ON THE ORNSTEIN-UHLENBECK OPERATOR IN L^2 SPACES WITH RESPECT TO INVARIANT MEASURES

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ABSTRACT. We consider a class of elliptic and parabolic differential operators with unbounded coefficients in $\mathbb{R}^n$, and we study the properties of the realization of such operators in suitable weighted L^2 spaces.

1. INTRODUCTION

There is a very wide literature on boundary value problems for linear elliptic and parabolic equations in bounded domains in $\mathbb{R}^n$. A big part of the results can be extended easily to unbounded domains, provided the coefficients of the differential operators are bounded.

A comprehensive approach to the case of unbounded coefficients in $\mathbb{R}^n$ may be found in [1], [2] and [3]. Under appropriate hypotheses, they are able to work in suitably weighted spaces. A typical simple example which is not in general covered by their results is the Ornstein-Uhlenbeck operator

$$(1.1) \quad \mathcal{A}u = \frac{1}{2} \sum_{i,j=1}^n q_{ij} D_{ij} u + \sum_{i,j=1}^n b_{ij} x_i D_j u = \frac{1}{2} \text{Tr}(QD^2 u) + \langle Bx, Du \rangle, \quad x \in \mathbb{R}^n,$$

and the associated semigroup

$$(1.2) \quad \begin{cases} (T(t)\varphi)(x) &= \frac{1}{(2\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\langle Q_t^{-1}y, y \rangle/2} \varphi(e^{tB}x - y) dy, \quad t > 0, \\ T(0)\varphi &= \varphi. \end{cases}$$

Here $Q = [q_{ij}]_{i,j=1,\dots,n}$ is any symmetric positive definite matrix, and B is any nonzero matrix. Q_t is the matrix defined by

$$(1.3) \quad Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \quad t \geq 0,$$

where e^{sB^*} is the exponential of the transpose matrix B^* .

Besides their own mathematical interest, operators with unbounded coefficients arise in stochastic perturbations of ODE's. Consider for instance the linear equation

Received by the editors February 20, 1995.

1991 *Mathematics Subject Classification*. Primary 35J15, Secondary 35K10.

Key words and phrases. Elliptic operators with unbounded coefficients, semigroups, interpolation spaces.

in $\mathbb{R}^n$

$$u'(t) = Bu(t),$$

perturbed by the noise $\sqrt{Q} dW_t$, where W_t is a standard n dimensional Brownian motion. Then the semigroup $T(t)$ defined in (1.2) is the Markov semigroup associated to the stochastic differential equation

$$dX = BX dt + \sqrt{Q} dW_t,$$

since we have

$$(T(t)\varphi)(x) = \mathbb{E}[\varphi(X(t, x))]$$

for a large class of initial data φ . See e.g. [6, Ch. 5]. It would be obviously important (see again [6]) to study also stochastic perturbations of nonlinear systems

$$u'(t) = Bu(t) + F(u(t)),$$

which give nonlinear coefficients to the operator $\mathcal{A}$. Surprisingly, the literature deals essentially with the case where F is bounded. Our work can be considered as a first step in the study of the more general case of a Lipschitz continuous F . Moreover, at the end of the paper we consider also an example in which $\mathcal{A}$ has variable nonlinear coefficients.

In the paper [4] we have described the properties of the realizations of $\mathcal{A}$ and $T(t)$ in spaces of continuous and bounded functions in $\mathbb{R}^n$. Here we study the realizations of $\mathcal{A}$ and $T(t)$ in a L^2 space with respect to a suitable measure. Besides the usual Lebesgue measure, which will be considered in a forthcoming paper [9], an appropriate measure in the study of a dynamical system is its invariant measure, which exists and is unique under suitable assumptions, see [6]. We assume that all the eigenvalues of the matrix B have negative real part, so that there exist $C > 0$, $\omega > 0$ such that

$$(1.4) \quad \|e^{tB}\| \leq Ce^{-\omega t}, \quad t \in \mathbb{R}.$$

Therefore, the matrix

$$(1.5) \quad Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*} ds$$

is well defined. We consider the Gaussian weight associated to the matrix Q_∞ ,

$$(1.6) \quad \mu(x) = \frac{1}{(2\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}x, x \rangle/2}, \quad x \in \mathbb{R}^n.$$

The weighted space L_μ^2 is defined by

$$(1.7) \quad L_\mu^2 = \left\{ f : \mathbb{R}^n \mapsto \mathbb{C} \text{ measurable} \mid \|f\|_{L_\mu^2} = \left(\int_{\mathbb{R}^n} |f(x)|^2 \mu(x) dx \right)^{1/2} < \infty \right\}.$$

Similarly, the Sobolev weighted space H_μ^s , $s > 0$, is the subspace of L_μ^2 consisting of all the functions f such that $x \mapsto f(x)\mu(x)^{1/2}$ belongs to $H^s(\mathbb{R}^n)$. If s is integer, it coincides with the space of all the functions $f \in H_{loc}^s(\mathbb{R}^n)$ such that $D^\beta f$ belongs to L_μ^2 for every multi-index β such that $0 \leq |\beta| \leq s$.

We shall see that the semigroup defined by (1.2) is analytic, strongly continuous, positivity preserving, and it is a contraction semigroup in L_μ^2 . Moreover, the

measure $\mu(x)dx$ is invariant for $T(t)$, in the sense that

$$(1.8) \quad \int_{\mathbb{R}^n} (T(t)\varphi)(x)\mu(x)dx = \int_{\mathbb{R}^n} \varphi(x)\mu(x)dx \quad \forall t \geq 0, \varphi \in L^2_\mu.$$

The main result of the paper is the characterization of the domain of the realization A of $\mathcal{A}$ in L^2_μ . We prove that $D(A) = H^2_\mu$. The embedding $H^2_\mu \subset D(A)$ is easy, while proving that $D(A) \subset H^2_\mu$ is more delicate. Indeed, due to the strong decay of the weight $\mu(x)$ as $|x| \rightarrow \infty$, there are difficulties in treating differential operators in L^2_μ by the usual methods. We use a technique similar to the one employed in [8] to get optimal Schauder type estimates: we show that for every $\alpha \in (0, 1)$, $D(A)$ is continuously embedded in the interpolation space

$$(H^\alpha_\mu, H^{2+\alpha}_\mu)_{1-\alpha/2, 2} = H^2_\mu.$$

This is done by using the representation formula for the resolvent $R(\lambda, A)$,

$$R(\lambda, A)\varphi = \int_0^\infty e^{-\lambda t} T(t)\varphi dt, \quad \lambda > 0,$$

and optimal estimates for $\|T(t)\|_{L(L^2_\mu, H^\alpha_\mu)}$, $\|T(t)\|_{L(L^2_\mu, H^{2+\alpha}_\mu)}$, which are obtained with the aid of the explicit representation formula (1.2) and interpolation arguments.

