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AN ω_2-MINIMAL BOOLEAN ALGEBRA

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Abstract. For every linear order L we define a notion of L-minimal Boolean algebra and then give a consistent example of an ω_2-minimal algebra. The Stone space X of our algebra contains a point $\{*\}$ such that $X - \{*\}$ is an example of a countably tight, initially $\aleph_1$-compact, non-compact space. This answers a question of Dow and van Douwen.

1. Introduction

In this paper we define, for a linear order L, a notion of an L-minimal Boolean algebra and construct an ω_2-minimal algebra. L-minimal algebras are, when L is an ordinal, a special kind of minimally generated Boolean algebras, see [5] and [6]. Moreover for an ordinal κ, the Stone space of a κ-minimal algebra can be identified with $\kappa + 1 = \{\alpha : \alpha \leq \kappa\}$ with a topology which is right separated, i.e., the initial segments are open. We denote the maximal point of $\kappa + 1$ by $\{*\}$.

We deal with the following generalization of compactness:

Definition 1. A topological space X is $\aleph_1$-compact if every open cover of X of cardinality $\leq \aleph_1$ has a finite subcover.

It is easy to see that a space X is initially $\aleph_1$-compact if and only if every subset $A \subseteq X$ of cardinality $\leq \aleph_1$ has a complete accumulation point, i.e., a point p such that $|U \cap A| = |A|$ for every open U containing p.

Note that the ordinal space $\omega_2 = \{\xi : \xi < \omega_2\}$ with the order topology is an example of initially $\aleph_1$-compact space which is not compact. However this space contains points of uncountable character and, more generally, of uncountable tightness. This led to the following question. Dow and van Douwen asked whether every initially $\aleph_1$-compact space of countable tightness is compact and proved that the answer is positive under CH. Later it was proved that the answer is positive in the Cohen model, Dow [4], and under PFA, Fremlin and Nyikos [2].

The main result of this paper is to give a consistent counterexample to the above question. More precisely we construct by forcing an ω_2-minimal Boolean algebra B such that its Stone space has countable tightness and there are no ω_1 nor ω sequences converging to the distinguished point $\{*\}$. Recall that we say that a set A is a κ sequence converging to a point p if $|A| = \kappa$ and for every open set U containing p we have $|A - U| < \kappa$. Note that if X is the Stone space of the algebra we construct, then $X - \{*\}$ is initially $\aleph_1$-compact. Indeed, suppose that $A \subseteq X - \{*\}$ has cardinality $\leq \aleph_1$. Since X is compact, A has a complete accumulation point $p \in X$. If there is such a point not equal to $\{*\}$, then it is in $X - \{*\}$ and we are

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done. Otherwise $\{*\}$ is the unique complete accumulation point of A, which implies that A converges to $\{*\}$, a contradiction. Note that, since $\{*\}$ is non-isolated, the space $X - \{*\}$ is non-compact.

The algebra we construct can be seen as a generalization of an example of Baumgartner and Shelah of a thin-very tall superatomic Boolean algebra. In particular we use the Δ function as defined in [1] and prove some of its additional properties which are necessary for the proof of the non-existence of sequences converging to $\{*\}$.

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2. L-minimal algebras

In this section for every linear order L we associate a class of Boolean algebras, L-minimal algebras. It turns out that, for an ordinal κ the Stone space of a κ-minimal algebra can be identified with $\kappa + 1$. For a generalization to partial orders see [7].

Definition 2. Let $(L, <)$ be a linear order. We say that a Boolean algebra B is L-minimal if there is an ideal $\{a_x : x \in L\} \subseteq B$ with the following properties:

(1) B is generated by $\{a_x : x \in L\}$,
(2) if $x_1, \ldots, x_n < y$, then $a_y - \bigcup_{i \leq n} a_{x_i} \neq \emptyset$,
(3) if $x < y$, then $a_x \cap a_y \in B_x$, where B_x is the subalgebra of B generated by $\{a_z : z \leq x\}$.

Lemma 2.1. *Let κ be an ordinal, B a κ-minimal Boolean algebra generated by $\{a_x : x \in \kappa\}$ and let X be its Stone space. Then there is an isomorphism H between X and $\kappa + 1$ with some right separated topology.*

Proof. Let F be an ultrafilter on B. Define $H(F)$ to be the minimal ordinal $\alpha < \kappa$ such that $a_\alpha \in F$ and let $H(F) = \kappa$ if there is no such α. To prove that H is a bijection we define its inverse.

We claim that for every $\alpha \in \kappa$, the set $A_\alpha = \{a_\beta^{-1} : \beta < \alpha\} \cup \{a_\alpha\}$ generates an ultrafilter on B. Since B is minimal it follows that A_α has the finite intersection property. Moreover, A_α generates an ultrafilter on the algebra B_α. Let a be any element of B. Then a is a Boolean combination of some elements a_x with $x \in \kappa$. But, since B is minimal, the intersection $a_x \cap a_\alpha$ is in the algebra B_α. Also, since $a_x^{-1} \cap a_\alpha = a_\alpha - (a_x \cap a_\alpha)$ is in B_α, it follows that $a \cap a_\alpha$ is in B_α. Hence the ultrafilter on B_α generated by A_α decides $a \cap a_\alpha$, i.e., there is an element of this ultrafilter contained in or disjoint from $a \cap a_\alpha$. Therefore it also decides a. This proves that H is a bijection between X and $\kappa + 1$.

Using the above correspondence we can identify X with $\kappa + 1$. If $\gamma \in \omega_2$, then a typical neighborhood of γ is $U_\gamma(\alpha_1, \ldots, \alpha_k) = a_\gamma \cap \bigcap \{a_{\alpha_i}{}^{-1} : i \leq k\}$, where $\alpha_1, \ldots, \alpha_k < \gamma$. If $\beta < \gamma$, then $\beta \in a_\gamma$ iff $a_\gamma \cap a_\beta$ is in the ultrafilter generated by A_β. Note that if $\beta > \gamma$, then $\beta \notin a_\gamma$, so X is right separated.

