FINITE-DIMENSIONAL LATTICE-SUBSPACES OF $C(\Omega)$ AND CURVES OF $\mathbb{R}^n$

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ABSTRACT. Let $x_1, \dots, x_n$ be linearly independent positive functions in $C(\Omega)$, let X be the vector subspace generated by the x_i and let β denote the curve of $\mathbb{R}^n$ determined by the function $\beta(t) = \frac{1}{z(t)}(x_1(t), x_2(t), \dots, x_n(t))$, where $z(t) = x_1(t) + x_2(t) + \dots + x_n(t)$. We establish that X is a vector lattice under the induced ordering from $C(\Omega)$ if and only if there exists a convex polygon of $\mathbb{R}^n$ with n vertices containing the curve β and having its vertices in the closure of the range of β . We also present an algorithm which determines whether or not X is a vector lattice and in case X is a vector lattice it constructs a positive basis of X . The results are also shown to be valid for general normed vector lattices.

1. INTRODUCTION

It is well known that each separable Banach space is isometric to a closed subspace of $C[0, 1]$. By a minor modification of the existing proof of the universality of $C[0, 1]$, we can show (see [12]) that $C[0, 1]$ is also a universal Banach lattice. More precisely, we can show that each separable Banach lattice is order isomorphic to a closed lattice-subspace of $C[0, 1]$, i.e., it is order isomorphic to a closed subspace of $C[0, 1]$ which with the induced ordering is a vector lattice in its own right. Since the class of vector sublattices is not enough for this representation, the class of lattice-subspaces seems to be the proper class of subspaces for studying Banach lattices. The structure of lattice-subspaces has not been yet systematically studied. In [8] it is shown that a subspace X of a vector lattice is a lattice-subspace if and only if there exists a positive projection from the vector sublattice generated by X onto X . In [11] and [12] the existence of positive bases in lattice-subspaces has been studied. A recent survey of lattice-subspaces and positive projections as well as some new results on lattice-subspaces can be found in [1]. Recently, lattice-subspaces have been employed in economics where they appear naturally in incomplete markets and the theory of finance [4, 5].

Now let E be an ordered Banach space with positive cone E_+ . A sequence $\{e_n\}$ of E is a *positive basis* if $\{e_n\}$ is a (Schauder) basis and

$$E_+ = \left\{ x = \sum_{i=1}^{\infty} \lambda_i e_i \in E : \lambda_i \in \mathbb{R}_+ \text{ for each } i \right\}.$$

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A positive basis $\{e_n\}$ of E is unique in the sense that if $\{b_n\}$ is another positive basis of E , then each element of $\{b_n\}$ is a positive multiple of an element of $\{e_n\}$. If $\{e_n\}$ is a positive basis of E , then the following statements are equivalent.

- (i) The basis $\{e_n\}$ is unconditional.
- (ii) The cone E_+ is generating and normal.
- (iii) E is a locally solid vector lattice (i.e., E is a Banach lattice with respect to an equivalent norm).

For a proof see [13, Theorem 16.3, p. 473] and [6, Theorems 3.5.2 and 4.1.5]. The cone E_+ is generating if $E = E_+ - E_+$ and E_+ is normal (or self-allied) if there exists some $c \in \mathbb{R}_+$ such that: $0 \leq x \leq y$ implies $\|x\| \leq c\|y\|$. If $\{e_n\}$ is a positive unconditional basis of E , then the lattice operations in E are given by

$$x \vee y = \sum_{i=1}^{\infty} (\lambda_i \vee \mu_i) e_i \quad \text{and} \quad x \wedge y = \sum_{i=1}^{\infty} (\lambda_i \wedge \mu_i) e_i,$$

for each $x = \sum_{i=1}^{\infty} \lambda_i e_i$, $y = \sum_{i=1}^{\infty} \mu_i e_i \in E$.

If E is an n -dimensional space and the cone E_+ is closed and generating, then, by the Choquet-Kendall Theorem [7] (see also [9, Theorem 3.11, p. 30]), E is a vector lattice if and only if a base B for E_+ is an $(n-1)$ -dimensional simplex (i.e., B is a convex polygon with n vertices). In such a case, if $b_1, b_2, \dots, b_n$ are the vertices of B , then the set $\{b_1, b_2, \dots, b_n\}$ is a positive basis for E . Therefore, for finite-dimensional spaces the Choquet-Kendall Theorem can be stated as follows.

Proposition 1.1 (Choquet-Kendall). *A finite-dimensional ordered vector space E with a closed and generating cone E_+ is a vector lattice if and only if E has a positive basis.*

If E is 2-dimensional and E_+ is closed and generating, then E is a vector lattice. This is true because each base B for E_+ is a closed line segment. Therefore, B is a simplex.

Now let X be a subspace of a partially ordered vector space E . The cone $X \cap E_+$ is called the *induced cone* of X and the ordering defined in X by this cone is called the *induced ordering*. An *ordered subspace* of E is a subspace of E ordered by the induced cone. A *lattice-subspace* of E is an ordered subspace of E which is also a vector lattice (Riesz space).¹

If X is a lattice-subspace, then for each $x, y \in X$ we denote by $x \nabla y$ (resp. $x \Delta y$) the supremum (resp. infimum) of $\{x, y\}$ in X . It is clear that $x \Delta y \leq x \wedge y$ and $x \vee y \leq x \nabla y$, whenever $x \wedge y$ and $x \vee y$ exist.

Now let E be a vector lattice. If X is a lattice-subspace and $x \nabla y = x \vee y$, $x \Delta y = x \wedge y$ for all $x, y \in X$, then X is a sublattice (Riesz subspace) of E . If X is the range of a positive projection $P : E \rightarrow E$, then X is a lattice-subspace with $x \nabla y = P(x \vee y)$, $x \Delta y = P(x \wedge y)$ for each $x, y \in X$. For notation and terminology not explained here, we refer the reader to [3, 6, 9].

2. LATTICE-SUBSPACES OF $C(\Omega)$ WITH POSITIVE BASES

In this paper, we shall denote by Ω a compact Hausdorff topological space and by $C(\Omega)$ the Banach lattice of continuous real valued functions defined on Ω .

¹The term "lattice-subspace" has been introduced in [10]. In [8] a lattice-subspace is called a "quasi-sublattice."

Let Y be a closed subspace of $C(\Omega)$ with a basis $\{b_n\}$. Fix $t \in \Omega$ and $m \in \mathbb{N}$. If $b_m(t) \neq 0$ and $b_n(t) = 0$ for each $n \neq m$, then we shall say that the point t is an m -node (or simply a node) of the basis $\{b_n\}$. If for each n there exists an n -node t_n of the basis $\{b_n\}$, then we shall say that $\{b_n\}$ is a *basis of Y with nodes* and that $\{t_n\}$ is a *sequence of nodes* of $\{b_n\}$. If $\dim Y = n$ and for each $m \in \{1, 2, \dots, n\}$ there exists an m -node t_m of the basis of Y , then we shall say that $\{b_1, b_2, \dots, b_n\}$ is a basis of Y with nodes and that the points $t_1, t_2, \dots, t_n$ are nodes of the basis $\{b_1, b_2, \dots, b_n\}$.

Recall that the *support* of a function $x \in C(\Omega)$, in symbols $\text{supp } x$, is the closure of the set $\{t \in \Omega : x(t) > 0\}$.