By a similar procedure it is possible to prove that for every $\theta \in (0, 1)$ the domain of the realization of $\mathcal{A}$ in H^θ_μ is $H^{2+\theta}_\mu$.

We consider also the case of matrices Q , B depending on x , with continuous coefficients, such that the limits $\lim_{|x| \rightarrow \infty} Q(x) = Q$, $\lim_{|x| \rightarrow \infty} B(x) = B$ exist, and Q , B satisfy the above assumptions. μ is again the Gaussian weight associated to the matrix Q_∞ defined in (1.5). By using a suitable localization procedure we show that also in this case the domain of the realization A of $\mathcal{A} = \text{Tr}(Q(\cdot)D^2) + \langle B(\cdot), D \rangle$ in L^2_μ is H^2_μ , and that A generates an analytic semigroup in L^2_μ .

Once optimal regularity results for elliptic equations have been established, from the general theory of analytic semigroups one gets easily optimal regularity results for parabolic equations,

$$(1.9) \quad \begin{cases} u_t = \mathcal{A}u + f, & 0 < t < T, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Precisely, one shows that for every $u_0 \in H^1_\mu$ and for every $f \in L^2((0, T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$ the solution u of (1.9) is such that u , u_t , $D_i u$, $D_{ij} u$ ($i, j = 1, \dots, n$) belong to $L^2((0, T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$.

2. THE SPACES L^2_μ AND H^s_μ

The space L^2_μ has been defined in (1.7). If there is no danger of confusion we shall write $\|f\|$ instead of $\|f\|_{L^2_\mu}$ for every $f \in L^2_\mu$.

We may assume, possibly changing coordinates, that

$$Q_\infty = \text{diag}(\lambda_1, \dots, \lambda_n),$$

with $\lambda_i > 0$ for $i = 1, \dots, n$. Let Γ be the set of all multi-indexes $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_i \in \mathbb{N} \cup \{0\}$. For every $\gamma \in \Gamma$ let H_γ be the Hermite polynomial in $\mathbb{R}^n$ associated to the matrix Q_∞ , defined by

$$H_\gamma(x) = \prod_{k=1}^n H_{\gamma_k} \left(\frac{x_k}{\sqrt{\lambda_k}} \right), \quad x \in \mathbb{R}^n,$$

where for every nonnegative integer r , H_r is the one dimensional r -th Hermite polynomial

$$H_r(x) = \frac{(-1)^r}{\sqrt{r!}} e^{x^2/2} \frac{d^r}{dx^r} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

It is not hard to see that the set $\{H_\gamma : \gamma \in \Gamma\}$ is an orthonormal basis in L_μ^2 .

For $s \in \mathbb{N}$ we define the spaces H_μ^s as

$$H_\mu^s = \{f \in L_\mu^2 : \forall |\beta| \leq s \exists D^\beta f \in L_\mu^2\}, \quad \|f\|_{H_\mu^s} = \sum_{|\beta| \leq s} \|D^\beta f\|_{L_\mu^2}.$$

For notational convenience we set also

$$H_\mu^0 = L_\mu^2.$$

Using the equalities

$$D_h H_\gamma(x) = \sqrt{\frac{\gamma_h}{\lambda_h}} H_{\gamma_h-1} \left(\frac{x_h}{\sqrt{\lambda_h}} \right) \prod_{k \neq h} H_{\gamma_k} \left(\frac{x_k}{\sqrt{\lambda_k}} \right), \quad \text{if } \gamma \in \Gamma, \gamma_h > 0$$

$$D_h H_\gamma(x) = 0, \quad \text{if } \gamma \in \Gamma, \gamma_h = 0,$$

($h = 1, \dots, n$) one checks that a function $\varphi \in L_\mu^2$, $\varphi = \sum_{\gamma \in \Gamma} \varphi_\gamma H_\gamma$, is differentiable with respect to x_h with derivative $D_h \varphi \in L_\mu^2$ if and only if

$$\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \gamma_h < \infty,$$

in which case we have

$$D_h \varphi(x) = \sum_{\gamma \in \Gamma, \gamma_h > 0} \varphi_\gamma \sqrt{\frac{\gamma_h}{\lambda_h}} H_{\gamma_h-1} \left(\frac{x_h}{\sqrt{\lambda_h}} \right) \prod_{k \neq h} H_{\gamma_k} \left(\frac{x_k}{\sqrt{\lambda_k}} \right),$$

and

$$\|D_h \varphi\|_{L_\mu^2}^2 = \frac{1}{\lambda_h} \sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \gamma_h.$$

Moreover one can see that for every $s \in \mathbb{N}$ a function φ belongs to H_μ^s if and only if

$$\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \left(\sum_{h=1}^n \gamma_h \right)^s < \infty,$$

and that the norm

$$\varphi \mapsto \left(\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \left(\sum_{h=1}^n \gamma_h \right)^s \right)^{1/2}$$

is equivalent to the norm of H_μ^s .

A useful property of the space H_μ^1 is the following.

Lemma 2.1. *If φ is differentiable with respect to x_h and $D_h\varphi \in L_\mu^2$ then $x \mapsto \psi(x) = x_h\varphi(x)$ belongs to L_μ^2 , with norm less than $\text{const} \cdot (\|\varphi\|_{L_\mu^2} + \|D_h\varphi\|_{L_\mu^2})$. Consequently, if $a : \mathbb{R}^n \mapsto \mathbb{R}$ is any linear function, the mapping $\varphi \mapsto a\varphi$ is bounded from H_μ^1 to L_μ^2 .*

Proof. It is sufficient to show that for every polynomial φ we have

$$\int_{\mathbb{R}^n} (\psi(x))^2 \mu(x) dx \leq C(\|\varphi\|^2 + \|D_h\varphi\|^2).$$

If φ is a polynomial, then

$$\begin{aligned} \int_{\mathbb{R}^n} (x_h\varphi(x))^2 \mu(x) dx &= \int_{\mathbb{R}^n} x_h\varphi(x)^2 (-\lambda_h/2)(D_h\mu(x)) dx \\ &= \frac{\lambda_h}{2} \int_{\mathbb{R}^n} (2x_h\varphi(x)D_h\varphi(x) + \varphi(x)^2)\mu(x) dx \\ &\leq \frac{\lambda_h}{2} \|\varphi\|^2 + \lambda_h \left(\int_{\mathbb{R}^n} (x_h\varphi(x))^2 \mu(x) dx \right)^{1/2} \|D_h\varphi\| \\ &\leq \frac{\lambda_h}{2} \|\varphi\|^2 + \frac{1}{2} \int_{\mathbb{R}^n} (x_h\varphi(x))^2 \mu(x) dx + \frac{\lambda_h^2}{2} \|D_h\varphi\|^2. \end{aligned}$$

