PACKING DIMENSION AND CARTESIAN PRODUCTS

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ABSTRACT. We show that for any analytic set A in $\mathbf{R}^d$, its packing dimension $\dim_{\mathbf{p}}(A)$ can be represented as $\sup_B \{\dim_{\mathbf{H}}(A \times B) - \dim_{\mathbf{H}}(B)\}$, where the supremum is over all compact sets B in $\mathbf{R}^d$, and $\dim_{\mathbf{H}}$ denotes Hausdorff dimension. (The lower bound on packing dimension was proved by Tricot in 1982.) Moreover, the supremum above is attained, at least if $\dim_{\mathbf{p}}(A) < d$. In contrast, we show that the dual quantity $\inf_B \{\dim_{\mathbf{p}}(A \times B) - \dim_{\mathbf{p}}(B)\}$, is at least the “lower packing dimension” of A , but can be strictly greater. (The lower packing dimension is greater than or equal to the Hausdorff dimension.)

1. INTRODUCTION

Marstrand’s product theorem ([9]) asserts that if $A, B \subset \mathbf{R}^d$ then

$$\dim_{\mathbf{H}}(A \times B) \geq \dim_{\mathbf{H}}(A) + \dim_{\mathbf{H}}(B),$$

where “ $\dim_{\mathbf{H}}$ ” denotes Hausdorff dimension. Refining earlier results of Besicovitch and Moran [3], Tricot [14] showed that

$$(1) \quad \dim_{\mathbf{H}}(A \times B) - \dim_{\mathbf{H}}(B) \leq \dim_{\mathbf{p}}(A),$$

where “ $\dim_{\mathbf{p}}$ ” denotes packing dimension (see the next section for background). Tricot [15] and also Hu and Taylor [6] asked if this is sharp, i.e., given A , can the right-hand side of (1) be approximated arbitrarily well by appropriate choices of B on the left-hand side? Our first result, proved in Section 3, states that this is possible.

Theorem 1.1. *For any analytic set A in $\mathbf{R}^d$*

$$(2) \quad \sup_B \{\dim_{\mathbf{H}}(A \times B) - \dim_{\mathbf{H}}(B)\} = \dim_{\mathbf{p}}(A),$$

where the supremum is over all compact sets $B \subset \mathbf{R}^d$.

The proof actually shows that, when $\dim_{\mathbf{p}}(A) < d$, the supremum is attained at some compact set B of Hausdorff dimension $d - \dim_{\mathbf{p}}(A)$. We remark that there are compact sets $A \subset \mathbf{R}^d$ for which the supremum is attained *only* with sets B of dimension $d - \dim_{\mathbf{p}}(A)$.

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The sets which have equal Hausdorff and packing dimensions have been singled out by Tricot [14] (who called them “dimension-regular”) and by Taylor [12] because of their good behavior in Cartesian products and probabilistic applications. Theorem 1.1 shows that these are the only compact sets which have these good properties universally.

A related question concerns the behavior of packing dimension under products. Tricot [14] showed that

$$(3) \quad \dim_{\mathbf{P}}(E \times F) \geq \dim_{\mathbf{H}}(E) + \dim_{\mathbf{P}}(F).$$

Hu and Taylor [6] asked if all sets $E \subset \mathbf{R}$ satisfy

$$(4) \quad \inf_F \{ \dim_{\mathbf{P}}(E \times F) - \dim_{\mathbf{P}}(F) \} = \dim_{\mathbf{H}}(E) ?$$

The answer turns out to be more delicate. For a bounded set E in a metric space, denote by $N(E, \epsilon)$ the maximal cardinality of a collection of disjoint closed balls of radius ϵ with centers in E . Let

$$(5) \quad \underline{\dim}_{\mathbf{M}}(E) = \liminf_{\epsilon \rightarrow 0} \frac{\log N(E, \epsilon)}{|\log \epsilon|}$$

be the **lower Minkowski dimension** of E . Following Mattila [10], we call the “regularization” of this index the **lower packing dimension**, denoted $\underline{\dim}_{\mathbf{P}}$, although it is not constructed using packing measures. It is defined, for any set E in a metric space, by

$$(6) \quad \underline{\dim}_{\mathbf{P}}(E) = \inf_{E \subset \bigcup_j E_j} \sup_j \underline{\dim}_{\mathbf{M}}(E_j),$$

where the infimum is over all countable collections of bounded sets $\{E_j\}$ whose union contains E .

It is easy to see that $\dim_{\mathbf{H}}(E) \leq \underline{\dim}_{\mathbf{P}}(E)$ for any set E . A set E where the inequality is strict was constructed by Tricot [14]; A simpler example, due to B. Weiss, was described in Benjamini and Peres ([1], pp. 587).