Theorem 2.1. *For a closed ordered subspace Y of $C(\Omega)$ having a basis $\{b_n\}$ consisting of positive functions we have the following.*

- (i) *If $\{b_n\}$ is a positive basis of Y , then*
 - (a) *for each m there exists a sequence $\{\omega_\nu\}$ of Ω such that $\lim_{\nu \rightarrow \infty} \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} = 0$ for each $i \neq m$, and*
 - (b) *there exists a sequence $\{t_n\}$ of Ω with $t_n \in \text{supp } b_n$ and $b_m(t_n) = 0$ for $m \neq n$.*
- (ii) *If $\{t_n\}$ is a sequence of nodes of $\{b_n\}$, then $\{b_n\}$ is a positive basis of Y and for each $x = \sum_{i=1}^{\infty} \lambda_i b_i \in Y$ we have $\lambda_i = \frac{x(t_i)}{b_i(t_i)}$ for each i .*

Proof. (i) For each k let $z_k = -\frac{1}{k}b_m + \sum_{i=1, i \neq m}^k b_i$. Since $\{b_n\}$ is a positive basis, $z_k \notin Y_+$ and so there exists some $\omega_k \in \Omega$ (depending on m) such that $z_k(\omega_k) < 0$ or

$$0 \leq \sum_{\substack{i=1 \\ i \neq m}}^k \frac{b_i(\omega_k)}{b_m(\omega_k)} < \frac{1}{k}$$

for each k . Also, let t_m be a limit point of the sequence $\{\omega_\nu\}$. Now (a) and (b) follow by letting $k \rightarrow \infty$.

(ii) Let $\{t_n\}$ be a sequence of nodes of $\{b_n\}$. It is easy to see that if $x = \sum_{i=1}^{\infty} \lambda_i b_i \in X$, then $\lambda_n = \frac{x(t_n)}{b_n(t_n)}$. In particular, $x \in X_+$ implies $\lambda_i \in \mathbb{R}_+$ for each i which shows that $\{b_n\}$ is also a positive basis. $\square$

Proposition 2.2. *For a closed lattice-subspace Y of $C(\Omega)$ with a positive basis $\{b_n\}$ the following statements are equivalent.*

1. *Y is a sublattice of $C(\Omega)$.*
2. *If $b_m(t) > 0$ for some m and t , then t is an m -node of the basis $\{b_n\}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that Y is a sublattice of $C(\Omega)$ and that $b_m(t) > 0$ and let $n \neq m$. Then $e_m \wedge e_n = e_m \Delta e_n = 0$. This implies $b_n(t) = 0$ and hence t is an m -node of the basis $\{b_n\}$.

(2) $\Rightarrow$ (1) Let $x = \sum_{i=1}^{\infty} \lambda_i b_i$, $y = \sum_{i=1}^{\infty} \mu_i b_i$ and let $b_m(t) > 0$. Then t is an m -node of $\{b_n\}$ which implies $b_n(t) = 0$ for each $n \neq m$. Therefore,

$$\begin{aligned} (x \nabla y)(t) &= \sum_{i=1}^{\infty} (\lambda_i \vee \mu_i) b_i(t) = (\lambda_m \vee \mu_m) b_m(t) \\ &= \left[\frac{x(t)}{b_m(t)} \vee \frac{y(t)}{b_m(t)} \right] b_m(t) = x(t) \vee y(t) \\ &= (x \vee y)(t). \end{aligned}$$

If $b_i(t) = 0$ for all i , then $(x \vee y)(t) = (x \nabla y)(t) = 0$ is also true. Thus, $x \nabla y = x \vee y$, and so Y is a sublattice of $C(\Omega)$. $\square$

Proposition 2.3. *Let Y be an n -dimensional subspace of $C(\Omega)$ and let $b_1, b_2, \dots, b_n$ in Y_+ . Then $\{b_1, \dots, b_n\}$ is a positive basis of Y if and only if for each $1 \leq m \leq n$ there exists a sequence $\{\omega_\nu\}$ of Ω satisfying $\lim_{\nu \rightarrow \infty} \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} = 0$ for each $i \neq m$.*

Proof. We show the “only if” part. So, assume that the vectors $b_1, b_2, \dots, b_n$ satisfy the stated property. We must show that $\{b_1, \dots, b_n\}$ is a positive basis. To this end, assume $x = \sum_{i=1}^n \lambda_i b_i \in Y_+$. Then from

$$0 \leq \frac{x(\omega_\nu)}{b_m(\omega_\nu)} = \sum_{i=1}^n \lambda_i \frac{b_i(\omega_\nu)}{b_m(\omega_\nu)} \xrightarrow{\nu \rightarrow \infty} \lambda_m,$$

we have $\lambda_m \geq 0$ for each m . Also, if $x = 0$, then as before we see that $\lambda_m = 0$ for each m . Hence $\{b_1, b_2, \dots, b_n\}$ is a positive basis of Y . $\square$

Proposition 2.4. *Let $\{b_1, \dots, b_n\}$ be a positive basis of an n -dimensional lattice-subspace Y of $C(\Omega)$. For a function $x = \sum_{i=1}^n \lambda_i b_i \in Y$, we have the following.*

- (i) *If a point t_i is an i -node of the basis, then $\lambda_i = \frac{x(t_i)}{b_i(t_i)}$.*
- (ii) *If $\{\omega_\nu\}$ is a sequence of Ω such that $\lim_{\nu \rightarrow \infty} \frac{b_j(\omega_\nu)}{b_i(\omega_\nu)} = 0$ for each $j \neq i$, then we have $\lambda_i = \lim_{\nu \rightarrow \infty} \frac{x(\omega_\nu)}{b_i(\omega_\nu)}$.*

Proof. If the point t_i is an i -node, then $x(t_i) = \lambda_i b_i(t_i)$ and the validity of (i) follows. For (ii) notice that $\lim_{\nu \rightarrow \infty} \frac{x(\omega_\nu)}{b_i(\omega_\nu)} = \lim_{\nu \rightarrow \infty} \sum_{j=1}^n \lambda_j \frac{b_j(\omega_\nu)}{b_i(\omega_\nu)} = \lambda_i$. $\square$

Theorem 2.5 ([12]). *Let E be a Banach lattice with a positive basis. Then there exists a closed lattice-subspace Z of $C[0, 1]$ with positive basis $\{b_n\}$ having node-intervals (i.e., besides $\{b_n\}$ being a positive basis of Z there exists a sequence $\{J_n\}$ of intervals of $[0, 1]$ satisfying $b_n(t) > 0$ for $t \in J_n$ and $b_n(t) = 0$ for all $t \in J_m$ with $m \neq n$) and an onto order-isomorphism $T : E \rightarrow Z$ such that*

$$\frac{1}{8} \|x\| \leq \|Tx\| \leq \|x\|$$

for all $x \in E$.

Now let Y be a closed lattice-subspace of $C(\Omega)$ with a positive basis. As we shall see in Example 3.1(iii) below, in general Y need not have a positive basis with nodes. However, according to Theorem 2.5, the space Y is order-isomorphic to a closed lattice-subspace Z of $C[0, 1]$ which has a positive basis with node-intervals (and therefore Z also has a positive basis with nodes).

3. FINITE-DIMENSIONAL LATTICE-SUBSPACES OF $C(\Omega)$

For our discussion here, we shall fix n linearly independent positive functions $x_1, x_2, \dots, x_n$ of $C(\Omega)$. The ordered subspace of $C(\Omega)$ generated by these functions will be denoted by X or by $[x_1, x_2, \dots, x_n]$, i.e.,

$$X = [x_1, x_2, \dots, x_n].$$

We now state the main problem of our study.

- *When is X a lattice-subspace of $C(\Omega)$? Or equivalently, when does X have a positive basis?*

FIGURE 1

We shall answer this question by completely characterizing the lattice-subspaces. As a matter of fact, we shall not only characterize the lattice-subspaces X but we shall also present an algorithm of determining the positive basis of X . This is important since once the positive basis has been found, we can determine the coordinates of the elements of X and the lattice operations in X ; see Examples 4.3 and 4.5 below.