Therefore,

$$\|\psi\|^2 \leq \lambda_h(\|\varphi\|^2 + \lambda_h\|D_h\varphi\|^2),$$

and the statement follows. $\square$

In the next sections we shall use an explicit characterization of the interpolation spaces $(L_\mu^2, H_\mu^s)_{\theta,2}$. To this aim we define the spaces H_μ^s for $s > 0$ not integer. We set

$$\begin{cases} H_\mu^s = \{f \in L_\mu^2 : x \mapsto f(x) \exp(-\langle Q_\infty x, x \rangle/4) \in H^s(\mathbb{R}^n)\}, \\ \|f\|_{H_\mu^s} = \|f \exp(-\langle Q_\infty^2 \cdot, \cdot \rangle/4)\|_{H^s(\mathbb{R}^n)}. \end{cases}$$

We define the strongly continuous semigroups $T_h(t)$, $h = 1, \dots, n$, in L_μ^2 by

$$T_h(t)\varphi(x) = \varphi(x + te_h) \exp(-(t^2 + 2tx_h)/4\lambda_h), \quad x \in \mathbb{R}^n, \quad t \geq 0.$$

The infinitesimal generator A_h of $T_h(t)$ is the operator defined by

$$\begin{cases} D(A_h) = \{\varphi \in L_\mu^2 : \exists D_h(\varphi e^{-x_h^2/4\lambda_h}), \quad x \mapsto e^{x_h^2/4\lambda_h} D_h(\varphi e^{-x_h^2/4\lambda_h}) \in L_\mu^2\}, \\ A_h\varphi(x) = e^{x_h^2/4\lambda_h} D_h(\varphi(x) e^{-x_h^2/4\lambda_h}) = D_h\varphi - \frac{x_h}{2\lambda_h} \varphi. \end{cases}$$

Lemma 2.2. *For every $h = 1, \dots, n$ and $m \in \mathbb{N}$ we have*

$$D(A_h^m) = \{\varphi \in L_\mu^2 : \exists \partial^m / \partial x_h^m \varphi \in L_\mu^2\},$$

and the graph norm of A_h^m is equivalent to

$$\varphi \mapsto \sum_{k=0}^m \|\partial^k / \partial x_h^k \varphi\|_{L_\mu^2}.$$

Proof. Let us prove that the statement holds for $m = 1$. If $\varphi \in L_\mu^2$ is differentiable with respect to x_h with derivative in L_μ^2 , by Lemma 2.1 φ belongs to $D(A_h)$. Conversely, if φ is a polynomial then

$$\begin{aligned} A_h \varphi(x) &= \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \left(\sum_{m \geq 0} \varphi_{\tilde{\gamma}, m+1} \sqrt{m+1} (1/\sqrt{\lambda_h} + 1/\lambda_h) - \sum_{m \geq 2} \varphi_{\tilde{\gamma}, m-1} \sqrt{m}/\lambda_h \right) \\ &\quad \cdot H_m \left(\frac{x_h}{\sqrt{\lambda_h}} \right) \prod_{k=1}^{n-1} H_{\gamma_k} \left(\frac{x_k}{\sqrt{\lambda_k}} \right), \end{aligned}$$

where $\tilde{\Gamma}$ is the set of all multi-indexes in $(\mathbb{N} \cup \{0\})^{n-1}$ and $\varphi_{\tilde{\gamma}, m+1}$ is the coefficient corresponding to the multi-index $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{h-1}, m, \tilde{\gamma}_h, \dots, \tilde{\gamma}_{n-1})$. Therefore,

$$\begin{aligned} \|A_h \varphi(x)\|^2 &= \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq 2} \left(\varphi_{\tilde{\gamma}, m+1} \sqrt{m+1} (1/\sqrt{\lambda_h} + 1/\lambda_h) - \varphi_{\tilde{\gamma}, m-1} \sqrt{m}/\lambda_h \right)^2 \\ &\quad + (1/\sqrt{\lambda_h} + 1/\lambda_h)^2 \sum_{\tilde{\gamma} \in \tilde{\Gamma}} (\varphi_{\tilde{\gamma}, 1}^2 + \sqrt{2} \varphi_{\tilde{\gamma}, 2}^2) \\ &= \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq 2} a_{\tilde{\gamma}, m}^2 + (1/\sqrt{\lambda_h} + 1/\lambda_h)^2 \sum_{\tilde{\gamma} \in \tilde{\Gamma}} (\varphi_{\tilde{\gamma}, 1}^2 + \sqrt{2} \varphi_{\tilde{\gamma}, 2}^2). \end{aligned}$$

For every $m \geq 2$ and $\varepsilon > 0$ we have

$$\begin{aligned} \left((1/\sqrt{\lambda_h} + 1/\lambda_h) \varphi_{\tilde{\gamma}, m+1} \sqrt{m+1} \right)^2 &= (a_{\tilde{\gamma}, m} + \varphi_{\tilde{\gamma}, m-1} \sqrt{m}/\lambda_h)^2 \\ &\leq (1 + 1/\varepsilon) a_{\tilde{\gamma}, m}^2 + (1 + \varepsilon) \varphi_{\tilde{\gamma}, m-1}^2 \frac{m}{\lambda_h^2}, \end{aligned}$$

so that for every $M \geq 2$

$$\begin{aligned} &(1/\sqrt{\lambda_h} + 1/\lambda_h)^2 \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq M} \varphi_{\tilde{\gamma}, m+1}^2 (m+1) \\ &\leq (1 + 1/\varepsilon) \|A_h \varphi\|^2 + C(M) \|\varphi\|^2 + (1 + \varepsilon) \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq M} \varphi_{\tilde{\gamma}, m-1}^2 \frac{m}{\lambda_h^2} \\ &\leq (1 + 1/\varepsilon) \|A_h \varphi\|^2 + C(M) \|\varphi\|^2 \\ &\quad + (1 + \varepsilon) \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \lambda_h^{-2} \sum_{m \geq M-2} \varphi_{\tilde{\gamma}, m+1}^2 (m+1) \frac{m+2}{m+1}, \end{aligned}$$

which implies

$$\begin{aligned} &\sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq M} \varphi_{\tilde{\gamma}, m+1}^2 (m+1) \\ &\leq C(\varepsilon, M) (\|A_h \varphi\|^2 + \|\varphi\|^2) + \frac{1 + \varepsilon}{(\sqrt{\lambda_h} + 1)^2} \frac{M+2}{M+1} \sum_{m \geq M-2} \varphi_{\tilde{\gamma}, m+1}^2 (m+1). \end{aligned}$$

Taking ε small and M large in such a way that

$$\frac{1 + \varepsilon}{(\sqrt{\lambda_h} + 1)^2} \frac{M + 2}{M + 1} < 1$$

we get

$$\sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq 0} \varphi_{\tilde{\gamma}, m+1}^2(m+1) \leq K(\|A_h \varphi\|^2 + \|\varphi\|^2).$$

Since the set of all polynomials is dense in $D(A_h)$, the statement is proved for $m = 1$. Arguing by recurrence, one can prove that the statement holds for every m . $\square$

The characterization of the domain of A_h^m allows us to characterize the interpolation spaces $(L_\mu^2, H_\mu^m)_{\theta, 2}$.

