QUASIADDITIVITY AND MEASURE PROPERTY OF CAPACITY AND THE TANGENTIAL BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

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Dedicated to Professor F.-Y. Maeda on the occasion of his sixtieth birthday

ABSTRACT. We show that if a set E is dispersely decomposed into subsets, then the capacity of E is comparable to the summation of the capacities of the subsets. From this fact it is derived that the Lebesgue measure of a certain expanded set is estimated by the capacity of E . These properties hold for classical capacities, L^p -capacities and energy capacities of general kernels. The estimation is applied to the boundary behavior of harmonic functions. We introduce a boundary thin set and show a fine limit type boundary behavior of harmonic functions. We show that a thin set does not meet essentially Nagel-Stein and Nagel-Rudin-Shapiro type approaching regions at almost all boundary points.

1. INTRODUCTION

Let $K(r) \not\equiv 0$ be a nonnegative nonincreasing lower semicontinuous (l. s. c.) function for $r > 0$. We assume that

$$\lim_{r \rightarrow 0} K(r) = \infty \text{ and } \lim_{r \rightarrow \infty} K(r) = 0.$$

For $x \in \mathbb{R}^N$ we define $K(x) = K(|x|)$, and assume that $K(x)$ is locally integrable on $\mathbb{R}^N$, in other words

$$\int_0^\infty K(t)t^{N-1}dt < \infty.$$

We define the capacity C_K by

$$C_K(E) = \inf\{\|\mu\| : K * \mu \geq 1 \text{ on } E\}.$$

Let $K_\alpha(r) = r^{\alpha-N}$ for $0 < \alpha < N$. This is the Riesz kernel of order α . If $K(r) = K_\alpha(r)$, then we write C_α for C_K and call it α -capacity.

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Let $h(r)$ be a positive nondecreasing function for $r > 0$ and $h(0) = 0$. Such a function is called a measure function. We define the content M_h by

$$M_h(E) = \inf\{\sum h(r_j) : E \subset \bigcup B(x_j, r_j)\},$$

where $B(x, r)$ stands for the open ball with center at x and radius r . If $h(r) = r^\beta$, then we write M_β for M_h and call it β -content.

It is well known that the capacity C_α and the content M_β are closely related:

(i) If $\beta > N - \alpha$, then $C_\alpha(E) = 0 \implies M_\beta(E) = 0$.

(ii) If $\beta \leq N - \alpha$, then $M_\beta(E) = 0 \implies C_\alpha(E) = 0$.

Since C_α is countably subadditive and homogeneous of degree $N - \alpha$, it follows that

$$C_\alpha(E) \leq AM_{N-\alpha}(E),$$

which readily yields (ii). It is noteworthy that we cannot put $\beta = N - \alpha$ in (i); there is a set E such that $C_\alpha(E) = 0$ and $M_{N-\alpha}(E) > 0$. Thus the C_α -capacity does not majorize the $(N - \alpha)$ -dimensional measure.

One of the main purposes of this paper is to compare C_α with a certain $(N - \alpha)$ -dimensional quantity. Hereafter we shall use the following notation. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive quantities f and g are comparable, written $f \approx g$, if and only if there exists a constant A such that $A^{-1}g \leq f \leq Ag$. By $|E|$ we denote the Lebesgue measure of E . For a proper subset E we put $\delta_E(x) = \text{dist}(x, E^c)$. Let us consider the set

$$\tilde{E}_\alpha = \bigcup_{x \in E} B(x, \delta_E(x)^{(N-\alpha)/N}).$$

It is easy to see that if $E = B(0, r)$ and $r > 0$ is small, then $\tilde{E}_\alpha$ is a ball with radius comparable to $r^{(N-\alpha)/N}$, so that

$$|\tilde{E}_\alpha| \approx r^{N-\alpha} \text{ for small } r > 0.$$

Thus $|\tilde{E}_\alpha|$ is, in a sense, an $(N - \alpha)$ -dimensional quantity. We shall show the following theorem.

Theorem 1. *Let U be a bounded set. Suppose $E \subset U$ and let $\tilde{E}_\alpha$ be as above. Then*

$$|\tilde{E}_\alpha| \leq AC_\alpha(E),$$

where $A > 0$ depends only on N , α and U .

Let g_α be the Bessel kernel. The Riesz and the Bessel kernels have the same asymptotics as $r \rightarrow 0$. However, $g_\alpha(r)$ decreases rapidly as $r \rightarrow \infty$ and hence g_α is integrable on $\mathbb{R}^N$. It is well known that

$$C_\alpha(E) \approx C_{g_\alpha}(E) \text{ for } E \subset U,$$

where U is a bounded set. Hence Theorem 1 follows from Theorem 2 below. Let us consider the capacity of a ball. Suppose $K * \mu \geq 1$ on $B(0, r)$. Then $|B(0, r)| \leq \|K\|_1 \|\mu\|$ by Fubini's theorem, so that

$$(1) \quad C_K(B(0, r)) \geq \|K\|_1^{-1} |B(0, r)|$$

by definition. We define a positive function $\eta(r)$ by $|B(0, \eta(r))| = C_K(B(0, r))$ and let $\eta^*(r) = \max\{\eta(r), 2r\}$. It follows from (1) that

$$(2) \quad \eta(r) \approx \eta^*(r) \geq Ar.$$

We put

$$\tilde{E}_K = \bigcup_{x \in E} B(x, \eta^*(\delta_E(x))).$$

This is a set enlarged from E if E is open.

Theorem 2. *Let K be an integrable kernel. Then*

$$|\tilde{E}_K| \leq AC_K(E),$$

where $A > 0$ depends only on N and K .

Theorem 2 comes from the following quasiadditive property of C_K : if a set E is decomposed into pieces E_j dispersely, then $C_K(E)$ is comparable to $\sum C_K(E_j)$. More precisely we have the following theorem.

Theorem 3. *Let K be an integrable kernel. Suppose $\{B(x_j, \eta^*(r_j))\}$ is disjoint and E is an analytic subset of $\bigcup B(x_j, r_j)$. Then*

$$C_K(E) \leq \sum C_K(E \cap B(x_j, r_j)) \leq AC_K(E),$$

where $A > 0$ depends only on N and K .

