REVERSION OF POWER SERIES AND THE EXTENDED RANEY COEFFICIENTS

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ABSTRACT. In direct as well as diagonal reversion of a system of power series, the reversion coefficients may be expressed as polynomials in the coefficients of the original power series. These polynomials have coefficients which are natural numbers (*Raney coefficients*). We provide a combinatorial interpretation for Raney coefficients. Specifically, each such coefficient counts a certain collection of ordered colored trees. We also provide a simple determinantal formula for Raney coefficients which involves multinomial coefficients.

Let $F_1, \dots, F_n$ be polynomials in variables $x_1, \dots, x_n$ with complex coefficients, where $n \geq 2$. Suppose, for each i , $F_i = x_i + \text{higher degree terms}$ and the Jacobian determinant of $F_1, \dots, F_n$ is equal to 1. Then the *Jacobian Conjecture* [1], [9] asserts, in this case, that $x_1, \dots, x_n$ are also polynomials in $F_1, \dots, F_n$ with complex coefficients. This long-standing conjecture has not been solved even for $n = 2$. Since it can be proved that $x_1, \dots, x_n$ are (formal) power series in $F_1, \dots, F_n$ with complex coefficients, the Jacobian Conjecture asserts that these power series are really polynomials. This provides the motivation for this paper.

Let $F_1, \dots, F_n$ be power series in variables $x_1, \dots, x_n$ of the form $F_i = x_i + \text{higher degree terms}$ with indeterminate coefficients for each i . It is known (e.g. [2, Chapter III, Section 4.4, Proposition 5, p. 219]) that $F = (F_1, \dots, F_n)$ has a (unique) compositional inverse, i.e., there exists $G = (G_1, \dots, G_n)$ where each G_i is a power series in variables $x_1, \dots, x_n$ such that $F \circ G = 1$ and $G \circ F = 1$, or equivalently, $F_i(G_1, \dots, G_n) = x_i$ and $G_i(F_1, \dots, F_n) = x_i$ for all i . There are various methods in the literature to find the coefficients of G_i . In this paper we shall present two new ones. Since each coefficient of G_i is a polynomial in the indeterminate coefficients of $F_1, \dots, F_n$, it is enough to find the coefficients of these polynomials. We will refer to these coefficients as the (*extended*) *Raney coefficients*. In the first method, generating functions in infinitely many variables are used to show that each Raney coefficient has a combinatorial interpretation as the number of colored trees in a certain collection (Theorems 2.4 and 2.5). In

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the second method, Jacobi's residue formula, in a more general setting (Theorem 1.4), is used to obtain a determinantal formula for each Raney coefficient (Corollary 2.7). Consequently this gives a formula for counting certain colored forests. These generalize Raney's results in [7] where $n = 1$.

1. FORMAL LAURENT SERIES AND JACOBI'S RESIDUE FORMULA

Let R be a commutative ring with identity and let $\mathbb{Z}$ denote the ring of integers. A *Laurent monomial* in variables $x_1, \dots, x_n$ is a power product $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ where each exponent j_i is in $\mathbb{Z}$. The *degree* of this monomial is the sum of its exponents. A *homogeneous Laurent polynomial* over R in variables $x_1, \dots, x_n$ is an R -linear combination of (finitely many) Laurent monomials of the same degree. A *formal Laurent series* in variables $x_1, \dots, x_n$ with coefficients in R is an expression of the form

$$(1.1) \quad F = \sum_{k=-\infty}^{\infty} f_k,$$

where each f_k is either 0 or a homogeneous Laurent polynomial of degree k in $x_1, \dots, x_n$ with coefficients in R , and there exists an integer p such that $f_k = 0$ for all $k < p$. In this case we shall call f_k the *homogeneous component* of F of degree k . The *order* of a nonzero formal Laurent series F is defined to be the smallest integer p such that $f_p \neq 0$. In this case, we shall call f_p the *initial part* of F . The set $R((x_1, \dots, x_n))$ ¹ of all formal Laurent series forms a commutative ring with identity under the obvious addition and multiplication. Note that if F and G have orders p and q respectively, then their product has order at least $p + q$. If, in addition, R is a field, then FG has order precisely $p + q$.

Remark. Let $R[[x_1, \dots, x_n]]_{x_1 x_2 \cdots x_n}$ denote the localization of the power series ring $R[[x_1, \dots, x_n]]$ at the multiplicative set $\{1, (x_1 x_2 \cdots x_n), (x_1 x_2 \cdots x_n)^2, \dots\}$. Note that if $n \geq 2$, then

$$R[[x_1, \dots, x_n]]_{x_1 x_2 \cdots x_n} \subsetneq R((x_1, \dots, x_n)),$$

as $\sum_{k=0}^{\infty} x_1^{2k+1} x_2^{-k}$ is not in the former but in the latter; although

$$R[[x_1]]_{x_1} = R((x_1)).$$

Lemma 1.1. *Let $F \in R((x_1, \dots, x_n))$ and let f_p be the initial part of F . Consider the following conditions.*

1. *F has a multiplicative inverse in $R((x_1, \dots, x_n))$.*
2. *f_p has a multiplicative inverse in $R((x_1, \dots, x_n))$.*
3. *f_p consists of a single term with invertible coefficient.*

Then $3 \Rightarrow 2 \Rightarrow 1$.

Proof. $3 \Rightarrow 2$. If $f_p = ax_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ with a invertible in R , then f_p has multiplicative inverse $a^{-1} x_1^{-j_1} x_2^{-j_2} \cdots x_n^{-j_n}$ in $R((x_1, \dots, x_n))$.