In the sequel, we shall denote by z the sum of $x_1, \dots, x_n$ (i.e., $z = \sum_{i=1}^n x_i$) and by β the function $\beta : \Omega \rightarrow \mathbb{R}^n$ such that

$$\beta(t) = \left(\frac{x_1(t)}{z(t)}, \frac{x_2(t)}{z(t)}, \dots, \frac{x_n(t)}{z(t)} \right)$$

for each $t \in \Omega$ with $z(t) > 0$. So, β defines a curve on the base $B = \{y \in \mathbb{R}_+^n : \sum_{i=1}^n y_i = 1\}$ for the cone $\mathbb{R}_+^n$. We shall refer to β as the *basic curve* of the vectors $x_1, x_2, \dots, x_n$. The reader should keep in mind that the cone X_+ of X is always generating and normal.

We start our study with several examples of ordered subspaces of $C(\Omega)$.

Example 3.1. (i) Let $\Omega = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 9\}$ and consider the functions $x_1, x_2 \in C(\Omega)$ defined by

$$x_1(u, v) = \begin{cases} 4(1 - u^2 - v^2) & \text{if } u^2 + v^2 \leq 1, \\ 0 & \text{if } u^2 + v^2 > 1, \end{cases}$$

$$x_2(u, v) = \begin{cases} 4[1 - u^2 - (v - 2)^2] & \text{if } u^2 + (v - 2)^2 \leq 1, \\ 0 & \text{if } u^2 + (v - 2)^2 > 1. \end{cases}$$

See Figure 1. Let X be the subspace of $C(\Omega)$ generated by x_1, x_2 . Then X as a 2-dimensional subspace, is a lattice-subspace of $C(\Omega)$. The points $(0, 0)$ and $(0, 2)$ are nodes of the basis $\{x_1, x_2\}$, and so $\{x_1, x_2\}$ is a positive basis of X . The space X is a sublattice of $C(\Omega)$ because each $(u, v) \in \Omega$ with $(x_1 + x_2)(u, v) > 0$ is a node of the basis $\{x_1, x_2\}$.

(ii) Let $\Omega = [0, 1]$, $x_1(t) = 1$, $x_2(t) = t$ and $X = [x_1, x_2]$. Then X is a lattice-subspace and the set $\{b_1(t) = 1 - t, b_2(t) = t\}$ is a basis of X with nodes the points $t_1 = 0$ and $t_2 = 1$. Therefore, $\{b_1, b_2\}$ is a positive basis. The space X is not a sublattice of $C(\Omega)$ because the point $t = \frac{1}{2}$ is not a node of the basis $\{b_1, b_2\}$. Also, it should be clear that $b_1 \nabla b_2 = \mathbf{1}$ and $b_1 \Delta b_2 = \mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ are the constant functions one and zero; see Figure 2.

FIGURE 2

(iii) Let $\Omega = [-1, 1]$, $x_1(t) = |t|$ and

$$x_2(t) = \begin{cases} \sqrt{|t|} & \text{if } -1 \leq t < 0, \\ t^2 & \text{if } 0 \leq t \leq 1. \end{cases}$$

Then X is a lattice-subspace of $C(\Omega)$ and $\{x_1, x_2\}$ is a basis of X since

$$\lim_{t \rightarrow 0^-} \frac{x_1(t)}{x_2(t)} = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \frac{x_2(t)}{x_1(t)} = 0.$$

The space X does not have a positive basis with nodes; see Figure 3.

(iv) Let $\Omega = [0, 1]$, $x_1(t) = 1$, $x_2(t) = t$, $x_3(t) = t^2$ and $X = [x_1, x_2, x_3]$. We claim that X is not a lattice-subspace. To see this, assume by way of contradiction that X is a lattice-subspace of $C[0, 1]$. Let $\{b_1, b_2, b_3\}$ be a positive basis of X . Then for each $a \in [0, 1]$ the function $x_a(t) = (t - a)^2$, as a positive element of X , is a positive linear combination of b_1, b_2, b_3 . This implies $b_i(a) = 0$ for at least one i and so $b_i = \mathbf{0}$ for at least one i , a contradiction. Therefore, X is not a lattice-subspace of $C[0, 1]$.

(v) Let X be the subspace of $C[0, 1]$ generated by the functions $x_1(t) = |1 - t|(2 - t)$, $x_2(t) = t(2 - t)$ and $x_3(t) = t|1 - t|$. The points $t_1 = 0$, $t_2 = 1$ and $t_3 = 2$ are nodes of the basis $\{x_1, x_2, x_3\}$. Hence, X is a lattice-subspace with a positive basis with nodes; Figure 4.

(vi) Let $\{t_\nu\}$, $\{\omega_\nu\}$ and $\{r_\nu\}$ be strictly increasing sequences of $[0, 1]$ all convergent to $\frac{1}{2}$ satisfying $t_\nu < \omega_\nu < r_\nu < t_{\nu+1}$ for each ν . Also, let x_1, x_2 and x_3 be elements of $C[0, 1]$ with $x_i(t) > 0$ for each $t \neq \frac{1}{2}$ and each $i = 1, 2, 3$ such that:

$$\begin{aligned} x_1(t_\nu) &= \frac{1}{2} - t_\nu, & x_1(\omega_\nu) &= \left(\frac{1}{2} - \omega_\nu\right)^2, & x_1(r_\nu) &= \left(\frac{1}{2} - r_\nu\right)^3, \\ x_2(t_\nu) &= \left(\frac{1}{2} - t_\nu\right)^3, & x_2(\omega_\nu) &= \left(\frac{1}{2} - \omega_\nu\right), & x_3(r_\nu) &= \left(\frac{1}{2} - r_\nu\right)^2, \\ x_3(t_\nu) &= \left(\frac{1}{2} - t_\nu\right)^2, & x_2(\omega_\nu) &= \left(\frac{1}{2} - \omega_\nu\right)^3, & x_3(r_\nu) &= \left(\frac{1}{2} - r_\nu\right). \end{aligned}$$

FIGURE 3

FIGURE 4

Then $x_i(\frac{1}{2}) = 0$ for each i . The graph of x_1, x_2 and x_3 over the interval $[t_\nu, t_{\nu+1}]$ might be as in Figure 5. Now note that

$$\lim_{\nu \rightarrow \infty} \frac{x_i(t_\nu)}{x_1(t_\nu)} = 0 \quad \text{for } i = 2, 3; \quad \lim_{\nu \rightarrow \infty} \frac{x_i(\omega_\nu)}{x_2(\omega_\nu)} = 0 \quad \text{for } i = 1, 3;$$

and

$$\lim_{\nu \rightarrow \infty} \frac{x_i(r_\nu)}{x_3(r_\nu)} = 0 \quad \text{for } i = 1, 2.$$

By Proposition 2.3, the set $\{x_1, x_2, x_3\}$ is a positive basis of X and so X is a lattice-subspace of $C[0, 1]$. Since $x_i(t) > 0$ for each $t \neq \frac{1}{2}$ and each i , it follows that X does not have a positive basis with nodes. $\square$

Recall that the Wronskian of the functions $\varphi_i \in C^{(m-1)}(a, b)$, $i = 1, 2, \dots, m$, is the determinant function:

$$W(\varphi_1, \varphi_2, \dots, \varphi_m)(t) = \det \begin{bmatrix} \varphi_1(t) & \varphi_2(t) & \dots & \varphi_m(t) \\ \varphi_1'(t) & \varphi_2'(t) & \dots & \varphi_m'(t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(m-1)}(t) & \varphi_2^{(m-1)}(t) & \dots & \varphi_m^{(m-1)}(t) \end{bmatrix}.$$

Proposition 3.2. Assume $[a, b] \subseteq (c, d)$ and $f_0, f_1, \dots, f_{m-1} \in C(c, d)$. If $m > 2$, then the vector space of all solutions of the linear differential equation

$$(*) \quad x^{(m)} + f_{m-1}(t)x^{(m-1)} + \dots + f_0(t)x = 0$$

is not a lattice-subspace of $C[a, b]$.

