INTEGER TRANSLATION OF MEROMORPHIC FUNCTIONS

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ABSTRACT. Let G be a given open set in the complex plane. We prove that there is an entire function such that its integer translations forms a normal family in a neighborhood of z exactly for z in G if and only if G is periodic with period 1, i.e., $z \pm 1 \in G$ for all $z \in G$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{S}$ be a space of functions defined on an open set G . For each positive integer n , we let T_n be an operator from $\mathcal{S}$ into itself. Our concern is the normality of the family $\{T_n(f): n = 0, 1, 2, \dots\}$ for a function $f \in \mathcal{S}$. In this paper, we consider the case $T_n(f(z)) = f(z + n)$, translation by n .

To discuss the normality of families of meromorphic functions, we need the following notion.

Definition 1.1. Let f be a meromorphic function in the complex plane $\mathbf{C}$. We define the *spherical derivative* of f at z by

$$\rho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

Theorem (Marty [6]). *A family $\mathcal{F}$ of functions $f(z)$, meromorphic in a domain D , is normal in D if and only if the set $\{\rho(f(z)): f \in \mathcal{F}\}$ is uniformly bounded on every compact subset of D .*

Now we state our definition and an example.

Definition 1.2. For a meromorphic function f , we define the set $\mathbf{N}(f)$ to be the set of all $z_0 \in \mathbf{C}$ such that the family $\{f(z + n): n = 0, 1, 2, \dots\}$ is normal in a neighborhood of z_0 .

Example 1.3. Let $f(z) = z \sin \pi z$; then $\mathbf{N}(f) = \mathbf{C} \setminus \mathbf{Z}$ where $\mathbf{Z}$ is the set of all integers.

Proof. For all integers n ,

$$f(z + n) = (z + n) \sin \pi(z + n) = (z + n) \sin \pi z;$$

thus we have

$$f(z + n) \begin{cases} = 0, & \text{if } z \in \mathbf{Z}, \\ \rightarrow \infty, & \text{if } z \notin \mathbf{Z} \text{ and } n \rightarrow \infty. \end{cases}$$

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Hence no subsequence of $\{f(z+n): n = 0, 1, 2, \dots\}$ converges on any neighborhood of any integer. But the sequence $\{f(z+n): n = 0, 1, 2, \dots\}$ converges uniformly to infinity on any compact subset which does not contain integers. These facts prove that $\mathbf{N}(f) = \mathbf{C} \setminus \mathbf{Z}$. $\square$

2. SOME PROPERTIES OF $\mathbf{N}(f)$

From the definition, the set $\mathbf{N}(f)$ is open in the complex plane $\mathbf{C}$ and is periodic with period 1.

The set $\mathbf{N}(f)$ may not be connected. For a certain meromorphic function f , the set $\mathbf{N}(f)$ has infinitely many components.

Example 2.1. Let $f(z) = \tan e^z$; then the set $\mathbf{N}(f)$ is not connected.

Proof. We let

$$I_c = \{z \in \mathbf{C}: \operatorname{Im} z = k\pi, k = 0, \pm 1, \pm 2, \dots\},$$

and we shall show that $\mathbf{N}(f) = \mathbf{C} \setminus I_c$. For $z = x + k\pi i \in I_c$, we have

$$|f(z+n)|^2 = |\tan e^{x+n+k\pi i}|^2 = \tan^2 e^{x+n}$$

and

$$|f'(z+n)| = |e^{x+n+k\pi i} \sec^2 e^{x+n+k\pi i}| = e^{x+n} \sec^2 e^{x+n}.$$

So for every real number x with $e^{x+n} \neq k\pi + \frac{\pi}{2}$ ($k = 0, \pm 1, \pm 2, \dots$),

$$\begin{aligned} \rho(f(z+n)) &= \frac{e^{x+n} \sec^2 e^{x+n}}{1 + \tan^2 e^{x+n}} \\ &= \frac{e^{x+n}}{\sin^2 e^{x+n} + \cos^2 e^{x+n}} \\ &= e^{x+n} \rightarrow \infty \end{aligned}$$

as $n \rightarrow \infty$. But for an integer l every neighborhood of $x + l\pi i$ contains a point $x' + l\pi i$ such that $e^{x'+n} \neq k\pi + \frac{\pi}{2}$. Hence by Marty's Theorem, $x + l\pi i \notin \mathbf{N}(f)$ for all real x .

Now for $z_0 \notin I_c$, we choose a positive number ε so that $B(z_0, \varepsilon) \cap I_c = \emptyset$. Let F be a compact subset in $B(z_0, \varepsilon)$; then for $z = x + iy \in F$, we have

$$\begin{aligned} \rho(f(z+n)) &= \frac{|(\tan e^{z+n})'|}{1 + |\tan e^{z+n}|^2} \\ (2.1) \quad &= \frac{|e^{z+n} \sec^2 e^{z+n}|}{1 + |\tan e^{z+n}|^2} \\ &= \frac{|e^{z+n}|}{|\sin e^{z+n}|^2 + |\cos e^{z+n}|^2}. \end{aligned}$$

For $z = x + iy$, it is easy to show

$$|\sin z|^2 + |\cos z|^2 = \frac{1}{2}(e^{2y} + e^{-2y})$$

and since

$$e^{z+n} = e^{x+n} \cos y + ie^{x+n} \sin y,$$

we can write

$$|\sin e^{z+n}|^2 + |\cos e^{z+n}|^2 = \frac{1}{2}(e^{2e^{x+n} \sin y} + e^{-2e^{x+n} \sin y}).$$

Thus from (2.1),

$$(2.2) \quad \begin{aligned} \rho(f(z+n)) &= \frac{2e^{x+n}}{e^{2e^{x+n} \sin y} + e^{-2e^{x+n} \sin y}} \\ &\leq \frac{2e^{x+n}}{e^{2|\sin y|e^{x+n}}}. \end{aligned}$$

We let $\delta = \min\{|\sin y| : z = x + iy \in F\}$; then $\delta > 0$. We define a function g on the real line $\mathbf{R}$ by

$$g(t) = \frac{2e^t}{e^{2\delta e^t}};$$

then we get

$$g'(t) = 2e^{t-2\delta e^t}(1 - 2\delta e^t).$$

So the positive continuous function g takes the maximum value $\frac{1}{\delta e}$ at $t = -\log 2\delta$. Hence from (2.2), we have

$$\rho(f(z+n)) \leq g(x+n) \leq \frac{1}{\delta e}$$

for all $z \in F$ and $n = 0, 1, 2, \dots$. Therefore z_0 belongs to the set $\mathbf{N}(f)$. This completes the proof. $\square$

Let $\mathcal{A}$ be the space of all entire functions equipped with the topology of uniform convergence on compact subset of the complex plane $\mathbf{C}$.

