Sp}(n)$, an analogue of the previous example. This is the set of special orthogonal quaternionic structures on $\mathbb{C}^{2n}$. We identify $\mathbb{R}^{4n} \cong \mathbb{C}^{2n}$ via a fixed orthogonal complex structure I on $\mathbb{R}^{4n}$. An orthogonal quaternionic structure on $\mathbb{C}^{2n}$ is given by $J \in \mathrm{SO}(4n)$ such that $J^2 = -\mathrm{Id}$ and $IJ = -JI$. As a map from $\mathbb{C}^{2n}$ to itself, J is complex anti-linear, i.e., $J(\lambda v) = \bar{\lambda}J(v)$. We show that the set of all orthogonal quaternionic structures can be written homogeneously as $\mathrm{U}(2n)/\mathrm{Sp}(n)$, and we call the submanifold $M = \mathrm{SU}(2n)/\mathrm{Sp}(n)$ the set of special orthogonal quaternionic structures. We first observe that if $I = \begin{pmatrix} 0 & -\mathrm{Id}_{2n} \\ \mathrm{Id}_{2n} & 0 \end{pmatrix}$ and if we identify $\mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{H}$ via $(u, v) \mapsto u + jv$, then multiplication by j on $\mathbb{C}^{2n} = \mathbb{R}^{4n}$ becomes

$$J_0 = \begin{pmatrix} 0 & -\mathrm{Id}_n & 0 & 0 \\ \mathrm{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_n \\ 0 & 0 & -\mathrm{Id}_n & 0 \end{pmatrix},$$

which is the standard orthogonal quaternionic structure. We want to show that $\mathrm{U}(2n)$ acts transitively on $\{J \in \mathrm{SO}(4n) \mid J^2 = -\mathrm{Id} \text{ and } IJ = -JI\}$, and the isotropy subgroup is $\mathrm{Sp}(n)$. Since $\mathrm{U}(2n) = \mathrm{GL}(2n, \mathbb{C}) \cap \mathrm{SO}(4n)$, A is unitary when $A \in \mathrm{SO}(4n)$ and $AI = IA$, and for A unitary, AJA^{-1} is a quaternionic structure if J is: $(AJA^{-1})(AJA^{-1}) = -\mathrm{Id}$ and $(AJA^{-1})I = AJIA^{-1} = -AIIA^{-1} = -I(AJA^{-1})$. Furthermore, IJ is also a quaternionic structure:

$$(IJ)(IJ) = -I(IJ)J = -(-\mathrm{Id})^2, \text{ and } (IJ)I = -I(IJ).$$

Given an orthogonal quaternionic structure J , we construct a unitary basis of $\mathbb{C}^{2n}$ in which J is represented by the matrix J_0 . Let $v_1 = e_1$, the first element of the standard basis. Let $v_{n+1} = Jv_1$, $v_{2n+1} = Iv_1$, and $v_{3n+1} = IJv_1$. Clearly $\{v_1, Jv_1, Iv_1, IJv_1\}$ is an orthonormal basis for a 4-plane invariant under I , J , and IJ . Choose v_2 to be any unit vector in the orthogonal complement, and repeat the process above, continuing up to v_n , v_{2n} , v_{3n} , v_{4n} . Notice this is a unitary basis for $\mathbb{R}^{4n}$, since $v_{2n+i} = Iv_i$ for all $1 \leq i \leq 2n$.

The isotropy subgroup of $\mathrm{U}(2n)$ corresponding to J_0 is all $A \in \mathrm{U}(2n)$ such that $AJ_0 = J_0A$. I.e., A commutes with I , J , and IJ : A is quaternionic linear. We embed $\mathrm{Sp}(n) \subset \mathrm{U}(2n)$ via $A + jB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. The image of this embedding is contained in $\mathrm{SU}(2n)$, and we now restrict ourselves to the orbit of $\mathrm{SU}(2n)$, which is the symmetric space of special orthogonal quaternionic structures on $\mathbb{R}^{4n}$, or $\mathrm{SU}(2n)/\mathrm{Sp}(n)$. This symmetric space is irreducible; up to scaling the symmetric metric is the unique $\mathrm{SU}(2n)$ -invariant metric, and it is Einstein. Notice that when $n = 2$, $\mathrm{SU}(4)/\mathrm{Sp}(2) = S^5$; for $n \geq 3$ the rank of the symmetric space is greater than one.

Let $\tilde{\mathfrak{p}}$ be the orthogonal complement to $\mathfrak{sp}(n)$ in $\mathfrak{su}(2n)$ with respect to the inner product $Q(X, Y) = -\frac{1}{2} \operatorname{tr}(XY)$.

$$\begin{aligned} \text{We have } \mathfrak{su}(2n) &= \left\{ \begin{pmatrix} X & Y \\ -\overline{Y}^t & Z \end{pmatrix} \mid X, Z \in \mathfrak{u}(n), Y \in \mathfrak{gl}(n, \mathbb{C}), \operatorname{tr} Z = -\operatorname{tr} X \right\} \\ \text{and } \mathfrak{sp}(n) &\cong \left\{ \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} \mid X \in \mathfrak{u}(n), Y = Y^t \in \mathfrak{gl}(n, \mathbb{C}) \right\}, \\ \text{hence } \tilde{\mathfrak{p}} &= \left\{ \begin{pmatrix} X & \overline{Y} \\ Y & -\overline{X} \end{pmatrix} \mid X \in \mathfrak{su}(n), Y = Y^t \in \mathfrak{gl}(n, \mathbb{C}) \text{ (and } X = 0) \right\}. \end{aligned}$$

The isotropy representation of $\operatorname{Sp}(n)$ on $\tilde{\mathfrak{p}}$ is $[\wedge^2 \nu_n - \operatorname{Id}]_{\mathbb{R}}$, where ν_n is the standard representation of $\operatorname{Sp}(n)$ on $\mathbb{H}^n \cong \mathbb{C}^{2n}$. (The representation $\wedge^2 \nu_n$ is the sum of a complex $(2n+1)(n-1)$ -dimensional irreducible representation and a one-dimensional trivial representation. We denote by $\wedge^2 \nu_n - \operatorname{Id}$ the non-trivial summand.) We write $[\wedge^2 \nu_n - \operatorname{Id}]_{\mathbb{R}}$ for the real representation whose complexification is $\wedge^2 \nu_n - \operatorname{Id}$.

The subgroup $\operatorname{SU}(2n-1) \subset \operatorname{SU}(2n)$ fixing e_1 acts transitively on M , just as in the previous example. The isotropy subgroup of $\operatorname{SU}(2n-1)$ corresponding to J_0 is

$$\begin{aligned} H &= \left\{ \begin{pmatrix} A & 0 & -\overline{B} \\ 0 & 1 & 0 \\ B & 0 & \overline{A} \end{pmatrix} \in \operatorname{SU}(2n-1) \mid A + jB \in \operatorname{Sp}(n-1, \mathbb{C}) \right\} \\ &= \operatorname{Sp}(n-1) \subset \operatorname{SU}(2n-2) \end{aligned}$$

fixing e_{2n+1} . In $\mathfrak{su}(2n-1)$, for $X, Y \in \mathfrak{gl}(n-1, \mathbb{C})$,

$$\mathfrak{h} = \mathfrak{sp}(n-1) = \left\{ \begin{pmatrix} X & 0 & -\overline{Y} \\ 0 & 0 & 0 \\ Y & 0 & \overline{X} \end{pmatrix} \mid X \in \mathfrak{u}(n-1), Y = Y^t \right\}.$$

We denote by $\mathfrak{p}$ the Q -orthogonal complement to $\mathfrak{sp}(n-1)$ in $\mathfrak{su}(2n-1)$:

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & -\overline{u}^t & \overline{Y} \\ u & z & v^t \\ Y & -\overline{v} & -\overline{X} \end{pmatrix} \mid X \in \mathfrak{u}(n-1), Y = -Y^t, u, v \in \mathbb{C}^{n-1}, z = -2 \operatorname{tr} X \right\}.$$

We have the following fibration of our symmetric space, which tells us how to decompose $\mathfrak{p}$ into irreducible $\operatorname{Ad} \operatorname{Sp}(n-1)$ -invariant subrepresentations:

$$\operatorname{SU}(2n-2)/\operatorname{Sp}(n-1) \rightarrow \operatorname{SU}(2n-1)/\operatorname{Sp}(n-1) \rightarrow \operatorname{SU}(2n-1)/\operatorname{SU}(2n-2) = S^{4n-3}.$$

From the fibration we see that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}'$, where $\mathfrak{p}_1$ is tangent to the fibre and the $\operatorname{Ad} \operatorname{Sp}(n-1)$ action on $\mathfrak{p}_1$ is $[\wedge^2 \nu_{n-1} - \operatorname{Id}]_{\mathbb{R}}$, and $\dim(\mathfrak{p}_1) = (2n-1)(n-2)$. The subspace $\mathfrak{p}'$ is tangent to the base, and $\operatorname{Ad} \operatorname{SU}(2n-2)$ acts on $\mathfrak{p}'$ by $[\mu_{2n-2}]_{\mathbb{R}} \oplus \operatorname{Id}$, which when restricted to $\operatorname{Sp}(n-1)$ is $[\nu_{n-1}]_{\mathbb{R}} \oplus \operatorname{Id}$. That is, $\mathfrak{p}' = \mathfrak{p}_2 \oplus \mathfrak{p}_3$; $\dim(\mathfrak{p}_2) = 4(n-1)$ and $\dim(\mathfrak{p}_3) = 1$ (the $\operatorname{Ad} \operatorname{Sp}(n-1)$ action on $\mathfrak{p}_3$ is trivial). The set of elements listed below gives a Q -orthogonal basis for $\mathfrak{p}$. We write E_{ij} for the skew-symmetric $(2n-1) \times (2n-1)$ matrix with 1 in the ij^{th} entry and -1 in the ji^{th} entry, and zeros elsewhere. We denote by F_{ij} the symmetric $(2n-1) \times (2n-1)$ matrix with 1 in both the ij^{th} and ji^{th} entries.

$$\begin{aligned}
\mathfrak{p}_1 &= \text{span}\{(E_{kl} - E_{n+k,n+l}), i(F_{kl} + F_{n+k,n+l}), \\
&\quad (E_{k,n+l} - E_{l,n+k}), i(F_{k,n+l} - F_{l,n+k}) \mid 1 \leq k < l \leq n-1\} \\
&\quad \oplus \text{span}\{i(F_{kk} - F_{n-1,n-1} + F_{n+k,n+k} - F_{2n-1,2n-1}) \mid 1 \leq k < n-1\}, \\
\mathfrak{p}_2 &= \text{span}\{E_{nk}, iF_{nk}, E_{n,n+k}, iF_{n,n+k} \mid 1 \leq k \leq n-1\}, \\
\mathfrak{p}_3 &= \text{span}\{\text{diag}(\eta, \dots, \eta, -2(n-1)\eta, \eta, \dots, \eta)\}, \text{ where } \eta = \frac{i}{\sqrt{(2n-1)(n-1)}}.
\end{aligned}$$

Since $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ are inequivalent irreducible representations of $\text{Sp}(n-1)$, any $\text{SU}(2n-1)$ -invariant metric on M must take the form

$$\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp x_3 Q|_{\mathfrak{p}_3}, \text{ with } x_i > 0 \text{ for } i = 1, 2, 3.$$

To find all $\text{SU}(2n-1)$ -invariant Einstein metrics on M , we solve for the critical points of the scalar curvature equation in terms of x_1, x_2 , and x_3 (restricting to unit volume). As in the previous case we use the formula given in [W-Z, (1.3)]:

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i \ j} \frac{x_k}{x_i x_j}.$$

In our example we have $b_i = 4(2n-1)$ for all i , since for $\mathfrak{su}(k)$, $-B(X, Y) = 2k \text{tr}(XY)$. And $d_1 = (2n-1)(n-1)$, $d_2 = 4(n-1)$, $d_3 = 1$. We find that

$$\begin{aligned}
[\mathfrak{p}_1, \mathfrak{p}_1] &\subset \mathfrak{sp}(n-1), & [\mathfrak{p}_1, \mathfrak{p}_2] &\subset \mathfrak{p}_2, \\
[\mathfrak{p}_2, \mathfrak{p}_2] &\subset \mathfrak{sp}(n-1) + \mathfrak{p}_1 + \mathfrak{p}_3, & [\mathfrak{p}_1, \mathfrak{p}_3] &= 0, \\
[\mathfrak{p}_3, \mathfrak{p}_3] &= 0, & [\mathfrak{p}_2, \mathfrak{p}_3] &\subset \mathfrak{p}_2.
\end{aligned}$$

Therefore $\binom{1}{2 \ 2}$, $\binom{3}{2 \ 2}$ (and rearrangements) are the only nonzero triples. We compute $\binom{1}{2 \ 2} = 4(2n-1)(n-2)$ and $\binom{3}{2 \ 2} = 4(2n-1)$. We now have the equation for the scalar curvature of M in terms of x_1, x_2 , and x_3 .

$$S = (2n-1) \left(\frac{4(n-1)(n-2)}{x_1} + \frac{8(n-1)}{x_2} - \frac{x_3}{x_2^2} - \frac{(n-2)x_1}{x_2^2} \right).$$

We normalize for volume 1 metrics: $\tilde{S} = S - \lambda(x_1^{d_1} x_2^{d_2} x_3 - 1)$.

$$\begin{aligned}
\frac{\partial \tilde{S}}{\partial x_1} &= -\frac{4(2n-1)(n-1)(n-2)}{x_1^2} - \frac{(2n-1)(n-2)}{x_2^2} - (2n-1)(n-2)\lambda x_1^{d_1-1} x_2^{d_2} x_3, \\
\frac{\partial \tilde{S}}{\partial x_2} &= \frac{8(2n-1)(n-1)}{x_2^2} + \frac{2(2n-1)((n-2)x_1 + x_3)}{x_2^3} - 4(n-1)\lambda x_1^{d_1} x_2^{d_2-1} x_3, \\
\frac{\partial \tilde{S}}{\partial x_3} &= -\frac{(2n-1)}{x_2^2} - \lambda x_1^{d_1} x_2^{d_2}.
\end{aligned}$$

Setting $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$ simultaneously is equivalent to

$$4(n-1)x_2^2 + x_1^2 = (2n-1)x_1 x_3 = 2(2n-1)x_1 x_2 - \frac{2n-1}{2n-2}((n-2)x_1^2 + x_1 x_3).$$

There is only one solution; it is $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{n}{2n-1}x_1$, unique up to scaling. This is not a new metric, rather it is the symmetric metric, which we knew must solve our equations. (It is $\text{SU}(2n)$ -invariant, hence $\text{SU}(2n-1)$ -invariant.) Thus the

only homogeneous Einstein metric on $M = \mathrm{SU}(2n-1)/\mathrm{Sp}(n-1) \cong \mathrm{SU}(2n)/\mathrm{Sp}(n)$ is the symmetric metric.