$2 \Rightarrow 1$. If f_p is invertible in $R((x_1, \dots, x_n))$, then

$$F = \sum_{k=p}^{\infty} f_k = f_p \left[1 + f_p^{-1} \sum_{k=p+1}^{\infty} f_k \right] = f_p [1 - H]$$

¹Thanks are due to Boo Barkee for this eye-catching notation.

where $H = (-1)f_p^{-1} \sum_{k=p+1}^{\infty} f_k$ is a formal Laurent series. Since $\text{order}(f_p^{-1}) = -p$, it follows that $\text{order}(H) \geq 1$. Hence $\text{order}(H^k) \geq k$ and therefore $1 + H + H^2 + \cdots \in R((x_1, \dots, x_n))$. Thus F is invertible with inverse $f_p^{-1}(1 + H + H^2 + \cdots)$. $\square$

Remark. In case R is a field the three conditions above are equivalent.

The *residue* of a formal Laurent series F in $R((x_1, \dots, x_n))$ is the coefficient of $x_1^{-1}x_2^{-1} \cdots x_n^{-1}$ in F . Note that residue is additive in the sense that if an infinite sum $\sum_{i \in I} G_i$ makes sense, then

$$(1.2) \quad \text{residue} \left(\sum_{i \in I} G_i \right) = \sum_{i \in I} \text{residue}(G_i).$$

In what follows we let

$$\frac{\partial(H_1, \dots, H_n)}{\partial(x_1, \dots, x_n)}$$

be the Jacobian determinant of $H_1, \dots, H_n \in R((x_1, \dots, x_n))$ with respect to $x_1, \dots, x_n$.

Lemma 1.2 ([9, pp. 471–472]). Let $\llbracket H_1, \dots, H_n \rrbracket = \frac{\partial(H_1, \dots, H_n)}{\partial(x_1, \dots, x_n)}$. Then

1. $\llbracket \cdot, \dots, \cdot \rrbracket$ is R -multilinear.
2. $\llbracket \cdot, \dots, \cdot \rrbracket$ is alternating, i.e., $\llbracket H_1, \dots, H_n \rrbracket = 0$ if there exist $i \neq j$ such that $H_i = H_j$.
3. $\llbracket \cdot, \dots, \cdot \rrbracket$ is anticommutative, i.e.,

$$\llbracket H_1, \dots, H_i, \dots, H_j, \dots, H_n \rrbracket = -\llbracket H_1, \dots, H_j, \dots, H_i, \dots, H_n \rrbracket.$$
4. (Product Rule) $\llbracket F_1 G_1, H_2, \dots, H_n \rrbracket = F_1 \llbracket G_1, H_2, \dots, H_n \rrbracket + G_1 \llbracket F_1, H_2, \dots, H_n \rrbracket$.
5. (Power Rule) For any integer m ,

$$\llbracket H_1^m, H_2, \dots, H_n \rrbracket = m H_1^{m-1} \llbracket H_1, H_2, \dots, H_n \rrbracket.$$

(Here we require that H_1 be invertible if m is negative.)

6. If H is invertible, then $\llbracket H^{-1}, H, H_3, \dots, H_n \rrbracket = 0$.

Lemma 1.3. $\text{residue} \left(\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right) = 0$.

Proof. By 1 of Lemma 1.2 and (1.2), it is enough to examine the case where each F_i consists of a single term:

$$F_i = a_i x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}.$$

Factoring $a_i x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}}$ from the i^{th} row of the Jacobian matrix for all i , and then factoring x_j^{-1} from the j^{th} column for all j , we have

$$\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} = a_1 \cdots a_n \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} x_1^{-1+\sum b_{i1}} \cdots x_n^{-1+\sum b_{in}}.$$

Suppose $\sum b_{i1} = \cdots = \sum b_{in} = 0$. Then the displayed determinant is 0 and therefore the residue is 0. Otherwise, at least one of the x_i 's has exponent $\neq -1$ and so the residue is 0 by definition. $\square$

Theorem 1.4 (Jacobi's Residue Formula). *Suppose, for each $i = 1, \dots, n$, $F_i \in R((x_1, \dots, x_n))$ is of the form*

$$F_i = a_i x_1^{b_{i1}} x_2^{b_{i2}} \cdots x_n^{b_{in}} + \text{higher degree terms}$$

with a_i invertible in R . Then

$$\begin{aligned} & \text{residue} \left(F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n} \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right) \\ &= \begin{cases} \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix}, & \text{if } e_1 = e_2 = \cdots = e_n = -1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. We first consider the case $e_1 = e_2 = \cdots = e_n = -1$. Following the proof of Lemma 1.3, we have

$$\begin{aligned} \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} &= a_1 \cdots a_n \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix} x_1^{-1+\sum b_{i1}} \cdots x_n^{-1+\sum b_{in}} \\ &\quad + \text{higher degree terms.} \end{aligned}$$

Since a_i is invertible, by the proof of Lemma 1.1, we have

$$F_i^{-1} = a_i^{-1} x_1^{-b_{i1}} x_2^{-b_{i2}} \cdots x_n^{-b_{in}} + \text{higher degree terms}$$

for each i . Hence the result follows.