Theorem 2.2. *If $\mathcal{F}$ is a family of entire functions which is dense in $\mathcal{A}$, then the family $\mathcal{F}$ is not normal on any open set in $\mathbf{C}$.*

Proof. For a given open set Ω and $z_0 \in \Omega$, we choose a positive number ε so that $\overline{B}(z_0, \varepsilon) \subset \Omega$ is a compact subset. By the density of $\mathcal{F}$, there exists a sequence $\{g_n(z)\} \subset \mathcal{F}$, such that

$$|g_n(z) - n(z - z_0)| < \frac{1}{n}$$

for all $z \in \overline{B}(z_0, \varepsilon)$. So we have

$$|g_n(z_0)| < \frac{1}{n} \rightarrow 0$$

as $n \rightarrow \infty$.

On the other hand, for $z \in \overline{B}(z_0, \varepsilon) \setminus \{z_0\}$,

$$|g_n(z)| > n|z - z_0| - \frac{1}{n} \rightarrow \infty$$

as $n \rightarrow \infty$. Hence the family $\mathcal{F}$ is not normal on Ω . $\square$

An entire function f whose translates are dense in $\mathcal{A}$ is called a *universal entire function*. In 1929, the existence of a universal entire function was proved by G. D. Birkhoff [1]. For a proof of the following theorem, see [5, pp. 60–61].

Theorem (Birkhoff [1]). *There is an entire function whose integer translations are dense in the set $\mathcal{A}$ of all entire functions.*

Birkhoff's Theorem and Theorem 2.2 have the following consequence.

Corollary 2.3. *There is an entire function f such that $\mathbf{N}(f) = \emptyset$.*

In Definition 1.2, we considered positive integer translations only. At this time we allow the negative integer translations also.

Definition 2.4. Let f be a meromorphic function in the complex plane $\mathbf{C}$. We define as set $\mathbf{N}_B(f)$ by the set of all $z_0 \in \mathbf{C}$ such that the family $\{f(z+n): n = 0, \pm 1, \pm 2, \dots\}$ is normal at z_0 .

By the same reasons as in $\mathbf{N}(f)$, the set $\mathbf{N}_B(f)$ is a periodic open set with period 1. And clearly $\mathbf{N}_B(f)$ is a subset of $\mathbf{N}(f)$. But $\mathbf{N}_B(f)$ is different from $\mathbf{N}(f)$ for a certain meromorphic function f .

Theorem 2.5. *There is an entire function f such that $\mathbf{N}_B(f) \neq \mathbf{N}(f)$.*

Proof. We shall show that the entire function

$$f(z) = \frac{e^{z^2}}{\Gamma(z)}$$

satisfies the property by proving $0 \in \mathbf{N}(f) \setminus \mathbf{N}_B(f)$.

(i) $0 \notin \mathbf{N}_B(f)$: Let $0 < \varepsilon < \frac{1}{10}$ and n be a positive integer. Since

$$z\Gamma(z) = \Gamma(z+1),$$

we have

$$\begin{aligned} \Gamma(\varepsilon - n) &= \frac{\Gamma(\varepsilon - n + 1)}{\varepsilon - n} \\ &= \frac{\Gamma(\varepsilon)}{(\varepsilon - n)(\varepsilon - n + 1) \cdots (\varepsilon - 1)} \end{aligned}$$

and

$$(2.3) \quad |f(\varepsilon - n)| = \left| \frac{e^{(\varepsilon - n)^2} (\varepsilon - n)(\varepsilon - n + 1) \cdots (\varepsilon - 1)}{\Gamma(\varepsilon)} \right|.$$

The right-hand side of (2.3) tends to infinity as $n \rightarrow \infty$. But $f(-n) = 0$ for all positive integers. Hence 0 does not belong to $\mathbf{N}_B(f)$.

(ii) $0 \in \mathbf{N}(f)$: Since

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{m=1}^{\infty} \frac{z-1}{m(z+m-1)}$$

where

$$\begin{aligned} \gamma &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) \\ &= 0.5772157 \cdots, \end{aligned}$$

we get

$$\begin{aligned} f'(z) &= \frac{e^{z^2}}{\Gamma(z)} \left(2z - \frac{\Gamma'(z)}{\Gamma(z)} \right) \\ &= \frac{e^{z^2}}{\Gamma(z)} \left(2z + \gamma - \sum_{m=1}^{\infty} \frac{z-1}{m(z+m-1)} \right). \end{aligned}$$

Let K be a compact subset of $B(0, \frac{1}{10})$. Then for $z \in K$ and positive integer n ,

$$\begin{aligned} |f'(z+n)| &= \left| \frac{e^{(z+n)^2}}{\Gamma(z+n)} \left(2(z+n) + \gamma - \sum_{m=1}^{\infty} \frac{z+n-1}{m(z+n+m-1)} \right) \right| \\ &\leq \left| \frac{e^{(z+n)^2}}{\Gamma(z+n)} \right| \left(2(n+1) + \gamma + n \sum_{m=1}^{\infty} \frac{1}{m^2} \right) \\ &= \left| \frac{e^{(z+n)^2}}{\Gamma(z+n)} \right| \left(2(n+1) + \gamma + \frac{\pi^2}{6}n \right). \end{aligned}$$