2.1. **Finite amalgamations.** We are going to construct our ω_2-minimal algebra by forcing with finite approximations, L-minimal Boolean algebras for finite $L \subseteq \omega_2$. Below, we introduce some notation and define a minimal amalgamation of two isomorphic algebras.

Let $L = \{x_1, \ldots, x_k, <\}$ be a finite linear order. Suppose that B is an L-minimal Boolean algebra generated by $\{a_x : x \in L\}$. Of course B is atomic with the atoms $\{b_0, b_1, \ldots, b_k\}$, where $\{b_{x_i}\} = a_{x_i} - \bigcup\{a_{x_j} : j < i\}$ for $i = 1, \ldots, k$ and $\{b_0\}$ is the complement of $\bigcup\{a_x : x \in L\}$. Since b_0 is definable from $\{b_1, \ldots, b_k\}$ we often abuse the notation by saying that $\{b_1, \ldots, b_k\}$ are the atoms of B. We do not distinguish strictly between b and $\{b\}$. Since each element of the algebra B is a set of atoms it follows that B is completely determined by the truth values of the statements $b_y \in a_x$, $y, x \in L$. Since B is minimal it follows that if $b_y \in a_x$, then $y \leq x$ and we always have $b_x \in a_x$.

Definition 3. Let L be a linear order and let B be a L minimal algebra generated by $\{a_x : x \in L\}$. Let $K \subseteq L$. We say that K generates a subalgebra of B if $\{a_x : x \in K\}$ generates a K-minimal algebra. Equivalently: for $x < y$ in K, the intersection $a_x \cap a_y$ is in the algebra generated by $\{a_v : v \leq x, v \in K\}$.

Note that if K is an initial part of L, then K generates a subalgebra of B.

Definition 4. Let L, L' be two finite linear orders of the same size, $\Delta = L \cap L'$ has the same position in L and L', i.e., the order isomorphism between L and L' is constant on Δ. Let B be an L-minimal algebra generated by $\{a_x : x \in L\}$ with the atoms $\{b_x : x \in L\}$ and let B' be an L'-minimal algebra generated by $\{a'_x : x \in L'\}$ with the atoms $\{b'_x : x \in L'\}$ such that the order isomorphism between L and L' rises to the isomorphism between B and B'. Assume that Δ generates a subalgebra of B (thus also B'). Let $M = L \cup L'$ be a linear order extending both K and K'.

The minimal amalgamation of B and B' is an M-minimal algebra C generated by $\{c_x : x \in M\}$ with the atoms $\{d_\alpha : \alpha \in M\}$ defined as follows. For $x \in L - \Delta$ let $D_x = \{d_x\}$, for $z \in \Delta$ let $D_z = \{d_y : y \in L', b'_y \in a'_z - \bigcup\{a'_v : v \in \Delta, v < z\}\}$ and $D'_z = \{d_x : x \in L, b_x \in a_z - \bigcup\{a_v : v \in \Delta, v < z\}\}$. For $y \in L' - \Delta$ let $D_y = \{d_y\}$. Define c_α:

(1) $c_\alpha = \bigcup\{D_v : v \in L,\ b_v \in a_\alpha\}$, for $\alpha \in L - \Delta$,
(2) $c_\alpha = \bigcup\{D'_w : w \in L',\ b'_w \in a'_\alpha\}$, for $\alpha \in L' - \Delta$,
(3) $c_\alpha = \bigcup\{D_x \cup D'_x : x \in \Delta,\ b_x \in a_\alpha\}$, for $\alpha \in \Delta$.

Lemma 2.2. *Let B, B' be as above. Then the minimal amalgamation of B and B' exists.*

Proof. We have to prove that the minimal amalgamation is well defined. It follows from the following claims.

Claim 2.3. If $x_i, x_j \in L$, $i < j$, then $D_{x_i} \cap D_{x_j} = \emptyset$.

Proof. The only nontrivial case is when $x_i, x_j \in \Delta$. Suppose that $d_y \in D_{x_i} \cap D_{x_j}$. Then $b'_y \in a'_{x_j} - a'_{x_i}$ and also $b'_y \in a'_{x_i}$, a contradiction.

Similarly we have the following.

Claim 2.4. If $y_i, y_j \in L'$, $i < j$, then $D'_{y_i} \cap D'_{y_j} = \emptyset$.

Claim 2.5. If $z_i, z_j \in \Delta$, $i < j$, then $(D_{z_i} \cup D'_{z_i}) \cap (D_{z_j} \cup D'_{z_j}) = \emptyset$.

Lemma 2.6. *Let B, B' be as above, C the minimal amalgamation of B and B'. Then L generates a subalgebra of C which is isomorphic to B and L' generates a subalgebra of C which is isomorphic to B'.*

Proof. We prove that the restriction of C to L is isomorphic with B. The other case is symmetric. First we prove the following claim.

Claim 2.7. For $x \leq y$ in K, if $b_x \in a_y$, then $D_x \subseteq c_y$ and if $b_x \notin a_y$, then $D_x \cap c_y = \emptyset$.

Proof. Assume $b_x \in a_y$. The only non-trivial case is when $y \in \Delta$ and $x \in L - \Delta$. Then $D_x = \{d_x\}$. Let $z \in \Delta$ be minimal such that $b_x \in a_z$. Then $z \leq y$. It follows that $d_x \in D'_z$. We are going to show that $D'_z \subseteq c_y$. It is enough to show that $b_z \in a_y$. This follows from the fact that Δ generates a subalgebra of B. Indeed, since $b_x \in a_y \cap a_z$ and $a_z - \bigcup\{a_v : v \in \Delta, v < z\}$ is an atom in the subalgebra of B generated by Δ, we must have that $a_z - \bigcup\{a_v : v \in \Delta, v < z\} \subseteq a_y$. In particular $b_z \in a_y$, and we are done.