Consider now the remaining case. Permuting the F_i and using 3 of Lemma 1.2, we may assume that $e_1 \neq -1, \dots, e_j \neq -1$, but $e_{j+1} = \cdots = e_n = -1$, for some j , $1 \leq j \leq n$. Setting $G_i = \frac{1}{e_i+1} F_i^{e_i+1}$ and using 5, 1 and, successively, 4, 6 and 3 of Lemma 1.2, we have

$$\begin{aligned} & F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n} [F_1, \dots, F_n] \\ &= \frac{1}{e_1+1} \cdots \frac{1}{e_j+1} F_{j+1}^{-1} \cdots F_n^{-1} [F_1^{e_1+1}, \dots, F_j^{e_j+1}, F_{j+1}, \dots, F_n] \\ &= F_{j+1}^{-1} \cdots F_n^{-1} [G_1, \dots, G_j, F_{j+1}, \dots, F_n] \\ &= F_{j+2}^{-1} \cdots F_n^{-1} [F_{j+1}^{-1} G_1, \dots, G_j, F_{j+1}, \dots, F_n] \\ &= F_{j+3}^{-1} \cdots F_n^{-1} [F_{j+1}^{-1} F_{j+2}^{-1} G_1, \dots, G_j, F_{j+1}, \dots, F_n] \\ &\quad \dots \dots \dots \\ &= [F_{j+1}^{-1} F_{j+2}^{-1} F_{j+3}^{-1} \cdots F_n^{-1} G_1, \dots, G_j, F_{j+1}, \dots, F_n]. \end{aligned}$$

The result now follows from Lemma 1.3. $\square$

Corollary 1.5. *Suppose, for each $i = 1, \dots, n$, $F_i \in R((x_1, \dots, x_n))$ is of the form*

$$F_i = a_i x_i + \text{higher degree terms}$$

with a_i invertible in R . Suppose also that for nonnegative integers $p_1, \dots, p_n$,

$$(1.3) \quad x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} = \sum_{l_1, \dots, l_n} d_{l_1, \dots, l_n}^{p_1, \dots, p_n} F_1^{l_1} F_2^{l_2} \cdots F_n^{l_n},$$

where $d_{l_1, \dots, l_n}^{p_1, \dots, p_n} \in R$. Then

$$d_{m_1, \dots, m_n}^{p_1, \dots, p_n} = \text{residue} \left(x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} F_1^{-m_1-1} \cdots F_n^{-m_n-1} \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right).$$

Proof.

$$\begin{aligned} & \text{residue} \left(x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} F_1^{-m_1-1} \cdots F_n^{-m_n-1} \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right) \\ &= \text{residue} \left(\sum_{l_1, \dots, l_n} d_{l_1, \dots, l_n}^{p_1, \dots, p_n} F_1^{l_1-m_1-1} \cdots F_n^{l_n-m_n-1} \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right) \quad \text{by (1.3)} \\ &= \text{residue} \left(d_{m_1, \dots, m_n}^{p_1, \dots, p_n} F_1^{-1} \cdots F_n^{-1} \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right) \quad \text{by (1.2) and Theorem 1.4} \\ &= d_{m_1, \dots, m_n}^{p_1, \dots, p_n} \quad \text{by Theorem 1.4. } \square \end{aligned}$$

2. POWER SERIES REVERSION IN TWO VARIABLES

A *2-colored tree* is a (finite) rooted tree in which every node has either color 1 or color 2 and the children of each node are linearly ordered in such a way that nodes of color 1 always precede those of color 2. A (p, q) -*forest* F is an ordered set of p 2-colored trees with color-1 roots followed by q 2-colored trees with color-2 roots. For instance, a $(1, 0)$ -forest is simply a 2-colored tree whose root has color 1.

Let $\mathbb{N}$ denote the set of nonnegative integers and let $\alpha = (\alpha_{ij})_{j=0,1,2,\dots}^{i=0,1,2,\dots}$, $\beta = (\beta_{ij})_{j=0,1,2,\dots}^{i=0,1,2,\dots}$ be two matrices over $\mathbb{N}$ with only finitely many nonzero entries. We shall call the ordered pair (α, β) the *inventory* of F if α_{ij} equals the number of color-1 nodes in F with i children of color 1 and j children of color 2, and β_{ij} equals the number of color-2 nodes in F with i children of color 1 and j children of color 2.

Let

$$\sigma(\alpha) = \sum_{i,j} \alpha_{ij}, \quad \sigma_1(\alpha) = \sum_{i,j} i\alpha_{ij},$$

and

$$\sigma_2(\alpha) = \sum_{i,j} j\alpha_{ij}.$$

Example 2.0. Consider the 2-colored tree T in Figure 2.1. A round node indicates that it is of color 1 (or female) while a square node indicates that it is of color 2 (or male). Then α_{ij} counts the number of female nodes with i daughters and j sons, and β_{ij} counts the number of male nodes with i daughters and j sons.

FIGURE 2.1

The inventory of the tree T is

$$\begin{aligned}
 (\alpha, \beta) &= \begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \dots & \vdots & \beta_{00} & \beta_{01} & \beta_{02} & \dots \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & \dots & \vdots & \beta_{10} & \beta_{11} & \beta_{12} & \dots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \dots & \vdots & \beta_{20} & \beta_{21} & \beta_{22} & \dots \\ \dots & \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots \end{pmatrix} \\
 &= \begin{pmatrix} 6 & 0 & & \vdots & 2 & 0 & & \\ 0 & 1 & \mathbf{0} & \vdots & 1 & 0 & \mathbf{0} & \\ 2 & 1 & & \vdots & 0 & 1 & & \\ \mathbf{0} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} & & & \end{pmatrix}.
 \end{aligned}$$