5. $G_2^+(\mathbb{R}^7)$

The Grassmann manifold of oriented two-planes through the origin in $\mathbb{R}^7$ is generally written homogeneously $G_2^+(\mathbb{R}^7) \cong \mathrm{SO}(7)/\mathrm{SO}(2)\mathrm{SO}(5)$. It is irreducible: the symmetric metric is not only Einstein, it is the only $\mathrm{SO}(7)$ -invariant metric. We will show that this Grassmannian manifold can also be written homogeneously as $G_2/\mathrm{U}(2)$ and we find it carries two non-symmetric G_2 -invariant Einstein metrics.

First we must see how $G_2 \subset \mathrm{SO}(7)$, following [M, p.190]. We identify $\mathbb{R}^8$ with the Cayley numbers, or octonians, the normed division algebra $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$. Then G_2 is the set of automorphisms of $\mathbb{O}$. Any automorphism of the Cayley numbers must take 1 to itself and must preserve the inner product, so elements of G_2 also preserve $\mathrm{Im}(\mathbb{O})$, the space of imaginary Cayley numbers, the orthogonal complement to 1. In this way we see $G_2 \subset \mathrm{SO}(7)$. To see that G_2 acts transitively on $G_2^+(\mathbb{R}^7)$, we use the following observation [M, p.186].

Lemma 5.1. *Given three imaginary orthogonal unit octonians: v_1, v_2 , and $v_3 \in \{v_1, v_2, v_1v_2\}^\perp$, there exists a unique automorphism A of $\mathbb{O}$ with $A(i) = v_1$, $A(j) = v_2$, and $A(\varepsilon) = v_3$.*

Using the lemma we take any $P = \mathrm{span}\{v, w\}$ to the oriented two-plane $P_0 = \mathrm{span}\{i, j\}$, where we take v and w an orthonormal basis for P . We must find the isotropy subgroup fixing P_0 . It will be useful to recall the following well known fact.

Lemma 5.2. *The quotient $G_2/\mathrm{SU}(3) \cong S^6$.*

Proof. By the previous lemma, G_2 acts transitively on $S^6(1) \subset \mathrm{Im}(\mathbb{O})$. We need to show that the isotropy subgroup H_v of G_2 corresponding to any $v \in S^6$ is $\mathrm{SU}(3)$. We observe first that $A(v) = v$ implies the map L_v is a complex structure on $\mathbb{O}$ and on $V = \mathrm{span}\{1, v\}^\perp$. This shows that $H_v \subset \mathrm{U}(V) \cong \mathrm{U}(3)$. Furthermore, $\dim(H_v) = 8$. Since G_2 and S^6 are connected and simply connected, H_v is connected.

Consider the homomorphism $\det : H_v \rightarrow S^1$; it must be either trivial or onto. If it is trivial, $H_v \cong \mathrm{SU}(3)$. If it is onto, then let H' denote the kernel. Then H' is a normal subgroup of H_v of dimension seven, and if H'_0 is the connected component of the identity, $\mathrm{rank}(H'_0) \leq \mathrm{rank}(\mathrm{SU}(3)) = 2$. But the only compact connected seven-dimensional Lie groups, up to finite cover, are T^7 , $S^3 \times T^4$, and $S^3 \times S^3 \times S^1$, and each of these has rank > 2 :

$$\mathrm{rank}(T^7) = 7, \quad \mathrm{rank}(S^3 \times T^4) = 5, \quad \mathrm{rank}(S^3 \times S^3 \times S^1) = 3.$$

This shows $H' = H_v$, and thus $H_v \cong \mathrm{SU}(3)$. □

We are now ready to describe the isotropy subgroup H corresponding to the oriented two-plane P_0 . Since G_2 and $G_2^+(\mathbb{R}^7)$ are connected and simply connected, we know H is connected. If we have $A \in G_2$ such that $A(P_0) = P_0$ (with orientation), then $A(i) = i \cos \theta - j \sin \theta$, $A(j) = i \sin \theta + j \cos \theta$, thus $A(k) = A(i)A(j) = k$. The isotropy subgroup $H \subset \{A \in G_2 \mid A(k) = k\} \cong \mathrm{SU}(3)$, and since the ij -plane is a complex line with respect to our complex structure L_k , H must preserve this complex line and the complex two-plane perpendicular to it. Hence $H \subset S(\mathrm{U}(1)\mathrm{U}(2)) \subset \mathrm{SU}(3)$. A dimension count shows $H = S(\mathrm{U}(1)\mathrm{U}(2)) \cong \mathrm{U}(2)$.

If we look on the Lie algebra level, and we take $\{i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon\}$ as our basis for $\text{Im}(\mathbb{O})$, then using the automorphism property we can write $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ in the following way:

$$\begin{aligned} \mathfrak{g}_2 = \text{span}\{ & E_{12} + E_{56}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}, E_{46} + E_{57} + 2E_{13}, \\ & E_{67} - E_{45} + 2E_{23}, E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} \\ & + 2E_{34}, E_{17} + E_{24} + 2E_{35}, E_{14} + E_{27}, -E_{15} + E_{26}, E_{16} + E_{25}, -E_{17} + E_{24}\}. \end{aligned}$$

We use the inner product on $\mathfrak{g}_2$ given by $Q(X, Y) = -\frac{1}{2} \text{tr}(XY)$, in which the basis for $\mathfrak{g}_2$ above is orthogonal. The subalgebra $\mathfrak{h}$ corresponds to $H \cong \text{U}(2)$:

$$\mathfrak{u}(2) \cong \text{span}\{2E_{12} + E_{56} - E_{47}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}\}.$$

The isotropy representation of $\text{U}(2)$ is the action of $\text{Ad } \text{U}(2)$ on $\mathfrak{p}$, the Q -orthogonal complement of $\mathfrak{u}(2)$ in $\mathfrak{g}_2$. We can use the following fibrations to decompose $\mathfrak{p}$ into its irreducible $\text{Ad } \text{U}(2)$ representations: First,

$$\mathbb{C}P^2 \cong \text{SU}(3)/\text{U}(2) \rightarrow \text{G}_2/\text{U}(2) \rightarrow \text{G}_2/\text{SU}(3) \cong \text{S}^6.$$

The tangent space to the base is isomorphic to $[\mu_3]_{\mathbb{R}}$; when restricted to $\text{U}(2)$ it gives $[(\mu_1 \hat{\otimes} \text{Id}) \oplus (\text{Id} \hat{\otimes} \mu_2)]_{\mathbb{R}} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Thus, $\mathfrak{p}_1 = [\mu_1 \hat{\otimes} \text{Id}]_{\mathbb{R}}$, $\mathfrak{p}_2 = [\text{Id} \hat{\otimes} \mu_2]_{\mathbb{R}}$. The tangent space to the fibre is $\mathfrak{p}_3 = [\mu_1 \hat{\otimes} \mu_2]_{\mathbb{R}}$. We have $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$.

$$\begin{aligned} \mathfrak{p}_1 &= \text{span}\{E_{46} + E_{57} + 2E_{13}, E_{67} - E_{45} + 2E_{23}\}, \\ \mathfrak{p}_2 &= \text{span}\{E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} + 2E_{34}, \\ & \quad E_{17} + E_{24} + 2E_{35}\}, \\ \mathfrak{p}_3 &= \text{span}\{E_{14} + E_{27}, E_{26} - E_{15}, E_{16} + E_{25}, E_{24} - E_{17}\}. \end{aligned}$$

We have a second fibration of our manifold: we claim $\text{U}(2) \subset \text{SO}(4) \subset \text{G}_2$. Before showing this, we briefly discuss the embedding $\text{SO}(4) \subset \text{G}_2$ and the irreducible symmetric space $\text{G}_2/\text{SO}(4)$.