FIGURE 5

Proof. Suppose that the vector space L of all solutions of $(*)$ is a lattice-subspace having a positive basis $\{b_1, b_2, \dots, b_m\}$. Choose $t_j \in [a, b]$, $j = 1, \dots, m$, such that $b_i(t_j) = 0$ for each $i \neq j$. Since $W(b_1, b_2, \dots, b_m)(t_i) \neq 0$, we infer that $b_i(t_i) > 0$. This implies that $t_i \neq t_j$ for $i \neq j$ and so (in view of $m > 2$) there exists some $t_k \in (a, b)$. But then for each $i \neq k$, the function b_i attains a local minimum at t_k , which implies that $b'_i(t_k) = 0$ for each $i \neq k$. In turn, the latter implies $W(b_1, b_2, \dots, b_m)(t_k) = 0$, a contradiction. $\square$

Corollary 3.3. Assume $[a, b] \subseteq (c, d)$ and $\varphi_1, \varphi_2, \dots, \varphi_m \in C^{(m)}(c, d)$, where $m > 2$. If $W(\varphi_1, \varphi_2, \dots, \varphi_m)(t) \neq 0$ for each $t \in (c, d)$, then the subspace generated by the functions $\varphi_1, \dots, \varphi_m$ is not a lattice-subspace of $C[a, b]$.

Theorem 3.4. The following statements are equivalent.

1. The subspace X has a positive basis with nodes.
2. The subspace X is the range of a positive projection $P : C(\Omega) \rightarrow C(\Omega)$.

In particular, if $\{b_1, b_2, \dots, b_n\}$ is a positive basis with nodes the points $t_1, t_2, \dots, t_n$, then the operator $P : C(\Omega) \rightarrow C(\Omega)$, defined by

$$P(x) = \sum_{i=1}^n \frac{x(t_i)}{b_i(t_i)} b_i,$$

is a positive projection with range X .

Proof. (1) $\Rightarrow$ (2) Clearly, if $\{b_1, b_2, \dots, b_n\}$ is a positive basis with nodes the points $t_1, t_2, \dots, t_n$, then the operator $P : C(\Omega) \rightarrow C(\Omega)$, defined by

$$P(x) = \sum_{i=1}^n \frac{x(t_i)}{b_i(t_i)} b_i,$$

is a positive projection with range X .

(2) $\Rightarrow$ (1) Since X is the range of a positive projection, we know that X is a lattice-subspace of $C(\Omega)$. Let $\{b_1, b_2, \dots, b_n\}$ be a positive basis of X . Then there are $t_1, t_2, \dots, t_n$ in Ω such that $b_i(t_j) = 0$ for each $i \neq j$. Let $\psi_i = \varphi_i \circ P$, where φ_i denotes the i th coefficient functional of the basis $\{b_1, \dots, b_n\}$. By the Riesz Representation Theorem, for each i there exists a unique Borel regular measure μ_i such that

$$\psi_i(x) = \int_{\Omega} x d\mu_i$$

for each $x \in C(\Omega)$. Let $b = \sum_{i=1}^n b_i$, $u_i = b - b_i$, $A = b^{-1}((0, \infty))$, $A_i = b_i^{-1}((0, \infty))$ and $B_i = u_i^{-1}((0, \infty))$. Since

$$\varphi_i(b_i) = \int_A b_i d\mu_i = 1 \quad \text{and} \quad \varphi_i(u_i) = \int_A u_i d\mu_i = 0,$$

it follows that the measure μ_i restricted to A is supported by a subset S_i of $A \setminus B_i$ such that $S_i \cap A_i \neq \emptyset$. Now notice that if $t_i \in S_i \cap A_i$, then the points $t_1, \dots, t_n$ are nodes of the basis $\{b_1, b_2, \dots, b_n\}$. $\square$

In Banach spaces each finite-dimensional subspace is complemented. The above result shows that an order-analogue result is not valid in Banach lattices. For instance, the lattice-subspace of Example 3.1(iii) is not positively complemented

because it does not have a positive basis with nodes. On the other hand, the lattice-subspace X of Example 3.1(v) has a positive basis with nodes and therefore is positively complemented. As a matter of fact, a positive projection $P : C[0, 2] \rightarrow C[0, 2]$ with range X is given by $P(x) = \frac{x(0)}{2}x_1 + x(1)x_2 + \frac{x(2)}{2}x_3$.

Proposition 3.5. *Let X be a lattice-subspace of $C(\Omega)$. If $z(t) > 0$ for each $t \in \Omega$, then X has a positive basis with nodes (and therefore X is the range of a positive projection).*

Proof. Let $\{b_1, b_2, \dots, b_n\}$ be a positive basis of X . Then there exist $t_1, t_2, \dots, t_n \in \Omega$ such that $b_j(t_i) = 0$ for each $j \neq i$. If $b_i(t_i) = 0$, then $z(t_i) = 0$, a contradiction. Hence $b_i(t_i) > 0$ for each i and so the points $t_1, t_2, \dots, t_n$ are nodes of $\{b_1, b_2, \dots, b_n\}$. $\square$

We shall denote by $D(\beta)$ the domain and by $R(\beta)$ the range of the basic curve β of $x_1, x_2, \dots, x_n$. As usual, if K is a subset of a topological space F , we shall denote by $\text{Int}(K)$ the interior of K , by $\bar{K}$ the closure of K and by ∂K the boundary of K . Also (whenever F is a linear topological space) we shall denote by $\text{co } K$ the convex hull of K , by $\overline{\text{co}} K$ the closed convex hull of K (i.e., the closure of $\text{co } K$). If A is a matrix, then we shall denote by A^T the transpose of A .

Theorem 3.6. *The following statements are equivalent.*

- (i) X is a lattice-subspace of $C(\Omega)$.
- (ii) *There exist n linearly independent vectors $P_1, P_2, \dots, P_n$ of $\mathbb{R}^n$, belonging to the closure of the range of β such that for each $t \in D(\beta)$ the vector $\beta(t)$ is a convex combination of the vectors of $P_1, \dots, P_n$, i.e., $R(\beta) \subseteq \text{co}\{P_1, \dots, P_n\}$.*

If (ii) is true, $P_i = \lim_{\nu \rightarrow \infty} \beta(\omega_{i\nu})$ for each i , A is the $n \times n$ matrix whose i th column is the vector P_i and $b_1, b_2, \dots, b_n$ are the functions defined by the formula

$$(1) \quad (b_1, b_2, \dots, b_n)^T = A^{-1}(x_1, x_2, \dots, x_n)^T,$$

then X has the following properties:

- (a) *The set $\{b_1, b_2, \dots, b_n\}$ is a positive basis of X . In addition, if t_i is a limit point of the sequence $\{\omega_{i\nu} : \nu = 1, 2, \dots\}$, then $t_i \in \text{supp } b_i$ and $b_k(t_i) = 0$ for each $k \neq i$.*
- (b) *The closed convex hull of $R(\beta)$ and the convex polygon with vertices the points $P_1, P_2, \dots, P_n$ coincide.*
- (c) *If $P_k = \beta(t_k)$, then t_k is a k -node of the basis $\{b_1, \dots, b_n\}$.*
- (d) *If $\Omega \subseteq \mathbb{R}^m$, $P_k = \beta(t_k)$ for some interior point t_k of Ω and the functions x_i are C^2 -functions in a neighborhood of t_k , then*

$$D_j \beta(t_k) = \mathbf{0}, \quad j = 1, 2, \dots, m,$$

where D_j denotes the operator of the j th partial derivative.