The following proposition is proved in Section 4.

Proposition 1.2. *For any compact set E in $\mathbf{R}^d$,*

$$(7) \quad \inf_F \{ \dim_{\mathbf{P}}(E \times F) - \dim_{\mathbf{P}}(F) \} \geq \underline{\dim}_{\mathbf{P}}(E)$$

where the infimum is over all compact sets $F \subset \mathbf{R}^d$. There exist compact sets $E \subset \mathbf{R}$ for which the inequality in (7) is strict.

2. BACKGROUND ON PACKING DIMENSION AND TREES

For background on Hausdorff measures and dimension we refer to Falconer [4]. A more naive notion is the **upper Minkowski dimension** (sometimes called the “upper box dimension”) defined for any set E in a metric space by

$$(8) \quad \overline{\dim}_{\mathbf{M}}(E) = \limsup_{\epsilon \rightarrow 0} \frac{\log N(E, \epsilon)}{|\log \epsilon|}.$$

(The lower and upper Minkowski dimensions are infinite if E is not totally bounded.) Tricot [13], [14] introduced **packing dimension**, which plays a dual role to Hausdorff dimension in many settings. For our present purpose, the representation of

packing dimension which is convenient to use as a definition, is as a regularization of upper Minkowski dimension:

$$(9) \quad \dim_p(A) = \inf_{A \subset \bigcup_j A_j} \sup_j \overline{\dim}_m(A_j),$$

where the infimum is over all countable covers of A . (See Tricot [14], Proposition 2, or Falconer [4], Proposition 3.8.)

Part (i) of the next lemma is due to Tricot [14] (see also Falconer [4]); Part (ii) for trees can be found in Benjamini and Peres ([2], Proposition 4.2(b)); the general version given is in Falconer and Howroyd [5] and in Mattila and Mauldin [11].

Lemma 2.1. (i) *Let E be a closed set in a complete metric space. If any open set V which intersects E satisfies $\overline{\dim}_m(E \cap V) \geq \alpha$, then $\dim_p(E) \geq \alpha$.*

(ii) *Let E be a subset of a separable metric space, with $\dim_p(E) > \alpha$. Then there is a (relatively closed) nonempty subset $\tilde{E}$ of E , such that $\dim_p(\tilde{E} \cap V) > \alpha$ for any open set V which intersects $\tilde{E}$.*

(iii) *The analogues of (i) and (ii) with $\underline{\dim}_m$ and $\underline{\dim}_p$ in place of $\overline{\dim}_m$ and $\dim_p$ respectively, are valid.*

Proof. (i) See Tricot [14] or Falconer [4], Proposition 3.6. (ii) Define $\tilde{E}$ to be the set of points $x \in E$ such that every neighborhood W of x satisfies $\dim_p(E \cap W) > \alpha$. Clearly $\tilde{E}$ is relatively closed in E . If $\tilde{E}$ was empty, then E would be covered by relatively open sets $E \cap W$ of packing dimension at most α . Passing to a countable subcover (using separability) would yield a contradiction. To verify that $\tilde{E}$ has the required property, let V be any open set which intersects $\tilde{E}$. By the definition of $\tilde{E}$, the intersection $V \cap (E \setminus \tilde{E})$ is covered by the union of the open sets W such that $\dim_p(E \cap W) \leq \alpha$, and using separability again,

$$\alpha < \dim_p(E \cap V) \leq \max\{\dim_p(\tilde{E} \cap V), \alpha\}.$$

This completes the proof.

(iii) The proofs of the previous two parts transfer. $\square$

Tree Notation. We will represent closed subsets of the unit cube by infinite trees, and use finite trees as tools in our constructions. A (rooted) **tree** T is a connected acyclic graph with a distinguished vertex ρ designated as the root. It may be finite or infinite, but every vertex must have finite degree. For every vertex v we write $|v|$ for its **level**, i.e., the number of edges between it and the root. Neighbors of v at level $|v| + 1$ are called **children** of v . More generally, if v is on the path from the root to a vertex w , then w is called a **descendant** of v , and v is an **ancestor** of w . If a vertex of T has no children, it is called a **leaf** of T . If all the leaves of a finite tree T are at level n , we say that T has **uniform depth** n . A function μ on the vertices of T that satisfies Kirchhoff's law

$$(10) \quad \mu(v) = \sum \{\mu(w) : w \text{ a child of } v\}$$

for every vertex v of T which is not a leaf, is called a **flow** on T .