Theorem 2 and Theorem 3 have counterparts in L^p -capacity theory. Let $1 < p < \infty$. We define

$$C_{K,p}(E) = \inf\{\|f\|_p^p : K * f \geq 1 \text{ on } E\}.$$

Let us consider the capacity of a ball. Suppose $K * f \geq 1$ on $B(0, r)$. Then $\|\chi_{B(0,r)}\|_p \leq \|K\|_1 \|f\|_p$, so that

$$(3) \quad C_{K,p}(B(0, r)) \geq \|K\|_1^{-p} |B(0, r)|$$

by definition. We define a positive function $\eta_p(r)$ by

$$|B(0, \eta_p(r))| = C_{K,p}(B(0, r))$$

and let $\eta_p^*(r) = \max\{\eta_p(r), 2r\}$. It follows from (3) that

$$(4) \quad \eta_p(r) \approx \eta_p^*(r) \geq Ar.$$

We put

$$\tilde{E}_{K,p} = \bigcup_{x \in E} B(x, \eta_p^*(\delta_E(x))).$$

Theorem 4. *Let K be an integrable kernel. Then*

$$|\tilde{E}_{K,p}| \leq AC_{K,p}(E),$$

where $A > 0$ depends only on N , K and p .

Theorem 5. *Let K be an integrable kernel. Suppose $\{B(x_j, \eta_p^*(r_j))\}$ is disjoint and E is an analytic subset of $\bigcup B(x_j, r_j)$. Then*

$$C_{K,p}(E) \leq \sum C_{K,p}(E \cap B(x_j, r_j)) \leq AC_{K,p}(E),$$

where $A > 0$ depends only on N , K and p .

We note that the quasiadditivity in Theorem 3 and Theorem 5 is different from the quasiadditivity considered in [3] and [4]. Theorem 2 and Theorem 4 have an application to the tangential boundary behavior of harmonic functions. We shall later generalize these theorems, in connection with Nagel-Stein approach regions ([8]). We shall introduce a notion of “thin sets” and combine it with the generalized version of Theorem 2 and Theorem 4 to obtain precise description of the tangential boundary behavior of harmonic functions given as the Poisson integral of certain potentials.

The plan of this paper is as follows. We prove Theorem 3 and Theorem 5 in Section 2 and Section 3, respectively. In Section 4 we prove Theorem 2 and Theorem 4. In Section 5 we introduce the Nagel-Stein approach region and generalize Theorem 2 and Theorem 4. The boundary behavior of harmonic function will be considered in Sections 6–10. We shall introduce a new notion of boundary thinness in Section 6.

Hereafter, we let K be an integrable kernel, unless otherwise specified.

2. PROOF OF THEOREM 3

We shall need to give a lower estimate for the capacity. To this end we use the dual definition of C_K .

Theorem A. *Let E be an analytic set. Then*

$$C_K(E) = \sup\{\|\mu\| : \mu \text{ is concentrated on } E, K * \mu \leq 1 \text{ on } \mathbb{R}^N\}.$$

We prepare an elementary lemma.

Lemma 1. *Let $0 < 2r \leq R$. Suppose $x \notin B(x_0, R)$ and let $\rho = \text{dist}(x, B(x_0, r))$. Then*

$$|B(x, \rho) \cap B(x_0, R)| \geq A|B(x_0, R)|,$$

where A depends only on the dimension.

Proof. Let x_1 be the point on the line segment connecting x_0 and x such that $|x_1 - x_0| = \frac{3}{4}R$. It is easy to see that

$$\rho = |x - x_0| - r = |x - x_1| + \frac{3}{4}R - r \geq |x - x_1| + \frac{1}{4}R.$$

We observe that

$$B(x_1, \frac{1}{4}R) \subset B(x, \rho) \cap B(x_0, R),$$

since if $y \in (x_1, \frac{1}{4}R)$, then

$$|x - y| < |x - x_1| + \frac{1}{4}R \leq \rho \text{ and } |x_0 - y| < |x_0 - x_1| + \frac{1}{4}R = R.$$

Thus the required inequality follows.

Proof of Theorem 3. The countable subadditivity yields

$$C_K(E) \leq \sum C_K(E \cap B(x_j, r_j)).$$

Let us prove $\sum C_K(E \cap B(x_j, r_j)) \leq AC_K(E)$. For simplicity we write E_j for $E \cap B(x_j, r_j)$. Let ε be an arbitrary positive number. By Theorem A we can find a measure μ_j such that

$$\begin{aligned} \mu_j &\text{ is concentrated on } E_j, \\ K * \mu_j &\leq 1 \text{ on } \mathbb{R}^N, \\ \|\mu_j\| &\geq C_K(E_j) - 2^{-j}\varepsilon. \end{aligned}$$

Let $\mu = \sum \mu_j$. Observe that μ is concentrated on E . We claim

$$(5) \quad K * \mu \leq A \text{ on } \mathbb{R}^N.$$

If we have (5), then Theorem A yields

$$C_K(E) \geq A\|\mu\| = A \sum \|\mu_j\| \geq A \sum C_K(E_j) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the required inequality follows.

For the proof of (5) we put

$$d\mu'_j = \frac{\|\mu_j\|}{|B(x_j, \eta^*(r_j))|} \chi_{B(x_j, \eta^*(r_j))} dx$$

and $\mu' = \sum \mu'_j$. This kind of measure was introduced by Borichev [5]. Since

$$\|\mu_j\| \leq C_K(E_j) \leq C_K(B(x_j, r_j)) = |B(x_j, \eta(r_j))| \leq |B(x_j, \eta^*(r_j))|,$$

it follows that

$$d\mu' \leq \sum \chi_{B(x_j, \eta^*(r_j))} dx \leq dx,$$

where we have used the disjointness of $B(x_j, \eta^*(r_j))$ in the last inequality. Hence

$$(6) \quad K * \mu' \leq K * 1 = \int_{\mathbb{R}^N} K dx < \infty \text{ on } \mathbb{R}^N.$$

Let us compare $K * \mu$ and $K * \mu'$. Suppose first $x \notin B(x_j, \eta^*(r_j))$. We apply Lemma 1 with $x_0 = x_j$, $R = \eta^*(r_j)$ and $r = r_j$. Let $\rho_j = \text{dist}(x, B(x_j, r_j))$. We have

$$\begin{aligned} K * \mu'_j(x) &\geq \int_{B(x, \rho_j) \cap B(x_j, \eta^*(r_j))} K(x-y) d\mu'_j(y) \\ &\geq K(\rho_j) \mu'_j(B(x, \rho_j) \cap B(x_j, \eta^*(r_j))) \\ &= K(\rho_j) \|\mu_j\| \frac{|B(x, \rho_j) \cap B(x_j, \eta^*(r_j))|}{|B(x_j, \eta^*(r_j))|} \\ &\geq AK(\rho_j) \|\mu_j\|. \end{aligned}$$