Lemma 5.3. *The quotient $\text{G}_2/\text{SO}(4)$ is the space of quaternionic subalgebras of the Cayley numbers, $\mathbb{O}$.*

Proof. We have $\text{SO}(4) \cong (\text{Sp}(1) \times \text{Sp}(1))/\{(q_1, q_2) \simeq (-q_1, -q_2)\}$, and it acts on $\mathbb{O} \cong \mathbb{H} \oplus \mathbb{H}\varepsilon$ by

$$(q_1, q_2) : a + b\varepsilon \mapsto q_1 a \bar{q}_1 + (q_2 b \bar{q}_1)\varepsilon.$$

A calculation shows that $\text{SO}(4) \subset \text{G}_2$, and this embedding of $\text{SO}(4)$ in G_2 can also be described as the subgroup of G_2 which leaves the subalgebra $\mathbb{H} \cong \text{span}\{1, i, j, k\}$ invariant. $\square$

Since $\text{U}(2)$ is the subgroup of G_2 preserving the plane spanned by i and j , elements of $\text{U}(2)$ take 1 to itself and k to itself, hence they preserve $\text{span}\{1, i, j, k\}$. This also shows $\text{U}(2) \subset \text{SO}(4) \cap \text{SU}(3)$. We also have $\text{SO}(4) \cap \text{SU}(3) \subset \text{U}(2)$: under G_2 , $1 \mapsto 1$; under $\text{SO}(4)$, $\text{span}\{1, i, j, k\} \mapsto \text{span}\{1, i, j, k\}$; under $\text{SU}(3)$, $k \mapsto k$. Thus $\text{SO}(4) \cap \text{SU}(3) = \text{U}(2)$. Our second fibration is

$$\text{SO}(4)/\text{U}(2) \rightarrow \text{G}_2/\text{U}(2) \rightarrow \text{G}_2/\text{SO}(4).$$

Here $\mathfrak{p}_1$ is tangent to the fibre, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are tangent to the base: From the two fibrations we obtain the following Lie bracket relations among the $\mathfrak{p}_i$'s: $[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{u}(2)$, $[\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{p}_2 \oplus \mathfrak{p}_3$, $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{u}(2) \oplus \mathfrak{p}_1$, and $[\mathfrak{p}_3, \mathfrak{p}_3] \subset \mathfrak{u}(2)$.

Since $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$ are mutually inequivalent, any G_2 -invariant metric on our space is of the form $\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp x_3 Q|_{\mathfrak{p}_3}$, with $x_i > 0$, for $i = 1, 2, 3$.

As in the previous cases we can write the scalar curvature on $G_2^+(\mathbb{R}^7)$ as function of x_1, x_2 , and x_3 via the formula given in [W-Z]:

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{ij} \frac{x_k}{x_i x_j}.$$

When we compute the non-zero Lie bracket relations between the $\mathfrak{p}_i$'s we find that $\binom{3}{12} = 4$ and $\binom{1}{22} = \frac{16}{3}$. Also we compute (using the basis elements for $\mathfrak{g}_2$) $b_i = 8$ for each i , and of course $d_1 = 2$, $d_2 = 4$, and $d_3 = 4$. This gives us

$$S = 8\left(\frac{1}{x_1} + \frac{2}{x_2} + \frac{2}{x_3}\right) - \frac{4}{3}\left(\frac{x_1}{x_2^2} + \frac{2}{x_1}\right) - 2\left(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2}\right).$$

Then $\tilde{S} = S - \lambda(x_1^2 x_2^4 x_3^4 - 1)$ includes our boundary condition, volume = 1. Einstein metrics on $G_2/U(2)$ will be critical points of $\tilde{S}$.

$$\begin{aligned} \frac{\partial \tilde{S}}{\partial x_1} &= -\frac{16}{3x_1^2} - \frac{4}{3x_2^2} + 2\left(\frac{x_2}{x_1^2 x_3} + \frac{x_3}{x_1^2 x_2} - \frac{1}{x_2 x_3}\right) - 2\lambda x_1 x_2^4 x_3^4, \\ \frac{\partial \tilde{S}}{\partial x_2} &= -\frac{16}{x_2^2} + \frac{8x_1}{3x_2^3} + 2\left(\frac{x_1}{x_2^2 x_3} + \frac{x_3}{x_1 x_2^2} - \frac{1}{x_1 x_3}\right) - 4\lambda x_1^2 x_2^3 x_3^4, \\ \frac{\partial \tilde{S}}{\partial x_3} &= -\frac{16}{x_3^2} + 2\left(\frac{x_1}{x_2 x_3^2} + \frac{x_2}{x_1 x_3^2} - \frac{1}{x_1 x_2}\right) - 4\lambda x_1^2 x_2^4 x_3^3. \end{aligned}$$

Now we look for all solutions to $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$. We solve these using Maple, and we find the following:

$$\begin{aligned} &\text{either } x_2 = \frac{1}{2}x_1 \text{ and } x_3 = \frac{3}{2}x_1 \\ &\text{or } x_2 = \zeta x_1 \text{ and } x_3 = \left(\frac{7}{120}\zeta^4 - \frac{7}{60}\zeta^3 - \frac{151}{96}\zeta^2 + \frac{39}{10}\zeta - \frac{21}{40}\right)x_1, \\ &\text{where } \zeta \text{ is a root of } 56\zeta^5 - 532\zeta^4 + 1570\zeta^3 - 1891\zeta^2 + 776\zeta - 60 = 0. \end{aligned}$$

We need both a positive solution to this polynomial in order for $x_2 > 0$, and also

$$\left(\frac{7}{120}\zeta^4 - \frac{7}{60}\zeta^3 - \frac{151}{96}\zeta^2 + \frac{39}{10}\zeta - \frac{21}{40}\right) > 0, \text{ so that } x_3 > 0.$$

There are exactly three real solutions to the quintic polynomial. We give the approximate values for x_2 and x_3 , setting $x_1 = 1$:

$$\begin{aligned} x_2 &= 0.09953, & x_3 &= -.15252, \\ x_2 &= 0.59713, & x_3 &= 1.22554, \\ x_2 &= 5.35063, & x_3 &= 5.25153. \end{aligned}$$

We must eliminate the first of the solutions from the quintic, since it gives a negative value for x_3 . The solution $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{3}{2}x_1$ is the symmetric metric; that is, we will see that it is $\text{SO}(7)$ -invariant, after projection. If we denote by $\tilde{\mathfrak{p}}$ the Q -orthogonal complement to $\mathfrak{so}(2) \oplus \mathfrak{so}(5)$ in $\mathfrak{so}(7)$, then $\tilde{\mathfrak{p}} = \{E_{ij} \mid 1 \leq i \leq 2, 3 \leq j \leq 7\}$. When we project $\mathfrak{p}$ to $\tilde{\mathfrak{p}}$, we see that in $\mathfrak{p}_1$, the basis element $\frac{1}{\sqrt{6}}(2E_{13} + E_{46} + E_{57})$ projects to $\frac{2}{\sqrt{6}}E_{13}$, i.e., an element of norm $= \sqrt{x_1}$ projects to an element of norm $= \sqrt{\frac{2}{3}}$. In $\mathfrak{p}_2$, the basis element $\frac{1}{\sqrt{6}}(2E_{36} - E_{14} + E_{27})$ projects to $\frac{1}{\sqrt{6}}(-E_{13} + E_{27})$,

so $\text{norm} = \sqrt{x_2}$ is projected to $\text{norm} = \sqrt{\frac{1}{3}}$. In $\mathfrak{p}_3$, basis element $\frac{1}{\sqrt{2}}(E_{14} + E_{27})$ is already an element of $\tilde{\mathfrak{p}}$, so $\text{norm} = \sqrt{x_3}$ is projected to $\text{norm} = 1$. This shows that the symmetric metric in our basis satisfies $\frac{x_2}{x_1} = \frac{1}{2}$ and $\frac{x_3}{x_1} = \frac{3}{2}$. Thus we end up with two new G_2 -homogeneous Einstein metrics on the Grassmannian $G_2^+(\mathbb{R}^7)$.