Now we focus on *infinite* trees without leaves. An infinite self-avoiding path in a tree T , starting at the root of T , is called a **ray**. The set of all rays is called the **boundary** of T and denoted ∂T . For two distinct rays $x, y \in \partial T$ we define their *distance* to be

$$(11) \quad \text{dist}(x, y) = 2^{-n} \text{ if } x \text{ and } y \text{ have exactly } n \text{ edges in common.}$$

FIGURE 1. A set K and the initial levels of an associated tree $T(K)$.

This makes ∂T a compact metric space. For any vertex v , denote by $[v]$ the set of rays going through v . There is a one-to-one correspondence between measures on ∂T and flows on T (where the μ -measure of the set $[v]$ is $\mu(v)$), since the additivity axiom for measures corresponds to Kirchhoff's law (10) for flows. Probability measures correspond to **unit flows**. A set of vertices Π that intersects every ray of T is called a **cut-set**. Equivalently, Π is a cut-set iff $\{[v] : v \in \Pi\}$ is a cover of ∂T .

The closed dyadic subcubes of the unit cube in $\mathbf{R}^d$ have a natural tree structure when ordered by inclusion, and the resulting tree is regular with 2^d children to every vertex; we denote this tree by $\Gamma^{(2^d)}$.

We employ the canonical mapping $\mathcal{R}$ from the boundary of $\Gamma^{(2^d)}$ to the cube $[0, 1]^d$. Formally, label the edges from each vertex of $\Gamma^{(2^d)}$ to its children in a one-to-one manner with the vectors in $\Omega = \{0, 1\}^d$. Then the boundary $\partial\Gamma^{(2^d)}$ is identified with the sequence space $\Omega^{\mathbf{Z}^+}$ and we define the binary representation map $\mathcal{R} : \Omega^{\mathbf{Z}^+} \rightarrow [0, 1]^d$ by

$$(12) \quad \mathcal{R}(\omega_1, \omega_2, \dots) = \sum_{n=1}^{\infty} 2^{-n} \omega_n.$$

Similarly, a vertex v at level k of $\Gamma^{(2^d)}$ is identified with a finite sequence $(\omega_1, \dots, \omega_k) \in \Omega^k$ and we write $\mathcal{R}(v)$ for the cube of side 2^{-k} obtained as the image under $\mathcal{R}$ of all sequences in $\Omega^{\mathbf{Z}^+}$ with prefix $(\omega_1, \dots, \omega_k)$.

With the notation above, let T be an infinite subtree of the regular 2^d -ary tree $\Gamma^{(2^d)}$, with the same root and without leaves. We may identify ∂T with a closed subset of $\Omega^{\mathbf{Z}^+}$. The image of this set under $\mathcal{R}$ is a compact set $K \subset [0, 1]^d$. We say that T is **associated** with K and denote it by $T(K)$ even though it is sometimes not uniquely determined by K . (See Figure 1.) (For instance, the one-point set $\{1/2\}$ has three subtrees of $\Gamma^{(2^d)}$ associated with it, two of them consisting of single rays and the other consisting of two disjoint rays.)

Given any compact set $K \subset [0, 1]^d$, the dyadic cubes which intersect K form a subtree of $\Gamma^{(2^d)}$, and this is a natural choice for $T(K)$, but not all leafless subtrees of $\Gamma^{(2^d)}$ can arise in this way.

Given a closed set $K \subset [0, 1]^d$, and an associated tree $T = T(K)$, let $|T_n(K)|$ denote the number of vertices of the tree $T(K)$ at level n .

Every closed dyadic cube intersecting K is either the image under $\mathcal{R}$ of a vertex in T or adjacent to such an image cube of the same size. Therefore, the number of closed dyadic cubes of side length 2^{-n} which intersect K is at least $|T_n(K)|$ and

at most $3^d \cdot |T_n(K)|$. It is a standard fact that the upper and lower Minkowski dimensions can be calculated by counting dyadic cubes; comparing this with the the definition (11) of the metric on ∂T gives

$$\overline{\dim}_M(K) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 |T_n(K)| = \overline{\dim}_M(\partial T(K))$$

and

$$\underline{\dim}_M(K) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log_2 |T_n(K)| = \underline{\dim}_M(\partial T(K)).$$