Obviously, $K * \mu_j(x) \leq K(\rho_j) \|\mu_j\|$, whence

$$(7) \quad K * \mu_j(x) \leq AK * \mu'_j(x).$$

Now suppose $x \notin \bigcup B(x_j, \eta^*(r_j))$. Then (7) holds for all j . Hence we have from (6)

$$K * \mu(x) = \sum K * \mu_j(x) \leq A \sum K * \mu'_j(x) \leq AK * \mu'(x) \leq A.$$

Suppose $x \in \bigcup B(x_j, \eta^*(r_j))$, say $x \in B(x_{j_0}, \eta^*(r_{j_0}))$ for some j_0 . Then, by the disjointness of $\{B(x_j, \eta^*(r_j))\}$ we have $x \notin \bigcup_{j \neq j_0} B(x_j, \eta^*(r_j))$. Hence (6) and (7) yield

$$\sum_{j \neq j_0} K * \mu_j(x) \leq A \sum_{j \neq j_0} K * \mu'_j(x) \leq AK * \mu'(x) \leq A.$$

Since μ_{j_0} is an equilibrium measure, it follows that $K * \mu_{j_0} \leq 1$ on $\mathbb{R}^N$. Thus

$$K * \mu(x) = K * \mu_{j_0}(x) + \sum_{j \neq j_0} K * \mu_j(x) \leq A.$$

Therefore (5) follows. The theorem is proved.

3. PROOF OF THEOREM 5

First we prepare fundamental facts for $C_{K,p}$. We have the dual definition ([6, Theorem 14]). Let $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem B. *Let E be an analytic set. Then*

$$C_{K,p}(E) = \sup\{\|\mu\|^p : \mu \text{ is concentrated on } E, \|K * \mu\|_q \leq 1\}.$$

Proof of Theorem 5. The proof will be carried out in a fashion similar to that of Theorem 3. It is sufficient to show that $\sum C_{K,p}(E \cap B(x_j, r_j)) \leq AC_{K,p}(E)$ in case the left hand side is positive. For simplicity we write E_j for $E \cap B(x_j, r_j)$. Let ε be an arbitrary number such that

$$0 < \varepsilon < \sum C_{K,p}(E_j).$$

By Theorem B we can find a measure μ_j such that

$$\begin{aligned} \mu_j &\text{ is concentrated on } E_j, \\ \|K * \mu_j\|_q &= 1, \\ C_{K,p}(E_j) - 2^{-j}\varepsilon &\leq \|\mu_j\|^p \leq C_{K,p}(E_j). \end{aligned}$$

Let $\mu_j^* = C_{K,p}(E_j)^{1/q} \mu_j$ and let $\mu^* = \sum \mu_j^*$. We observe that

$$(8) \quad C_{K,p}(E_j) - 2^{-j}\varepsilon \leq \|\mu_j^*\| \leq \|K * \mu_j^*\|_q^q = C_{K,p}(E_j).$$

We claim

$$(9) \quad \|K * \mu^*\|_q^q \leq A \sum C_{K,p}(E_j).$$

If we have (9), then

$$\tilde{\mu} = \left(\sum C_{K,p}(E_j) \right)^{-1/q} \mu^*$$

satisfies $\|K * \tilde{\mu}\|_q \leq A$. Obviously $\tilde{\mu}$ is concentrated on E and by (8)

$$\|\tilde{\mu}\| \geq \left(\sum C_{K,p}(E_j) \right)^{-1/q} \left(\sum C_{K,p}(E_j) - \varepsilon \right).$$

Since ε is arbitrary, it follows from Theorem B that

$$C_{K,p}(E) \geq A \sum C_{K,p}(E_j).$$

Thus the required inequality follows.

For the proof of (9) we put

$$d\mu'_j = \frac{\|\mu_j^*\|}{|B(x_j, \eta_p^*(r_j))|} \chi_{B(x_j, \eta_p^*(r_j))} dx$$

and $\mu' = \sum \mu'_j$. By (8)

$$(10) \quad \|\mu'\| = \|\mu^*\| \leq \sum C_{K,p}(E_j).$$

Let $d\mu' = f dx$. Then, we observe from the disjointness of $B(x_j, \eta_p^*(r_j))$ that

$$f = \sum \chi_{B(x_j, \eta_p^*(r_j))} \leq 1.$$

Hence by (10)

$$(11) \quad \begin{aligned} \|K * \mu'\|_q^q &= \|K * f\|_q^q \leq \|K\|_1^q \|f\|_q^q = A \|f\|_q^q \\ &\leq A \|f\|_\infty^{q-1} \|f\|_1 \leq A \|f\|_1 = A \|\mu'\| \leq A \sum C_{K,p}(E_j). \end{aligned}$$

Let us compare $K * \mu^*$ and $K * \mu'$. Applying Lemma 1, we obtain

$$(12) \quad K * \mu_j^*(x) \leq A K * \mu'_j(x) \text{ for } x \notin B(x_j, \eta_p^*(r_j)).$$

Thus

$$K * \mu^*(x) \leq AK * \mu'(x) \text{ for } x \notin \bigcup B(x_j, \eta_p^*(r_j)).$$

Suppose $x \in \bigcup B(x_j, \eta_p^*(r_j))$, say $x \in B(x_{j_0}, \eta_p^*(r_{j_0}))$ for some j_0 . Then, by the disjointness of $\{B(x_j, \eta_p^*(r_j))\}$ we have $x \notin \bigcup_{j \neq j_0} B(x_j, \eta_p^*(r_j))$. Hence (12) yields

$$K * \mu^*(x) \leq K * \mu_{j_0}^*(x) + A \sum_{j \neq j_0} K * \mu_j'(x) \leq K * \mu_{j_0}^*(x) + AK * \mu'(x).$$

It follows from (8) and (11) that

$$\begin{aligned} \|K * \mu^*\|_q^q &= \int_{\mathbb{R}^N \setminus \bigcup B(x_j, \eta_p^*(r_j))} (K * \mu^*)^q dx + \int_{\bigcup B(x_j, \eta_p^*(r_j))} (K * \mu^*)^q dx \\ &\leq A \|K * \mu'\|_q^q + A \sum \int_{B(x_j, \eta_p^*(r_j))} (K * \mu_j^*)^q dx \\ &\leq A \sum C_{K,p}(E_j) + A \sum \|K * \mu_j^*\|_q^q \\ &\leq A \sum C_{K,p}(E_j). \end{aligned}$$

Therefore (9) follows. The theorem is proved.