So we can write

$$\begin{aligned} \rho(f(z+n)) &\leq \frac{\left| \frac{e^{(z+n)^2}}{\Gamma(z+n)} \right| \left(2(n+1) + \gamma + \frac{\pi^2}{6}n \right)}{1 + \left| \frac{e^{(z+n)^2}}{\Gamma(z+n)} \right|^2} \\ (2.4) \quad &\leq \left| \frac{\Gamma(z+n)}{e^{(z+n)^2}} \right| \left(2(n+1) + \gamma + \frac{\pi^2}{6}n \right). \end{aligned}$$

By Stirling's formula,

$$(2.5) \quad \left| \frac{\Gamma(z+n)}{e^{(z+n)^2}} \right| \sim \frac{|\sqrt{2\pi}(z+n)^{z+n-1/2}e^{-(z+n)}|}{|e^{(z+n)^2}|}.$$

But for all $z \in K \subset B(0, \frac{1}{10})$, we have

$$\operatorname{Re}((z+n)^2 + (z+n)) \geq n^2 + \frac{n}{2}$$

and

$$|z+n|^{z+n-1/2} < e^{n \log(n+1)}.$$

Thus we have

$$\begin{aligned} \frac{1}{n} - \frac{|(z+n)^{z+n-1/2}e^{-(z+n)}|}{|e^{(z+n)^2}|} &> \frac{1}{n} - \frac{e^{n \log(n+1)}}{e^{n^2 + \frac{n}{2}}} = \frac{e^{n^2 + \frac{n}{2}} - ne^{n \log(n+1)}}{ne^{n^2 + \frac{n}{2}}} \\ &> \frac{e^{n^2} - e^{n \log(n+1)}}{e^{n^2 + \frac{n}{2}}} > 0. \end{aligned}$$

With (2.5), for sufficiently large n we obtain an inequality,

$$(2.6) \quad \left| \frac{\Gamma(z+n)}{e^{(z+n)^2}} \right| < \frac{\sqrt{2\pi}}{n}.$$

So we can write (2.4) as

$$\rho(f(z+n)) \leq \frac{\sqrt{2\pi}}{n} \left(2(n+1) + \gamma + \frac{\pi^2}{6}n \right) < 16$$

for all $z \in K$ and every positive integer n . Hence by Marty's theorem $0 \in \mathbf{N}(f)$. This completes the proof of the theorem. $\square$

3. MAIN RESULTS

Now we shall prove our main result. We know that, for every meromorphic function f , the set $\mathbf{N}(f)$ is a periodic open set with period 1. To prove the existence part we shall construct an entire function with the help of the Weierstrass factorization theorem.

Theorem 3.1. *Let G be an open set in the complex plane $\mathbf{C}$. Then there exists an entire function f such that $\mathbf{N}(f) = G$ if and only if G is periodic with period 1.*

Theorem 3.1 is also true for the set $\mathbf{N}_B(f)$. We can prove it by a slight modification of the proof of Theorem 3.1.

Theorem 3.2. *Let G be an open set in the complex plane $\mathbf{C}$. Then there exists an entire function f such that $\mathbf{N}_B(f) = G$ if and only if G is periodic with period 1.*

Proof of Theorem 3.1. If $G = \mathbf{C}$, then we take a constant function for f . We assume G is a proper subset of $\mathbf{C}$. Since G is a periodic open set with period 1, it is enough to check for the set $\{z \in G: 0 \leq \operatorname{Re} z < 1\}$ to show $\mathbf{N}(f) = G$. Because of its length we divide the proof into three steps and put each step as a section.

3.1. Construction of an entire function. Let G be a periodic open set in the complex plane $\mathbf{C}$ with period 1. For each positive integer i , we let

$$W_i = \{w_{ij}: j \geq i\}$$

be a countable dense subset of $R_i = \{z \in G^c: 0 \leq \operatorname{Re} z < 1, i-1 \leq |\operatorname{Im} z| < i\}$ (for some i , the set W_i can be empty or finite). We let $W = \bigcup_{i=1}^{\infty} W_i$ and

$$(3.1) \quad G_0^c = \{z \in G^c: 0 \leq \operatorname{Re} z < 1\}.$$

Then W is a countable dense subset of G_0^c .

From the set W , we form a sequence $Z = \{z_{ij}\}$ in G^c as in Table 3.1 so that it has no finite limit point. In the construction of the sequence Z , for some positive

TABLE 3.1. Sequence $Z = \{z_{ij}\}$

C_1	C_2	C_3	$\dots$
$z_{11} = w_{11} + 2$	$z_{12} = w_{11} + (2^3 + 1)!$ $z_{22} = w_{12} + (2^3 + 2)!$ $z_{32} = w_{22} + (2^3 + 3)!$	$z_{13} = w_{11} + (3^3 + 1)!$ $z_{23} = w_{12} + (3^3 + 2)!$ $z_{33} = w_{22} + (3^3 + 3)!$ $z_{43} = w_{13} + (3^3 + 4)!$ $z_{53} = w_{23} + (3^3 + 5)!$ $z_{63} = w_{33} + (3^3 + 6)!$	$\dots$

integer i if the set W_i contains only finitely many elements, for instance, suppose that $W_i = \{w_{ij}: i \leq j \leq k\}$, and if z_{st} is a translation of w_{il} where $l > k$, then we skip the term z_{st} . So if we let $\|C_j\|$ be the number of the elements in the j th column of the Table 3.1, then we have

$$(3.2) \quad \|C_j\| \leq \frac{j(j+1)}{2}.$$

And for each $z_{ij} \in C_j$, we have

$$(3.3) \quad \operatorname{Re} z_{ij} < 1 + \left(j^3 + \frac{j(j+1)}{2}\right)!$$

and

$$(3.4) \quad (j^3 + 1)! \leq |z_{ij}| < j + 1 + \left(j^3 + \frac{j(j+1)}{2}\right)!$$

We define a function g on the complex plane $\mathbf{C}$ by

$$(3.5) \quad g(z) = \prod_{j=1}^{\infty} \prod_{z_{ij} \in C_j} \left(1 - \frac{z}{z_{ij}}\right).$$

From (3.2) and (3.4) we have

$$\sum_{j=1}^{\infty} \sum_{z_{ij} \in C_j} \frac{1}{|z_{ij}|} < \sum_{j=1}^{\infty} \frac{j(j+1)}{2} \frac{1}{(j^3 + 1)!} < e.$$

Therefore the function g is entire and its only zeros are at the points $z_{ij} \in Z$.