Assume now that $b_x \notin a_y$. Again the non-trivial case is when $y \in \Delta$ and $x \in L - \Delta$. It is enough to show that if $b_z \in a_y$ for $z \in \Delta$, then $d_x \notin D_z \cup D'_z$. Suppose that $d_x \in D_z \cup D'_z$. Then $d_x \in D'_z$ and hence $b_x \in a_z - \bigcup\{a_v : v \in \Delta, v < z\}$. As above, since Δ generates a subalgebra of B, it follows that $a_z - \bigcup\{a_v : v \in \Delta, v < z\} \subseteq a_y$. Thus $b_x \in a_y$, a contradiction.

Now we finish the proof of the lemma. The restriction of C to L has atoms $c_\alpha - \bigcup\{c_x : x \in K, x < \alpha\}$, for $\alpha \in L$. It follows from the claim and the definition of c_x that the above difference is in fact equal to D_α. Hence, by the claim, C restricted to L is isomorphic to B.

The following lemma gives an explicit formula for the intersection of two generators of the minimal amalgamation algebra. It is used in section 4.

Lemma 2.8. *If $x \in L - \Delta$, $y \in L' - \Delta$ and $x < y$, then*

$$\begin{aligned} c_x \cap c_y = {} & (c_x \cap \bigcup\{c_z : z < x, z \in \Delta\}) \cap (c_y \cap \bigcup\{c_z : z < x, z \in \Delta\}) \\ & \cup \bigcup\{\{(c_w - \bigcup\{c_v : v < w, v \in \Delta\}) \cap c_x\} : x < w < y, w \in \Delta, \\ & \qquad \text{and } d_w \in c_y\}. \end{aligned}$$

Proof. Suppose that $d \in c_x \cap c_y$, is an atom. If $d \in \bigcup\{c_z : z < x, z \in \Delta\}$, then we are done. Suppose that d is disjoint from the above set. Then $d = d_\alpha$ for some $\alpha \in L - \Delta$, $\alpha \leq x$. Since $d \in c_y$ there is $w \in \Delta, x < w < y$ such that $d \in D'_w \subseteq c_y$. Then $d \in c_x \cap (c_w - \bigcup\{c_v ; v < w, v \in \Delta\})$ and also $d_w \in c_y$.

Conversely, assume that $w \in \Delta$ is such that $x < w < y$, $d_w \in c_y$ and $d \in c_x \cap (c_w - \bigcup\{c_v ; v < w, v \in \Delta\})$. We have to show that $d \in c_y$. Note that $D'_w \subseteq c_y$ as $d_w \in c_y \cap D'_w$. So it is enough to show that $d \in D'_w$. Note that $(c_w - \bigcup\{c_v ; v < w, v \in \Delta\}) = D_w \cup D'_w$, and $d \notin D_w$. Therefore $d \in D'_w$. This finishes the proof.

3. A remark on the Δ-function

A function with the Δ-property, or a Δ-function, is a function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ with the following properties:

(1) $f\{x, y\} \subseteq \min\{x, y\} + 1$ and $\min\{x, y\} \in f\{x, y\}$.
(2) For all uncountable sets D of finite subsets of ω_2 there are $a, b \in D$, $a \neq b$ and $\forall x \in a - b\ \forall y \in b - a\ \forall z \in a \cap b$

(a) if $x, y > z$, then $z \in f\{x, y\}$,
(b) if $y > z$, then $f\{x, z\} \subseteq f\{x, y\}$,
(c) if $x > z$, then $f\{y, z\} \subseteq f\{x, y\}$.

Note that there is a slight difference between the above definition of a Δ-function and the definition in [1], namely we added the condition $\min\{\alpha, \beta\} \in f\{\alpha, \beta\}$. It is easy to see that if f is a Δ-function as defined in [1] and we put $\bar{f}\{\alpha, \beta\} = f\{\alpha, \beta\} \cup \{\min\{\alpha, \beta\}\}$ for all α, β, then $\bar{f}$ is a Δ-function in the above sense. It has been shown in [1], that a Δ-function can be forced by a σ-closed ω_2-cc poset P.

Let us recall the definition of the poset P. Let H be a family of functions h such that for some $a \in [\omega_2]^{\leq\omega}$, $h : [a]^2 \to [a]^{\leq\omega}$ and $h\{\alpha, \beta\} \subseteq \min\{\alpha, \beta\}$, and we also add $\min\{\alpha, \beta\} \in h\{\alpha, \beta\}$. Suppose that g is another function in H, with the domain $[b]^2$ such that there are sets x, y, z with $a = x \cup y$, $b = x \cup z$, $\forall\alpha \in x$, $\forall\beta \in y$, $\forall\tau \in z$ $\alpha < \beta < \tau$, and suppose that there is an order-preserving mapping $\pi : a \to b$ which lifts to an isomorphism of h with g. Then define $f \in H$ to be the maximal amalgamation of h and g if $\mathrm{dom}(f) = [a \cup b]^2$, $h, g \subseteq f$ and for every $\alpha \in a - b$, $\beta \in b - a$, $f\{\alpha, \beta\} = (a \cup b) \cap (\min\{\alpha, \beta\} + 1)$.

Conditions in P are certain countable subsets p of H with the property that $\bigcup p \in p$, $\bigcup p$ is called the base of p. If p, q are such sets, then $p \leq q$ iff $\mathrm{base}(q) \in p$ and $q = \{h \in p : h \subseteq \mathrm{base}(q)\}$. For $\alpha \in \omega_1$ define $P_\alpha \subseteq H$ as follows. P_0 consists of sets of the form $\{h\}$, where $\mathrm{dom}(h) = \{\{\alpha, \beta\}\}$ for some $\{\alpha, \beta\} \in [\omega_2]^2$ and $h\{\alpha, \beta\} = \min\{\alpha, \beta\}$.