The β -dimensional Hausdorff content of a set E is

$$\mathcal{H}_\infty^\beta(E) = \inf \left\{ \sum_j (\text{diam } E_j)^\beta \mid E \subset \bigcup_j E_j \right\},$$

the infimum being over all countable covers of E . The Hausdorff dimension of E is

$$\dim_H(E) = \inf \{ \beta > 0 \mid \mathcal{H}_\infty^\beta(E) = 0 \}.$$

For purposes of computing the Hausdorff dimension, it suffices to estimate the Hausdorff content using coverings by dyadic cubes. Such a covering corresponds to a cut-set Π of the tree $T(K)$ and we get

$$(13) \quad \dim_H(K) = \inf \left\{ \beta \mid \inf_{\Pi} \sum_{v \in \Pi} 2^{-\beta|v|} = 0 \right\}.$$

Call the inner sum a “ β -dimensional cut-set sum.” To verify (13), observe the preimage under $\mathcal{R}$ of a closed dyadic cube of side-length 2^{-j} is covered by at most 3^d sets of the form $[v]$ with v at level j of $\Gamma^{(2^d)}$.

The mass distribution principle (see Proposition 4.2 in Falconer ([4]) says that if K supports a positive measure μ' satisfying a Hölder estimate $\mu'(F) \leq C(\text{diam } F)^\beta$ for any set F and some constant C , then $\dim_H(K) \geq \beta$. It suffices to assume the Hölder estimate for dyadic cubes. In terms of the tree, this estimate means that $T(K)$ supports a flow μ so that the flow going through any vertex v satisfies $\mu(v) \leq C2^{-\beta|v|}$.

The product tree. Given two compact sets A, B in $[0, 1]^d$, we can construct a tree associated to the Cartesian product $A \times B \subset [0, 1]^{2d}$ from trees $T(A)$ and $T(B)$. In general, given two trees T^1, T^2 , the product tree $T^1 * T^2$ has the vertex set

$$\{(v_1, v_2) : v_i \in T^i \text{ for } i = 1, 2 \text{ and } |v_1| = |v_2|\},$$

the root $(\text{root}(T^1), \text{root}(T^2))$, and the adjacency relation:

the vertex (v_1, v_2) is adjacent to (u_1, u_2) iff v_i is adjacent to u_i for $i = 1, 2$.

It is easy to see that $T(A) * T(B)$ is associated with $A \times B$.

3. PROOF OF THEOREM 1.1

Given an analytic set A in $\mathbf{R}^d$, denote $\alpha = \dim_p(A)$. Clearly, we may assume that $\alpha > 0$; we will first prove the theorem under the further assumption $\alpha < d$. We will construct a compact set B in $\mathbf{R}^d$ of Hausdorff dimension $d - \alpha$ so that $\dim_H(A \times B) = d$.

Let $0 < \tilde{\alpha} < \alpha$. Recent results of Joyce and Preiss [7] imply that A contains a subset A_0 which is closed in $\mathbf{R}^d$ with $\dim_p(A_0) > \tilde{\alpha}$. By partitioning and translating if necessary, we may assume that A_0 is contained in the unit cube.

FIGURE 2. Maximal branching followed by no branching

Applying Lemma 2.1(ii) to the boundary of the tree $\partial T(A_0)$, we obtain a subtree $T(\tilde{A})$ with $\tilde{A} \subset A_0$ such that any open set V on the boundary of this subtree has packing dimension greater than $\tilde{\alpha}$. This implies that for any vertex v of the tree $T(\tilde{A})$, we have

$$(14) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 |T_n(\tilde{A}, v)| \geq \tilde{\alpha},$$

where $|T_n(\tilde{A}, v)|$ is the number of descendants w of v such that $|w| = n$.

We will consider truncations $T^k(\tilde{A})$ of the tree $T(\tilde{A})$ to certain levels N_k , and use them to inductively construct a nested increasing sequence of trees Γ^k , whose limit Γ is associated to an approximation $\tilde{B}$ of the promised set B . Simultaneously, we will describe directed unit flows from the root to the boundaries of the product trees $T^k(\tilde{A}) * \Gamma^k$ which satisfy uniform Hölder estimates of order d_k close to d . This will suffice to establish the variational principle (2). A further limiting argument will be needed to show that the supremum there is attained.

First step. By (14), there is a level N_1 such that $T(\tilde{A})$ has more than $2^{\tilde{\alpha}N_1}$ vertices at level N_1 . Construct Γ^1 of uniform depth N_1 , where every vertex in the first $\lceil N_1(1 - \alpha/d) \rceil$ level has 2^d children, and every vertex in the remaining $\lfloor N_1\alpha/d \rfloor$ levels has a single child (see Figure 2). The leaves of Γ^1 form a cut-set Π_1 of cardinality at most $2^{N_1(d-\alpha)+d}$, so the corresponding $d - \alpha$ dimensional cut-set sum satisfies

$$\sum_{v \in \Pi_1} 2^{(\alpha-d)|v|} \leq 2^d.$$

Now consider the **uniform unit flow** μ on the product tree $T^1(\tilde{A}) * \Gamma^1$ which assigns each vertex at level N_1 the same mass.