Proof. We assume that (ii) and the other assumptions are true. We shall establish that (a), (b), (c) and (d) (and therefore (i)) are true.

Since $\{x_1, x_2, \dots, x_n\}$ is a basis of X , it follows that $\{b_1, b_2, \dots, b_n\}$ is likewise a basis of X . Let

$$P_i = (a_{i1}, a_{i2}, \dots, a_{in}), \quad i = 1, 2, \dots, n.$$

Since each P_i is a vector of the base $B = \{y \in \mathbb{R}_+^n : \sum_{r=1}^n y_r = 1\}$, it follows that $\sigma_i = \sum_{j=1}^n a_{ij} = 1$ for each i . Therefore

$$z = \sum_{i=1}^n x_i = \sum_{i=1}^n \sigma_i b_i = \sum_{i=1}^n b_i.$$

Let $\beta(t) = \sum_{i=1}^n \xi_i(t) P_i$ be the expansion of $\beta(t)$ relative to the basis $\{P_1, P_2, \dots, P_n\}$. Then

$$\frac{1}{z(t)}(x_1(t), x_2(t), \dots, x_n(t))^T = A(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T,$$

and in view of (1) we get

$$(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T = \frac{1}{z(t)}(b_1(t), b_2(t), \dots, b_n(t))^T.$$

Since $\beta(t)$ is a convex combination of the vectors P_i , we get $\xi_i(t) \in \mathbb{R}_+$ and so $b_i(t) \in \mathbb{R}_+$ for each i . Thus, $b_i \in X_+$ for each i . From (1), we have

$$(\beta(t))^T = A\left(\frac{b_1(t)}{z(t)}, \frac{b_2(t)}{z(t)}, \dots, \frac{b_n(t)}{z(t)}\right)^T.$$

Replacing t by $\omega_{i\nu}$ and taking limits, we get

$$(2) \quad (a_{i1}, a_{i2}, \dots, a_{in})^T = A(\eta_{i1}, \eta_{i2}, \dots, \eta_{in})^T,$$

where $\eta_{ij} = \lim_{\nu \rightarrow \infty} \frac{b_j(\omega_{i\nu})}{z(\omega_{i\nu})}$.

Since the solution of the system (2) is unique, we have $\eta_{ii} = 1$ and $\eta_{ij} = 0$ for $j \neq i$. Since $z(\omega_{i\nu}) > 0$ for each ν and $\eta_{ii} = 1$, we have $b_i(\omega_{i\nu}) > 0$ for each ν . Therefore,

$$\lim_{\nu \rightarrow \infty} \left(\frac{b_i}{z}\right)(\omega_{i\nu}) = \lim_{\nu \rightarrow \infty} \frac{1}{1 + \left(\sum_{j=1, j \neq i}^n \frac{b_j}{b_i}\right)(\omega_{i\nu})} = 1.$$

Hence

$$(3) \quad \lim_{\nu \rightarrow \infty} \left(\frac{b_j}{b_i}\right)(\omega_{i\nu}) = 0 \quad \text{for each } j \neq i,$$

and so, from Proposition 2.3, the set $\{b_1, \dots, b_n\}$ is a positive basis of X . In other words, X is a lattice-subspace.

Now let t_i be an accumulation point of the sequence $\{\omega_{i\nu} : \nu = 1, 2, \dots\}$. We have shown that $b_i(\omega_{i\nu}) > 0$ for each ν and therefore $t_i \in \text{supp } b_i$. Also, from (3), we see that $b_j(t_i) = 0$ for each $j \neq i$. Thus, the validity of (a) has been established.

By our assumptions, $R(\beta) \subseteq \text{co}\{P_1, P_2, \dots, P_n\}$ and so

$$\overline{\text{co}} R(\beta) \subseteq \overline{\text{co}}\{P_1, \dots, P_n\} = \text{co}\{P_1, \dots, P_n\}.$$

Since $P_i \in \overline{R(\beta)} \subseteq \overline{\text{co}} R(\beta)$, we get $\text{co}\{P_1, \dots, P_n\} \subseteq \overline{\text{co}} R(\beta)$, and the validity of (b) follows.

To establish (c) let $P_k = \beta(t_k)$. Without loss of generality, we can assume that $\omega_{k\nu} = t_k$ for each ν . Then $b_k(t_k) > 0$. Also, by (a), we have $b_j(t_k) = 0$ for each $j \neq k$. Hence t_k is a k -node of the basis $\{b_1, \dots, b_n\}$, and therefore statement (c) is true.

Finally, assume that the hypotheses of claim (d) are valid. Then t_k is a k -node and so $b_\mu(t_k) = 0$ for each $\mu \neq k$. Since t_k is an interior point of Ω , for each $\mu \neq k$

the function b_μ attains a local minimum at the point t_k . This implies $D_j b_\mu(t_k) = 0$ for each j and all $\mu \neq k$. Now let $x_\mu = \sum_{i=1}^n c_{\mu i} b_i$. Then $x_\mu(t_k) = c_{\mu k} b_k(t_k)$ and $D_j x_\mu(t_k) = c_{\mu k} D_j b_k(t_k)$. Hence

$$\begin{aligned} z(t_k) D_j x_\mu(t_k) - x_\mu(t_k) D_j z(t_k) \\ = \left(\sum_{r=1}^n c_{rk} b_k(t_k) \right) c_{\mu k} D_j b_k(t_k) - c_{\mu k} b_k(t_k) \left(\sum_{r=1}^n c_{rk} D_j b_k(t_k) \right) = 0, \end{aligned}$$

and the validity of (d) also follows.

(i) $\Rightarrow$ (ii) Suppose that X is a lattice-subspace of $C(\Omega)$ and that $\{b_1, b_2, \dots, b_n\}$ is a positive basis of X . Then, by Proposition 2.3, for each i there exists a sequence $\{\omega_{i\nu}\}$ such that $\lim_{\nu \rightarrow \infty} \frac{b_j(\omega_{i\nu})}{b_i(\omega_{i\nu})} = 0$ for each $j \neq i$. Let

$$(4) \quad x_i = \sum_{j=1}^n \lambda_{ij} b_j.$$

Since $\{b_1, \dots, b_n\}$ is a positive basis, we see that $\lambda_{ij} \in \mathbb{R}_+$ for all i and j . Moreover, we have $z = \sum_{i=1}^n x_i = \sum_{i=1}^n \sigma_i b_i$, where $\sigma_i = \sum_{j=1}^n \lambda_{ji}$. Hence,

$$\lim_{\nu \rightarrow \infty} \left(\frac{x_j}{z} \right) (\omega_{i\nu}) = \lim_{\nu \rightarrow \infty} \left(\frac{\sum_{k=1}^n \lambda_{jk} \frac{b_k}{b_i}}{\sum_{k=1}^n \sigma_k \frac{b_k}{b_i}} \right) (\omega_{i\nu}) = \frac{\lambda_{ji}}{\sigma_i},$$

from which it follows that

$$\lim_{\nu \rightarrow \infty} \beta(\omega_{i\nu}) = \left(\frac{\lambda_{1i}}{\sigma_i}, \frac{\lambda_{2i}}{\sigma_i}, \dots, \frac{\lambda_{ni}}{\sigma_i} \right) = P_i.$$