Let $\alpha = \beta + 1$. Then $p \in P_\alpha$ iff $\exists q, r \in P_\beta$, $q \neq r$ and if $g = \mathrm{base}(q)$, $\mathrm{dom}(g) = [a]^2$, $h = \mathrm{base}(r)$, $\mathrm{dom}(h) = [b]^2$ then there are sets x, y, z with $a = x \cup y$, $b = x \cup z$, $\forall\alpha \in x$, $\forall\beta \in y$, $\forall\tau \in z$ $\alpha < \beta < \tau$ and there is order preserving bijection $\pi : a \to b$ which lifts to an isomorphism of q with r, and $p = q \cup r \cup \{f\}$ where f is the maximal amalgamation of g and h.

If α is a limit, then $p \in P_\alpha$ iff $p = \bigcup\{p_n : n \in \omega\} \cup \{\bigcup p\}$, where $p_0 \geq p_1 \geq \dots$, each $p_n \in P_{\alpha_n}$ and $\langle \alpha_n : n \in \omega \rangle$ is an increasing sequence cofinal in α. Let $P = \bigcup\{P_\alpha : \alpha \in \omega_1\}$, ordered by $\leq$.

For $h \in H$ define the support of h, $\mathrm{supp}(h)$, to be the set a such that $\mathrm{dom}(h) = [a]^2$ and for $p \in P$ let $\mathrm{supp}(p) = \mathrm{supp}(\mathrm{base}(p))$.

Let G be P-generic, then $\bigcup\bigcup G$ is a Δ-function on a set A cofinal in ω_2. We can identify A with ω_2 by an order preserving bijection. We denote the generic function by f.

We will need the following property of the Δ-function f.

Lemma 3.1. *Let $\gamma \in \omega_2$, $\mathrm{cf}(\gamma) = \omega_1$. Let $B \in [\gamma]^\omega$, $E \in [\omega_2 - \gamma]^{<\omega}$ and $x \in E$. Then there is $A \in [\gamma]^\omega$ and $x' < \gamma$ such that:*
(1) *$B \subseteq A$,*
(2) *$x' > \sup(A)$,*
(3) *$f\{x, y\} = f\{x', y\}$ for every $y \in A$.*
(4) *for every $v \in E$, $A \subseteq f\{v, x'\}$,*
(5) *for every $y \in A$, $f\{x', y\} \subseteq A \cup \{x'\}$.*

Proof. Let $p \in P$ be any condition. We can assume that p decides γ, x, E and, since P is σ-closed, also B, i.e., all these sets are in the ground model. We can assume that $B \cup E \cup \{\gamma\} \subseteq \mathrm{supp}(p)$. Our intention is to find a condition $q \in P$, A and x' such that the maximal amalgamation of q and p forces the conclusion.

Let $A = \mathrm{supp}(p) \cap \gamma$, $\delta = \sup(A)$. Let $C = \mathrm{supp}(p) \cap (\omega_2 - \gamma)$. Let $C' \subseteq \gamma - \delta$ be a set of the same order type as C and let $\pi : C \to C'$ be the order preserving bijection. Then π lifts to a order preserving bijection from $\operatorname{supp}(p)$ onto $A \cup C'$ which is constant on A. Using π we can define a condition q such that $\operatorname{supp}(q) = A \cup C'$ and π lifts to an isomorphism between p and q. Let $x' = \pi(x)$ and let $r \in Q$ be the maximal amalgamation of p and q. Then r forces the required properties: (1) is obvious, (2),(3),(4) and (5) follow from the definition of a maximal amalgamation.

Lemma 3.2. *Let $\gamma \in \omega_2$, $\operatorname{cf}(\gamma) = \omega_1$. Let $B \in [\gamma]^\omega$, $E \in [\omega_2 - \gamma]^{<\omega}$, $E = \{x_1, \ldots, x_k\}$. Then there is $A \in [\gamma]^\omega$ including B, and $E' \in [\gamma - \sup(A)]^k$, $E' = \{x'_1, \ldots, x'_k\}$ and $F \in [[\gamma]^2]^k$ such that $F = \{\{y_i^1, y_i^2\} : i \leq k\}$ and letting $D = E' \cup F$, i.e., D is the set of ordinals appearing in E' or in F, we have*

1. *For all $i, j \leq k$, $\sup(A) < x'_i < y_j^1 < y_j^2$.*
2. *For $i < j$, $y_i^2 < y_j^1$.*
3. *If $z \in D \cup A$ and $y \in A$, then $f\{z, y\} \subseteq A$.*
4. *For $i \leq k$, if z is any element of D with a subscript i, then $f\{x_i, y\} = f\{z, y\}$ for every $y \in A$.*
5. *For $i \leq k$, for every $v \in D$, $(A \cup \{z : z \in D, z < v\}) \subseteq f\{x_i, v\}$.*
6. *For every $v, w \in D$, if $v < w$, then $(A \cup \{z : z \in D, z < v\}) \subseteq f\{v, w\}$.*

Proof. Let $p \in P$ be such that $\operatorname{supp}(p)$ includes B, E and $\{\gamma\}$. Let $A = \operatorname{supp}(p) \cap \gamma$, $\delta = \sup(A)$. By induction we define elements $\{z_i : i \leq 3k\}$ of $E' \cup F$ together with conditions p_i and q_i for $i \leq 3k$ such that:

(a) $p_1 = p$,
(b) q_i is obtained from p_i as in the proof of the previous lemma, i.e., $\operatorname{supp}(q_i) \subseteq \gamma$ and there is an order preserving bijection $\pi_i : \operatorname{supp}(p_i) \to \operatorname{supp}(q_i)$, constant on $\operatorname{supp}(p_i) \cap \gamma$, which lifts to an isomorphism of p_i with q_i,
(c) p_{i+1} is the maximal amalgamation of p_i and q_i
(d) if $i \leq k$, then put $z_i = \pi_i(x_i)$, if $i = k+(2l-1)$ or $i = k+2l$ for $l = 1, 2, \ldots, k$, then put $z_i = \pi_i(x_l)$.