Let (v, w) be a vertex at level $j \leq N_1$ of this product tree. Observe that v has at most $2^{d(N_1-j)}$ descendants at level N_1 of $T(\tilde{A})$. If $j > \lceil N_1(1 - \alpha/d) \rceil$ then w

has only one descendant at level N_1 of Γ^1 , so that

$$(15) \quad \mu(v, w) \leq 2^{d(N_1-j)} \cdot 1 \cdot 2^{-\tilde{\alpha}N_1} \cdot 2^{-dN_1(1-\alpha/d)} = 2^{-dj+(\alpha-\tilde{\alpha})N_1}.$$

Denoting

$$(16) \quad \tilde{d} := d - \frac{\alpha - \tilde{\alpha}}{1 - \alpha/d},$$

we deduce from (15) that the Hölder estimate

$$(17) \quad \mu(v, w) \leq 2^{-\tilde{d}|(v,w)|}$$

holds if $j = |(v, w)| > \lceil N_1(1 - \alpha/d) \rceil$. On the other hand, an improved Hölder estimate (with d in place of $\tilde{d}$) clearly holds for smaller values of j , since Γ_1 has full branching at these levels. We conclude that (17) is valid for all vertices (v, w) of $T^1(\tilde{A}) * \Gamma^1$. For use in the induction below, we define $N_0 = 0$ and $d_0 = d_1 = \tilde{d}$.

The inductive step. We assume that we have constructed a tree Γ^k of uniform depth N_k , in which every vertex has at most 2^d children. Also, we assume there is a unit flow μ on the product tree $T^k(\tilde{A}) * \Gamma^k$ which satisfies $\mu(v, w) \leq 2^{-d_{k-1}j}$ for any $N_{k-1} \leq j \leq N_k$ and any vertex (v, w) at level j of the product tree; for leaves (v, w) of Γ^k , we assume the estimate $\mu(v, w) \leq 2^{-d_k N_k}$ holds.

The construction of Γ^{k+1} from Γ^k is done in four stages:

1. To each leaf of Γ^k attach a full 2^d -ary tree of depth kN_k to obtain a tree $\Gamma^{k,1}$ of uniform depth $(k+1)N_k$.
2. To each leaf of $\Gamma^{k,1}$ attach a copy of $T^k(\tilde{A})$. This yields a tree $\Gamma^{k,2}$ of uniform depth $(k+2)N_k$, where each leaf of $\Gamma^{k,2}$ corresponds to some leaf of $T^k(\tilde{A})$ (this is a many-to-one correspondence).
3. Now we mimic the first step of the induction. For each leaf u of $T^k(\tilde{A})$, choose an integer $n(u)$ such that u has at least $2^{n(u)\tilde{\alpha}}$ descendants at level $|u| + n(u)$, and $n(u) \geq (k^2 + k + 1)N_k$. This is possible by (14). For future reference, we denote by $T^{k,3}$ the subtree of $T(\tilde{A})$ obtained by appending to every vertex u at level N_k its descendants in the next $n(u)$ levels. To each vertex of $\Gamma^{k,2}$ corresponding to u , append a tree of uniform depth $m(u) = n(u) - (k+1)N_k$, where every vertex in the first $\lceil m(u)(1 - \alpha/d) \rceil$ levels has 2^d children, and every vertex in the remaining $\lfloor m(u)\alpha/d \rfloor$ levels has a single child. This defines $\Gamma^{k,3}$. The leaves of $\Gamma^{k,3}$ form a cut-set Π_{k+1} such that the corresponding β -dimensional cut-set sum satisfies

$$(18) \quad \sum_{w \in \Pi_{k+1}} 2^{-\beta|w|} \leq 2^{d(k+1)N_k} \cdot \sum_{u \in \partial T^k(\tilde{A})} 2^{m(u)(d-\alpha)+d} \cdot 2^{-\beta(N_k+n(u))},$$

where the first factor $2^{d(k+1)N_k}$ is a bound on the number of leaves of $\Gamma^{k,1}$. The number of leaves of $T^k(\tilde{A})$ is at most 2^{dN_k} . Using this and the inequality $m(u) \leq n(u)$, we see that the cut-set sum (18) is bounded by

$$(19) \quad 2^{d(k+1)N_k} \cdot \sum_{u \in \partial T^k(\tilde{A})} 2^d \cdot 2^{n(u)(d-\alpha-\beta)} \leq 2^{d(k+2)N_k+d} \cdot 2^{k^2 N_k(d-\alpha-\beta)}.$$

where the last inequality used $n(u) > k^2 N_k$. For any fixed $\beta > d - \alpha$, these bounds tend to zero as $k \rightarrow \infty$.