Finally, we define a function f on the complex plane $\mathbf{C}$ by

$$(3.6) \quad f(z) = e^z g(z).$$

Then f is an entire function whose zero set is equal to that of g .

We shall show that f is a suitable function.

3.2. For all $z \in G$, we have $z \in \mathbf{N}(f)$. For $z_0 \in G$ (we assume $0 \leq \operatorname{Re} z_0 < 1$), $0 < \varepsilon < \frac{1}{10}$, let $F = \overline{B}(z_0, \varepsilon)$ be a compact subset of G . Since G^c is a closed set, there is a positive number ε_0 such that

$$(3.7) \quad d(F, G^c) = \varepsilon_0.$$

Let n be a sufficiently large positive integer and suppose that k is the largest integer such that

$$(3.8) \quad 1 + \left(k^3 + \frac{k(k+1)}{2}\right)! < n.$$

We fix n for a moment and estimate the value of $|g(z+n)|$, for $z \in F$, and then we show $|f(n+z)| \rightarrow \infty$ uniformly on the compact set F as $n \rightarrow \infty$.

To estimate the value $|g(z+n)|$, for $z \in F$, we consider the following four cases. Throughout this paper C_j denotes the j th column of Table 3.1.

Case 1. $j < k$

For each $z_{ij} \in C_j$ and $z \in F$, we have

$$\begin{aligned} |z_{ij} - (z+n)| &> \operatorname{Re}(z+n - z_{ij}) \\ &> n - 2 - \left(j^3 + \frac{j(j+1)}{2}\right)! \end{aligned}$$

by (3.3). Thus from the inequalities (3.4) and (3.8), we obtain

$$\begin{aligned} \left| 1 - \frac{z+n}{z_{ij}} \right| &= \frac{1}{|z_{ij}|} |z_{ij} - (z+n)| \\ &> \frac{n-2 - (j^3 + \frac{j(j+1)}{2})!}{j+1 + (j^3 + \frac{j(j+1)}{2})!} \\ &> \frac{(k^3 + \frac{k(k+1)}{2})! - 1 - (j^3 + \frac{j(j+1)}{2})!}{j+1 + (j^3 + \frac{j(j+1)}{2})!} \\ &> 1. \end{aligned}$$

Hence we can conclude

$$(3.9) \quad \prod_{j=1}^{k-1} \prod_{z_{ij} \in C_j} \left| 1 - \frac{z+n}{z_{ij}} \right| > 1.$$

Case 2. $j = k$

If $z_{ik} = w_{kk} + (k^3 + \frac{k(k+1)}{2})! \in C_k$, then for all $z \in F$ we have

$$|z_{ik} - (z+n)| \geq \varepsilon_0$$

by (3.7). Thus by (3.4), we get

$$(3.10) \quad \left| 1 - \frac{z+n}{z_{ik}} \right| \geq \frac{\varepsilon_0}{k+1 + (k^3 + \frac{k(k+1)}{2})!}.$$

Now we suppose $z_{ik} \neq w_{kk} + (k^3 + \frac{k(k+1)}{2})!$. Then from Table 3.1, we get

$$\operatorname{Re} z_{ik} < 1 + \left(k^3 + \frac{k(k+1)}{2} - 1 \right)!$$

and

$$|z_{ik}| < k+1 + \left(k^3 + \frac{k(k+1)}{2} - 1 \right)!$$

Hence for each $z_{ik} \in C_k$ ($z_{ik} \neq w_{kk} + (k^3 + \frac{k(k+1)}{2})!$), we have

$$\begin{aligned} \left| 1 - \frac{z+n}{z_{ik}} \right| &= \frac{1}{|z_{ik}|} |z_{ik} - (z+n)| \\ &\geq \frac{n - \operatorname{Re} z_{ik} - 1}{|z_{ik}|} \\ &\geq \frac{n-2 - (k^3 + \frac{k(k+1)}{2} - 1)!}{k+1 + (k^3 + \frac{k(k+1)}{2} - 1)!}. \end{aligned}$$

But we defined $k \geq 2$ in (3.8) so that $1 + (k^3 + \frac{k(k+1)}{2})! < n$. Thus

$$(3.11) \quad \left| 1 - \frac{z+n}{z_{ik}} \right| > \frac{(k^3 + \frac{k(k+1)}{2})! - (k^3 + \frac{k(k+1)}{2} - 1)! - 1}{k+1 + (k^3 + \frac{k(k+1)}{2} - 1)!} > 1.$$

From (3.10) and (3.11), we conclude

$$(3.12) \quad \prod_{z_{ik} \in C_k} \left| 1 - \frac{z+n}{z_{ik}} \right| > \frac{\varepsilon_0}{k+1 + (k^3 + \frac{k(k+1)}{2})!}$$

for all $z \in F$.

Case 3. $j = k + 1$

We shall show that

$$(3.13) \quad \prod_{x_{ij} \in C_{k+1}} \left| 1 - \frac{z+n}{z_{ij}} \right| > \varepsilon_0^2 \frac{[(k+1)^3 + 2]! - ((k+1)^3 + 1)!}{\left[k+2 + \left((k+1)^3 + \frac{(k+1)(k+2)}{2} \right)! \right]^{\frac{(k+1)(k+2)}{2} - 2}}$$

for all $z \in F$.