Lemma 2.1. *If (α, β) is the inventory of a (p, q) -forest F , then*

$$\begin{aligned}
 (2.1) \quad p &= \sigma(\alpha) - \sigma_1(\alpha + \beta), \\
 q &= \sigma(\beta) - \sigma_2(\alpha + \beta).
 \end{aligned}$$

Proof. Note that $\sum \alpha_{ij}$ is the number of color-1 nodes in F and $\sum \beta_{ij}$ is the number of color-2 nodes in F . Since the number of children of color 1 is equal to $\sum i\alpha_{ij} + \sum i\beta_{ij}$, the first identity follows from the fact that roots are the only nodes that are not children. The second identity can be proved similarly. $\square$

Remark. The converse of the above is true if $pq \neq 0$ (see Corollary 3.4). To see that the condition $pq \neq 0$ is necessary, let $p = 1$, $q = 0$, $\alpha_{00} = 1$, $\beta_{01} = 1$ with all other α_{ij} 's and β_{ij} 's being zero. Then (2.1) is satisfied but (α, β) is not the inventory of any $(1, 0)$ -forest.

Given a pair (α, β) of matrices as before, define $R(\alpha, \beta)$ to be the number of (p, q) -forests with inventory (α, β) where p and q are defined by (2.1). For $p \geq 0$ and $q \geq 0$, we shall denote by $\mathcal{F}_{p,q}$ the set of all (α, β) satisfying (2.1). Let X and

Y be the generating functions for the number of $(1, 0)$ -forests and the number of $(0, 1)$ -forests respectively, both classified by inventories. Then

$$(2.2) \quad \begin{aligned} X &= \sum_{\mathcal{F}_{1,0}} R(\alpha, \beta) \prod a_{ij}^{\alpha_{ij}} \prod b_{ij}^{\beta_{ij}}, \\ Y &= \sum_{\mathcal{F}_{0,1}} R(\alpha, \beta) \prod a_{ij}^{\alpha_{ij}} \prod b_{ij}^{\beta_{ij}}. \end{aligned}$$

The sum in the first equation should be over all inventories of $(1, 0)$ -forests; however, it can be extended to $\mathcal{F}_{1,0}$ for if (α, β) is in $\mathcal{F}_{1,0}$ but is not the inventory of any $(1, 0)$ -forest, then $R(\alpha, \beta) = 0$. Both X and Y are formal power series in infinitely many variables a_{ij}, b_{ij} , where $i, j = 0, 1, 2, \dots$.

Lemma 2.2. $X^p Y^q$ is the generating function for the number of (p, q) -forests classified by inventories. In fact,

$$X^p Y^q = \sum_{\mathcal{F}_{p,q}} R(\alpha, \beta) \prod a_{ij}^{\alpha_{ij}} \prod b_{ij}^{\beta_{ij}}.$$

Proof. Let (α, β) be the inventory of a (p, q) -forest. Then $R(\alpha, \beta)$ is equal to $\sum R(\alpha_1, \beta_1) \cdots R(\alpha_{p+q}, \beta_{p+q})$ where the sum is over all sequences $(\alpha_1, \beta_1), \dots, (\alpha_{p+q}, \beta_{p+q})$ such that $\alpha = \sum \alpha_k, \beta = \sum \beta_k$, and (α_k, β_k) is the inventory of a $(1, 0)$ -forest for $k = 1, \dots, p$, (α_k, β_k) is the inventory of a $(0, 1)$ -forest for $k = p+1, \dots, p+q$. Clearly, the sum can be extended to all sequences such that $\alpha = \sum \alpha_k, \beta = \sum \beta_k$, and (α_k, β_k) is in $\mathcal{F}_{1,0}$ for $k = 1, \dots, p$, (α_k, β_k) is in $\mathcal{F}_{0,1}$ for $k = p+1, \dots, p+q$. Thus it follows from (2.2) that $X^p Y^q$ is the generating function for the number of (p, q) -forests. $\square$

Lemma 2.3. X and Y of (2.2) satisfy the following functional equations

$$(2.3) \quad \begin{aligned} X &= \sum a_{ij} X^i Y^j, \\ Y &= \sum b_{ij} X^i Y^j. \end{aligned}$$

Proof. Using Lemma 2.2, we see that $a_{pq} X^p Y^q$ is the generating function for the number of $(1, 0)$ -forests the removal of whose roots produces (p, q) -forests. Then the first equation follows by partitioning the set of all $(1, 0)$ -forests into classes by the number p of color-1-children and the number q of color-2-children of the roots. The proof for the second is similar. $\square$

Theorem 2.4 (Direct Reversion). Suppose

$$(2.4) \quad \begin{aligned} F &= x - \sum_{i+j \geq 2} a_{ij} x^i y^j, \\ G &= y - \sum_{i+j \geq 2} b_{ij} x^i y^j, \end{aligned}$$

where the a_{ij} and the b_{ij} are indeterminates. Then, for $p, q \geq 0$,

$$x^p y^q = \sum_{l,m} e_{l,m}^{p,q} F^l G^m$$

where

$$e_{l,m}^{p,q} = \sum R(\alpha, \beta) \prod_{i+j \geq 2} a_{ij}^{\alpha_{ij}} \prod_{i+j \geq 2} b_{ij}^{\beta_{ij}}$$

with the sum indexed by all (α, β) in $\mathcal{F}_{p,q}$ satisfying the condition that $l = \alpha_{00}$, $m = \beta_{00}$, and $\alpha_{10} = \alpha_{01} = \beta_{10} = \beta_{01} = 0$