Remark 5.4. Notice none of these is a fibration metric, since a metric of the first fibration would require $x_1 = x_2$ and a metric of the second fibration would require $x_2 = x_3$. Many examples of Einstein metrics have been obtained by using fibrations over Einstein spaces and with Einstein fibres [Bes, ch.9]. In our example we have two fibrations; in each the fibre and base space are isotropy irreducible, so they must be Einstein. However, in each fibration the isotropy representation of the base, when restricted to $U(2)$, decomposes into the sum of two irreducible subrepresentations. Recall $Q(X, Y) = -\frac{1}{2} \text{tr} XY$ is our comparison metric. The Casimir constant corresponding to Q and to the restriction to $U(2)$ of the isotropy representations for these homogeneous spaces differs on these two subrepresentations. This implies that the O'Neill tensor restricted to horizontal vectors is not a scalar multiple of Q . Using Besse's Proposition 9.70 [Bes, p.253] we see that therefore our fibrations cannot give rise to Einstein metrics.

To see that we have three non-isometric solutions we compare the scale-invariant product: $(S)^{\frac{d}{2}}(V)^{\frac{1}{2}}$, where S is the scalar curvature and V is the volume, and d is the dimension of our homogeneous space, for our three metrics. If we let $x_2 = \frac{1}{2}x_1$ and $x_3 = \frac{3}{2}x_1$, we obtain $S = \frac{100}{3x_1}$, and $V = \frac{3^4}{2^8}x_1^{10}$, so $(S)^5(V)^{\frac{1}{2}} = \frac{2^6 5^{10}}{3^3}$. This is approximately 2.3148×10^7 . The second solution gives $(S)^5(V)^{\frac{1}{2}} \cong 2.3044 \times 10^7$, and the third solution gives $(S)^5(V)^{\frac{1}{2}} \cong 1.5836 \times 10^7$.

We note that these have been found previously in [A] and [K]. They observe that one of the three metrics is Kähler and the other two are not Kähler for any complex structure on M . Neither author observes that the Kähler Einstein metric is the symmetric metric.

6. $G_3^+(\mathbb{R}^8)$

We can write the Grassmann manifold of oriented three-planes in $\mathbb{R}^8$ homogeneously $G_3^+(\mathbb{R}^8) \cong \text{SO}(8)/\text{SO}(3)\text{SO}(5)$, but we can also write it as $G_3^+(\mathbb{R}^8) \cong \text{Spin}(7)/\text{SO}(4)$. With respect to $\text{SO}(8)$, $G_3^+(\mathbb{R}^8)$ is irreducible, and therefore the symmetric metric is Einstein and it is the unique $\text{SO}(8)$ -invariant metric, up to scaling. However, we find there are two more $\text{Spin}(7)$ -invariant Einstein metrics which are not symmetric.

We first describe how $\text{Spin}(7)$ sits inside $\text{SO}(8)$. We again identify $\mathbb{R}^8$ with the Cayley numbers $\mathbb{O}$; from Murakami [M] we know

$$\text{Spin}(7) = \{A \in \text{SO}(8) \mid \exists B \in \text{SO}(8) \text{ such that } B(x)A(y) = A(xy) \ \forall x, y \in \mathbb{O}\}.$$

In this definition notice $B(1) = 1$, so $B \in \text{SO}(7)$ and if A corresponds to B , $-A$ corresponds to B as well, which shows $\text{Spin}(7)$ is indeed a double cover of $\text{SO}(7)$. We also remark that that $\{L_a \mid a \in \text{Im}(\mathbb{O}), |a| = 1\} \subset \text{Spin}(7)$. (The corresponding B is conjugation by a .) We need to show that $C_a(x)L_a(y) = L_a(xy)$ for any $x, y \in \mathbb{O}$. Because a is a unit imaginary octonian, $a^{-1} = -a$, thus $C_a(x)L_a(y) = (axa^{-1})(ay) = -(axa)(ay)$. Then the first Moufang identity tells us that $-(axa)(ay) = -a(x(aay))$, and using that $aa = -1$, we have $-a(x(aay)) = a(xy) = L_a(xy)$.

It is also convenient to identify two subgroups of $\text{Spin}(7)$, they are G_2 , the automorphisms of the Cayley numbers, and $\text{SU}(4)$, complex linear maps with respect to L_i . We see $G_2 \subset \text{Spin}(7)$ by letting $B = A$ in the definition of $\text{Spin}(7)$. Murakami shows how to see that $\text{SU}(4) \subset \text{Spin}(7)$ with the following lemma [M].

Lemma 6.1. *In $\text{SO}(8)$, $\text{U}(4) = \{A \in \text{SO}(8) \mid iA(x) = A(ix) \ \forall x, y \in \mathbb{O}\}$ and $\text{U}(4) \cap \text{Spin}(7) = \text{SU}(4)$.*

Proof. Let $p^+ : \text{Spin}(7) \rightarrow \text{SO}(7)$ be the homomorphism sending $A \mapsto B$, for A and B in the definition of $\text{Spin}(7)$. For every $A \in \text{U}(4) \cap \text{Spin}(7)$ we have $B(i) = i$, so $p^+(\text{U}(4) \cap \text{Spin}(7)) \subset \text{SO}(6)$. And for every $B \in \text{SO}(6)$ ($B(i) = i$), the corresponding A must be in $\text{U}(4) \cap \text{Spin}(7)$, hence

$$\text{SO}(6) \subset p^+(\text{U}(4) \cap \text{Spin}(7)).$$

Furthermore, p^+ is a local isomorphism, thus $\text{U}(4) \cap \text{Spin}(7)$ is a 15-dimensional connected Lie group with a simple Lie algebra. Observe that $\text{SU}(4)$ is the commutator subgroup of $\text{U}(4)$. Since its Lie algebra is simple, $\text{U}(4) \cap \text{Spin}(7)$ is its own commutator subgroup, thus $\text{U}(4) \cap \text{Spin}(7) \subset \text{SU}(4)$, and a dimension count tells us that these subgroups are equal. $\square$

We embed $\text{SU}(4) \subset \text{SO}(8)$ via $A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. To embed $\text{SU}(4)$ into $\text{SO}(8)$ in this way, we want $\mathbb{R}^4 \oplus \mathbb{R}^4 \cong \mathbb{C}^4$ with $(u, v) \mapsto u + iv$; this restricts our choice of ordered bases for $\mathbb{O}$: we choose $\{1, j, \varepsilon, j\varepsilon, i, k, i\varepsilon, -k\varepsilon\}$. The intersection of our two subgroups is $G_2 \cap \text{SU}(4) = \text{SU}(3)$, and this time $\text{SU}(3)$ in G_2 fixes i instead of k .