Proposition 2.3. *For every $m \in \mathbb{N}$ and $\theta \in (0, 1)$ we have*

$$(L_\mu^2, H_\mu^m)_{\theta, 2} = H_\mu^{\theta m},$$

with equivalence of the respective norms.

Proof. Following [11, §1.13.3] we set

$$K^m = \bigcap_{0 \leq k \leq m, 1 \leq h \leq n} D(A_h^k).$$

By Lemma 2.2 we have $K^m = H_\mu^m$. By [11, Thm. 1.13.6.1] we have

$$(L_\mu^2, H_\mu^m)_{\theta, 2} = \{f \in L_\mu^2 : |||f||| < \infty\},$$

where

$$|||f||| = \|f\|_{L_\mu^2} + \left(\int_{[0,1]^n} |y|^{-2\theta m - n} \left\| \left(\prod_{s=1}^n T_s(y_s) - I \right)^m f \right\|_{L_\mu^2}^2 dt \right)^{1/2}.$$

For every $y \in \mathbb{R}^n$ we have

$$\left(\prod_{s=1}^n T_s(y_s) f \right)(x) = f(x+y) e^{-\langle Q_\infty^{-1} y, y \rangle + 2\langle Q_\infty^{-1} y, x \rangle / 4}, \quad x \in \mathbb{R}^n,$$

so that

$$\left[\left(\prod_{s=1}^n T_s(y_s) - I \right)^m f \right](x) e^{-\langle Q_\infty^{-1} x, x \rangle / 4} = \sum_{k=0}^m \binom{m}{k} (-1)^k \varphi(x + ky),$$

where

$$\varphi(x) = f(x) e^{-\langle Q_\infty^{-1} x, x \rangle / 4}.$$

By [11, §2.5.1] we get $|||f||| < \infty$ iff $\varphi \in H^{\theta m}(\mathbb{R}^n)$, and the statement follows. $\square$

3. PROPERTIES OF $T(t)$

The measure $\mu(x)dx$ is invariant for the semigroup $T(t)$, in the sense specified by the following lemma.

Lemma 3.1. *For every $f \in L_\mu^1$ and $t > 0$ we have*

$$(3.1) \quad \int_{\mathbb{R}^n} (T(t)f)(x) \mu(x) dx = \int_{\mathbb{R}^n} f(x) \mu(x) dx.$$

Proof. Since the set of the functions $x \mapsto e^{i\langle h, x \rangle}$ is dense in L^1_μ , it is sufficient to prove that for every $h \in \mathbb{R}^n$ we have

$$(3.2) \quad \int_{\mathbb{R}^n} (T(t)e^{i\langle h, \cdot \rangle})(x) \mu(x) dx = \int_{\mathbb{R}^n} e^{i\langle h, x \rangle} \mu(x) dx.$$

To this aim we remark that

$$(T(t)e^{i\langle h, \cdot \rangle})(x) = e^{i\langle h, e^{tB}x \rangle - \langle Q_t h, h \rangle / 2}, \quad t > 0,$$

and

$$\int_{\mathbb{R}^n} e^{i\langle h, x \rangle} \mu(x) dx = e^{-\langle Q_\infty h, h \rangle / 2}.$$

So, for $t > 0$ we have

$$\int_{\mathbb{R}^n} (T(t)e^{i\langle h, \cdot \rangle})(x) \mu(x) dx = e^{-\langle \langle Q_t h, h \rangle + \langle e^{tB} Q_\infty e^{tB^*} h, h \rangle \rangle / 2}.$$

Taking into account that

$$\begin{aligned} Q_t + e^{tB} Q_\infty e^{tB^*} &= \int_0^t e^{sB} Q e^{sB^*} ds + \int_0^\infty e^{(t+s)B} Q e^{(t+s)B^*} ds \\ &= \int_0^t e^{sB} Q e^{sB^*} ds + \int_t^\infty e^{sB} Q e^{sB^*} ds = Q_\infty, \end{aligned}$$

(3.2) follows. $\square$

Estimates for $T(t)f$ and its derivatives are provided by the following lemma.

Lemma 3.2. *For every $f \in L^2_\mu$ and $t > 0$ we have*

$$(3.3) \quad \|T(t)f\|_{L^2_\mu} \leq \|f\|_{L^2_\mu},$$

$$(3.4) \quad \|D^\beta T(t)f\|_{L^2_\mu} \leq \frac{C}{t^{|\beta|/2}} \|f\|_{L^2_\mu}, \quad |\beta| \leq 3.$$

Proof. Using the Hölder inequality in formula (1.2) we get

$$|(T(t)f)(x)|^2 \leq (T(t)f^2)(x), \quad x \in \mathbb{R}^n,$$

and (3.3) follows from (3.1).

Moreover, setting $\mu_t(y) = (2\pi)^{-n/2} (\det Q_t)^{-1/2} e^{-\langle Q_t^{-1}y, y \rangle / 2}$, for every $t > 0$ we have

$$(DT(t)f)(x) = - \int_{\mathbb{R}^n} e^{tB^*} Q_t^{-1} y f(e^{tB}x + y) \mu_t(y) dy.$$

By the Hölder inequality,

$$\begin{aligned} \|D_i T(t)f\|^2 &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle Q_t^{-1/2} e^{tB} e_i, Q_t^{-1/2} y \rangle|^2 \mu_t(y) dy \right. \\ &\quad \left. \cdot \int_{\mathbb{R}^n} (f(e^{tB}x + y))^2 \mu_t(y) dy \right) \mu(x) dx \\ &\leq |Q_t^{-1/2} e^{tB} e_i|^2 \int_{\mathbb{R}^n} (T(t)f^2)(x) \mu(x) dx = |Q_t^{-1/2} e^{tB} e_i|^2 \|f\|_{L^2_\mu}^2 \leq Ct^{-1} \|f\|_{L^2_\mu}^2, \end{aligned}$$

so that (3.4) holds for $|\beta| = 1$.