Now put $x'_i = z_i$ for $i \leq k$, $y_l^1 = z_{k+2l-1}$, $y_l^2 = z_{k+2l}$ for $l \leq k$. It is easy to see that that the conditions of the lemma hold by the definition of a maximal amalgamation. Let us check for example (6). So suppose that $i < j$. We have to show that $(A \cup \{z_l : l < i\}) \subseteq f\{z_i, z_j\}$. Note that $z_j = \pi_j(x)$ for some $x \in E$. Moreover $f\{z_i, z_j\} = f\{z_i, x\}$. So by (5) we are done.

4. The construction

Let us define now a ccc poset Q which forces an ω_2-minimal Boolean algebra A generated by $\{a_\alpha : \alpha \in \omega_2\}$. A pair (B, L) is a condition in Q if $L = \{x_1, \ldots, x_k\}$ is a subset of ω_2, and B is a L-minimal Boolean algebra generated by $\{c_x : x \in L\}$.

For every $i, j \leq k$ the element $c_{x_i} \cap c_{x_j}$ is in the Boolean algebra generated by $\{c_{x_m} : x_m \leq \min\{x_i, x_j\}$ and $x_m \in f\{x_i, x_j\}\}$.

A condition (B', L') extends (B, L) if $L \subseteq L'$ and L generates a subalgebra of B' which is isomorphic to B.

Note that Q forces an ω_2-minimal algebra A.

Lemma 4.1. *The forcing Q satisfies the ccc.*

Proof. Let $\{(B_\alpha, L_\alpha) : \alpha \in \omega_1\}$ be an uncountable subset of Q. By thinning out we can assume that $\{L_\alpha : \alpha \in \omega_1\}$ form a Δ-system with the root Δ; for every $\alpha \neq \beta$, the structures (L_α, Δ) and (L_β, Δ) are isomorphic and the isomorphism lifts to the isomorphism between B_α and B_β. Let $A \in [\omega_2]^\omega$ be such that A is closed under f and $\Delta \subseteq A$. Let α, β, $\alpha \neq \beta$ be such that $L_\alpha \cap A = \Delta$ and $L_\beta \cap A = \Delta$ and L_α, L_β satisfy conditions (a), (b), (c) of the definition of a Δ-function. Note that it follows that Δ generates subalgebras of both B_α and B_β.

Let $L = L_\alpha \cup L_\beta$ and let C be the minimal amalgamation of B_α and B_β. Suppose that C is generated by $\{c_x : x \in L\}$. We have to prove that (C, L) is a condition in Q. To see this it is enough to prove that if $x \in L_\alpha - \Delta$, $y \in L_\beta - \Delta$, $x < y$, then the intersection $c_x \cap c_y$ is in the algebra generated by $\{c_v : v \leq \min\{x, y\}, v \in L \cap f\{x, y\}\}$. Assume e.g. that $x < y$. Recall now Lemma 2.8. For $z \in \Delta$, $z < x$, the intersection $c_x \cap c_z$ is in the algebra generated by $\{c_v : v \leq z, v \in L_\alpha \cap f\{z, x\}\}$, and also $c_y \cap c_z$ is in the algebra generated by $\{c_v : v \leq z, v \in L_\beta \cap f\{z, y\}\}$. Note that we have that $f\{x, z\} \cup f\{y, z\} \subseteq f\{x, y\}$. Therefore the above intersections are in the algebra generated by $\{c_v : v \leq x, v \in L \cap f\{x, y\}\}$. Similarly we show that for $w \in \Delta, x < w < y$ we have $c_x \cap (c_w - \bigcup\{c_v : v < w, v \in \Delta\})$ is in the algebra generated by $\{c_v : v \leq x, v \in L \cap f\{x, y\}\}$. This follows from the fact that $f\{x, v\} \subseteq f\{x, y\}$ for $v \in \Delta$, $v < y$. This finishes the proof of the lemma.

5. Topological properties of the Stone space of A

Let X be the Stone space of A. As we showed in the introduction we can identify X with $\omega_2 \cup \{*\}$.

We use the following notation: if $q \in Q$, then $q = (B(q), L(q))$ and $B(q)$ is generated by $\{a_x(q) : x \in L(q)\}$ with the atoms $\{d_x(q) : x \in L(q)\}$.

Lemma 5.1. *$X - \{*\}$ has countable tightness.*

Proof. Suppose that $\gamma \in \omega_2$ and $\gamma \in \mathrm{cl}(D)$, where $D \subseteq \omega_2$. We can assume that $D \subseteq \gamma$. We want to find a countable set $E \subseteq D$ such that $\gamma \in \mathrm{cl}(E)$. Suppose that there is no such set. Then we can find an ω_1-sequence of points in D and neighborhoods of γ, $\{(\gamma_\xi, U_\xi) : \xi \in \omega_1\}$, such that

(1) $\gamma \in \mathrm{cl}(\{\gamma_\xi : \xi \in \omega_1\})$,
(2) for every $\zeta \in \omega_1$, $\{\gamma_\xi : \xi < \zeta\} \cap U_\zeta = \emptyset$.