We check that $\text{Spin}(7)$ acts transitively on $G_3^+(\mathbb{R}^8)$: Let $P_0 = \text{span}\{i, j, k\}$, an oriented three-plane through the origin. Let P be given by $\text{span}\{v_1, v_2, v_3\}$, where v_1, v_2 and v_3 are ordered orthonormal vectors. Without loss of generality we may assume $v_1, v_2 \in \text{Im}(\mathbb{O})$, since P is a three-plane, so $\dim(P \cap \text{Im}(\mathbb{O})) \geq 2$. We know from the previous example that we can find an element $A \in G_2$ such that $A(i) = v_1$, $A(j) = v_2$. Let $x = A^{-1}(v_3)$. Observe that

$$\begin{aligned} \langle x, i \rangle &= \langle A^{-1}(v_3), i \rangle = \langle v_3, v_1 \rangle = 0, \\ \langle x, j \rangle &= \langle A^{-1}(v_3), j \rangle = \langle v_3, v_2 \rangle = 0. \end{aligned}$$

We claim there exists $A' \in \text{Spin}(7)$ such that $A'(i) = i$, $A'(j) = j$, and $A'(k) = x$. This is because the subgroup of $\text{Spin}(7)$ fixing one unit octonian is conjugate to G_2 , and the subgroup of $\text{Spin}(7)$ fixing two orthonormal octonians is conjugate to $\text{SU}(3)$, which acts transitively on $S^5(1) \subset \text{span}\{i, j\}^\perp$. The composition $A \circ A' \in \text{Spin}(7)$ is our map taking $i \mapsto v_1$, $j \mapsto v_2$, and $k \mapsto v_3$, so that P_0 goes to P , and this shows $\text{Spin}(7)$ acts transitively on $G_3^+(\mathbb{R}^8)$.

Next we must determine the isotropy subgroup H of P_0 . We claim that $H \subset G_2$. To see this, we note that if $A(P_0) = P_0$, then if B is the element of $\text{SO}(7)$ in the definition of $\text{Spin}(7)$ corresponding to A , since $B(i)A(j) = A(k)$ we know that $B(i) \in \text{span}\{i, j, k\}$. Thus $A(1) = -B(i)A(i) \in \text{span}\{1, i, j, k\}$, and furthermore $A(1) \perp P_0$, so $A(1) = \pm 1$. Since $\text{Spin}(7)$ and $G_3^+(\mathbb{R}^8)$ are connected and simply connected, we know H is connected, hence $A(1) = 1$. From the definition of $\text{Spin}(7)$ it follows that $A = B$ and this implies $H \subset G_2$. Furthermore, any element of H takes 1 to itself and preserves the standard quaternionic subalgebra $\text{span}\{1, i, j, k\}$. Thus $H \subset \text{SO}(4) \subset G_2$, and by a dimension count $H = \text{SO}(4)$.

Now we are ready to find the isotropy representation. On the Lie algebra level we have

$$\begin{aligned} \mathfrak{spin}(7) = & \text{span}\{E_{ij} + E_{4+i,4+j}, E_{i,4+j} + E_{j,4+i} \mid 1 \leq i < j \leq 4\} \\ & \oplus \text{span}\{E_{i,4+i} - E_{48} \mid 1 \leq i \leq 3\} \\ & \oplus \text{span}\{E_{27} - E_{45}, E_{23} + E_{58}, E_{24} - E_{57}, E_{28} + E_{35}, \\ & E_{56} - E_{78}, 2E_{25} - E_{38} + E_{47}\}. \end{aligned}$$

The subalgebra corresponding to the isotropy subgroup is $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$:

$$\begin{aligned} \mathfrak{h} = & \text{span}\{E_{37} - E_{48}, E_{34} + E_{78}, E_{38} + E_{47}\} \\ & \oplus \text{span}\{2E_{56} + E_{34} - E_{78}, 2E_{26} - E_{48} - E_{37}, 2E_{25} - E_{38} + E_{47}\}. \end{aligned}$$

Notice each copy of $\mathfrak{su}(2)$ is an ideal in $\mathfrak{h}$ and its basis vectors above are orthogonal with respect to the inner product on $\mathfrak{spin}(7)$ given by $Q(X, Y) = -\frac{1}{2} \text{tr}(XY)$. As usual we denote by $\mathfrak{p}$ the Q -orthogonal complement of $\mathfrak{h}$ in $\mathfrak{spin}(7)$.

There are two fibrations of our symmetric space $\text{Spin}(7)/\text{SO}(4)$. The first is

$$G_2/\text{SO}(4) \rightarrow \text{Spin}(7)/\text{SO}(4) \rightarrow \text{Spin}(7)/G_2 \cong S^7.$$

Let $\mathfrak{p}'$ be the subspace tangent to the fibre; let $\mathfrak{p}''$ be the subspace tangent to the base. Let θ_k denote the unique irreducible complex representation of $\mathfrak{su}(2)$ in k dimensions; the first fibration tells us that $\mathfrak{p}' = [\theta_2 \hat{\otimes} \theta_4]_{\mathbb{R}}$, since this is the representation of the symmetric space $G_2/\text{SO}(4)$ [W, p.287]. The isotropy representation of $\text{Spin}(7)/G_2$ is the seven-dimensional representation of $G_2 \subset \text{SO}(7)$. We restrict this representation to $\text{SO}(4)$, to see that $\mathfrak{p}'' = [\rho_3 \hat{\otimes} \text{Id}] \oplus [\theta_2 \hat{\otimes} \theta_2]_{\mathbb{R}}$, where ρ_3 denotes the standard representation of $\mathfrak{so}(3)$ on $\mathbb{R}^3$. We let $\mathfrak{p}_1 = [\rho_3 \hat{\otimes} \text{Id}]$, and $\mathfrak{p}_2 = [\theta_2 \hat{\otimes} \theta_2]_{\mathbb{R}}$, and $\mathfrak{p}_3 = [\theta_2 \hat{\otimes} \theta_4]_{\mathbb{R}}$. We have $\dim(\mathfrak{p}_1) = 3$, $\dim(\mathfrak{p}_2) = 4$, and $\dim(\mathfrak{p}_3) = 8$.

For the second fibration, we first need some explanation.

Lemma 6.2. *The compact group $(\text{Spin}(4) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id})\}$ satisfies $\text{SO}(4) \subset (\text{Spin}(4) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id})\} \subset \text{Spin}(7)$.*

Proof. Recall, for any n , $\text{Spin}(n)$ is the simply connected double cover of $\text{SO}(n)$; let $\pi : \text{Spin}(n) \rightarrow \text{SO}(n)$ denote the two-fold homomorphism. Because $\pi_1(\text{SO}(k)) \rightarrow \pi_1(\text{SO}(n))$ is a surjection, $\pi(\text{Spin}(k)) = \text{SO}(k)$ for all $k \leq n$. Observe that $\ker(\pi) = \{\pm(\text{Id}, \text{Id})\} = \text{Spin}(k) \cap \text{Spin}(n-k)$, thus it is $(\text{Spin}(k) \times \text{Spin}(n-k))/\{\pm(\text{Id}, \text{Id})\} \subset \text{Spin}(n)$. In $\text{Spin}(7)$ we can consider the subgroup $(\text{Spin}(3) \times \text{Spin}(4))/\{\pm(\text{Id}, \text{Id})\}$. We know $\text{Spin}(4) \cong \text{Spin}(3) \times \text{Spin}(3)$ (and $\text{Spin}(3) \cong \text{SU}(2)$). Thus $\text{Spin}(7)$ has a subgroup $(\text{Spin}(3) \times \text{Spin}(3) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id}, \text{Id})\}$. Our isotropy subgroup $\text{SO}(4) \subset \text{Spin}(7)$ is exactly $(\Delta \text{Spin}(3) \times \text{Spin}(3))/\{\pm(\text{Id}, \text{Id})\}$, where $\Delta \text{Spin}(3)$ is the diagonal subgroup of $\text{Spin}(3) \times \text{Spin}(3)$ isomorphic to $\text{Spin}(3)$. This can be seen via the restriction to $\text{SO}(4)$ of the homomorphism from $\text{Spin}(7)$ to $\text{SO}(7)$ taking A to B (A and B in the definition of $\text{Spin}(7)$). $\square$