Let $z_{\alpha j}, z_{\beta j} \in C_j$ be the first and second nearest points to the compact set $F+n$; then from (3.7) and Table 3.1, we have

$$|z_{ij} - (z+n)| \geq \begin{cases} \varepsilon_0, & \text{if } z_{ij} = z_{\alpha j} \text{ or } z_{\beta j}, \\ (j^3 + 2)! - (j^3 + 1)!, & \text{if } z_{ij} \neq z_{\alpha j} \text{ and } z_{\beta j}. \end{cases}$$

So by (3.2) and (3.4), we have

$$(3.14) \quad \prod_{z_{ij} \in C_j} \left| 1 - \frac{z+n}{z_{ij}} \right| \geq \frac{\prod_{z_{ij} \in C_j} |z_{ij} - (z+n)|}{\left(j+1 + \left(j^3 + \frac{j(j+1)}{2} \right)! \right)^{\frac{j(j+1)}{2}}} > \varepsilon_0^2 \frac{((j^3 + 2)! - (j^3 + 1)!)^{\frac{j(j+1)}{2} - 2}}{\left(j+1 + \left(j^3 + \frac{j(j+1)}{2} \right)! \right)^{\frac{j(j+1)}{2}}}.$$

Put $j = k + 1$ into (3.14); then it becomes (3.13).

Case 4. $j > k + 1$

We define a sequence $\{\alpha_k\}$ by

$$(3.15) \quad a_k = \min_{\substack{z \in F \\ n \in N(k)}} \left| \prod_{j=k+2}^{\infty} \prod_{z_{ij} \in C_j} \left(1 - \frac{z+n}{z_{ij}} \right) \right|$$

where

$$N(k) = \left\{ n: 1 + \left(k^3 + \frac{k(k+1)}{2} \right)! < n \leq 1 + \left((k+1)^3 + \frac{(k+1)(k+2)}{2} \right)! \right\}.$$

Since $F \subset G$, by the periodicity of G , we have $F+n \subset G$ for all positive integers n . Hence $|g(z+n)| \neq 0$ for all $z \in F$. Thus $a_k > 0$ for all positive integers k .

We shall show that $\{a_k\}$ is eventually an increasing sequence. For convenience, we write $\min_{z, n_1}$ and $\min_{z', n_2}$ instead of $\min_{\substack{z \in F \\ n_1 \in N(k)}}$ and $\min_{\substack{z' \in F \\ n_2 \in N(k+1)}}$ (and likewise for max). To prove that a_k is increasing, we examine the ratio:

$$\begin{aligned}
 \frac{a_{k+1}}{a_k} &\geq \frac{\min_{z', n_2} \left| \prod_{j=k+3}^{\infty} \prod_{z'_{ij} \in C_j} \left(1 - \frac{z' + n_2}{z'_{ij}}\right) \right|}{\max_{z', n_1} \left| \prod_{j=k+2}^{\infty} \prod_{z_{ij} \in C_j} \left(1 - \frac{z + n_1}{z_{ij}}\right) \right|} \\
 &\geq \min_{z, n_1} \prod_{z_{ij} \in C_{k+2}} \left| \frac{z_{ij}}{z_{ij} - (z + n_1)} \right| \times \frac{\min_{z', n_2} \prod_{z'_{ij} \in C_{k+3}} \left| \frac{z'_{ij} - (z' + n_2)}{z'_{ij}} \right|}{\max_{z, n_1} \prod_{z_{ij} \in C_{k+3}} \left| \frac{z_{ij} - (z + n_1)}{z_{ij}} \right|} \\
 &\quad \times \frac{\min_{z', n_2} \prod_{z'_{ij} \in C_{k+4}} \left| \frac{z'_{ij} - (z' + n_2)}{z'_{ij}} \right|}{\max_{z, n_1} \prod_{z_{ij} \in C_{k+4}} \left| \frac{z_{ij} - (z + n_1)}{z_{ij}} \right|} \times \cdots \\
 (3.16) \quad &\geq \min_{z, n_1} \prod_{z_{ij} \in C_{k+2}} \left| \frac{z_{ij}}{z_{ij} - (z + n_1)} \right| \times \frac{\min_{z', n_2} \prod_{z'_{ij} \in C_{k+3}} |z'_{ij} - (z' + n_2)|}{\max_{z, n_1} \prod_{z_{ij} \in C_{k+3}} |z_{ij} - (z + n_1)|} \\
 &\quad \times \frac{\min_{z', n_2} \prod_{z'_{ij} \in C_{k+4}} |z'_{ij} - (z' + n_2)|}{\max_{z, n_1} \prod_{z_{ij} \in C_{k+4}} |z_{ij} - (z + n_1)|} \times \cdots \\
 &= \min_{z, n_1} \prod_{z_{ij} \in C_{k+2}} \left| \frac{z_{ij}}{z_{ij} - (z + n_1)} \right| \times \min_{z', n_2} \prod_{z'_{ij} \in C_{k+3}} |z'_{ij} - (z' + n_2)| \\
 &\quad \times \frac{\min_{z', n_2} \prod_{z'_{ij} \in C_{k+4}} |z'_{ij} - (z' + n_2)|}{\max_{z, n_1} \prod_{z_{ij} \in C_{k+3}} |z_{ij} - (z + n_1)|} \\
 &\quad \times \frac{\min_{z', n_2} \prod_{z'_{ij} \in C_{k+5}} |z'_{ij} - (z' + n_2)|}{\max_{z, n_1} \prod_{z_{ij} \in C_{k+4}} |z_{ij} - (z + n_1)|} \times \cdots.
 \end{aligned}$$