Proof. Setting $a_{01} = a_{10} = b_{01} = b_{10} = 0$, $X = x$, $Y = y$, $a_{00} = F$, $b_{00} = G$ in (2.2), and using Lemma 2.3, we see that the following satisfies (2.4).

$$(2.5) \quad \begin{aligned} x &= \sum_{l,m} \left(\sum R(\alpha, \beta) \prod_{i+j \geq 2} a_{ij}^{\alpha_{ij}} \prod_{i+j \geq 2} b_{ij}^{\beta_{ij}} \right) F^l G^m, \\ y &= \sum_{l,m} \left(\sum R(\alpha, \beta) \prod_{i+j \geq 2} a_{ij}^{\alpha_{ij}} \prod_{i+j \geq 2} b_{ij}^{\beta_{ij}} \right) F^l G^m, \end{aligned}$$

where the first inner sum is indexed by all (α, β) in $\mathcal{F}_{1,0}$ and the second by $\mathcal{F}_{0,1}$, and in both cases, satisfying the conditions: $l = \alpha_{00}$, $m = \beta_{00}$ and $\alpha_{10} = \alpha_{01} = \beta_{10} = \beta_{01} = 0$. By [2, Chapter III, Section 4.4, Proposition 5, pp. 219–220], a right compositional inverse is also a left inverse in formal power series reversion, so (2.4) also satisfies (2.5). Now the result follows from Lemma 2.2. $\square$

Theorem 2.5 (Diagonal Reversion). *Suppose*

$$F = \frac{x}{1 + \sum_{i+j \geq 1} a_{ij} x^i y^j}, \quad G = \frac{y}{1 + \sum_{i+j \geq 1} b_{ij} x^i y^j},$$

where the a_{ij} and the b_{ij} are indeterminates. Then, for $p, q \geq 0$,

$$x^p y^q = \sum_{l,m} d_{l,m}^{p,q} F^l G^m$$

where

$$d_{l,m}^{p,q} = \sum R(\alpha, \beta) \prod_{i+j \geq 1} a_{ij}^{\alpha_{ij}} \prod_{i+j \geq 1} b_{ij}^{\beta_{ij}}$$

with the sum indexed by all (α, β) in $\mathcal{F}_{p,q}$ satisfying the conditions $l = \sigma(\alpha)$ and $m = \sigma(\beta)$.

Proof. Substitute $a_{00}a_{ij}$ for a_{ij} , $b_{00}b_{ij}$ for b_{ij} for all $i+j \geq 1$, and then set $a_{00} = F$, $b_{00} = G$ in (2.2). Now proceed as in the proof of Theorem 2.4. $\square$

In the following we shall find a formula for $R(\alpha, \beta)$ thus determining $e_{l,m}^{p,q}$ and $d_{l,m}^{p,q}$ in Theorems 2.4 and 2.5 respectively. For any matrix $\alpha = (\alpha_{ij})_{j=0,1,2,\dots}^{i=0,1,2,\dots}$ over $\mathbb{N}$ with a finite number of nonzero entries, we define the multinomial and the “modified” multinomial coefficients by

$$\begin{aligned} M(\alpha) &= \binom{\sigma(\alpha)}{\alpha_{00}, \dots, \alpha_{ij}, \dots} = \frac{\sigma(\alpha)!}{\prod (\alpha_{ij}!)}, \\ \widetilde{M}(\alpha) &= \begin{cases} 1, & \text{if all } \alpha_{ij} = 0, \\ \frac{1}{\sigma(\alpha)} M(\alpha), & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem 2.6. Suppose F and G are as in Theorem 2.5. Then

$$d_{l,m}^{p,q} = \sum \left| \begin{pmatrix} M(\alpha) & 0 \\ 0 & M(\beta) \end{pmatrix} - \begin{pmatrix} \widetilde{M}(\alpha) & 0 \\ 0 & \widetilde{M}(\beta) \end{pmatrix} \begin{pmatrix} \sigma_1(\alpha) & \sigma_2(\alpha) \\ \sigma_1(\beta) & \sigma_2(\beta) \end{pmatrix} \right| \\ \times \prod_{i+j \geq 1} a_{ij}^{\alpha_{ij}} \prod_{i+j \geq 1} b_{ij}^{\beta_{ij}}$$

with the sum indexed by all (α, β) in $\mathcal{F}_{p,q}$ satisfying the conditions $l = \sigma(\alpha)$ and $m = \sigma(\beta)$.

Proof. It suffices to prove

$$(2.6) \quad x^p y^q \begin{vmatrix} F^{-l-1} \frac{\partial F}{\partial x} & F^{-l-1} \frac{\partial F}{\partial y} \\ G^{-m-1} \frac{\partial G}{\partial x} & G^{-m-1} \frac{\partial G}{\partial y} \end{vmatrix} \\ = \sum_{\substack{\alpha, \beta \\ \sigma(\alpha)=l \\ \sigma(\beta)=m}} x^{\sigma_1(\alpha+\beta)-\sigma(\alpha)+p-1} y^{\sigma_2(\alpha+\beta)-\sigma(\beta)+q-1} \\ \times \begin{vmatrix} M(\alpha) - \widetilde{M}(\alpha)\sigma_1(\alpha) & 0 - \widetilde{M}(\alpha)\sigma_2(\alpha) \\ 0 - \widetilde{M}(\beta)\sigma_1(\beta) & M(\beta) - \widetilde{M}(\beta)\sigma_2(\beta) \end{vmatrix} \mathbf{A}^\alpha \mathbf{B}^\beta,$$

since the conclusion of Theorem 2.6 follows from taking the residue of (2.6) and using Corollary 1.5.