We obtain the following fibration:

$$\begin{aligned} S^3 \cong \frac{\text{Spin}(4) \times \text{Spin}(3)}{\{\pm(\text{Id}, \text{Id})\}} \Big/ \frac{\text{Spin}(3) \times \Delta \text{Spin}(3)}{\{\pm(\text{Id}, \text{Id})\}} & \longrightarrow \text{Spin}(7)/\text{SO}(4) \cong G_3^+(\mathbb{R}^8) \\ & \downarrow \\ \text{Spin}(7) \Big/ \frac{\text{Spin}(4) \times \text{Spin}(3)}{\{\pm(\text{Id}, \text{Id})\}} & \cong G_3^+(\mathbb{R}^7). \end{aligned}$$

In the second fibration, it is $\mathfrak{p}_1$ which is the subspace tangent to the fibre, while $\mathfrak{p}_2 \oplus \mathfrak{p}_3$ is the subspace tangent to the base.

$$\begin{aligned}\mathfrak{p}_1 &= \text{span}\{3E_{12} - E_{34} + E_{56} + E_{78}, 3E_{15} - E_{26} - E_{37} - E_{48}, \\ &\quad 3E_{16} + E_{25} + E_{38} - E_{47}\}, \\ \mathfrak{p}_2 &= \text{span}\{3E_{13} + E_{24} + E_{57} - E_{68}, 3E_{14} - E_{23} + E_{58} + E_{67}, \\ &\quad 3E_{17} - E_{28} + E_{35} + E_{46}, 3E_{18} + E_{27} - E_{36} + E_{45}\}, \\ \mathfrak{p}_3 &= \text{span}\{E_{23} + E_{67}, E_{24} + E_{68}, E_{27} + E_{36}, E_{28} + E_{46}, 2E_{57} - E_{24} + E_{68}, \\ &\quad 2E_{58} + E_{23} - E_{67}, 2E_{35} + E_{28} - E_{46}, 2E_{45} - E_{27} + E_{36}\}.\end{aligned}$$

Since each of the $\mathfrak{p}_i$'s has a different dimension, they are inequivalent representations. This means any $\text{Spin}(7)$ -invariant metric on $G_3^+(\mathbb{R}^8)$ is determined by an inner product on $\mathfrak{p}$ satisfying

$$\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp x_3 Q|_{\mathfrak{p}_3}, \text{ for } x_1, x_2, x_3 > 0.$$

From the fibrations we obtain the following Lie bracket relations:

$$\begin{aligned}[\mathfrak{p}_1, \mathfrak{p}_1] &\subset \mathfrak{h} \oplus \mathfrak{p}_1, & [\mathfrak{p}_1, \mathfrak{p}_2] &\subset \mathfrak{p}_2 \oplus \mathfrak{p}_3, \\ [\mathfrak{p}_2, \mathfrak{p}_2] &\subset \mathfrak{h} \oplus \mathfrak{p}_1, & [\mathfrak{p}_3, \mathfrak{p}_3] &\subset \mathfrak{h}.\end{aligned}$$

Recall the scalar curvature formula from [W-Z]:

$$S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \binom{k}{i \ j} \frac{x_k}{x_i x_j}.$$

On $\text{Spin}(7)$ we find that $b_i = 10$ for $i = 1, 2, 3$, and we know $d_1 = 3$, $d_2 = 4$, and $d_3 = 8$. From the Lie bracket relations we know $\binom{1}{11}$, $\binom{1}{22}$, $\binom{1}{23} \neq 0$; all other triples (except rearrangements) are zero. We find that $\binom{1}{11} = 2$, $\binom{1}{22} = 4$, and $\binom{3}{12} = 8$.

We now have the scalar curvature function in x_1, x_2, x_3 :

$$S = \frac{25}{2x_1} + \frac{20}{x_2} + \frac{40}{x_3} - \frac{x_1}{x_2^2} - 4 \left(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right).$$

We want the critical points of the scalar curvature function with the constraint equation of volume 1: $\tilde{S} = S - \lambda(x_1^3 x_2^4 x_3^8 - 1)$.

$$\begin{aligned}\frac{\partial \tilde{S}}{\partial x_1} &= -\frac{25}{2x_1^2} - \frac{1}{x_2^2} - \frac{4}{x_2 x_3} + \frac{4x_2}{x_1^2 x_3} + \frac{4x_3}{x_1^2 x_2} - 3\lambda x_1^2 x_2^4 x_3^8, \\ \frac{\partial \tilde{S}}{\partial x_2} &= -\frac{20}{x_2^2} + \frac{2x_1}{x_2^3} + \frac{4x_1}{x_2^2 x_3} - \frac{4}{x_1 x_3} + \frac{x_3}{x_1 x_2^2} - 4\lambda x_1^3 x_2^3 x_3^8, \\ \frac{\partial \tilde{S}}{\partial x_3} &= -\frac{40}{x_3^2} + \frac{4x_1}{x_2 x_3^2} + \frac{4x_2}{x_1 x_3^2} - \frac{4}{x_1 x_2} - 8\lambda x_1^3 x_2^4 x_3^7.\end{aligned}$$

A solution to $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$ is equivalent to the simultaneous solution of the following two polynomials:

$$\begin{aligned}10x_1 x_2^2 - 10x_1 x_2 x_3 + x_1^2 x_2 + x_1^2 x_3 + 3x_2 x_3^2 - 3x_2^3 &= 0, \\ -11x_1^2 x_2 + 11x_2 x_3^2 + 5x_2^3 - 25x_2^2 x_3 + 30x_1 x_2^2 - 2x_1^2 x_3 &= 0.\end{aligned}$$

We obtain three solutions, using Maple. The first is the symmetric solution: $x_1 = \frac{3}{4}x_3$, $x_2 = \frac{1}{4}x_3$. Let $\tilde{\mathfrak{p}}$ denote the Q -orthogonal complement to $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ in

$\mathfrak{so}(8)$; $\tilde{\mathfrak{p}} = \text{span}\{E_{ij} \mid i = 2, 5, 6, j = 1, 3, 4, 7, 8\}$. (Of course we take $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ corresponding to P_0 .) We must project $\mathfrak{p}_1, \mathfrak{p}_2$, and $\mathfrak{p}_3$ to $\tilde{\mathfrak{p}}$. In $\mathfrak{p}_1$, we take the basis element $\frac{1}{2\sqrt{3}}(3E_{12} - E_{34} + E_{56} + E_{78})$ of norm $= \sqrt{x_1}$. It is projected to $-\frac{\sqrt{3}}{2}E_{12}$ in $\tilde{\mathfrak{p}}$, an element of norm $= \frac{\sqrt{3}}{2}$. In $\mathfrak{p}_2$ we take the basis element

$$\frac{1}{2\sqrt{3}}(3E_{13} + E_{24} + E_{57} - E_{68}) \text{ of norm } = \sqrt{x_2},$$

which projects to $\frac{1}{2\sqrt{3}}(E_{24} + E_{57} - E_{68})$ in $\tilde{\mathfrak{p}}$, an element of norm $= \frac{1}{2}$. Finally, the element $\frac{1}{\sqrt{2}}(E_{23} + E_{67})$ is already in $\tilde{\mathfrak{p}}$, so norm $= \sqrt{x_3}$ corresponds to norm $= 1$. Hence $x_1 = \frac{3}{4}x_3, x_2 = \frac{1}{4}x_3$ is indeed the symmetric metric.