FIGURE 3. The four stages of the construction

4. Finally, denote by N_{k+1} the maximum depth of $\Gamma^{k,3}$, and to each leaf w of $\Gamma^{k,3}$ append a tree of uniform depth $N_{k+1} - |w|$ with 2^d children at every level. This yields Γ^{k+1} .

The construction is illustrated in Figure 3. The dashed boxes represent subtrees with full growth (2^d children at each vertex).

Now that we have constructed the tree Γ^{k+1} we want to construct a flow μ on the product tree $T^{k+1}(\tilde{A}) * \Gamma^{k+1}$ which extends the flow μ constructed previously on $T^k(\tilde{A}) * \Gamma^k$. (Rather than call the flow constructed at each step μ_k , we simply refer to them all as μ since the definitions are consistent.) It is most convenient to do this in four stages following the construction of the tree.

1. Suppose (v, w) is a leaf of $T^k(\tilde{A}) * \Gamma^k$. By our induction hypothesis the flow satisfies the Hölder estimate

$$\mu(v, w) \leq 2^{-d_k N_k}.$$

Let $T^{k,1}$ be the truncation of $T(\tilde{A})$ to level $(k+1)N_k$ (so it has the same depth as $\Gamma^{k,1}$). We define the flow on $T^{k,1} * \Gamma^{k,1}$ by dividing the mass assigned by μ to each vertex of the smaller tree $T^k(\tilde{A}) * \Gamma^k$ equally among its descendants at level $(k+1)N_k$ of the extended tree $T^{k,1} * \Gamma^{k,1}$. Since $\Gamma^{k,1}$ has full branching at these levels (i.e., each vertex has 2^d children) it is clear that the resulting flow satisfies

$$\mu(v, w) \leq 2^{-d_k j},$$

for vertices in level j such that $N_k \leq j \leq (k+1)N_k$.

2. *This is the only stage where we do not divide the mass equally among descendants.* Let $T^{k,2}$ be the truncation of $T(\tilde{A})$ to level $(k+2)N_k$ (so it has the same depth as $\Gamma^{k,2}$). For each leaf (v, w) of $T^{k,1} * \Gamma^{k,1}$, let u be the leaf of $T^k(\tilde{A})$ which is an ancestor of v . Divide the mass of (v, w) evenly among the leaves (v', w') of $T^{k,2} * \Gamma^{k,2}$ which satisfy: v' is a descendant of v and w' is the unique descendant of w corresponding to u . At worst, we have concentrated all the mass of (v, w) onto a single descendant, so for leaves (v', w') of $T^{k,2} * \Gamma^{k,2}$ we get

$$\mu(v', w') \leq \mu(v, w) \leq 2^{-dkN_k} \leq 2^{-d_{k+1}(k+2)N_k},$$

where we define

$$(20) \quad d_{k+1} := \min\left\{\frac{dk}{k+2}, \tilde{d}\right\}.$$

Thus at level $(k+2)N_k$, the flow μ satisfies a Hölder estimate of order d_{k+1} .

3. Next, we extend the definition of μ to the product tree $T^{k,3} * \Gamma^{k,3}$. The two trees in the product do not have uniform depth. Let v be a leaf of $T^{k,2}$ which is a descendant of a leaf u of T^k , and let w be a leaf of $\Gamma^{k,2}$ which corresponds to u . Divide the mass $\mu(v, w)$ equally among the descendants of (v, w) at level $N_k + n(u)$. Since the flow μ is Hölder of order $d_{k+1} \leq \tilde{d}$ at (v, w) , the argument in step 1 of the induction implies that the same estimate holds for all descendants of (v, w) down to level $N_k + n(u)$.
4. Finally, to define the flow on $T^{k+1} * \Gamma^{k+1}$, use equal division again; since Γ^{k+1} has full branching here, the Hölder estimate of order d_{k+1} established at the previous stage persists.

This completes the inductive step.