Now let A be the $n \times n$ matrix with columns the vectors $P_1, P_2, \dots, P_n$. Then, from (4), we see that

$$(5) \quad (x_1, x_2, \dots, x_n)^T = A(\sigma_1 b_1, \sigma_2 b_2, \dots, \sigma_n b_n)^T.$$

So, the vectors $P_1, \dots, P_n$ are linearly independent since $\{x_1, \dots, x_n\}$ and $\{b_1, \dots, b_n\}$ are both bases of X . Let $\beta(t) = \sum_{i=1}^n \xi_i(t) P_i$ be the expansion of $\beta(t)$ relative to the basis $\{P_1, \dots, P_n\}$. Then

$$(\beta(t))^T = A(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T$$

and from (5) we get

$$(\xi_1(t), \xi_2(t), \dots, \xi_n(t))^T = \frac{1}{z(t)} (\sigma_1 b_1(t), \sigma_2 b_2(t), \dots, \sigma_n b_n(t))^T.$$

Hence $\xi_i(t) \in \mathbb{R}_+$ for each i and $\sum_{i=1}^n \xi_i(t) = 1$, and the proof is finished. $\square$

Now we define the subset $E(\beta)$ of $\overline{R(\beta)}$ as follows: If there exists a subset G of $\overline{R(\beta)}$ consisting of n linearly independent vectors such that $R(\beta) \subseteq \text{co } G$, then we put $E(\beta) = G$, otherwise we put $E(\beta) = \emptyset$. We shall refer to the set $E(\beta)$ as the *extreme subset* of β .

From the preceding definition the following result should be immediate.

Proposition 3.7. *The subspace X satisfies the properties*

1. X is a lattice-subspace if and only if $E(\beta) \neq \emptyset$.
2. If $\beta(t) \in E(\beta)$, then t is a node of the positive basis of X .
3. X has a positive basis with nodes if and only if $E(\beta)$ is a nonempty subset of $R(\beta)$.

Now let $P \in E(\beta)$. If $P \notin R(\beta)$, then by Theorem 3.6, we have $P = \lim_{\nu \rightarrow \infty} \beta(t_\nu)$, where $\{t_\nu\}$ is a sequence of $D(\beta)$ having all its limit points in the boundary $\partial D(\beta)$ of $D(\beta)$. Accordingly, we define the *limit set* $L(\beta)$ of the curve β as follows:

$$L(\beta) = \{P \in \mathbb{R}^n : \exists \{t_\nu\} \subseteq D(\beta) \text{ with its limit points in } \partial D(\beta), P = \lim_{\nu \rightarrow \infty} \beta(t_\nu)\}.$$

It should be clear that $\overline{R(\beta)} \setminus R(\beta) \subseteq L(\beta)$.

Example 3.8. (i) Let x_1, x_2 be the functions defined in Example 3.1(iii), β be the basic curve of x_1, x_2 and X be the lattice-subspace generated by x_1, x_2 . It is easy to show that the range $R(\beta)$ of β is the open line segment joining the points $P_1 = (1, 0)$ and $P_2 = (0, 1)$. The extreme subset of β is the set $E(\beta) = \{P_1, P_2\}$ because $P_1, P_2 \in \overline{R(\beta)}$ and each $\beta(t)$ is a convex combination of P_1 and P_2 . Also, the limit set of β is $L(\beta) = \{P_1, P_2\}$ because $\partial D(\beta) = \{0\}$ and $\lim_{t \rightarrow 0^-} \beta(t) = P_2$ and $\lim_{t \rightarrow 0^+} \beta(t) = P_1$.

According to Proposition 3.7, $E(\beta) \neq \emptyset$ implies that X is a lattice-subspace and $E(\beta) \not\subseteq R(\beta)$ guarantees that X does not have a positive basis with nodes. See Figure 6.

(ii) Let x_1, x_2 and x_3 be the functions introduced in Example 3.1(vi), β be the basic curve of x_1, x_2, x_3 and X the subspace generated by x_1, x_2, x_3 . Then $D(\beta) = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and so $\partial D(\beta) = \{\frac{1}{2}\}$.

Now let the sequences $\{t_\nu\}$, $\{\omega_\nu\}$ and $\{r_\nu\}$ be as in Example 3.1(vi). Then

$$\lim_{\nu \rightarrow \infty} \beta(t_\nu) = P_1(1, 0, 0), \quad \lim_{\nu \rightarrow \infty} \beta(\omega_\nu) = P_2(0, 1, 0),$$

FIGURE 6

FIGURE 7

and

$$\lim_{\nu \rightarrow \infty} \beta(r_\nu) = P_3(0, 0, 1).$$

Therefore $P_1, P_2, P_3 \in L(\beta)$. Also $E(\beta) = \{P_1, P_2, P_3\}$ because $P_i \in \overline{R(\beta)}$ for each i and $\beta(t)$ is a convex combination of the P_i . See Figure 7.

4. THE CASE $\Omega \subseteq \mathbb{R}^m$

Suppose now that Ω is a compact subset of $\mathbb{R}^m$. Again, we shall study in this special case when the space X generated by the linearly independent positive elements $x_1, \dots, x_n$ of $C(\Omega)$ is a lattice-subspace. If t is an interior point of Ω and $\beta(t) \in E(\beta)$, then the point t is a root of the equations

$$(6) \quad D_j \beta(t) = 0, \quad j = 1, \dots, m.$$

We shall denote by $I(\beta)$ the images of the roots of (6), i.e.,

$$I(\beta) = \{\beta(t) : t \in \text{Int}(\Omega) \cap D(\beta) \text{ and } t \text{ is a root of equations (6)}\}.$$

Also, we shall denote by $\beta(\partial\Omega)$ the set $\beta(\partial\Omega) = \{\beta(t) : t \in \partial\Omega\}$.

Proposition 4.1. *If the functions $x_1, \dots, x_n$ are C^2 -functions in the set $\text{Int}(\Omega) \cap D(\beta)$, then $E(\beta) \subseteq L(\beta) \cup I(\beta) \cup \beta(\partial\Omega)$.*

Proof. Let $P \in E(\beta)$. If $P \in R(\beta)$, then there exists some t such that $P = \beta(t)$. If $t \in \text{Int}(\Omega)$, then by Theorem 3.6, we get $P \in I(\beta)$. If $t \notin \text{Int}(\Omega)$, then $t \in \partial\Omega$ and so $P \in \beta(\partial\Omega)$. If $P \notin R(\beta)$, then $P \in \overline{R(\beta)} \setminus R(\beta) \subseteq L(\beta)$ and the proof of the proposition is finished. $\square$

We continue with one more definition. Any subset of $L(\beta) \cup I(\beta) \cup \beta(\partial\Omega)$ consisting of n linearly independent vectors will be called a *possible extreme subset* of β . In order to study when X is a lattice-subspace, we shall determine the set $L(\beta) \cup I(\beta) \cup \beta(\partial\Omega)$ and shall investigate when one of the possible extreme subsets of β is indeed an extreme subset of β .

In case $\Omega = [a, b] \subset \mathbb{R}$, we make the following remark. The set $\beta(\partial\Omega)$ is known because $\partial\Omega = \{a, b\}$. If $D(\beta) = [a, b]$, then $\partial D(\beta) = \emptyset$ and so $L(\beta) = \emptyset$. If we assume that

$$D(\beta) = [a, \omega_1) \cup (t_1, \omega_2) \cup \dots \cup (t_{n-1}, \omega_n) \cup (t_n, b]$$

and the limits

$$R_i = \lim_{t \rightarrow t_i^+} \beta(t), \quad Q_i = \lim_{t \rightarrow \omega_i^-} \beta(t)$$

exist for each i , then $L(\beta) = \{R_1, R_2, \dots, R_n\} \cup \{Q_1, \dots, Q_n\}$.