Note that this can be arranged since the character of γ is ω_1. Now we argue in the model V^P. Suppose that $q \in Q$ forces the above situation. We can assume that γ is already determined by q. Let $\{q_\xi : \xi \in \omega_1\}$ be a set of conditions in Q, extending q such that for each ξ there is $\{\rho^1_\xi, \ldots, \rho^k_\xi\}$ such that

$$q_\xi \Vdash U_\xi = U_\gamma(\rho^1_\xi, \ldots, \rho^k_\xi) = \bigcap\{a^{-1}_{\rho^i_\xi} : i \leq k\} \cap a_\gamma.$$

By further extending the conditions q_ξ we can assume that $\{\rho^1_\xi, \ldots, \rho^k_\xi\} \subseteq L(q_\xi)$. Now recall the proof of the ccc. We can find an uncountable subcollection W of $\{q_\xi : \xi \in \omega_1\}$ such that any two conditions in W can be minimally amalgamated. Moreover there exists a finite set $\Delta \subseteq \omega_2$ such that for every $q_\xi, q_\zeta \in W$, if $\xi < \zeta$, then for all $x \in L(q_\xi) - \Delta$, $y \in L(q_\zeta) - \Delta$, $a_x \cap a_y \subseteq \bigcup\{a_z : z \in \Delta, z < \min\{x, y\}\}$. Note that this implies that the minimal amalgamation of q_ξ and q_ζ forces that $U^{-1}_\xi \cap U^{-1}_\zeta \subseteq V$, where $V = a^{-1}_\gamma \cup \bigcup\{a_z : z \in \Delta, z < \gamma\}$. It follows, by (2), that the minimal amalgamation of q_ξ and q_ζ forces that $\{\gamma_\tau : \tau < \xi\} \subseteq U^{-1}_\xi \cap U^{-1}_\zeta \subseteq V$. Note that the definition of V does not depend on ξ, ζ. Therefore, for every q_μ and q_ν in W, if $\mu < \nu$, then the minimal amalgamation of q_μ and q_ν forces that $\{\gamma_\tau : \tau < \mu\} \subseteq V$. Consider now any uncountable set of pairwise disjoint pairs of elements of W and let Z be the set of the corresponding minimal amalgamations.

Since Q satisfies the ccc we can assume that H is Q-generic such that $H \cap Z$ is uncountable. Then in the extension we have that $\{\gamma_\xi : \xi \in \omega_1\} \subseteq V$. This contradicts (1) as $\gamma \notin V$.

Lemma 5.2. *There are no ω_1-sequences in X converging to $\{*\}$.*

Proof. Suppose that $D = \{\gamma_\xi : \xi \in \omega_1\}$ is an ω_1-sequence in X converging to $\{*\}$, i.e. every neighborhood of $\{*\}$ contains all but countably many elements of D. As before, working in the model V^P, we can find conditions $\{q_\xi : \xi \in \omega_1\}$ such that every q_ξ determines γ_ξ to be some ordinal δ_ξ and $\delta_\xi \in L(q_\xi)$. Let $\rho \in \omega_2$ be such that $\rho > \sup(L(q_\xi))$ for each $\xi \in \omega_1$. We extend each condition q_ξ to a condition q'_ξ defined as follows. We put $L(q'_\xi) = L(q_\xi) \cup \{\rho\}$. We form $B(q'_\xi)$ by adding one new element c_ρ to $B(q_\xi)$ and putting $c_x \subseteq c_\rho$ for every $x \in L_\xi$.

Note that q'_ξ is well defined as $c_\alpha \cap c_\rho = c_\alpha$ and $\alpha \in f\{\alpha, \rho\}$. It is easy to see that $q'_\xi \Vdash \gamma_\xi = \delta_\xi$ and $\delta_\xi \in c_{\delta_\xi} \subseteq c_\rho$. Since Q is ccc it follows that uncountably many elements of $\{\gamma_\xi : \xi \in \omega_1\}$ are contained in c_ρ. But c_ρ is a clopen subset of X disjoint from $\{*\}$, a contradiction.

Before the proof of the next lemma we need some more observations about the relationship between the forcing Q and the Δ-function f.

Lemma 5.3. *Let $A \subseteq \omega_2$ be closed under f, i.e., $f\{x, y\} \in A$ for every $x, y \in A$. Suppose that $q_1 = (B_1, L_1)$ and $q_2 = (B_2, L_2)$ are two conditions in Q such that $L_1 \subseteq A$, $D = L_2 - A$ is such that if $y \in A$ and $z \in D$, then $y < z$ and $A \cup D$ is closed under f. Suppose that a condition $q = (B, L)$ extends both q_1 and q_2. Then the restriction of q to $A \cup D$ is a condition in Q extending q_1 and q_2.*

Proof. Let $q = (B, L)$ be a condition in Q extending q_1 and q_2. Let $q' = (B', L')$ be a restriction of q to $A \cup D$. Let $x < y$ be in L'. We have to show that $a_x \cap a_y$ is in the algebra generated by $\{a_z : z \in L' \cap f\{x, y\}\}$.

Assume first that $x \in A$. Then, as q is a condition, we have that $a_x \cap a_y$ is in the algebra generated by $\{a_v : v \in L \cap f\{x, y\}\}$. By our assumption it follows that $f\{x, y\} \subseteq A$. Hence $L' \cap f\{x, y\} = L \cap f\{x, y\}$ and we are done.

If $x \in D$, then $x, y \in L_2$ and since q_2 is a condition it follows that $a_x \cap a_y$ is in the algebra generated by $\{a_z : z \in L_2 \cap f\{x, y\}\}$. Now note that $L_2 \subseteq L'$.

Lemma 5.4. *There are no ω-sequences in X converging to $\{*\}$.*

Proof. Suppose that $\{x_n : n \in \omega\} \subseteq \omega_2$ is a sequence converging to $\{*\}$. We work in the model V^P. For $n \in \omega$ let A_n be a maximal antichain in Q that determines x_n, i.e., for each $u \in A_n$ there is some $\alpha_n \in L(u)$ such that $u \Vdash x_n = \alpha_n$. Let $\gamma \in \omega_2$, $\mathrm{cf}(\gamma) = \omega_1$ be such that for every n, for every $u \in A_n$, $L(u) \subseteq \gamma$.