To estimate the second order derivatives we remark that for every regular $\varphi \in L_\mu^2$ we have

$$DT(t)\varphi = e^{tB^*}T(t)D\varphi, \quad t > 0.$$

It follows that for $i, j = 1, \dots, n$ we have

$$\begin{aligned} \|D_{ij}T(t)f\| &= \|D_i(D_jT(t/2)T(t/2)f)\| = \|D_i(e^{tB^*/2}T(t/2)DT(t/2)f)_j\| \\ &\leq C\|DT(t/2)f\|_{(L_\mu^2)^n}\|DT(t/2)\|_{L(L_\mu^2, (L_\mu^2)^n)} \leq Ct^{-1}\|f\|, \end{aligned}$$

and (3.4) follows for $|\beta| = 2$. The proof for $|\beta| = 3$ is similar. $\square$

From (3.3) and (3.4) it follows by interpolation that for $0 < t < 1$, $0 < \alpha \leq 3$

$$(3.5) \quad \|T(t)\|_{L(L_\mu^2, H_\mu^\alpha)} \leq Ct^{-\alpha/2}.$$

This estimate is not optimal for t near 0, and it will be improved later. However, we are going to use it in the next proposition to characterize the interpolation spaces $D_A(\theta, 2)$, A being the infinitesimal generator of $T(t)$.

It is not difficult to see that the semigroup $T(t)$ is analytic in L_μ^2 . In [4, §2.2] one can find a proof which is an adaptation to the finite dimensional case of a result for equations in infinitely many variables due to [7]. Here we give a simple direct proof.

Proposition 3.3. *The semigroup $T(t)$ defined in (1.2) is analytic in L_μ^2 .*

Proof. By estimates (3.4) with $|\beta| = 2$, for every $f \in L_\mu^2$ and $t > 0$ we have

$$\left\| \sum_{i,j=1}^n q_{ij}D_{ij}T(t)f \right\|_{L_\mu^2} \leq \frac{C}{t} \|f\|_{L_\mu^2}.$$

Moreover, due to Lemma 2.1, $x \mapsto \langle Bx, DT(t)f \rangle$ belongs to L_μ^2 and by estimates (3.4) we have

$$\|\langle B \cdot, DT(t)f \rangle\|_{L_\mu^2} \leq C\|T(t)f\|_{H_\mu^1} \leq C(1 + t^{-1/2} + t^{-1})\|f\|_{L_\mu^2}.$$

Therefore, for every $f \in L_\mu^2$ and $t > 0$

$$\frac{\partial}{\partial t} T(t)f = \mathcal{A}T(t)f \in L_\mu^2, \quad \left\| \frac{\partial}{\partial t} T(t)f \right\|_{L_\mu^2} \leq C(1 + t^{-1/2} + t^{-1})\|f\|_{L_\mu^2}.$$

It follows that $t \mapsto T(t)f$ is differentiable for $t > 0$ with values in L_μ^2 , and

$$\left\| \frac{d}{dt} T(t)f \right\|_{L_\mu^2} = \|\mathcal{A}T(t)f\|_{L_\mu^2} \leq C(1 + t^{-1/2} + t^{-1})\|f\|_{L_\mu^2}, \quad t > 0.$$

Since $\|T(t)f\|_{L_\mu^2} \leq \|f\|_{L_\mu^2}$ by (3.3), then $T(t)$ is an analytic semigroup. $\square$

We recall that if X is any Banach space and $A : D(A) \subset X \mapsto X$ generates an analytic semigroup $T(t)$ in X , for $0 \leq \theta < 1$ the space $D_A(\theta, 2)$ is defined by

$$\begin{cases} D_A(\theta, 2) = \{f \in X : [f]_{\theta,2} = \int_0^1 t^{1-2\theta} \|\mathcal{A}T(t)f\|_X^2 dt < \infty\}, \\ \|f\|_{D_A(\theta,2)} = \|f\|_X + [f]_{\theta,2}. \end{cases}$$

It is well known that for $\theta \in (0, 1)$ the space $D_A(\theta, 2)$ coincides with the interpolation space $(X, D(A))_{\theta,2}$, with equivalence of the respective norms. In the next proposition we characterize such spaces.

Proposition 3.4. *We have*

$$D_A(\theta, 2) = \begin{cases} L_\mu^2, & \theta = 0, \\ H_\mu^{2\theta}, & 0 < \theta < 1, \end{cases}$$

with equivalence of the respective norms.

Proof. Let $0 < \theta < 1$. Since H_μ^2 is continuously embedded in $D(A)$ by Lemma 2.1, then $H_\mu^{2\theta} = (L_\mu^2, H_\mu^2)_{\theta, 2}$ is continuously embedded in $D_A(\theta, 2)$. We are going to show that for every $\alpha \in (0, 1)$, $H_\mu^{2\alpha}$ belongs to the class J_α between L_μ^2 and $D(A)$, i.e.

$$(3.6) \quad \|f\|_{H_\mu^{2\alpha}} \leq C \|f\|_{L_\mu^2}^{1-\alpha} \|f\|_{D(A)}^\alpha, \quad \forall f \in D(A).$$

Indeed, given any $f \in D(A)$, for $\lambda > 0$ we have, due to (3.5),

$$\begin{aligned} \|f\|_{H_\mu^{2\alpha}} &= \left\| \int_0^\infty e^{-\lambda t} T(t) (\lambda f - Af) dt \right\|_{H_\mu^{2\alpha}} \leq C \int_0^\infty e^{-\lambda t} t^{-\alpha} dt \|\lambda f - Af\|_{L_\mu^2} \\ &\leq C' (\lambda^\alpha \|f\|_{L_\mu^2} + \lambda^{\alpha-1} \|Af\|_{L_\mu^2}). \end{aligned}$$

Taking the minimum for $\lambda > 0$ we get (3.6). Then we may apply the Reiteration Theorem ([11, §1.10]) to get, for $0 < \theta < \alpha < 1$,

$$D_A(\theta, 2) = (L_\mu^2, D(A))_{\theta, 2} \subset (L_\mu^2, H_\mu^{2\alpha})_{\theta/\alpha, 2} = H_\mu^{2\theta},$$

and the statement follows for $0 < \theta < 1$.

Let now $\theta = 0$. We remark preliminarily that if X is a Banach space and A generates a bounded analytic semigroup $T(t)$ in X , then $X = D_A(0, 2)$ if and only if $D(A) = (X, D(A^2))_{1/2, 2}$. Indeed, we may replace A by $\tilde{A} = A - I$, $T(t)$ by $\tilde{T}(t) = T(t)e^{-t}$ and we get $D_A(0, 2) = D_{\tilde{A}}(0, 2)$, $(X, D(\tilde{A}^2))_{1/2, 2} = (X, D(A^2))_{1/2, 2}$. By [11, §1.14.5],

$$(X, D(\tilde{A}^2))_{1/2, 2} = \{f \in X : |||f||| = \int_0^1 t \|\tilde{A}^2 T(t)f\|^2 dt < \infty\},$$

and the norm $|||\cdot|||$ is equivalent to the norm of $(X, D(A^2))_{1/2, 2}$. Therefore, $(X, D(\tilde{A}^2))_{1/2, 2} = D(\tilde{A})$ means that a function f belongs to $D(\tilde{A})$ if and only if $|||f|||$ is finite. Since $\tilde{A}$ is invertible, applying $\tilde{A}^{-1}$ we get that a function g belongs to X if and only if

$$\int_0^1 t \|\tilde{A} T(t)g\|^2 dt < \infty,$$

which means that $g \in D_{\tilde{A}}(0, 2)$.