The second and third solutions are $x_2 = \eta x_3$, and

$$x_1 = \left(-\frac{629}{1980} + \frac{5689}{660}\eta - \frac{3799}{165}\eta^2 + \frac{13559}{495}\eta^3 - \frac{392}{33}\eta^4 \right) x_3,$$

for η a positive root of the polynomial

$$4704t^5 - 11788t^4 + 10400t^3 - 3315t^2 - 398t + 289.$$

This polynomial has three real roots, of which two are positive and yield two positive values for x_1 and x_2 in terms of x_3 . We give the approximate values, setting $x_3 = 1$:

$$\begin{aligned} x_1 &= -.241854, & x_2 &= -4.177304, \\ x_1 &= .425179, & x_2 &= .902192, \\ x_1 &= 1.100300, & x_2 &= .369813. \end{aligned}$$

These are two new Einstein metrics on $G_3^+(\mathbb{R}^8)$.

Remark 6.3. None of these is a fibration metric, since the first fibration required $x_1 = x_2$, and the second required $x_2 = x_3$. Just as in the previous example in both fibrations the fibre and base space are isotropy irreducible, therefore Einstein. However, the isotropy representation of each base, when restricted to $\text{SO}(4)$, again decomposes into the sum of two irreducible subrepresentations, where the Casimir constant corresponding to Q and to the restriction to $\text{SO}(4)$ of the isotropy representations for these homogeneous spaces differs. Again this implies the O'Neill tensor restricted to horizontal vectors is not a scalar multiple of Q . Using Besse's Proposition 9.70 [Bes, p.253] we know our fibrations cannot give rise to Einstein metrics.

We verify that they are all distinct by estimating the (scale-invariant) product: $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}}$, where S is the scalar curvature of the metric and V is the volume of the metric. For the symmetric metric, $S = \frac{90}{x_3}$ and $V = \frac{3^3}{2^{14}}x_3^{15}$, and so $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}} \cong 1.84200 \times 10^{13}$. For the two new metrics, we find that $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}} \cong 1.80936 \times 10^{13}$, and $(S)^{\frac{15}{2}}(V)^{\frac{1}{2}} \cong 1.61159 \times 10^{13}$, respectively. This shows that they are non-isometric.

7. APPENDIX

Oniščik in fact lists more triples of Lie algebras $(\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'')$, but the extra triples can be obtained by combining the information given on page 155. For example, in addition to $\text{SO}(2n)/\text{SO}(2n-1) = \text{SU}(n)/\text{SU}(n-1)$ and $\text{SO}(4n)/\text{SO}(4n-1) = \text{Sp}(n)/\text{Sp}(n-1)$, he lists $\text{SU}(2n)/\text{SU}(2n-1) = \text{Sp}(n)/\text{Sp}(n-1)$, which follows

from the inclusions $\mathrm{Sp}(n) \subset \mathrm{SU}(2n) \subset \mathrm{SO}(4n)$. Many of the triples on his list come from the subgroups of $\mathrm{SO}(8)$: In addition to $\mathrm{SO}(7) \subset \mathrm{SO}(8)$, we have

$$\begin{aligned}\mathrm{Sp}(2) &\subset \mathrm{Sp}(2) \mathrm{U}(1) \subset \mathrm{Sp}(2) \mathrm{Sp}(1) \subset \mathrm{SO}(8), \\ \mathrm{U}(2) &\subset \mathrm{SU}(3) \subset \mathrm{SU}(4) \subset \mathrm{U}(4) \subset \mathrm{SO}(8), \\ \text{and } \mathrm{SO}(4) &\subset \mathrm{G}_2 \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8).\end{aligned}$$

We note that $\mathrm{SO}(8)$ contains two copies of $\mathrm{Spin}(7)$ and that there is an outer automorphism of $\mathrm{SO}(8)$ of order three, called the triality automorphism, which interchanges the $\mathrm{Spin}(7)$'s, and on the Lie algebra level, it interchanges the $\mathfrak{spin}(7)$'s and the standard embedding of $\mathfrak{so}(7)$. This yields equalities like the following:

$$\begin{aligned}\mathrm{SO}(8)/\mathrm{Spin}(7) &= \mathrm{SO}(6)/\mathrm{SU}(3) \text{ (with double cover } \mathrm{SO}(8)/\mathrm{SO}(7) = \mathrm{SU}(4)/\mathrm{SU}(3)) \\ \mathrm{SO}(8)/\mathrm{Spin}(7) &= \mathrm{SO}(5)/\mathrm{SU}(2) \text{ (with double cover } \mathrm{SO}(8)/\mathrm{SO}(7) = \mathrm{Sp}(2)/\mathrm{Sp}(1)).\end{aligned}$$

We include some intersections of subgroups of $\mathrm{SO}(8)$ and related equalities:

$$\begin{aligned}\mathrm{G}_2 &= \mathrm{SO}(7) \cap \mathrm{Spin}(7) \text{ implies } \mathrm{SO}(8)/\mathrm{SO}(7) = \mathrm{Spin}(7)/\mathrm{G}_2; \\ \mathrm{Sp}(1) \mathrm{Sp}(1) &= \mathrm{SO}(7) \cap \mathrm{Sp}(2) \mathrm{Sp}(1) \text{ implies } \mathrm{SO}(8)/\mathrm{SO}(7) \\ &= \mathrm{Sp}(2) \mathrm{Sp}(1)/\mathrm{Sp}(1) \mathrm{Sp}(1); \\ \mathrm{SU}(3) &= \mathrm{SO}(7) \cap \mathrm{SU}(4) \text{ implies } \mathrm{SO}(8)/\mathrm{SO}(7) = \mathrm{SU}(4)/\mathrm{SU}(3).\end{aligned}$$

Here are the non-symmetric homogeneous spaces on Oniřćik's list [O1]:

$$\begin{aligned}\mathrm{SO}(7)/\mathrm{SO}(5) &= \mathrm{G}_2/\mathrm{SU}(2) &&= V_2(\mathbb{R}^7) \\ \mathrm{SO}(8)/\mathrm{SO}(6) &= \mathrm{Spin}(7)/\mathrm{SU}(3) &&= V_2(\mathbb{R}^8) \\ \mathrm{SO}(8)/\mathrm{SO}(5) &= \mathrm{Spin}(7)/\mathrm{SU}(2) &&= V_3(\mathbb{R}^8) \\ \mathrm{SO}(8)/\mathrm{SO}(2) \mathrm{SO}(5) &= \mathrm{Spin}(7)/\mathrm{SO}(2) \mathrm{SU}(2) \\ \mathrm{SO}(16)/\mathrm{Spin}(9) &= \mathrm{SO}(15)/\mathrm{Spin}(7) \\ \mathrm{SO}(2n)/\mathrm{SU}(n) &= \mathrm{SO}(2n-1)/\mathrm{SU}(n-1) \\ \mathrm{SO}(4n)/\mathrm{Sp}(n) &= \mathrm{SO}(4n-1)/\mathrm{Sp}(n-1) \\ \mathrm{SO}(4n)/\mathrm{Sp}(n) \mathrm{U}(1) &= \mathrm{SO}(4n-1)/\mathrm{Sp}(n-1) \mathrm{U}(1) \\ \mathrm{SO}(4n)/\mathrm{Sp}(n) \mathrm{Sp}(1) &= \mathrm{SO}(4n-1)/\mathrm{Sp}(n-1) \mathrm{Sp}(1).\end{aligned}$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, 209 S. 33RD STREET, PHILADELPHIA, PENNSYLVANIA 19104-6395

Current address: Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755

E-mail address: `megan@math.upenn.edu`