4. PROOF OF THEOREM 2 AND THEOREM 4

Let $C_{K,1}(E) = C_K(E)$, $\eta_1(r) = \eta(r)$, $\eta_1^*(r) = \eta^*(r)$ and $\tilde{E}_{K,1} = \tilde{E}_K$. Hereafter we let $1 \leq p < \infty$. Then we can prove Theorem 2 and Theorem 4 in the same way with the help of Theorem 3 and Theorem 5.

Proof of Theorem 2 and Theorem 4. Take an arbitrary compact subset F of $\tilde{E}_{K,p}$. By the usual covering lemma we can find $x_j \in E$ such that

$$\begin{aligned} F &\subset \bigcup B(x_j, 5\eta_p^*(r_j)), \\ \{B(x_j, \eta_p^*(r_j))\} &\text{ is disjoint,} \\ r_j &= \delta_E(x_j). \end{aligned}$$

Let $E' = \bigcup B(x_j, r_j)$. By definition this is a subset of E . We apply Theorem 3 and Theorem 5 for $B(x_j, r_j)$ and E' . We obtain

$$\sum C_{K,p}(B(x_j, r_j)) \leq AC_{K,p}(E') \leq AC_{K,p}(E).$$

On the other hand we have

$$|F| \leq \sum |B(x_j, 5\eta_p^*(r_j))| = A \sum |B(x_j, \eta_p^*(r_j))|.$$

It follows from (2) and (4) that

$$(13) \quad |B(0, \eta_p^*(r))| \approx |B(0, \eta_p(r))| = C_{K,p}(B(0, r)).$$

Hence

$$|F| \leq AC_{K,p}(E).$$

Since F is an arbitrary compact subset of $\tilde{E}_{K,p}$, the required inequality follows. The theorems are proved.

5. GENERALIZATION

In this section we let $1 \leq p < \infty$ and use the convention for $C_{K,p}(E)$, $\eta_p(r)$, $\eta_p^*(r)$ and $\tilde{E}_{K,p}$ as in the last section. Let Ω be a set in $\mathbb{R}_+^{N+1}$ with $\overline{\Omega} \cap \partial\mathbb{R}_+^{N+1} = \{0\}$. For simplicity we assume that $\Omega \supset \{(0, y) : y > 0\}$. Put $\Omega(y) = \{x : (x, y) \in \Omega\}$. We say that Ω satisfies the Nagel-Stein condition (abbreviated to (NS)), if

- (i) $|\Omega(y)| \leq Ay^N$ with $A = A(\Omega)$;
- (ii) there is $\alpha > 0$ such that

$$(x_1, y_1) \in \Omega \text{ and } |x - x_1| < \alpha(y - y_1) \implies (x, y) \in \Omega.$$

For E we put

$$\tilde{E}_{K,p;\Omega} = \bigcup_{x \in E} (x - \Omega(\eta_p^*(\delta_E(x)))) .$$

Theorem 6. *Let Ω satisfy (NS). Then*

$$|\tilde{E}_{K,p;\Omega}| \leq AC_{K,p}(E),$$

where $A > 0$ depends only on N , K , p and Ω .

Proof. The proof will be carried out in the same way as in the proof of Theorem 2 and Theorem 4 if we use the covering lemma due to Nagel-Stein [8, pp. 90–92]. But, for the completeness we give a proof.

Let $Q(y) = \bigcup_{x \in \Omega(y)} B(x, y)$. Obviously, $\Omega(y) \subset Q(y)$. In particular,

$$\tilde{E}_{K,p;\Omega} \subset \bigcup_{x \in E} (x - Q(\eta_p^*(\delta_E(x)))) .$$

For a moment we fix $y > 0$. By [8, Lemma 1 (d)] we find points $u_1, \dots, u_M \in \Omega(y)$ so that $Q(y) \subset \bigcup_{j=1}^M B(u_j, 3y)$. Note that the number M is independent of y . We put

$$\tilde{Q}(y) = \bigcup_{j,k=1}^M \{u_j - u_k + \bigcup_{i=1}^M B(u_i, 9y)\}.$$

We observe that

$$(14) \quad |\tilde{Q}(y)| \leq Ay^N,$$

where $A > 0$ is independent of y .

Take an arbitrary compact subset F of $\tilde{E}_{K,p;\Omega}$. We follow the same argument as in [8, pp. 90–92] and find points $x_1, \dots, x_J \in E$ such that

$$(15) \quad F \subset \bigcup_{j=1}^J x_j - \tilde{Q}(y_j), \quad y_j = \eta_p^*(\delta_E(x_j));$$

$$\{x_j - Q(y_j)\}_{j=1}^J \text{ is disjoint.}$$

We assumed that $\Omega \supset \{(0, y) : y > 0\}$ and hence $Q(y) \supset B(0, y)$. Hence, in particular, $\{B(x_j, y_j)\}_{j=1}^J$ is disjoint. Let

$$E' = \bigcup_{j=1}^J B(x_j, \delta_E(x_j)).$$

Obviously, E' is a subset of E . We apply Theorem 3 and Theorem 5 to E' to obtain

$$\sum C_{K,p}(B(x_j, \delta_E(x_j))) \leq AC_{K,p}(E') \leq AC_{K,p}(E).$$

On the other hand, by (13), (14) and (15) we have

$$|F| \leq A \sum y_j^N = A \sum |B(x_j, \eta_p^*(\delta_E(x_j)))| \leq A \sum C_{K,p}(B(x_j, \delta_E(x_j))).$$

Hence

$$|F| \leq AC_{K,p}(E).$$

Since F is an arbitrary compact subset of $\tilde{E}_{K,p;\Omega}$, the required inequality follows. The theorem is proved.