So, it is sufficient to prove that $(X, D(A^2))_{1/2, 2} = D(A)$. To this aim, we remark that $T(t)$ is a contraction semigroup so that A is m-accretive and it admits bounded imaginary powers (see e.g. [10, §2]). Consequently, by [11, §1.15.3] we get $D(A) = [X, D(A^2)]_{1/2}$ (complex interpolation). Since in our case X and $D(A^2)$ are Hilbert spaces, then $[X, D(A^2)]_{1/2} = (X, D(A^2))_{1/2, 2}$ with equivalence of the norms. The statement follows. $\square$

Once the spaces H_μ^α have been characterized as interpolation spaces we may improve estimates (3.5). We shall state just the estimates we need for the sequel. We shall use the following lemma.

Lemma 3.5. *Let A be the generator of an analytic semigroup $T(t)$ in a Banach space X , and let $0 \leq \theta < 1$. Then for every $f \in D_A(\theta, 2)$ and $\theta < \alpha < 1$*

$$\int_0^1 t^{2(\alpha-\theta)-1} \|T(t)f\|_{D_A(\alpha,2)}^2 dt \leq C \|f\|_{D_A(\theta,2)}.$$

Proof. Since $\|T(t)\|_{L(D_A(\theta,2),X)}$ is bounded in $(0,1)$, it is sufficient to prove that $\int_0^1 t^{2(\alpha-\theta)-1} [T(t)f]_{D_A(\alpha,2)}^2 dt$ is bounded by $C \|f\|_{D_A(\theta,2)}^2$. Indeed,

$$\begin{aligned} \int_0^1 t^{2(\alpha-\theta)-1} [T(t)f]_{D_A(\alpha,2)}^2 dt &= \int_0^1 t^{2(\alpha-\theta)-1} \int_0^1 \xi^{1-2\alpha} \|AT(t+\xi)f\|^2 d\xi \\ &= \int_0^1 t^{2(\alpha-\theta)-1} \int_t^{t+1} (s-t)^{1-2\alpha} \|AT(s)f\|^2 ds \\ &\leq \int_0^2 \|AT(s)f\|^2 \int_0^s t^{2(\alpha-\theta)-1} (s-t)^{1-2\alpha} dt ds \\ &= \int_0^2 s^{1-2\theta} \|AT(s)f\|^2 \int_0^1 \sigma^{2(\alpha-\theta)-1} (1-\sigma)^{1-2\alpha} d\sigma ds \\ &= C [f]_{D_A(\theta,2)}^2. \quad \square \end{aligned}$$

Corollary 3.6. *Let $0 \leq \theta < \alpha < 1$, and let $T(t)$ be the semigroup defined in (1.2). There exists $C > 0$ such that for every $f \in H_\mu^\theta$ we have*

$$(3.7) \quad \int_0^1 t^{\alpha-\theta-1} \|T(t)f\|_{H_\mu^\alpha}^2 dt \leq C \|f\|_{H_\mu^\theta}^2,$$

$$(3.8) \quad \int_0^1 t^{\alpha-\theta+1} \|T(t)f\|_{H_\mu^{2+\alpha}}^2 dt \leq C \|f\|_{H_\mu^\theta}^2.$$

Proof. Estimate (3.7) is an immediate consequence of Proposition 3.4 and Lemma 3.5.

To prove (3.8) we remark that Proposition 3.4 and estimate (3.5) imply that $\|T(t)\|_{L(H_\mu^\alpha, H_\mu^{2+\alpha})} \leq Ct^{-1}$ for $0 < \alpha < 1$, $0 < t < 1$, so that

$$\begin{aligned} &\int_0^1 t^{\alpha-\theta+1} \|T(t)f\|_{H_\mu^{2+\alpha}}^2 dt \\ &\leq \int_0^1 t^{\alpha-\theta+1} \|T(t/2)\|_{L(H_\mu^\alpha, H_\mu^{2+\alpha})}^2 \|T(t/2)f\|_{H_\mu^\alpha}^2 dt \\ &\leq C \int_0^1 t^{\alpha-\theta-1} \|T(t/2)f\|_{H_\mu^\alpha}^2 dt \leq C \|f\|_{H_\mu^\theta}^2. \quad \square \end{aligned}$$

4. CHARACTERIZATION OF THE DOMAIN OF A

The main result of the paper is the following theorem.

Theorem 4.1. *Let $\mathcal{A}$ be the operator defined by (1.1) and let A, A_θ be the realizations of $\mathcal{A}$ in L_μ^2 and in H_μ^θ , respectively ($0 < \theta < 1$). Then*

$$D(A) = H_\mu^2, \quad D(A_\theta) = H_\mu^{2+\theta},$$

with equivalence of the respective norms.

Proof. Let us prove the embeddings $\subset$. By Proposition 3.4, $L_\mu^2 = D_A(0, 2)$ and $H_\mu^\theta = D_A(\theta/2, 2)$. So, it is sufficient to show that if $f \in D(A)$ is such that $Af \in D_A(\theta/2, 2)$ with $0 \leq \theta < 1$, then $f \in H_\mu^{2+\theta}$. By Proposition 2.3 and the Reiteration Theorem ([11, §1.10]) we have

$$H_\mu^{2+\theta} = (H_\mu^\alpha, H_\mu^{2+\alpha})_{1-(\alpha-\theta)/2, 2}, \quad \forall \alpha \in (\theta, 1).$$

So, we fix once and for all $\alpha \in (\theta, 1)$ and we prove that

$$(4.1) \quad f \in (H_\mu^\alpha, H_\mu^{2+\alpha})_{1-(\alpha-\theta)/2, 2}.$$

Let $\lambda > 0$, and set $\varphi = \lambda f - Af$. Then

$$f = \int_0^\infty e^{-\lambda t} T(t) f dt.$$

We recall that if X, Y are Banach spaces such that $Y \subset X$, the interpolation space $(X, Y)_{1-(\alpha-\theta)/2, 2}$ is the set of all $f \in X$ such that the function

$$\xi \mapsto k(\xi, f) = \xi^{(-3+\alpha-\theta)/2} \inf_{f=a+b, a \in X, b \in Y} (\|a\|_X + \xi \|b\|_Y)$$

belongs to $L^2(0, 1)$, and the $(X, Y)_{1-(\alpha-\theta)/2, 2}$ norm is equivalent to $\|k(\cdot, f)\|_{L^2(0, 1)}$. In our case it is convenient to split f as $f = a(\xi) + b(\xi)$, where

$$a(\xi) = \int_0^\xi e^{-\lambda t} T(t) f dt, \quad b(\xi) = \int_\xi^\infty e^{-\lambda t} T(t) f dt, \quad 0 \leq \xi \leq 1.$$