In general, the set $D(\beta) = z^{-1}((0, \infty))$, as an open subset of $[a, b]$, is the union of at most countably many disjoint open intervals of $[a, b]$ and therefore $\partial D(\beta)$ is at most countable. Finally, we note that a necessary condition for X to be a lattice-subspace of $C[a, b]$ is that the Wronskian of the functions $x_1, x_2, \dots, x_n$ have at least one root in the interval $[a, b]$ (Corollary 3.3).

So, in order to determine whether X is a lattice-subspace we must follow the steps of the algorithm below:

1. Does the Wronskian of the functions $x_1, \dots, x_n$ have at least one root in the interval $[a, b]$?
2. Determine the sets $\beta(\partial\Omega)$, $L(\beta)$, $I(\beta)$ and the possible extreme subsets of β .

3. Is one of the possible extreme subsets an extreme subset of β ?
4. Determine a positive basis of X .

Example 4.2. Let $\Omega = [0, 1]$ and $x_i(t) = t^i, i = 0, 1, \dots, k$. Then the space generated by the functions x_i is not a lattice-subspace of $C[0, 1]$ since the Wronskian does not have any root in $[0, 1]$. $\square$

Example 4.3. Let $\Omega = [0, 2]$, $x_1(t) = t^2 - 2t + 2$, $x_2(t) = -t^3 + 2t^2 - t + 2$, $x_3(t) = t^3 - 3t^2 + 3t$ and X be the subspace of $C[0, 2]$ generated by x_1, x_2, x_3 .

The Wronskian of these functions $W(t) = -24(t-1)^2$ has a root in the interval $[0, 2]$. Therefore, according to Step 2, we must determine the possible extreme subsets of β . We have $z(t) = x_1(t) + x_2(t) + x_3(t) = 4$ and $\beta(t) = \frac{1}{4}(x_1(t), x_2(t), x_3(t))$ is the basic curve of the triplet. Since $D(\beta) = [0, 2]$, we see that $L(\beta) = \emptyset$. Also,

$$\beta(\partial\Omega) = \{\beta(0) = (\frac{1}{2}, \frac{1}{2}, 0), \beta(2) = (\frac{1}{2}, 0, \frac{1}{2})\}.$$

Now the equation $\beta'(t) = \mathbf{0}$ has one solution in the interval $(0, 2)$ (namely $r = 1$) and so

$$I(\beta) = \{\beta(1) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})\}.$$

Hence

$$\beta(\partial\Omega) \cup L(\beta) \cup I(\beta) = \{P_1(\frac{1}{2}, \frac{1}{2}, 0), P_2(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}), P_3(\frac{1}{2}, 0, \frac{1}{2})\},$$

and so the set $\{P_1, P_2, P_3\}$ is the only possible extreme subset of β .

We claim that $\{P_1, P_2, P_3\}$ is an extreme subset of β . To this end, suppose that

$$(7) \quad \beta(t) = \xi_1(t)P_1 + \xi_2(t)P_2 + \xi_3(t)P_3$$

is the expansion of $\beta(t)$ relative to the basis $\{P_1, P_2, P_3\}$. We must show that $\xi_i(t) \geq 0$ for each i . From (7), it follows that

$$\xi_1(t) = \frac{1}{2}(t-1)^2(2-t), \quad \xi_2(t) = t(2-t) \quad \text{and} \quad \xi_3(t) = \frac{1}{2}t(t-1)^2,$$

from which it follows that $\xi_i(t) \geq 0$ for each i and each $t \in [0, 2]$. Since $\beta(t)$ belongs to the base $B = \{y \in \mathbb{R}_+^3 : y_1 + y_2 + y_3 = 1\}$ of $\mathbb{R}_+^3$ we observe that $\xi_1(t) + \xi_2(t) + \xi_3(t) = 1$, which shows that $\{P_1, P_2, P_3\}$ is an extreme subset of β .

A positive basis $\{b_1, b_2, b_3\}$ of X is the following:

$$\begin{aligned} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ -2 & 2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 2(x_1 - x_3) \\ 2(-x_1 + x_2 + x_3) \\ x_1 - x_2 + x_3 \end{bmatrix}; \end{aligned}$$

hence $b_1(t) = 2(t-1)^2(2-t)$, $b_2(t) = 4t(2-t)$ and $b_3(t) = 2t(t-1)^2$. The points $t_1 = 0$, $t_2 = 1$ and $t_3 = 2$ are nodes for the basis $\{b_1, b_2, b_3\}$. See Figure 8.

Now, we can determine the coefficients of the points of X and the lattice operations of X . For example,

$$x_1 = \frac{x_1(0)}{b_1(0)}b_1 + \frac{x_1(1)}{b_2(1)}b_2 + \frac{x_1(2)}{b_3(2)}b_3 = \frac{1}{2}b_1 + \frac{1}{4}b_2 + \frac{1}{2}b_3,$$

and similarly $x_2 = \frac{1}{2}b_1 + \frac{1}{2}b_2$. Therefore,

$$x_1 \nabla x_2 = \frac{1}{2}b_1 + \frac{1}{2}b_2 + \frac{1}{2}b_3 \quad \text{and} \quad x_1 \Delta x_2 = \frac{1}{2}b_1 + \frac{1}{4}b_2.$$

FIGURE 8

FIGURE 9

A positive projection P with range X is given by

$$P(x) = \frac{x(t_1)}{b_1(t_1)}b_1 + \frac{x(t_2)}{b_2(t_2)}b_2 + \frac{x(t_3)}{b_3(t_3)}b_3 = \frac{x(0)}{4}b_1 + \frac{x(1)}{4}b_2 + \frac{x(2)}{4}b_3,$$

for each $x \in C[0, 2]$. $\square$

Example 4.4. Let $\Omega = [-1, 2]$, $x_1(t) = t^2(t-1)^2$, $x_2(t) = t^4(t-1)^2$ and $x_3(t) = t^4(t-1)^4$; see Figure 9. Also let X be the subspace of $C[-1, 2]$ generated by x_1, x_2, x_3 . A computation shows that the Wronskian of these functions is $W(t) = 4t^8(t-1)^5(4t^2 - 7t + 3)$ which has roots in the interval $[-1, 2]$. Notice that $z(t) = x_1(t) + x_2(t) + x_3(t) = x_1(t)a(t)$, where $a(t) = 1 + t^2 + t^2(t-1)^2$ and

$$\beta(t) = \left(\frac{1}{a(t)}, \frac{t^2}{a(t)}, \frac{t^2(t-1)^2}{a(t)} \right), \quad t \in [-1, 0) \cup (0, 1) \cup (1, 2],$$

is the basic curve of x_1, x_2, x_3 . Also,

$$\beta(\partial\Omega) = \left\{ \beta(-1) = P_1 \left(\frac{1}{6}, \frac{1}{6}, \frac{4}{9} \right), \beta(2) = P_2 \left(\frac{1}{9}, \frac{4}{9}, \frac{4}{9} \right) \right\}$$

and $\lim_{t \rightarrow 0} \beta(t) = P_3(1, 0, 0)$ and $\lim_{t \rightarrow 1} \beta(t) = P_4(\frac{1}{2}, \frac{1}{2}, 0)$. Therefore, $L(\beta) = \{P_3, P_4\}$. Now it is easy to show that the equation $\beta'(t) = \mathbf{0}$ does not have any root in the set $(-1, 2) \cap D(\beta)$. Therefore, $I(\beta) = \emptyset$ and hence

$$I(\beta) \cup L(\beta) \cup \beta(\partial\Omega) = \{P_1, P_2, P_3, P_4\}.$$

The possible extreme subsets of β are the following:

$$\begin{aligned} S_1 &= \{P_1, P_2, P_3\}, & S_2 &= \{P_1, P_2, P_4\}, \\ S_3 &= \{P_1, P_3, P_4\}, & \text{and } S_4 &= \{P_2, P_3, P_4\}. \end{aligned}$$

We can show that no set from the above candidates is an extreme subset of β . For instance if the set S_1 is an extreme subset of β , then let $\beta(t) = \xi_1(t)P_1 + \xi_2(t)P_2 + \xi_3(t)P_3$ and note that $\xi_1(t) = \frac{2t^3(t-2)}{a(t)} < 0$ for each $t \in (0, 2)$, a contradiction.