Remark 1. Let Ω be as in Theorem 6. Then

$$|\bigcup_{x \in E} (x + \Omega(\eta_p^*(\delta_E(x))))| \leq AC_{K,p}(E).$$

We record counterparts of Theorems 1–6 for energy capacity. The energy capacity for the kernel K is defined by

$$e_K(E) = \sup\{\|\mu\|^2 : \mu \text{ is concentrated on } E, \int K * \mu d\mu \leq 1\}.$$

We define $\eta_e(r)$ by

$$|B(0, \eta_e(r))| = e_K(B(0, r))$$

and let $\eta_e^*(r) = \max\{\eta_e(r), 2r\}$.

Theorem 7. *Let Ω satisfy (NS). Then*

$$\left| \bigcup_{x \in E} (x - \Omega(\eta_e^*(\delta_E(x)))) \right| \leq Ae_K(E).$$

where $A > 0$ is independent of E . In particular,

$$|\bigcup_{x \in E} B(x, \eta_e^*(\delta_E(x)))| \leq Ae_K(E).$$

Theorem 8. *Suppose $\{B(x_j, \eta_e^*(r_j))\}$ is disjoint and E is an analytic subset of $\bigcup B(x_j, r_j)$. Then*

$$e_K(E) \leq \sum e_K(E \cap B(x_j, r_j)) \leq Ae_K(E),$$

where $A > 0$ is independent of E .

We can prove Theorem 7 in exactly the same way as in the proof of Theorem 2 and Theorem 4, if we invoke Theorem 8. The proof of Theorem 8 is somewhat a mixture of those of Theorem 3 and Theorem 5. The details are left to the reader.

6. BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

In what follows we are interested in the boundary behavior of harmonic functions in $\mathbb{R}_+^{N+1}$. Hereafter we let $1 \leq p < \infty$. Following the idea in [2] and [5] we introduce the notion of thinness at the boundary. For a set $E \subset \mathbb{R}_+^{N+1}$ we put $E_t = \{(x, y) : 0 < y < t\}$ and $E^* = \bigcup_{(x, y) \in E} B(x, y)$. We recall that $B(x, y)$ is the N -dimensional ball with center at x and radius y , so that the set E^* is a set on the boundary $\mathbb{R}^N = \partial\mathbb{R}_+^{N+1}$. We shall combine the above notation and write

$$E_t^* = \bigcup_{(x, y) \in E, 0 < y < t} B(x, y).$$

Definition. Let $E \subset \mathbb{R}_+^{N+1}$. We say that E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$ if

$$\lim_{t \rightarrow 0} C_{K,p}(E_t^*) = 0.$$

Remark 2. If E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$, then the essential projection of E

$$\{x : \text{for any } t > 0 \text{ there is a positive number } y < t \text{ such that } (x, y) \in E\}$$

is of $C_{K,p}$ -capacity 0, and hence of measure 0.

We shall show the following theorems.

Theorem 9. Suppose Ω satisfies (NS). Let

$$\Omega_{K,p} = \{(x, y) : x \in \Omega(\eta_p^*(y))\}.$$

If E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$, then

$$\left| \bigcap_{t>0} \{x : (x + \Omega_{K,p}) \cap E_t \neq \emptyset\} \right| = 0.$$

In other words, for almost all $x \in \partial\mathbb{R}_+^{N+1}$, $x + \Omega_{K,p}$ lies eventually outside E , i.e., there is $t = t_x > 0$ such that $E_t \cap (x + \Omega_{K,p}) = \emptyset$.

Remark 3. It is not so difficult to see that $\eta_p^*(r)/r \rightarrow \infty$ as $r \rightarrow 0$ (cf. [1]). Hence $\Omega_{K,p}$ is a tangential region.

For a function f on $\partial\mathbb{R}_+^{N+1}$ we denote by $PI(f)$ its Poisson integral. More precisely, let $\varphi(x) = A_N(|x|^2 + 1)^{-(N+1)/2}$ and $\varphi_y(x) = y^{-N}\varphi(x/y)$, where $A_N > 0$ is such that $\|\varphi\|_1 = 1$. Then we see that

$$PI(f)(x, y) = \varphi_y * f(x).$$

Theorem 10. Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose $\overline{\Omega} \cap \partial\mathbb{R}_+^{N+1} = \{0\}$. If $p = 1$, then we assume that

$$(16) \quad K(r) \approx r^{-N} \int_0^r K(t)t^{N-1}dt \text{ for small } r > 0.$$

Suppose $f \in L^p(\mathbb{R}^N)$. Then there is a set $E \subset \mathbb{R}_+^{N+1}$ such that E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$ and that

$$(17) \quad \lim_{P \rightarrow x, P \in (x+\Omega) \setminus E} PI(K * f)(P) = K * f(x)$$

for $C_{K,p}$ -a.e. $x \in \partial\mathbb{R}_+^{N+1}$, i.e. there is a set $F \subset \partial\mathbb{R}_+^{N+1}$ such that $C_{K,p}(F) = 0$ and (17) holds for $x \in \partial\mathbb{R}_+^{N+1} \setminus F$.

As a corollary to Theorem 9 and Theorem 10 we have the following theorem. This is a generalization of [7, Theorem 2.9].

Theorem 11. Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose Ω satisfies (NS). If $p = 1$, then we assume that (16) holds. Suppose $f \in L^p(\mathbb{R}^N)$. Then

$$\lim_{P \rightarrow x, P \in x + \Omega_{K,p}} PI(K * f)(P) = K * f(x)$$

for almost all $x \in \partial \mathbb{R}_+^{N+1}$.

We note that Theorem 10 gives a sharp characterization of exceptional sets.