The limit of the trees Γ^k is a tree Γ which is associated to a compact set $\tilde{B}$ in the cube $[0, 1]^d$. Each cut-set of Γ corresponds to a cover of $\tilde{B}$ by dyadic cubes, so the cut-set sum estimates (19) in stage 3 of the inductive step imply that

$$(21) \quad \dim_{\text{H}}(\tilde{B}) \leq d - \alpha.$$

The flow μ extends to the product tree $T(\tilde{A}) * \Gamma$ and corresponds to a measure μ' on $\tilde{A} \times B$ which satisfies the estimate $\mu'(Q) \leq \text{diam}(Q)^{d_k}$ for any dyadic cube Q of side-length at most 2^{-N_k} . Since $d_k = \tilde{d}$ for all large k by (20), the mass distribution principle (see, e.g., Proposition 4.2 in Falconer [4]) implies that

$$(22) \quad \dim_{\text{H}}(\tilde{A} \times \tilde{B}) \geq \tilde{d}.$$

Under our standing assumption $\alpha = \dim_{\text{p}}(A) < d$, we can get $\tilde{d}$ arbitrarily close to d by choosing $\tilde{\alpha}$ close to α and using (16). Thus (21) and (22), combined with Tricot's upper bound (1), prove the variational formula (2) in this case.

Now we show that the supremum in (2) is **attained** when $\alpha < d$. For each integer m , taking $\tilde{\alpha} \geq \alpha - 1/m$, the construction above yields a compact set $\tilde{B}_m$ in $\mathbf{R}^d$ such that

$$(23) \quad \dim_{\text{H}}(\tilde{B}_m) \leq d - \alpha \quad \text{and} \quad \dim_{\text{H}}(A \times \tilde{B}_m) \geq \tilde{d}_m = d - \frac{d}{m(d - \alpha)}.$$

By translating and scaling, we may assume that each $\tilde{B}_m$ contains the origin and has diameter at most $1/m$. The union $B := \bigcup_{m=1}^{\infty} \tilde{B}_m$ is compact, and (23) implies

that

$$(24) \quad \dim_{\mathbf{H}}(B) \leq d - \alpha \quad \text{and} \quad \dim_{\mathbf{H}}(A \times B) \geq d.$$

By Tricot's upper bound (1), both inequalities in (24) must, in fact, be equalities. Thus $\dim_{\mathbf{P}}(A) = \dim_{\mathbf{H}}(A \times B) - \dim_{\mathbf{H}}(B)$, as desired.

For analytic sets $A \subset \mathbf{R}^d$ with $\dim_{\mathbf{P}}(A) = d$, the variational formula (2) follows easily from the validity of this formula for any closed subset A_0 of A with $d - \epsilon < \dim_{\mathbf{P}}(A_0) < d$, by taking ϵ small. (The existence of such subsets A_0 is a consequence of the work of Joyce and Preiss [7].) By a more involved recursive argument, we can show the supremum is attained in this case also, but since including this argument would lengthen the paper, we omit it. $\square$

Remark. The first occurrence we know of the expression

$$\sup_B \{ \dim_{\mathbf{H}}(A \times B) - \dim_{\mathbf{H}}(B) \}$$

is in the work of Kaufman [8].

4. PACKING DIMENSION OF PRODUCTS: PROOF OF PROPOSITION 1.2

We must show that any two compact sets E and F in $\mathbf{R}^d$ satisfy $\dim_{\mathbf{P}}(E \times F) \geq \underline{\dim}_{\mathbf{P}}(E) + \dim_{\mathbf{P}}(F)$. Assume that

$$(25) \quad \alpha < \underline{\dim}_{\mathbf{P}}(E) \quad \text{and} \quad \beta < \dim_{\mathbf{P}}(F).$$

Applying Lemma 2.1(ii) to the boundary of the tree $\partial T(F)$, we obtain a subtree $T(\tilde{F})$ with $\tilde{F} \subset F$ such that for any vertex w of the tree $T(\tilde{F})$, we have

$$(26) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 |T_n(\tilde{F}, w)| > \beta,$$

where $|T_n(\tilde{F}, w)|$ is the number of descendants u of w such that $|u| = n$. Similarly, by part (iii) of Lemma 2.1, there is a subset $\check{E}$ of E such that any vertex v of $T(\check{E})$ satisfies

$$(27) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log_2 |T_n(\check{E}, v)| \geq \alpha.$$

Consider any vertex (v, w) of the product tree $T(\check{E}) \times T(\tilde{F})$. For $n > |v| = |w|$ we have

$$|T_n(\check{E} \times \tilde{F}, (v, w))| = |T_n(\check{E}, v)| \cdot |T_n(\tilde{F}, w)|,$$

so the asymptotics (27) and (26) imply that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log_2 |T_n(\check{E} \times \tilde{F}, (v, w))| \geq \alpha + \beta.$$

Invoking part (i) of Lemma 2.1, we obtain $\dim_{\mathbf{P}}(E \times F) \geq \alpha + \beta$. Since α and β are only restricted by the assumption (25), this completes the proof of (7).