Let $H = 1 + \sum_{i+j \geq 1} a_{ij} x^i y^j$. Then $F = \frac{x}{H}$ and so,

$$\frac{\partial F}{\partial x} = \frac{1}{H} - \frac{x}{H^2} \frac{\partial H}{\partial x} = \frac{F}{x} - \frac{F^2}{x} \frac{\partial H}{\partial x}.$$

Hence

$$(2.7) \quad F^{-l-1} \frac{\partial F}{\partial x} = \frac{F^{-l}}{x} - \frac{F^{-l+1}}{x} \frac{\partial H}{\partial x}.$$

By binomial and multinomial expansions, we have

$$\left(1 + \sum_{i+j \geq 1} a_{ij} x^i y^j \right)^l = \sum_{k=0}^l \binom{l}{k} \left(\sum_{i+j \geq 1} a_{ij} x^i y^j \right)^k \\ = \sum_{k=0}^l \binom{l}{k} \sum_{\substack{\alpha' \\ \sigma(\alpha')=k}} \binom{k}{\alpha'} \prod_{i+j \geq 1} (a_{ij} x^i y^j)^{\alpha'_{ij}} \\ = \sum_{\substack{\alpha' \\ \sigma(\alpha') \leq l}} \binom{l}{l - \sigma(\alpha'), \alpha'_{01}, \alpha'_{10}, \dots, \alpha'_{ij}, \dots} \prod_{i+j \geq 1} (a_{ij} x^i y^j)^{\alpha'_{ij}}$$

where $\alpha' = (\alpha'_{ij})_{i+j \geq 1}$. Extend α' to $\alpha = (\alpha_{ij})_{i+j \geq 0}$ by letting $\alpha_{00} = l - \sigma(\alpha')$ and $\alpha_{ij} = \alpha'_{ij}$ for $i+j \geq 1$. Then $\sigma_1(\alpha) = \sigma_1(\alpha')$, $\sigma_2(\alpha) = \sigma_2(\alpha')$ and $\sigma(\alpha) = l$.

Thus the last expression equals

$$\sum_{\sigma(\alpha)=l} \binom{l}{\alpha_{00}, \alpha_{01}, \alpha_{10}, \dots, \alpha_{ij}, \dots} \prod_{i+j \geq 1} (a_{ij} x^i y^j)^{\alpha_{ij}}$$

and so

$$(2.8) \quad F^{-l} = \sum_{\sigma(\alpha)=l} M(\alpha) x^{\sigma_1(\alpha)-\sigma(\alpha)} y^{\sigma_2(\alpha)} \mathbf{A}^\alpha$$

where $\mathbf{A}^\alpha = \prod_{i+j \geq 1} a_{ij}^{\alpha_{ij}}$.

Now, let $\varepsilon(i, j)$ denote the matrix whose only nonzero entry is 1 and which occurs at the (i, j) -position. Then

$$\frac{\partial H}{\partial x} = \sum_{i+j \geq 1} i x^{i-1} y^j \mathbf{A}^{\varepsilon(i, j)}.$$

Using (2.8), we see that

$$\frac{F^{-l+1}}{x} \frac{\partial H}{\partial x} = \sum_{i+j \geq 1} \sum_{\sigma(\alpha')=l-1} i M(\alpha') x^{\sigma_1(\alpha')-\sigma(\alpha')-1+i-1} y^{\sigma_2(\alpha')+j} \mathbf{A}^{\alpha'} \mathbf{A}^{\varepsilon(i, j)}.$$

For each fixed pair (i, j) , let $\alpha = \alpha' + \varepsilon(i, j)$. Then $\alpha_{ij} \neq 0$, $\sigma_1(\alpha') + i = \sigma_1(\alpha)$, $\sigma_2(\alpha') + j = \sigma_2(\alpha)$, $\sigma(\alpha') + 1 = \sigma(\alpha)$, and $M(\alpha') = \alpha_{ij} \widetilde{M}(\alpha)$. Hence

$$\begin{aligned} \frac{F^{-l+1}}{x} \frac{\partial H}{\partial x} &= \sum_{i+j \geq 1} \sum_{\substack{\sigma(\alpha)=l \\ \alpha_{ij} \neq 0}} i \alpha_{ij} \widetilde{M}(\alpha) x^{\sigma_1(\alpha)-\sigma(\alpha)-1} y^{\sigma_2(\alpha)} \mathbf{A}^\alpha \\ &= \sum_{\sigma(\alpha)=l} \widetilde{M}(\alpha) \sigma_1(\alpha) x^{\sigma_1(\alpha)-\sigma(\alpha)-1} y^{\sigma_2(\alpha)} \mathbf{A}^\alpha. \end{aligned}$$

The last expression is obtained by noting the condition $\alpha_{ij} \neq 0$ for the inner sum is redundant as α_{ij} is a factor of the summand, and using the definition of $\sigma_1(\alpha)$ after the summation signs are interchanged. Hence it follows from (2.7) and (2.8) that

$$(2.9) \quad F^{-l-1} \frac{\partial F}{\partial x} = \sum_{\sigma(\alpha)=l} \left[M(\alpha) - \widetilde{M}(\alpha) \sigma_1(\alpha) \right] x^{\sigma_1(\alpha)-\sigma(\alpha)-1} y^{\sigma_2(\alpha)} \mathbf{A}^\alpha.$$