We can easily check that the first and second terms of the last equation of (3.16) are greater than 1. From (3.4) and Table 3.1, for all $z \in F$, $n_1 \in N(k)$ and $z_{ij} \in C_{k+l-1}$ ($l \geq 4$), we have

$$|z_{ij} - (z + n_1)| \leq |z_{ij}| \leq k + l + \left((k + l - 1)^3 + \frac{(k + l - 1)(k + l)}{2} \right)!$$

and for $z' \in F$, $n_2 \in N(k + 1)$ and $z'_{ij} \in C_{k+l}$ ($l \geq 4$), we have

$$\begin{aligned}
 |z'_{ij} - (z + n_2)| &\geq \operatorname{Re} |z'_{ij} - (z + n_2)| \\
 &\geq ((k + l)^3 + 1)! - 2 - \left[\left((k + 2)^3 + \frac{(k + 2)(k + 3)}{2} \right)! \right] \\
 &> ((k + l)^3 + 1)! - (k + 3)^3!.
 \end{aligned}$$

So we have an inequality,

$$\begin{aligned}
 & |z'_{ij} - (z' + n_2)| - |z_{ij} - (z + n_1)| \\
 & > ((k+l)^3 + 1)! - (k+3)^3! \\
 & - \left[k+l + \left((k+l-1)^3 + \frac{(k+l-1)(k+l)}{2} \right)! \right] \\
 (3.17) \quad & = (k+3)^3! \left[(k+l)^3 + 1 \right] \cdots ((k+3)^3 + 1) - 1 \\
 & - \left(k+l + (k+l-1)^3 + \frac{(k+l-1)(k+l)}{2} \right) \cdots ((k+3)^3 + 1) \Big].
 \end{aligned}$$

It is easy to check that

$$(k+l)^3 + 1 > k+l + (k+l-1)^3 + \frac{(k+l-1)(k+l)}{2}$$

for $l \geq 4$. Hence the right-hand side of (3.17) is greater than 0 and we have

$$\left| \frac{z'_{ij} - (z' + n_2)}{z_{ij} - (z + n_1)} \right| > 1$$

for all $z, z' \in F$, $n_1 \in N(k)$, $n_2 \in N(k+1)$, $z_{ij} \in C_{k+l-1}$ and $z'_{ij} \in C_{k+l}$. Since $\|C_{k+l}\| \geq \|C_{k+l-1}\|$, we have

$$\frac{\min_{z', n_2} \prod_{z'_{ij} \in C_{k+l}} |z'_{ij} - (z' + n_2)|}{\max_{z, n_1} \prod_{z_{ij} \in C_{k+l-1}} |z_{ij} - (z + n_1)|} > 1$$

for all $l \geq 4$.

Now we can conclude that $\frac{a_{k+1}}{a_k} > 1$. Therefore $\{a_k\}$ is a positive and eventually increasing sequence.

From (3.9), (3.12), (3.13) and (3.15), for all $z \in f$ and $n \in N(k)$, we can write

$$\begin{aligned}
 & |f(z+n)| = |e^{z+n} g(z+n)| \\
 & = |e^{z+n}| \times \prod_{j=1}^{k-1} \prod_{z_{ij} \in C_j} \left| 1 - \frac{z+n}{z_{ij}} \right| \times \prod_{z_{ij} \in C_k} \left| 1 - \frac{z+n}{z_{ij}} \right| \\
 & \quad \times \prod_{z_{ij} \in C_{k+1}} \left| 1 - \frac{z+n}{z_{ij}} \right| \times \left| \prod_{j=k+2}^{\infty} \prod_{z_{ij} \in C_j} \left(1 - \frac{z+n}{z_{ij}} \right) \right| \\
 (3.18) \quad & > a_k e^n \frac{\varepsilon_0}{k+1 + (k^3 + \frac{k(k+1)}{2})!} \\
 & \quad \times \varepsilon_0^2 \frac{[(k+1)^3 + 2]! - ((k+1)^3 + 1)!}{\left[k+2 + \left((k+1)^3 + \frac{(k+1)(k+2)}{2} \right)! \right]^{\frac{(k+1)(k+2)}{2}}} \\
 & > \varepsilon_0^3 a_k e^n \frac{[(k+1)^3 + 2]! - ((k+1)^3 + 1)!}{\left[k+2 + \left((k+1)^3 + \frac{(k+1)(k+2)}{2} \right)! \right]^{\frac{(k+1)(k+2)}{2} + 1}}.
 \end{aligned}$$

Here we will show that

$$\begin{aligned}
 (3.19) \quad & \log e^n - \left(\frac{(k+1)(k+2)}{2} + 1 \right) \log \left[k+2 + \left((k+1)^3 + \frac{(k+1)(k+2)}{2} \right)! \right] > 0.
 \end{aligned}$$

Since $n \in N(k)$, $1 + (k^3 + \frac{k(k+1)}{2})! < n$ by the definition of $N(k)$. We apply Stirling's formula to the left of (3.19); then we have

$$\begin{aligned}
 & \log e^n - \left(\frac{(k+1)(k+2)}{2} + 1 \right) \log \left[k+2 + \left((k+1)^3 + \frac{(k+1)(k+2)}{2} \right)! \right] \\
 & \geq \left(k^3 + \frac{k(k+1)}{2} \right)! - \frac{1}{2}(k+2)^2 \log(k+2)^3! \\
 & \sim \left(k^3 + \frac{k(k+1)}{2} \right)! - \frac{1}{2}(k+2)^2 \log \sqrt{2\pi(k+2)^3} e^{-(k+2)^3} (k+2)^{3(k+2)^3} \\
 & = \left(k^3 + \frac{k(k+1)}{2} \right)! + \frac{1}{2}(k+2)^5 \\
 & \quad - \frac{1}{2}(k+2)^2 (3(k+2)^3 \log(k+2) + \log \sqrt{2\pi(k+2)^3}) \\
 & > \left(k^3 + \frac{k(k+1)}{2} \right)! - \frac{3}{2}(k+2)^5 \log(k+2) \\
 & > 0
 \end{aligned}$$

for large enough k . This implies

$$(3.20) \quad |f(z+n)| > \varepsilon_0^3 a_k [((k+1)^3 + 2)! - ((k+1)^3 + 1)!]^{\frac{(k+1)(k+2)}{2} - 2}$$

and the right-hand side of (3.20) tends to infinity as $k \rightarrow \infty$. From (3.8), $k \rightarrow \infty$ as $n \rightarrow \infty$, so $|f(z+n)| \rightarrow \infty$ uniformly on the compact set F as $n \rightarrow \infty$. Therefore $z_0 \in \mathbf{N}(f)$.