Proposition 1. Suppose E is $C_{K,p}$ -thin at $\partial \mathbb{R}_+^{N+1}$. Then there is $f \in L^p(\mathbb{R}^N)$ such that

$$(18) \quad \inf_{(x,y) \in E, 0 < y < t} PI(K * f)(x, y) \rightarrow \infty \text{ as } t \rightarrow 0.$$

Finally we observe that the region $\Omega \subset \mathbb{R}_+^{N+1}$ in Theorem 10 is necessary. In fact, we have

Proposition 2. Let g be an integrable function and let $x_0 \in \partial \mathbb{R}_+^{N+1}$. Suppose there is a set E such that the essential projection of E is of measure 0 and

$$(19) \quad \lim_{P \rightarrow x_0, P \notin E} PI(g)(P) = \alpha.$$

Then

$$(20) \quad \operatorname{ess\,lim}_{x \rightarrow x_0} g(x) = \alpha.$$

Remark 2 and Proposition 2 assert that if $h(P) = PI(K * f)(P)$ approaches a limit as P approaches $x_0 \in \partial \mathbb{R}_+^{N+1}$ outside a set $C_{K,p}$ -thin at $\partial \mathbb{R}_+^{N+1}$, then $K * f$ must be essentially continuous at x_0 . Unless a point has positive $C_{K,p}$ -capacity, there exists $f \in L^p(\mathbb{R}^N)$ such that $K * f$ is essentially discontinuous at every point. In fact, assume that a point has null $C_{K,p}$ -capacity. Let E be a countable dense subset of $\mathbb{R}^N$. Then we see that $C_{K,p}(E) = 0$, so that there is a nonnegative function $f \in L^p(\mathbb{R}^N)$ such that $K * f = \infty$ on E . Since $K * f$ is lower semicontinuous, it follows that $\operatorname{ess\,lim\,sup}_{x \rightarrow x_0} K * f(x) = \infty$ for every $x_0 \in \mathbb{R}^N$. At almost every point, $K * f$ is discontinuous even in the extended sense, since $\operatorname{ess\,lim\,inf}_{x \rightarrow x_0} K * f(x) < \infty$ for a.e. $x_0 \in \mathbb{R}^N$ by the local integrability of $K * f$.

7. PROOF OF THEOREM 9

We begin by an elementary geometrical observation.

Lemma 2. Let $\Omega \subset \mathbb{R}_+^{N+1}$ and suppose $\Omega(y)$ is increasing, i.e., if $y_1 < y_2$, then $\Omega(y_1) \subset \Omega(y_2)$. Let $E \subset \mathbb{R}_+^{N+1}$ and let E^* be as in Section 7. Then

$$(21) \quad \{x \in \mathbb{R}^N : (x + \Omega) \cap E \neq \emptyset\} \subset \bigcup_{x \in E^*} (x - \Omega(\delta_{E^*}(x))),$$

where $\delta_{E^*}(x) = \operatorname{dist}(x, E^{*c})$.

Proof. We take an arbitrary point x_0 from the set in the left hand side of (21). Then $(x_0 + \Omega) \cap E \neq \emptyset$. Assume that $(x, y) \in (x_0 + \Omega) \cap E$. This implies

- (i) $x \in x_0 + \Omega(y)$ and hence $x_0 \in x - \Omega(y)$;
- (ii) $(x, y) \in E$ and hence $B(x, y) \subset E^*$ by definition, so that $\delta_{E^*}(x) \geq y$.

By the monotonicity of $\Omega(y)$ we have

$$x_0 \in x - \Omega(\delta_{E^*}(x)),$$

which yields (21). The lemma is proved.

Proof of Theorem 9. We apply Lemma 2 for $\Omega_{K,p}$ replacing Ω . Note that $\Omega_{K,p}(y) = \Omega(\eta_p^*(y))$. Then (21) becomes

$$\begin{aligned} \{x \in \mathbb{R}^N : (x + \Omega_{K,p}) \cap E \neq \emptyset\} &\subset \bigcup_{x \in E^*} (x - \Omega_{K,p}(\delta_{E^*}(x))) \\ &= \bigcup_{x \in E^*} (x - \Omega(\eta_p^*(\delta_{E^*}(x)))). \end{aligned}$$

Hence Theorem 6 yields

$$|\{x \in \mathbb{R}^N : (x + \Omega_{K,p}) \cap E \neq \emptyset\}| \leq AC_{K,p}(E^*).$$

Apply this inequality for E_t replacing E . Then the definition of thinness implies that

$$|\{x : (x + \Omega_{K,p}) \cap E_t \neq \emptyset\}| \leq AC_{K,p}(E_t^*) \rightarrow 0 \text{ as } t \rightarrow 0.$$

Thus the theorem follows.

8. PROOF OF THEOREM 10 IN CASE $1 < p < \infty$

In this section we let $1 < p < \infty$.

Lemma 3. *Let f be a nonnegative function in $L_+^p(\mathbb{R}^N)$ and let $\varepsilon > 0$. Put*

$$\begin{aligned} E(f, \varepsilon) &= \{(x, y) \in \mathbb{R}_+^{N+1} : PI(K * f)(x, y) > \varepsilon\}, \\ E^*(f, \varepsilon) &= \bigcup_{(x, y) \in E(f, \varepsilon)} B(x, y). \end{aligned}$$

Then

$$C_{K,p}(E^*(f, \varepsilon)) \leq A \left(\frac{\|f\|_p}{\varepsilon} \right)^p,$$

where A is independent of f and ε .

Proof. Let

$$E'(f, \varepsilon) = \bigcup_{(x, y) \in E(f, \varepsilon)} B(x, y) \times \{y\}.$$

Then the projection of $E'(f, \varepsilon)$ is equal to $E^*(f, \varepsilon)$. Since $PI(K * f)$ is a positive harmonic function, it follows from the Harnack inequality that

$$(22) \quad PI(K * f) \geq A\varepsilon \text{ on } E'(f, \varepsilon),$$

where A depends only on the dimension. Let us put

$$F(x) = \sup_{y>0} \varphi_y * f(x),$$

where φ_y is the Poisson kernel as defined in Section 7. Then, the maximal inequality shows that

$$(23) \quad \|F\|_p \leq A\|f\|_p$$

([9, Theorem 3.7]). By definition

$$K * F(x) \geq K * (\varphi_y * f)(x) = \varphi_y * (K * f)(x) = PI(K * f)(x, y) \text{ for all } y > 0.$$

Hence by (22)

$$K * F \geq A\varepsilon \text{ on } E^*(f, \varepsilon).$$

Thus, by (23),

$$C_{K,p}(E^*(f, \varepsilon)) \leq A \left(\frac{\|F\|_p}{\varepsilon} \right)^p \leq A \left(\frac{\|f\|_p}{\varepsilon} \right)^p.$$

The lemma follows.

In [2] we proved a theorem similar to Theorem 10 by using oscillation. Here, we present a more direct proof without using oscillation. By $\mathbf{B}(P, r)$ we denote the $(N+1)$ -dimensional open ball with center at P and radius r .