Since

$$\frac{\partial F}{\partial y} = 0 - \frac{F^2}{x} \frac{\partial H}{\partial y},$$

we may proceed as before to obtain

$$(2.10) \quad F^{-l-1} \frac{\partial F}{\partial y} = \sum_{\sigma(\alpha)=l} \left[0 - \widetilde{M}(\alpha) \sigma_2(\alpha) \right] x^{\sigma_1(\alpha)-\sigma(\alpha)} y^{\sigma_2(\alpha)-1} \mathbf{A}^\alpha.$$

Likewise, we have

$$(2.11) \quad G^{-m} = \sum_{\substack{\beta \\ \sigma(\beta)=m}} M(\beta) x^{\sigma_1(\beta)} y^{\sigma_2(\beta)-\sigma(\beta)} \mathbf{B}^\beta,$$

$$(2.11) \quad G^{-m-1} \frac{\partial G}{\partial x} = \sum_{\substack{\beta \\ \sigma(\beta)=m}} \left[0 - \widetilde{M}(\beta) \sigma_1(\beta) \right] x^{\sigma_1(\beta)-1} y^{\sigma_2(\beta)-\sigma(\beta)} \mathbf{B}^\beta,$$

$$(2.12) \quad G^{-m-1} \frac{\partial G}{\partial y} = \sum_{\substack{\beta \\ \sigma(\beta)=m}} \left[M(\beta) - \widetilde{M}(\beta) \sigma_2(\beta) \right] x^{\sigma_1(\beta)} y^{\sigma_2(\beta)-\sigma(\beta)-1} \mathbf{B}^\beta.$$

Now (2.6) follows from (2.9), (2.10), (2.11) and (2.12). $\square$

Corollary 2.7. For any $(\alpha, \beta) \in \mathcal{F}_{p,q}$,

$$R(\alpha, \beta) = \left| \begin{pmatrix} M(\alpha) & 0 \\ 0 & M(\beta) \end{pmatrix} - \begin{pmatrix} \widetilde{M}(\alpha) & 0 \\ 0 & \widetilde{M}(\beta) \end{pmatrix} \begin{pmatrix} \sigma_1(\alpha) & \sigma_2(\alpha) \\ \sigma_1(\beta) & \sigma_2(\beta) \end{pmatrix} \right|.$$

Proof. Compare the results of Theorems 2.5 and 2.6. $\square$

Remark. Using Corollary 2.7 we can now determine the $e_{l,m}^{p,q}$ in Theorem 2.4.

3. GENERALIZATIONS

Using essentially the same proofs, all results in Section 2 can be generalized to an arbitrary number of colors. In what follows, we shall indicate how this can be done.

An n -colored tree is a (finite) rooted tree in which every node has one of the n colors and the children of each node are linearly ordered in such a way that children of color i always precede those of color j if $i < j$. A $(p_1, \dots, p_n)$ -forest is an ordered set of n -colored trees with p_i trees of root-color i for $i = 1, \dots, n$ such that trees with root-color i always precede those of root-color j if $i < j$.

For $i = 1, \dots, n$, let $\alpha^{(i)} = (\alpha_{k_1, \dots, k_n}^{(i)})_{k_1, \dots, k_n \in \mathbb{N}}$ be an n -dimensional array over $\mathbb{N}$ with finitely many nonzero entries. If, for each i and for each n -tuple $(k_1, \dots, k_n)$, $\alpha_{k_1, \dots, k_n}^{(i)}$ equals the number of color i nodes in a $(p_1, \dots, p_n)$ -forest F with k_j children of color j for each $j = 1, \dots, n$, then $\widehat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)})$ is said to be the *inventory* of F .

For any n -dimensional array $\alpha = (\alpha_{k_1, \dots, k_n})_{k_1, \dots, k_n \in \mathbb{N}}$ over $\mathbb{N}$ with finitely many nonzero entries, define $\sigma(\alpha) = \sum \alpha_{k_1, \dots, k_n}$ and $\sigma_i(\alpha) = \sum k_i \alpha_{k_1, \dots, k_n}$, for each i .

As in the case of Lemma 2.1, if $\widehat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)})$ is the inventory of a $(p_1, \dots, p_n)$ -forest, then, for each i ,

$$(3.1) \quad p_i = \sigma(\alpha^{(i)}) - \sigma_i(\alpha^{(1)} + \dots + \alpha^{(n)}).$$

Given an n -tuple $\widehat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)})$ of n -dimensional arrays, let $R(\widehat{\alpha})$ be the number of $(p_1, \dots, p_n)$ -forests with inventory $\widehat{\alpha}$, where each p_i is defined by (3.1). For $p_1, \dots, p_n \in \mathbb{N}$, we shall also denote by $\mathcal{F}_{p_1, \dots, p_n}$ the set of all $\widehat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)})$ satisfying (3.1). For each i , let X_i be the generating function for the number of

$\mathbf{e}_i$ -forests classified by inventories. Here, $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$, the 1 being in the i^{th} position. Then

$$X_i = \sum_{\mathcal{F}_{\mathbf{e}_i}} R(\hat{\alpha}) \prod_{j=1}^n \prod_{k_1, k_2, \dots, k_n} a_{k_1, k_2, \dots, k_n}^{(j)} \alpha_{k_1, k_2, \dots, k_n}^{(j)}.$$

One can show, as in Lemma 2.2, that $X_1^{p_1} X_2^{p_2} \cdots X_n^{p_n}$ is the generating function for the number of $(p_1, \dots, p_n)$ -forests classified by inventories. Thus, as in Lemma 2.3, X_i satisfies the functional equation

$$X_i = \sum a_{k_1, \dots, k_n}^{(i)} X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n},$$

for $i = 1, \dots, n$.