Since a_γ is a clopen set disjoint from $\{*\}$, there is some condition $q \in Q$ and $m \in \omega$ such that $q \Vdash x_n \notin a_\gamma$ for every $n > m$. Let $q = (B(q), L(q))$ and let $B(q)$ be generated by $\{a_x(q) : x \in L(q)\}$ with the atoms $\{d_x(q) : x \in L(q)\}$. We can assume that $\gamma \in L(q)$. Then $L(q) = L' \cup E$, where $L' = L(q) \cap \gamma$, $E = L(q) - L'$. Let $\{x_1, \dots, x_k\}$ be an enumeration of E in the increasing order. Note that $x_1 = \gamma$. Let $A \in [\gamma]^\omega$ be such that A is closed under f and includes $L(q) \cap \gamma$ and $L(u)$ for every $u \in A_n$, $n \in \omega$. Let E', F be as in Lemma 3.2.

Let us now give the idea of the rest of the proof. Our intention is, of course, to find a condition $r \leq q$ and $n > m$ such that $r \Vdash x_n \in a_\gamma$. To do this we first find a condition s, such that $L(s) = L' \cup E'$, and $B(s)$ is isomorphic to $B(q)$ via bijection from E to E'.

Let U be a name for $\bigcup\{a_x : x \in L(s)\}$. If $p \in Q$ is any condition such that $L(s) \subseteq L(p)$, then by $U(p)$ we denote $\bigcup\{a_x(p) : x \in L(s)\}$. In particular $p \Vdash U(p) = U$. Next we find an auxiliary condition $t \leq s$ such that $L(t) = L(s) \cup \bigcup F$, $B(t)$ restricted to $L(s)$ is isomorphic to $B(s)$. Since t forces that U is a clopen set disjoint from $\{*\}$ and $\{x_n : n \in \omega\}$ converges to $\{*\}$, we can find a condition $t' \leq t$ and $n > m$ such that $t' \Vdash x_n \notin U$. We can assume that t' extends some condition $u \in A_n$, so there is some $\alpha_n \in L(t')$ such that $t' \Vdash \alpha_n = x_n$. By Lemma 5.3, since $L(u) \subseteq A$, we can assume that $L(t') - L(t) \subseteq A$. Let W be a name for $\bigcup\{a_x(t') : x \in L(t')\}$. For any condition p such that $L(t') \subseteq L(p)$, let $W(p) = \bigcup\{a_x(p) : x \in L(t')\}$.

Finally we define $r \leq t'$. Our intention is to define r such that r extends q and $W(r) - U(r) \subseteq a_\gamma$. Then $r \Vdash x_n \in a_\gamma$, a contradiction. The main part is to prove that r is a condition in Q. That is where the auxiliary condition t helps.

Continuation of the proof. We first verify that s and t are conditions in Q and then we define r and show that it has the required properties.

Recall that $L(s) = L' \cup E'$ and the bijection between $L(s)$ and $L(q)$, constant on L', lifts to an isomorphism of $B(s)$ with $B(q)$. Hence for $y < x$ in $L(s)$, if $y, x \in L'$, then $d_y(s) \in a_x(s)$ iff $d_y(q) \in a_x(q)$. If $y \in L', x \in E'$, then $x = x'_i$ for some $i \leq k$ and $d_y(s) \in a_{x'_i}(s)$ iff $d_y(q) \in a_{x_i}(q)$. Finally, if $y, x \in E'$, then $y = x'_j, x = x'_i$ and $d_{x'_j}(s) \in a_{x'_i}(s)$ iff $d_{x_j}(q) \in a_{x_i}(q)$.

We check that s is a condition in Q. Let $y < x$ in $L(s)$. We have to show that $a_x(s) \cap a_y(s)$ is in the algebra generated by $\{a_v(s) : v \leq y, v \in L(s) \cap f\{x, y\}\}$. If $x, y \in L'$, then we have nothing to do since the isomorphism is constant on L'. Assume that $y \in L', x = x'_i$. Then $a_y(s) \cap a_{x'_i}(s)$ has the same representation by $\{a_v(s) : v \leq y\}$ as $a_y(q) \cap a_{x_i}(q)$ by $\{a_v(q) : v \leq y\}$. Moreover, since q is a condition the intersection $a_y(q) \cap a_{x_i}(q)$ is in the algebra generated by $\{a_v(q) : v \in L(q) \cap f\{x_i, y\}\}$. By Lemma 3.2(4), we have $f\{x_i, y\} = f\{x'_i, y\}$, hence the intersection $a_y(s) \cap a_{x'_i}(s)$ is in the algebra generated by $\{a_v(s) : v \in L(s) \cap f\{x'_i, y\}\}$.

If $y = x'_j$, $x = x'_i$ with $j < i$, then by Lemma 3.2(6), $\{v : v \in L(s), v \leq y\} \subseteq f\{x'_i, x'_j\}$, hence $a_{x'_j}(s) \cap a_{x'_i}(s)$ is in the algebra generated by $\{a_v(s) : v \in L(s) \cap f\{x'_i, x'_j\}\}$ and we are done.

Now we define t, extending s. Let $L(t) = L(s) \cup \bigcup F$. We define $B(t)$ such that $B(t)$ restricted to $L(s)$ is isomorphic to $B(s)$. We define $a_{y_i^1}(t)$ and $a_{y_i^2}(t)$ for $i = 1, \ldots, k$ such that $a_{y_i^\mu}(t) \cap a_{y_j^\nu}(t) = U(t)$ for every $(\mu, i) \neq (\nu, j)$. Recall that $U(t) = \bigcup_{v \in L(s)} a_v(t)$. To check that t is a condition note that if $(\mu, i) \neq (\nu, j)$, then by Lemma 3.2(6), $L(s) \subseteq f\{y_i^\mu, y_j^\nu\}$.