Lemma 4. *Let $f \in L^p(\mathbb{R}^N)$ and let $\delta > 0$. Then there exist $E \subset \mathbb{R}_+^{N+1}$ and $F \subset \partial\mathbb{R}_+^{N+1}$ such that*

- (i) $C_{K,p}(E^*) < \delta$ and $C_{K,p}(F) < \delta$;
- (ii) *for each $\varepsilon > 0$ there is $r > 0$ such that*

$$(24) \quad \sup_{x \in \partial\mathbb{R}_+^{N+1} \setminus F} \left(\sup_{P \in \mathbf{B}(x, r) \cap \mathbb{R}_+^{N+1} \setminus E} |PI(K * f)(P) - K * f(x)| \right) < \varepsilon.$$

Proof. In view of the definition of $C_{K,p}$ and Lemma 3, we can choose a sequence $g_j \in C_0^\infty(\mathbb{R}^N)$ such that

$$C_{K,p}(F_j) < 2^{-j}\delta \text{ with } F_j = \{x : |K * g_j(x) - K * f(x)| \geq 2^{-j}\},$$

$$C_{K,p}(E_j^*) < 2^{-j}\delta \text{ with } E_j = \{P : |PI(K * g_j)(P) - PI(K * f)(P)| \geq 2^{-j}\}.$$

Let $F = \bigcup_{j=1}^\infty F_j$ and $E = \bigcup_{j=1}^\infty E_j$. Then $C_{K,p}(F) < \delta$ and $C_{K,p}(E^*) < \delta$. Moreover we observe that

$$\begin{aligned} K * g_j &\rightarrow K * f \text{ uniformly on } \partial\mathbb{R}_+^{N+1} \setminus F, \\ PI(K * g_j) &\rightarrow PI(K * f) \text{ uniformly on } \mathbb{R}_+^{N+1} \setminus E. \end{aligned}$$

Hence we can find j_0 such that

$$\begin{aligned} \sup_{x \in \partial\mathbb{R}_+^{N+1} \setminus F} |K * g_{j_0}(x) - K * f(x)| &< \frac{\varepsilon}{3}, \\ \sup_{P \in \mathbb{R}_+^{N+1} \setminus E} |PI(K * g_{j_0})(P) - PI(K * f)(P)| &< \frac{\varepsilon}{3}. \end{aligned}$$

Since $PI(K * g_{j_0})$ is uniformly continuous on $\mathbb{R}_+^{N+1}$ and has boundary values $K * g_{j_0}$, we can find $r > 0$ such that

$$\sup_{x \in \partial \mathbb{R}_+^{N+1}} \left(\sup_{P \in \mathbf{B}(x, r) \cap \mathbb{R}_+^{N+1}} |PI(K * g_{j_0})(P) - K * g_{j_0}(x)| \right) < \frac{\varepsilon}{3}.$$

The above three inequalities altogether imply (24). The lemma is proved

We prepare a geometrical observation for Ω . We assume that $\overline{\Omega} \cap \partial \mathbb{R}_+^{N+1} = \{0\}$, which implies that for each $r > 0$ there is $t > 0$ such that

$$\Omega \cap \{(x, y) : 0 < y < t\} \subset \mathbf{B}(0, r),$$

or equivalently

$$\Omega \setminus \mathbf{B}(0, r) \subset \{(x, y) : y \geq t\}.$$

Hence, if we have a decreasing sequence $r_j \downarrow 0$, then we can find $t_j \downarrow 0$ such that

$$(25) \quad t_j < r_{j+1}$$

and $\Omega \cap \{(x, y) : 0 < y < t_j\} \subset \mathbf{B}(0, r_{j+1})$, or equivalently

$$\Omega \setminus \mathbf{B}(0, r_{j+1}) \subset \{(x, y) : y \geq t_j\}.$$

Observe that

$$\Omega \cap \mathbf{B}(0, r_i) = \bigcup_{j=i}^{\infty} \Omega \cap \mathbf{B}(0, r_j) \setminus \mathbf{B}(0, r_{j+1}) \subset \bigcup_{j=i}^{\infty} \mathbf{B}(0, r_j) \cap \{(x, y) : y \geq t_j\}$$

By translation we have for $x \in \partial \mathbb{R}_+^{N+1}$

$$(26) \quad (x + \Omega) \cap \mathbf{B}(x, r_i) \subset \bigcup_{j=i}^{\infty} \mathbf{B}(x, r_j) \cap \{(x, y) : y \geq t_j\}.$$

Proof of Theorem 10. Take $\varepsilon_j \downarrow 0$. By Lemma 4 there are $E_j \subset \mathbb{R}_+^{N+1}$, $F_j \subset \partial \mathbb{R}_+^{N+1}$ and $r_j \downarrow 0$ such that

$$(27) \quad \sum_j C_{K,p}(E_j^*) < \infty \text{ and } \sum_j C_{K,p}(F_j) < \infty,$$

$$(28) \quad \sup_{x \in \partial \mathbb{R}_+^{N+1} \setminus F_j} \left(\sup_{P \in \mathbf{B}(x, r_j) \setminus E_j} |PI(K * f)(P) - K * f(x)| \right) < \varepsilon_j.$$

We choose $t_j \downarrow 0$ so that (25) and (26) hold. Let $E'_j = E_j \cap \{(x, y) : y \geq t_j\}$ and put $E = \bigcup_j E'_j$. Then (27) yields

$$C_{K,p}(E_{t_i}^*) \leq C_{K,p}\left(\bigcup_{j=i}^{\infty} E_j^*\right) \leq \sum_{j=i}^{\infty} C_{K,p}(E_j^*) \rightarrow 0 \text{ as } i \rightarrow \infty,$$

so that E is $C_{K,p}$ -thin at $\partial\mathbb{R}_+^{N+1}$. Again by (27) we see that $F = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} F_j$ satisfies $C_{K,p}(F) = 0$. Take $x \in \partial\mathbb{R}_+^{N+1} \setminus F$. We shall show that (17) holds. By definition there is $j_0 = j_0(x)$ such that $x \notin F_j$ for all $j \geq j_0$. By (28)

$$\sup_{P \in \mathbf{B}(x, r_j) \setminus E_j} |PI(K * f)(P) - K * f(x)| < \varepsilon_j$$

for all $j \geq j_0$. Let $i \geq j_0$. In view of (26) and the definition of E

$$\begin{aligned} & \sup_{P \in (x+\Omega) \cap \mathbf{B}(x, r_i) \setminus E} |PI(K * f)(P) - K * f(x)| \\ & \leq \sup_{j \geq i} \left(\sup_{P \in \mathbf{B}(x, r_j) \cap \{(x, y) : y \geq t_j\} \setminus E} |PI(K * f)(P) - K * f(x)| \right) \\ & \leq \sup_{j \geq i} \left(\sup_{P \in \mathbf{B}(x, r_j) \cap \{(x, y) : y \geq t_j\} \setminus E_j} |PI(K * f)(P) - K * f(x)| \right) \\ & \leq \sup_{j \geq i} \varepsilon_j = \varepsilon_i \rightarrow 0 \text{ as } i \rightarrow \infty. \end{aligned}$$

This implies (17). The theorem is proved.