By the Hardy-Young inequality and estimate (3.7) we get

$$\begin{aligned} \int_0^1 \xi^{\alpha-\theta-3} \|a(\xi)\|_{H_\mu^\alpha}^2 d\xi &\leq \int_0^1 \xi^{\alpha-\theta-3} \left(\int_0^\xi e^{-\lambda t} \|T(t) f\|_{H_\mu^\alpha} dt \right)^2 d\xi \\ &\leq \left(1 - \frac{\alpha-\theta}{2} \right)^{-2} \int_0^1 t^{\alpha-\theta-1} \|T(t) f\|_{H_\mu^\alpha}^2 dt \leq C \|f\|_{D_A(\theta/2, 2)}^2. \end{aligned}$$

By the Hardy-Young inequality and estimate (3.8) we get

$$\begin{aligned} \int_0^1 \xi^{\alpha-\theta-1} \|b(\xi)\|_{H_\mu^{2+\alpha}}^2 d\xi &\leq \int_0^1 \xi^{\alpha-\theta-1} \left(\int_\xi^\infty e^{-\lambda t} \|T(t) f\|_{H_\mu^{2+\alpha}} dt \right)^2 d\xi \\ &\leq \left(\frac{\alpha-\theta}{2} \right)^{-2} \int_0^\infty t^{\alpha-\theta+1} \|T(t) f\|_{H_\mu^{2+\alpha}}^2 dt \leq C \|f\|_{D_A(\theta/2, 2)}^2. \end{aligned}$$

Since

$$k(\xi, f) \leq \xi^{(-3+\alpha-\theta)/2} (\|a(\xi)\|_{H_\mu^\alpha} + \xi \|b(\xi)\|_{H_\mu^{2+\alpha}}),$$

we get

$$\|k(\cdot, f)\|_{L^2(0, 1)} \leq C \|f\|_{D_A(\theta/2, 2)},$$

and (4.1) follows.

The embedding $H_\mu^2 \subset D(A)$ is an obvious consequence of Lemma 2.1. Moreover, by Lemma 2.1, for every linear function a the mapping $\varphi \mapsto a\varphi$ is bounded from H_μ^1 to L_μ^2 . Consequently, it is bounded from H_μ^2 to H_μ^1 . By interpolation, it is bounded from $H_\mu^{\theta+1}$ to H_μ^θ for every $\theta \in (0, 1)$. Therefore, $\varphi \mapsto \langle B\cdot, D\varphi \rangle \in L(H_\mu^{\theta+2}, H_\mu^\theta)$, so that $H_\mu^{\theta+2}$ is continuously embedded in $D(A_\theta)$. $\square$

5. THE CASE OF COEFFICIENTS DEPENDING ON x

Let us consider now the case of coefficients q_{ij} , b_{ij} depending on x . The assumptions on the coefficients are the following:

$$(5.1) \quad \begin{cases} q_{ij}, b_{ij} \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), & q_{ij} = q_{ji}, \quad i, j = 1, \dots, n, \\ \sum_{i,j=1}^n q_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n, \end{cases}$$

for some $\nu > 0$. Moreover,

$$(5.2) \quad \exists \lim_{|x| \rightarrow \infty} q_{ij}(x) = q_{ij} \in \mathbb{R}, \quad \exists \lim_{|x| \rightarrow \infty} b_{ij}(x) = b_{ij} \in \mathbb{R}.$$

Setting $Q = [q_{ij}]_{i,j=1,\dots,n}$, $B = [b_{ij}]_{i,j=1,\dots,n}$ we assume that

$$(5.3) \quad B \text{ satisfies (1.4).}$$

Let $\mathcal{A}$ be the differential operator defined by

$$(5.4) \quad \mathcal{A}f = \frac{1}{2} \sum_{i,j=1}^n q_{ij}(x) D_{ij} f(x) + \sum_{i,j=1}^n b_{ij}(x) x_i D_j f(x), \quad x \in \mathbb{R}^n.$$

Let μ be the Gaussian weight associated to the matrix Q_∞ defined in (1.5). Define the operators A_∞ , $A : H_\mu^2 \mapsto L_\mu^2$ by

$$(A_\infty f)(x) = \text{Tr}(Q D^2 f(x)) + \langle Bx, Df(x) \rangle,$$

$$(Af)(x) = \text{Tr}(Q(x) D^2 f(x)) + \langle B(x)x, Df(x) \rangle = \mathcal{A}f(x).$$

Theorem 5.1. *Under assumptions (5.1), (5.2), (5.3), $A : D(A) = H_\mu^2 \mapsto L_\mu^2$ generates an analytic semigroup in L_μ^2 .*

Proof. As a first step we prove that there are $K, \omega > 0$ such that for $\text{Re } \lambda > \omega$ and for every $f \in H_\mu^2$

$$(5.5) \quad |\lambda| \|f\|_{L_\mu^2} + \|f\|_{H_\mu^2} \leq K \|\lambda f - Af\|_{L_\mu^2}.$$

For every $\varepsilon > 0$ let $R > 1$ be such that

$$|q_{ij}(x) - q_{ij}| + |b_{ij}(x) - b_{ij}| \leq \varepsilon \quad \text{for } |x| \geq R - 1.$$

Let θ be a smooth cutoff function such that

$$\begin{cases} \theta \equiv 0 \text{ in } B(0, R-1), & \theta \equiv 1 \text{ outside } B(0, R), \\ |D_i \theta| \leq 1, & |D_{ij} \theta| \leq 1, \quad i, j = 1, \dots, n. \end{cases}$$

Let $\text{Re } \lambda \geq 1$ and let $f \in H_\mu^2$. Then θf satisfies

$$\begin{aligned} \lambda(\theta f)(x) - A_\infty(\theta f)(x) &= \theta(x)(\lambda f(x) - Af(x)) \\ &\quad - \theta \left(\sum_{i,j=1}^n (q_{ij} - q_{ij}(x)) D_{ij} f(x) + \sum_{i,j=1}^n (b_{ij} - b_{ij}(x)) x_j D_i f(x) \right) \\ &\quad - \sum_{i,j=1}^n q_{ij}(f(x) D_{ij} \theta(x) + 2 D_i \theta(x) D_j f(x)) - \sum_{i,j=1}^n b_{ij} x_j f(x) D_i \theta(x) \end{aligned}$$

so that by Theorem 4.1 and Proposition 3.3

(5.6)

$$\begin{aligned} |\lambda| \|\theta f\|_{L_\mu^2} + \|\theta f\|_{H_\mu^2} &\leq C(\|\theta(\lambda f - Af)\|_{L_\mu^2} + \varepsilon\|f\|_{H_\mu^2} + \|f\|_{H_\mu^1}) \\ &\leq C'(\|\theta(\lambda f - Af)\|_{L_\mu^2} + \varepsilon\|f\|_{H_\mu^2} + C(\varepsilon)\|f\|_{L_\mu^2}). \end{aligned}$$