3.3. If $z \in G^c$, then $z \notin \mathbf{N}(f)$. Since W is a countable subset of G_0^c , for each $w_{st} \in W$ and $\delta > 0$, we can choose a number δ_0 , $0 < \delta_0 < \delta$, so that $w_{st} + \delta_0 \neq w_{kl}$ (and $\neq w_{kl} + 1$) for all $w_{kl} \in W$. From the construction of the sequence Z , t is the smallest integer such that C_t contains the translation of w_{st} . And the translations of w_{st} have the form

$$(3.21) \quad z_{\gamma n} = w_{st} + (n^3 + \gamma)!$$

where $n \geq t$ and $\gamma = \frac{t(t-1)}{2} + s$. We fix w_{st} and show $|f(z_{\gamma n} + \delta_0)| \rightarrow \infty$ as $n \rightarrow \infty$.

For each positive integer n , we let

$$(3.22.1) \quad I_n = \prod_{j=1}^{n-1} \prod_{z_{ij} \in C_j} \left(1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right),$$

$$(3.22.2) \quad J_n = \prod_{z_{ij} \in C_n} \left(1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right),$$

and

$$(3.22.3) \quad K_n = \prod_{j=n+1}^{\infty} \prod_{z_{ij} \in C_j} \left(1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right).$$

We fix n for a moment and write

$$(3.23) \quad \begin{aligned} g(z_{\gamma n} + \delta_0) &= \prod_{j=1}^{\infty} \prod_{z_{ij} \in C_j} \left(1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right) \\ &= I_n \times J_n \times K_n. \end{aligned}$$

We examine the moduli of I_n , J_n , and K_n .

(i) *Estimation of $|I_n|$* : Assume $\delta_0 < 1$; then by (3.4), for each $z_{ij} \in C_j$ where $1 \leq j < n$,

$$\begin{aligned} \left| \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right| &\geq \frac{n^3!}{j+1 + (j^3 + \frac{j(j+1)}{2})!} \\ &\geq \frac{n^3!}{n + ((n-1)^3 + \frac{n(n-1)}{2})!} \\ &> 2. \end{aligned}$$

So we obtain

$$\left| 1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right| \geq \left| 1 - \left| \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right| \right| > 1$$

and

$$(3.24) \quad |I_n| = \prod_{j=1}^{n-1} \prod_{z_{ij} \in C_j} \left| 1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}} \right| > 1.$$

(ii) *Estimation of $|J_n|$* : Since

$$\begin{aligned} &\min\{|z_{\gamma n} - z_{in}| : z_{in} \in C_n, z_{in} \neq z_{\gamma n}\} \\ &> |z_{1n} - z_{2n}| \geq \operatorname{Re}(z_{2n} - z_{1n}) \\ &> (n^3 + 2)! - ((n^3 + 1)! + 1) \\ &> (n^3 + 1)! \end{aligned}$$

and

$$\begin{aligned} |J_n| &= \prod_{z_{in} \in C_n} \left| 1 - \frac{z_{\gamma n} + \delta_0}{z_{in}} \right| \\ &= \left| 1 - \frac{z_{\gamma n} + \delta_0}{z_{\gamma n}} \right| \times \prod_{\substack{z_{in} \in C_n \\ z_{in} \neq z_{\gamma n}}} \left| 1 - \frac{z_{\gamma n} + \delta_0}{z_{in}} \right| \\ &\geq \frac{\delta_0}{|z_{\gamma n}|} \times \prod_{\substack{z_{in} \in C_n \\ z_{in} \neq z_{\gamma n}}} \left| \frac{z_{in} - (z_{\gamma n} + \delta_0)}{z_{in}} \right|, \end{aligned}$$

with (3.2) and (3.4), we obtain the following inequality

$$(3.25) \quad |J_n| > \delta_0 \frac{((n^3 + 1)!)^{\frac{n(n+1)}{2} - 1}}{\left(n + 1 + (n^3 + \frac{n(n+1)}{2})! \right)^{\frac{n(n+1)}{2}}}.$$

(iii) $\{|K_n|\}$ is an eventually increasing sequence: We consider the ratio (for convenience, we let $z_{\gamma\hat{n}} = z_{\gamma(n+1)}$):