9. PROOF OF THEOREM 10 IN CASE $p = 1$

In this section we let $p = 1$. In this case we cannot use the maximal inequality in the proof of Lemma 3. Instead, we estimate the kernel $K_y(x) = K * \varphi_y(x)$. Let

$$\kappa(r) = K\left(\frac{r}{2}\right) + \frac{1}{r+1} r^{-N} \int_0^r K(t) t^{N-1} dt$$

and let $\kappa(x) = \kappa(|x|)$ for $x \in \mathbb{R}^N$ as usual. The following lemma is a substitute of Lemma 3.

Lemma 5. *Let $f \in L_+^1(\mathbb{R}^N)$ and let $\varepsilon > 0$. Put*

$$\begin{aligned} E_1(f, \varepsilon) &= \{(x, y) \in \mathbb{R}_+^{N+1} : 0 < y < 1, PI(K * f)(x, y) > \varepsilon\}, \\ E_1^*(f, \varepsilon) &= \bigcup_{(x, y) \in E_1(f, \varepsilon)} B(x, y). \end{aligned}$$

Then

$$C_\kappa(E_1^*(f, \varepsilon)) \leq A \frac{\|f\|_1}{\varepsilon},$$

where A is independent of f and ε .

Proof. First we claim

$$K_y(x) \leq A\kappa(x) \text{ for } 0 < y < 1.$$

In fact we have

$$\begin{aligned}
K_y(x) &= A_N \int_{\mathbb{R}^N} K(x-z) \frac{y}{(|z|^2 + y^2)^{(N+1)/2}} dz \\
&\leq A_N \int_{|x-z| \geq |x|/2} K\left(\frac{|x|}{2}\right) \frac{y}{(|z|^2 + y^2)^{(N+1)/2}} dz \\
&\quad + A_N \int_{|x-z| < |x|/2} K(x-z) \frac{y}{((|x|/2)^2 + y^2)^{(N+1)/2}} dz \\
&\leq K\left(\frac{|x|}{2}\right) + A \frac{1}{|x|/y + 1} \frac{1}{|x|^N} \int_{|z| < |x|/2} K(z) dz \\
&\leq A\kappa(x).
\end{aligned}$$

In view of the claim and the Harnack inequality we have

$$\kappa * f(x) \geq A\varepsilon \text{ on } E_1^*(f, \varepsilon),$$

whence the required inequality follows.

We observe that if (16) holds, then $\kappa(r) \approx K(r)$ for small $r > 0$, so that

$$C_K(E) \approx C_\kappa(E) \text{ for } E \subset U,$$

where U is a bounded set. From this fact and Lemma 5 we can prove Theorem 10 in the same way as in Section 8 with a slight modification.

10. PROOF OF PROPOSITION 1 AND PROPOSITION 2

Proof of Proposition 1. By definition there is $t_j \downarrow 0$ such that

$$C_{K,p}(E_{t_j}^*) < 2^{-j}.$$

We find nonnegative functions $f_j \in L^p(\mathbb{R}^N) \cap C_0^\infty(\mathbb{R}^N)$ such that

$$K * f_j \geq 1 \text{ on } E_{t_j}^*$$

and $\|f_j\|_p^p < 2^{-j}$. Let $f = \sum f_j$. Then $f \in L^p(\mathbb{R}^N)$. We show that this f satisfies (18). Since $K * f_j$ is continuous, it follows that $PI(K * f_j)$ has continuous boundary values and

$$PI(K * f_j) \geq 1 \text{ on } E_{t_j}^*.$$

Observe that if $(x, y) \in E_{t_j}$, then $B(x, y) \subset E_{t_j}^*$. Since the harmonic measure of $B(x, y)$ evaluated at (x, y) is greater than a positive constant A depending only on the dimension, it follows that

$$PI(K * f_j) \geq A \text{ on } E_{t_j}.$$

Therefore

$$PI(K * f) \geq Aj \text{ on } E_{t_j}.$$

This implies (18). The proposition is proved.

Proof of Proposition 2. We prove the proposition by contradiction. Suppose (20) does not hold. Then there is $\varepsilon_0 > 0$ such that

$$(29) \quad |\{x \in B(x_0, r) : |g(x) - \alpha| > 2\varepsilon_0\}| > 0$$

for all $r > 0$. By (19) we see that there is $r_0 > 0$ such that

$$(30) \quad |PI(g)(P) - \alpha| < \varepsilon_0 \text{ for } P \in B(x_0, r_0) \cap \mathbb{R}_+^{N+1} \setminus E.$$

Let us consider the boundary values of $PI(g)$ on $B(x_0, r_0)$. It is well known that $PI(g)$ has nontangential boundary values g a.e. on $\partial\mathbb{R}_+^{N+1}$. Hence, by (29), we can find a set $F \subset B(x_0, r_0)$ such that $|F| > 0$ and

$$|\operatorname{nt} \lim_{P \rightarrow x} PI(g)(P) - \alpha| > 2\varepsilon_0 \text{ for } x \in F,$$

where $\operatorname{nt} \lim_{P \rightarrow x} PI(g)(P)$ means the nontangential limit of $PI(g)$ at x . In particular, there is a positive function $t(x)$ on F such that

$$|PI(g)(x, y) - \alpha| > \varepsilon_0 \text{ for } x \in F \text{ and } 0 < y < t(x),$$

or, in other words

$$|PI(g) - \alpha| > \varepsilon_0 \text{ on } A = \bigcup_{x \in F} I(x),$$

where $I(x) = \{x\} \times (0, t(x))$. By (30) we see that $A \cap B(x_0, r_0) \subset E$. Obviously, F is the essential projection of $A \cap B(x_0, r_0)$. Hence the essential projection of E has positive measure. This is a contradiction.

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