The conclusion here is that X is not a lattice-subspace of $C[-1, 2]$. $\square$

Example 4.5. Let $\Omega = \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\}$, $x_1(s, t) = s^2 + t^2$, $x_2(s, t) = 1 - e^{-(s^2+t^2)}$ and let X be the lattice-subspace of $C(\Omega)$ generated by x_1 and x_2 . We shall determine a positive basis of X . Notice that $z(s, t) = (x_1 + x_2)(s, t) = s^2 + t^2 + 1 - e^{-(s^2+t^2)}$ and that the basic curve of x_1 and x_2 is $\beta = (\frac{x_1}{z}, \frac{x_2}{z})$. For each $(s, t) \in \partial\Omega$, we have $\beta(s, t) = P_1(\frac{e}{2e-1}, \frac{e-1}{2e-1})$ and so $\beta(\partial\Omega) = \{P_1\}$. Also, $\partial D(\beta) = \{(0, 0)\}$. Since $\lim_{(s,t) \rightarrow (0,0)} \beta(s, t) = P_2(\frac{1}{2}, \frac{1}{2})$, we see that $L(\beta) = \{P_2\}$. Now the system of equations

$$\frac{\partial \beta(s, t)}{\partial s} = \mathbf{0} \quad \text{and} \quad \frac{\partial \beta(s, t)}{\partial t} = \mathbf{0}$$

does not have solutions in the set $\text{Int}(\Omega) \cap D(\beta)$ and so $I(\beta) = \emptyset$. This implies that extreme subset of β is $E(\beta) = \{P_1, P_2\}$. A positive basis $\{b_1, b_2\}$ of X is the following:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (4e - 2) \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{e-1}{2e-1} & \frac{e}{2e-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2e-1)(x_1 - x_2) \\ 2[(1-e)x_1 + ex_2] \end{bmatrix}.$$

Therefore

$$b_1(s, t) = (2e - 1)(s^2 + t^2 - 1 + e^{-(s^2+t^2)}) \quad \text{and} \\ b_2(s, t) = 2[(1 - e)(s^2 + t^2) + e - e^{1-(s^2+t^2)}].$$

The restriction of b_1 and b_2 in the s -axis is as in Figure 10.

FIGURE 10

Next, we shall determine the coefficients of x_1 relative to the basis $\{b_1, b_2\}$. Let $x_1 = \lambda_1 b_1 + \lambda_2 b_2$. Then $\lambda_1 = \frac{x_1(0,1)}{b_1(0,1)} = \frac{e}{2e-1}$ because the point $(0, 1)$ is a 1-node of the basis of X . Also, $\lim_{s \rightarrow 0} \frac{b_1(s,0)}{b_2(s,0)} = 0$, which (in view of Proposition 2.4) implies $\lambda_2 = \lim_{s \rightarrow 0} \frac{x_1(s,0)}{b_2(s,0)} = \frac{1}{2}$. Therefore, $x_1 = \frac{e}{2e-1}b_1 + \frac{1}{2}b_2$. $\square$

5. THE CASE $\Omega = \{1, 2, \dots, m\}$

Suppose now that $\Omega = \{1, 2, \dots, m\}$. Then $C(\Omega) = \mathbb{R}^m$ and X is the subspace of $\mathbb{R}^m$ generated by the linearly independent positive vectors

$$x_i = (x_i(1), x_i(2), \dots, x_i(m)), \quad i = 1, 2, \dots, n,$$

of $\mathbb{R}^m$. Then $z = x_1 + x_2 + \cdots + x_n$ and the basic curve of $x_1, x_2, \dots, x_n$ is the function $\beta : \Omega \rightarrow \mathbb{R}^m$ defined by

$$\beta(k) = \left(\frac{x_1(k)}{z(k)}, \frac{x_2(k)}{z(k)}, \dots, \frac{x_n(k)}{z(k)} \right) \quad \text{for each } k \text{ with } z(k) > 0.$$

Then $R(\beta) = \overline{R(\beta)}$. Therefore, if X is a lattice-subspace, then there exists $\{j_1, j_2, \dots, j_n\} \subseteq D(\beta)$ such that $E(\beta) = \{\beta(j_i) : i = 1, \dots, n\}$. So, in order to determine whether X is a lattice-subspace of $\mathbb{R}^m$ we find the subsets of $R(\beta)$ consisting of n linearly independent vectors (i.e., we find the possible extreme subsets of β) and examine if one of them is an extreme subset of β . As a consequence of Theorem 3.6 we have the following result—which was also proven in [1, Theorem 2.6].

Theorem 5.1. *The space X is a lattice-subspace of $\mathbb{R}^m$ if and only if there exist indices $j_1, j_2, \dots, j_n$ of $D(\beta)$ such that for each $k \in D(\beta)$ the vector $\beta(k)$ is a convex combination of the vectors $\beta(j_1), \beta(j_2), \dots, \beta(j_n)$.*

In this case, a positive basis $\{b_1, b_2, \dots, b_n\}$ of X , with nodes the points $j_1, j_2, \dots, j_n$ is given by the formula

$$(b_1, b_2, \dots, b_n)^T = A^{-1}(x_1, x_2, \dots, x_n)^T,$$

where A is the $n \times n$ matrix whose columns are the vectors $\beta(j_i)$, $i = 1, \dots, n$.

6. LATTICE-SUBSPACES OF NORMED LATTICES

Now let E be a normed vector lattice (normed Riesz space) and let U_+° denote the positive part of the closed unit ball of the norm dual E^* of E , i.e.,

$$U_+^\circ = \{f \in E^* : \|f\| \leq 1\}.$$

Suppose that U_+° is equipped with the weak* topology and let Ω be a weak* closed subset of U_+° defining the positive cone of E , i.e.,

$$E_+ = \{x \in E : f(x) \geq 0 \text{ for each } f \in \Omega\}.$$

For each $x \in E$ denote by $\hat{x}$ the function $\hat{x}(f) = f(x)$, $f \in \Omega$, and for any subspace X of E denote by $\hat{X}$ the subspace $\hat{X} = \{\hat{x} \in C(\Omega) : x \in X\}$ of $C(\Omega)$. Then, $x \mapsto T(x) = \hat{x}$ is a linear, one-to-one operator from E into $C(\Omega)$ such that T and T^{-1} are both positive.

Theorem 6.1. *A subspace X of E is a lattice-subspace if and only if $\hat{X}$ is a lattice-subspace of $C(\Omega)$.*

This study of lattice-subspaces is more interesting if there exists a sequence $\{f_n\}$ of U_+° defining the positive cone of E and Ω is the weak* closure of $\{f_n\}$. For instance, if E is a Banach lattice with positive basis, then we can take $\{f_n\}$ to be the sequence of the coefficient functionals of the basis and $\Omega = \{f_n : n \in \mathbb{N}\} \cup \{0\}$. On the other hand, if $E = l_\infty$, then we can take $\{f_n\}$ to be the sequence of functionals of the standard biorthogonal system of l_∞ and Ω to be the weak* closure of the sequence $\{f_n\}$. For details on determining Ω see [2, p. 448].

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