Now we construct a compact set in $[0, 1]$ for which the inequality in (7) is strict. For $i = 0, 1$, let E_i be a compact set in $[0, 1]$ such that every vertex at level n of the tree $T(E_i)$ has

$$\begin{cases} \text{one child} & \text{if } (4k + 2i)! \leq n < (4k + 2i + 1)! \text{ for some } k \geq 1, \\ \text{two children} & \text{otherwise.} \end{cases}$$

See Figure 4. Loosely speaking, the sets E_0 and E_1 are large at most scales, and the scales where they are small are disjoint. More precisely, if v is a vertex of $T(E_0)$

FIGURE 4. On disjoint scales, E_0 and E_1 have no branching.

with $|v| \leq (4k+1)!$, and if $n \in [(4k+2)!, (4k+4)!]$, then $|T_n(E_0, v)| \geq 2^{n-(4k+1)!}$. Defining $S_0 = \bigcup_{k=1}^{\infty} [(4k+2)!, (4k+4)!]$, we obtain

$$(28) \quad \liminf_{n \in S_0} \frac{1}{n} \log_2 |T_n(E_0, v)| = 1.$$

for any vertex v of $T(E_0)$. Similarly, defining $S_1 = \bigcup_{k=1}^{\infty} [(4k)!, (4k+2)!]$, we get

$$(29) \quad \liminf_{n \in S_1} \frac{1}{n} \log_2 |T_n(E_1, v)| = 1,$$

Let $E = E_0 \cup E_1$. We claim that

$$(30) \quad \dim_p(E \times F) \geq 1 + \dim_p(F) \text{ for any compact set } F \subset [0, 1].$$

(In fact this is an equality, but we only need one direction.)

To verify this claim, let $\beta < \dim_p(F)$. As before, there is a tree $T(\tilde{F})$ with $\tilde{F} \subset F$ such that for any vertex w of the tree $T(\tilde{F})$, the asymptotic relation (26) holds. Now we distinguish two cases.

CASE I: All vertices w of $T(\tilde{F})$ satisfy

$$(31) \quad \limsup_{n \in S_0} \frac{1}{n} \log_2 |T_n(\tilde{F}, w)| > \beta.$$

In this case, (28) implies that any vertex (v, w) of the product tree $T(E_0 \times \tilde{F})$ satisfies

$$\limsup_{n \in S_0} \frac{1}{n} \log_2 |T_n(E_0 \times \tilde{F}, (v, w))| > \beta.$$

By Lemma 2.1(i), this yields $\dim_p(E_0 \times \tilde{F}) \geq 1 + \beta$, establishing the claim (30).

CASE II: There is a vertex w of $T(\tilde{F})$ that does not satisfy (31). In this case, by (26), every descendant u of w must satisfy

$$(32) \quad \limsup_{n \in S_1} \frac{1}{n} \log_2 |T_n(\tilde{F}, u)| > \beta,$$

since the union $S_0 \cup S_1$ contains all integers greater than $4!$. Let $T(\tilde{F}, w)$ be the subtree of $T(\tilde{F})$ which consists of all ancestors and descendants of w . Applying Lemma 2.1(i) to the boundary of $T(E_1) * T(\tilde{F}, w)$ yields that $\dim_{\mathbb{P}}(E_1 \times \tilde{F}) \geq 1 + \beta$, completing the proof of the claim (30).

Since clearly $\underline{\dim}_{\mathbb{M}}(E_i) = 0$, for $i = 0, 1$, the definition of lower packing dimension gives $\underline{\dim}_{\mathbb{P}}(E) = 0$. Comparing this with the claim (30) shows that the inequality in Proposition 1.2 is strict for the set E constructed here. $\square$

Remark. Hu and Taylor (1994) used the notation “aDim(E)” for

$$\inf_F \{ \dim_{\mathbb{P}}(E \times F) - \dim_{\mathbb{P}}(F) \}.$$

It is easy to see that the sets E_0, E_1 in the preceding proof have $\text{aDim}(E_i) = 0$ for $i = 0, 1$, so the inequality $\text{aDim}(E) \geq 1$ established above (which is really an equality) shows that the index aDim is not stable under finite unions.

Question. Theorem 1.1 expresses the packing dimension of a set A in terms of Hausdorff dimensions. Is there an expression for the α -dimensional packing *measure* of A in terms of Hausdorff measures of sets associated to A ?

Remark. After obtaining our results, we were informed by Professor S. J. Taylor that similar results were obtained independently by Dr. Yimin Xiao in the special case $d = 1$. His paper will appear in Math. Proc. Camb. Phil. Soc.

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