$$\begin{aligned}
 \frac{k_{n+1}}{K_n} &= \prod_{j=n+1}^{\infty} \frac{\prod_{z'_{ij} \in C_{j+1}} \left(1 - \frac{z_{\gamma\hat{n}} + \delta_0}{z'_{ij}}\right)}{\prod_{z_{ij} \in C_j} \left(1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}}\right)} \\
 &= \prod_{z_{ij} \in C_{n+1}} \frac{z_{ij}}{z_{ij} - (z_{\gamma n} + \delta_0)} \times \frac{\prod_{z'_{ij} \in C_{n+2}} \left(1 - \frac{z_{\gamma\hat{n}} + \delta_0}{z'_{ij}}\right)}{\prod_{z_{ij} \in C_{n+2}} \left(1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}}\right)} \\
 &\quad \times \frac{\prod_{z'_{ij} \in C_{n+3}} \left(1 - \frac{z_{\gamma\hat{n}} + \delta_0}{z'_{ij}}\right)}{\prod_{z_{ij} \in C_{n+3}} \left(1 - \frac{z_{\gamma n} + \delta_0}{z_{ij}}\right)} \times \cdots \\
 (3.26) \quad &= \prod_{z_{ij} \in C_{n+1}} \frac{z_{ij}}{z_{ij} - (z_{\gamma n} + \delta_0)} \times \frac{\prod_{z'_{ij} \in C_{n+2}} (z'_{ij} - (z_{\gamma\hat{n}} + \delta_0))}{\prod_{z_{ij} \in C_{n+2}} (z_{ij} - (z_{\gamma n} + \delta_0))} \\
 &\quad \times \frac{\prod_{z'_{ij} \in C_{n+3}} (z'_{ij} - (z_{\gamma\hat{n}} + \delta_0))}{\prod_{z_{ij} \in C_{n+3}} (z_{ij} - (z_{\gamma n} + \delta_0))} \times \cdots \\
 &= \prod_{z_{ij} \in C_{n+1}} \frac{z_{ij}}{z_{ij} - (z_{\gamma n} + \delta_0)} \times \prod_{z'_{ij} \in C_{n+2}} (z'_{ij} - (z_{\gamma\hat{n}} + \delta_0)) \\
 &\quad \times \frac{\prod_{z'_{ij} \in C_{n+3}} (z'_{ij} - (z_{\gamma\hat{n}} + \delta_0))}{\prod_{z_{ij} \in C_{n+2}} (z_{ij} - (z_{\gamma n} + \delta_0))} \times \frac{\prod_{z'_{ij} \in C_{n+4}} (z'_{ij} - (z_{\gamma\hat{n}} + \delta_0))}{\prod_{z_{ij} \in C_{n+3}} (z_{ij} - (z_{\gamma n} + \delta_0))} \\
 &\quad \times \cdots.
 \end{aligned}$$

The moduli of the first and second parts of the last equation of (3.26) are greater than 1. And for $z'_{ij} \in C_{n+l+1}$ and $z_{ij} \in C_{n+l}$, $l \geq 2$, we have

$$\begin{aligned}
 \frac{|z'_{ij} - (z_{\gamma\hat{n}} + \delta_0)|}{|z_{ij} - (z_{\gamma n} + \delta_0)|} &> \frac{|z'_{ij}| - |z_{\gamma\hat{n}} + \delta_0|}{|z_{ij}|} \\
 &> \frac{((n+l+1)^3 + 1)! - \left[(n+1) + 1 + \left((n+1)^3 + \frac{(n+1)(n+2)}{2}\right)\right]}{n+l+1 + \left((n+l)^3 + \frac{(n+l)(n+l+1)}{2}\right)!} \\
 &> \frac{((n+l+1)^3 + 1)! - (n+2)^3!}{((n+l)^3 + (n+l)^2)!} \\
 &> 1
 \end{aligned}$$

for sufficiently large n . Since $\|C_{n+l+1}\| \geq \|C_{n+l}\|$,

$$\frac{\prod_{z'_{ij} \in C_{n+l+1}} |z'_{ij} - (z_{\gamma\hat{n}} + \delta_0)|}{\prod_{z_{ij} \in C_{n+l}} |z_{ij} - (z_{\gamma n} + \delta_0)|} > 1$$

for all $l \geq 2$. Hence we can conclude that $|\frac{K_{n+1}}{K_n}| > 1$ for sufficiently large n .

Now we can estimate the value $|f(z_{\gamma n} + \delta_0)|$. From (3.23), (3.24) and (3.25), we obtain

$$\begin{aligned}
 |f(z_{\gamma n} + \delta_0)| &= |e^{z_{\gamma n} + \delta_0} g(z_{\gamma n} + \delta_0)| \\
 &= |e^{z_{\gamma n} + \delta_0}| \cdot |I_n| \cdot |J_n| \cdot |K_n| \\
 (3.27) \quad &\geq e^{(n^3 + \gamma)! - 1} \delta_0 \frac{((n^3 + 1)!)^{\frac{n(n+1)}{2} - 1}}{\left(n + 1 + \left(n^3 + \frac{n(n+1)}{2}\right)!\right)^{\frac{n(n+1)}{2}}} |K_n|.
 \end{aligned}$$

Here we let

$$\alpha_n = \frac{e^{(n^3 + \gamma)! - 1}}{\left(n + 1 + \left(n^3 + \frac{n(n+1)}{2}\right)!\right)^{n(n+1)}}.$$

Then for large enough n , by Stirling's formula we get

$$\begin{aligned}
 \log \alpha_n &\geq (n^3 + \gamma)! - \frac{n(n+1)}{2} \log(n^3 + n^2)! \\
 &\sim (n^3 + \gamma)! - \frac{n(n+1)}{2} \log \sqrt{2\pi(n^3 + n^2)} e^{-(n^3 + n^2)} (n^3 + n^2)^{n^3 + n^2} \\
 &= (n^3 + \gamma)! + \frac{n(n+1)}{2} (n^3 + n^2) \\
 &\quad - \frac{n(n+1)}{2} [(n^3 + n^2) \log(n^3 + n^2) + \log \sqrt{2\pi(n^3 + n^2)}] \\
 &> (n^3 + \gamma)! - \frac{n^3(n+1)^2}{2} \log(n^3 + n^2) \\
 &> 0.
 \end{aligned}$$

So $\alpha_n > 1$ and the inequality (3.27) becomes

$$|f(z_{\gamma n} + \delta_0)| > \delta_0 \cdot |K_n| \cdot ((n^3 + 1)!)^{\frac{n(n+1)}{2} - 1}.$$

Since $\{|K_n|\}$ is an increasing sequence, $|f(z_{\gamma n} + \delta_0)| \rightarrow \infty$ as $n \rightarrow \infty$.

But for each positive integer n ,

$$\begin{aligned}
 f(z_{\gamma n}) &= f(w_{st} + (n^3 + \gamma)!) \\
 &= 0
 \end{aligned}$$

by the definition of the function f . So w_{st} does not belong to $\mathbf{N}(f)$. We choose w_{st} arbitrarily from the set W and W is a dense subset of G_0^c . Hence no point in G_0^c can be in $\mathbf{N}(f)$. $\square$

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