The function $(1 - \theta)f$ vanishes outside $B(0, R)$ and satisfies

$$\begin{aligned} \lambda((1 - \theta)f)(x) - A((1 - \theta)f)(x) &= (1 - \theta(x))(\lambda f(x) - Af(x)) \\ &\quad - \sum_{i,j=1}^n q_{ij}(x)(f(x)D_{ij}\theta(x) + D_i\theta(x)D_jf(x)) \\ &\quad - \sum_{i,j=1}^n b_{ij}(x)x_jf(x)D_i\theta(x). \end{aligned}$$

By the well known *a priori* estimates for elliptic equations with regular coefficients in bounded sets, if $\operatorname{Re} \lambda$ is large enough we have

$$\begin{aligned} |\lambda| \|(1 - \theta)f\|_{L^2(B(0,R))} + \|(1 - \theta)f\|_{H^2(B(0,R))} \\ \leq C(\|(1 - \theta)(\lambda f - Af)\|_{L^2(B(0,R))} + \|f\|_{H^1(B(0,R))} + R\|f\|_{L^2(B(0,R))}), \end{aligned}$$

so that for every $\delta > 0$

(5.7)

$$\begin{aligned} |\lambda| \|(1 - \theta)f\|_{L^2(B(0,R))} + \|(1 - \theta)f\|_{H^2(B(0,R))} \\ \leq C(\|(1 - \theta)(\lambda f - Af)\|_{L^2(B(0,R))} + \delta\|f\|_{H^2(B(0,R))} + C(\delta, R)\|f\|_{L^2(B(0,R))}). \end{aligned}$$

Using (5.6) and (5.7) we get

$$\begin{aligned} |\lambda| \|f\|_{L_\mu^2} + \|f\|_{H_\mu^2} &\leq |\lambda|(\|\theta f\|_{L_\mu^2} + C(R)\|(1 - \theta)f\|_{L^2(B(0,R))}) \\ &\quad + \|\theta f\|_{H_\mu^2} + C(R)\|(1 - \theta)f\|_{H^2(B(0,R))} \\ &\leq C'\varepsilon\|f\|_{H_\mu^2} + C_1(R)(\|\lambda f - Af\|_{L_\mu^2} + \delta\|f\|_{H_\mu^2} + C(\varepsilon, \delta, R)\|f\|_{L^2(B(0,R))}). \end{aligned}$$

Taking ε so small that $C'\varepsilon \leq 1/4$ and then δ so small that $C_1(R)\delta \leq 1/4$ we get

$$|\lambda| \|f\|_{L_\mu^2} + \|f\|_{H_\mu^2} \leq \frac{1}{2}\|f\|_{H_\mu^2} + K(\|\lambda f - Af\|_{L_\mu^2} + \|f\|_{L_\mu^2}),$$

and (5.5) follows.

To conclude, we remark that for $\operatorname{Re} \lambda$ large and for every $g \in L_\mu^2$, the equation

$$\lambda f - Af = g$$

has a unique solution $f \in H_\mu^2$. This can be seen using the continuity method: for every $\varepsilon \in [0, 1]$ consider the problem

$$(5.8) \quad \lambda f - (1 - \varepsilon)A_\infty f - \varepsilon Af = g.$$

Using the *a priori* estimate (5.5) and the fact that A_∞ generates an analytic semi-group it is not hard to see that the set of all ε 's such that (5.8) is uniquely solvable in H_μ^2 is open and closed in $[0, 1]$, so that it coincides with $[0, 1]$. Taking $\varepsilon = 1$ the statement follows. $\square$

6. OPTIMAL REGULARITY IN PARABOLIC PROBLEMS

We consider here the parabolic problem

$$(6.1) \quad \begin{cases} u_t = \mathcal{A}u + f, & 0 < t < T, \ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\mathcal{A}$ is the operator defined in (5.4). We assume that (5.1), (5.2) hold.

We recall that if A is the generator of an analytic semigroup in a Hilbert space H and $f \in L^2(0, T; H)$, $u_0 \in D_A(1/2, 2)$, then the problem

$$\begin{cases} u' = Au + f, & 0 < t < T, \\ u(0) = u_0, \end{cases}$$

has a unique solution $u \in L^2(0, T; D(A)) \cap H^1(0, T; H)$, and

$$\|u\|_{L^2(0, T; D(A))} + \|u\|_{H^1(0, T; H)} \leq C(\|u_0\|_{D_A(1/2, 2)} + \|f\|_{L^2(0, T; H)}).$$

Applying this result to problem (6.1) we get that if $u_0 \in H_\mu^1$ and f belongs to $L^2((0, T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$ then the solution u of (6.1) is such that u and the derivatives u_t , $D_i u$, $D_{ij} u$ ($i, j = 1, \dots, n$) belong to $L^2((0, T) \times \mathbb{R}^n)$ with respect to the measure $dt \times \mu(x)dx$.

REFERENCES

- [1] D.G. Aronson, P. Besala: *Parabolic equations with unbounded coefficients*, J. Diff. Eqns. **3** (1967), 1-14. MR **34**:7970
- [2] P. Besala: *On the existence of a fundamental solution for a parabolic differential equation with unbounded coefficients*, Ann. Polon. Math. **29** (1975), 403-409. MR **54**:10848
- [3] P. Cannarsa, V. Vespri: *Generation of analytic semigroups by elliptic operators with unbounded coefficients*, SIAM J. Math. Anal. **18** (1987), 857-872. MR **88d**:47053
- [4] G. Da Prato, A. Lunardi: *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*, J. Funct. Anal. **131** (1995), 94-114. CMP 95:16
- [5] S. Cerrai: *A Hille-Yosida theorem for weakly continuous semigroups*, Semigroup Forum **49** (1994), 349-367. MR **95f**:47058
- [6] M.I. Freidlin, A.D. Wentzell: *Random perturbations of dynamical systems*, Springer-Verlag, Berlin (1984). MR **85a**:60064
- [7] M. Fuhrman: *Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces*, preprint Dip. Mat. Politecnico di Milano (1993).
- [8] A. Lunardi: *Schauder estimates for a class of elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$* , preprint Sc. Norm. Sup. Pisa (1994).
- [9] A. Lunardi, V. Vespri: *Generation of smoothing semigroups in L^p spaces by elliptic operators with unbounded coefficients*, in preparation.
- [10] J. Prüss, H. Sohr: *On operators with bounded imaginary powers in Banach spaces*, Math. Z. **203** (1990), 429-452. MR **91b**:47030
- [11] H. Triebel: *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam (1978). MR **80i**:46032a

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