Recall that $t' \leq t$ is a condition such that $L(t') - L(t) \subseteq A \cap \gamma$. Finally we define r. Put $L(r) = L(t') \cup E$. The algebra $B(r)$ restricted to $L(t')$ is isomorphic to $B(t')$. We have to define $a_{x_i}(r)$ for $i = 1, \ldots, k$. Recall that $W(r) = \bigcup\{a_x(r) : x \in L(t')\}$. For $x \in L(q)$ we define auxiliary sets D_x as follows. If $x \in L'$, then put $D_x = a_x(r) - \bigcup\{a_y(r) : y < x, y \in L'\}$. Assume $x \in E$, i.e., $x = x_i$ for some i. Assume first that $i = 1$. Put $D_{x_1} = (a_{x'_1}(r) - \bigcup\{a_y(r) : y < x'_1, y \in L(s)\}) \cup (W(r) - U(r)) \cup \{d_{x_1}(r)\}$. For $i > 1$ define $D_{x_i}(r) = (a_{x'_i}(r) - \bigcup\{a_y(r) : y < x'_i, y \in L(s)\}) \cup \{d_{x_i}(r)\}$. Now define $a_{x_i}(r)$ for $i \leq k$. Let $a_{x_i}(r) = \bigcup\{D_x : x \in L(q), d_x(q) \in a_{x_i}(q)\}$.

Claim 5.5. For $x_i \in E$, $a_{x_i}(r) \cap U(r) = a_{x'_i}(r)$. Moreover $a_{x_i}(r) \cap W(r) = a_{x_i}(r) \cap U(r)$ if $d_{x_1}(q) \notin a_{x_i}(q)$ and $a_{x_i}(r) \cap W(r) = (a_{x_i}(r) \cap U(r)) \cup (W(r) - U(r))$ if $d_{x_1}(q) \in a_{x_i}(q)$.

Proof. Note that $a_{x_i}(r) \cap U(r)$ is the union of sets of the form $D'_x = a_x(r) - \bigcup\{a_y(r) : y < x, y \in L(s)\}$, for $x \in L(s)$. Since $B(r)$ restricted to $L(s)$ is isomorphic to $B(s)$ and the bijection between E' and E lifts to the isomorphism between $B(s)$ and $B(q)$, it follows that $a_{x'_i}(r)$ is also the union of sets D'_x for $x \in L(s)$. Moreover $D'_x \subseteq a_{x'_i}(r)$ if and only if $D'_x \subseteq a_{x_i}(r) \cap U(r)$. The second part is obvious by the definition of $a_{x_i}(r)$.

Consider now $B(r)$ restricted to $L(q)$. For $x \in L(q)$ the set $a_x(r) - \bigcup\{a_y(r) : y < x, y \in L(q)\}$ is equal to D_x. By the definition, $D_y \subseteq a_x(r)$ if $d_y(q) \in a_x(q)$, and $D_y \cap a_x(r) = \emptyset$, otherwise. Hence, the restriction of $B(r)$ to $L(q)$ is isomorphic to $B(q)$.

Finally we have to show that r is a condition in Q, i.e., we have to show that if $y < x$ in $L(r)$, then $a_x(r) \cap a_y(r)$ is in the algebra generated by $\{a_v(r) : v \in L(s) \cap f\{x, y\}\}$. Since $B(r)$ restricted to $L(t')$ is isomorphic to $B(t')$, we can assume that $x \in E$, i.e., $x = x_i$ for some $i \leq k$.

Assume first that $y \in L(t') - (E' \cup \bigcup F)$. By the definition, since $a_y(r) \subseteq W(r)$ for $y \in L(t')$ we have $a_{x_i}(r) \cap a_y(r)$ is equal to either $a_{x'_i}(r) \cap a_y(r)$ or $(a_{x'_i}(r) \cap a_y(r)) \cup (a_y(r) - U(r))$. Recall that $U(r) = a_{y_i^1}(r) \cap a_{y_i^2}(r)$. Hence $a_y(r) - U(r) = a_y(r) - ((a_y(r) \cap a_{y_i^1}(r)) \cap (a_y(r) \cap a_{y_i^2}(r)))$. Moreover, since $(L(t') - (E' \cup \bigcup F)) \subseteq A$, it follows by Lemma 3.2(4) that the sets $f\{y, x_i\}$, $f\{y, x'_i\}$, $f\{y, y_i^1\}$, $f\{y, y_i^2\}$ are equal, and we are done.

Assume that $y \in E'$. Then $y = x'_j$ for some $j \leq k$ and then, since $a_{x'_j}(r) \subseteq U(r)$, we have $a_{x_i}(r) \cap a_{x'_j}(r) = a_{x'_i}(r) \cap a_{x'_j}(r)$. But, by Lemma 3.2(5), we have $\{v : v \in L(t'), v \leq x'_j\} \subseteq f\{x_i, x'_j\}$ and of course $a_{x'_i}(r) \cap a_{x'_j}(r)$ is in the algebra generated by $\{a_v(r) : v \in L(t'), v \leq x'_j\}$.

Finally, if $y \in \bigcup F$, say $y = y_j^1$, then $a_{x_i}(r) \cap a_{y_j^1}(r)$ is equal either to $a_{x'_i}(r)$ or $a_{x'_i}(r) \cup (a_{y_j^1}(r) - U(r))$. In the first case note that $x'_i \in f\{x_i, y_j^1\}$. In the second case recall that $U(r) = \bigcup\{a_v(r) : v \in L(s)\}$ and $L(s) = L' \cup E'$. Hence $a_{y_j^1}(r) - U(r) = a_{y_j^1}(r) - \bigcup\{a_{y_j^1}(r) \cap a_v(r) : v \in L(s)\}$. By Lemma 3.2(5) it follows that $L(s) \subseteq f\{y_j^1, x_i\}$. Hence r is a condition in Q. This finishes the proof of the lemma.

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