We can now state our main results for the general case of arbitrary n .

Theorem 3.1 (Direct Reversion). *For each $i = 1, \dots, n$, let*

$$F_i = x_i - \sum_{k_1 + \dots + k_n \geq 2} a_{k_1, \dots, k_n}^{(i)} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n},$$

where the $a_{k_1, \dots, k_n}^{(i)}$ are indeterminates. Then, for $p_1, \dots, p_n \in \mathbb{N}$,

$$x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} = \sum_{r_1, \dots, r_n} e_{r_1, \dots, r_n}^{p_1, \dots, p_n} F_1^{r_1} F_2^{r_2} \cdots F_n^{r_n},$$

where

$$e_{r_1, \dots, r_n}^{p_1, \dots, p_n} = \sum R(\hat{\alpha}) \prod_{j=1}^n \prod_{k_1 + \dots + k_n \geq 2} a_{k_1, \dots, k_n}^{(j)} \alpha_{k_1, \dots, k_n}^{(j)},$$

with the sum indexed by all $\hat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)})$ in $\mathcal{F}_{p_1, \dots, p_n}$ satisfying the conditions that $r_i = \alpha_{0, \dots, 0}^{(i)}$ and $\alpha_{\mathbf{e}_j}^{(i)} = 0$ for all i and j .

Theorem 3.2 (Diagonal Reversion). *For each $i = 1, \dots, n$, let*

$$F_i = \frac{x_i}{1 + \sum_{k_1 + \dots + k_n \geq 1} a_{k_1, \dots, k_n}^{(i)} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}.$$

Then, for $p_1, \dots, p_n \in \mathbb{N}$,

$$x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} = \sum_{r_1, \dots, r_n} d_{r_1, \dots, r_n}^{p_1, \dots, p_n} F_1^{r_1} F_2^{r_2} \cdots F_n^{r_n},$$

where

$$d_{r_1, \dots, r_n}^{p_1, \dots, p_n} = \sum R(\hat{\alpha}) \prod_{j=1}^n \prod_{k_1 + \dots + k_n \geq 1} a_{k_1, \dots, k_n}^{(j)} \alpha_{k_1, \dots, k_n}^{(j)},$$

with the sum indexed by all $\hat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)})$ in $\mathcal{F}_{p_1, \dots, p_n}$ satisfying the conditions that $r_i = \sigma(\alpha^{(i)})$, for all i .

Theorem 3.3 (Extended Raney Coefficient). *For any $\hat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)}) \in \mathcal{F}_{p_1, \dots, p_n}$,*

$$R(\hat{\alpha}) = \left| \begin{pmatrix} M(\alpha^{(1)}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & M(\alpha^{(n)}) \end{pmatrix} \right| - \left| \begin{pmatrix} \widetilde{M}(\alpha^{(1)}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \widetilde{M}(\alpha^{(n)}) \end{pmatrix} \begin{pmatrix} \sigma_1(\alpha^{(1)}) & \dots & \sigma_n(\alpha^{(1)}) \\ \vdots & \ddots & \vdots \\ \sigma_1(\alpha^{(n)}) & \dots & \sigma_n(\alpha^{(n)}) \end{pmatrix} \right|.$$

Corollary 3.4. *Let $\hat{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(n)}) \in \mathcal{F}_{p_1, \dots, p_n}$. If all $p_i > 0$, then $R(\hat{\alpha}) > 0$.*

Proof. If each $p_i > 0$, then, by (3.1), none of $\alpha^{(1)}, \dots, \alpha^{(n)}$ consists entirely of 0's; hence, $M(\alpha^{(i)}) = \sigma(\alpha^{(i)})\widetilde{M}(\alpha^{(i)})$, for all i . Thus, by Theorem 3.3,

$$R(\hat{\alpha}) = \prod_{i=1, \dots, n} \widetilde{M}(\alpha^{(i)}) \times \left| \begin{pmatrix} \sigma(\alpha^{(1)}) - \sigma_1(\alpha^{(1)}) & -\sigma_2(\alpha^{(1)}) & \dots & -\sigma_n(\alpha^{(1)}) \\ -\sigma_1(\alpha^{(2)}) & \sigma(\alpha^{(2)}) - \sigma_2(\alpha^{(2)}) & \dots & -\sigma_n(\alpha^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma_1(\alpha^{(n)}) & -\sigma_2(\alpha^{(n)}) & \dots & \sigma(\alpha^{(n)}) - \sigma_n(\alpha^{(n)}) \end{pmatrix} \right|.$$

The result now follows from Lemma 3.5 below. $\square$

Lemma 3.5. *Suppose B is a square matrix of order n with real entries b_{ij} such that $b_{ii} \geq 0$, for all i , and $b_{ij} \leq 0$, for $i \neq j$. If all the column-sums of B are nonnegative, then so is the determinant of B . If, furthermore, all the column-sums of B are positive, then so is the determinant of B .*

Proof. By hypothesis, the j^{th} column-sum $p_j = \sum_{i=1}^n b_{ij} \geq 0$, for all j . Thus we can write B as a sum of two matrices with one of them diagonal, namely,

$$B = P + B' \\ = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ \mathbf{0} & & & p_n \end{pmatrix} + \begin{pmatrix} -\sum_{k \neq 1} b_{k1} & b_{12} & \dots & b_{1n} \\ b_{21} & -\sum_{k \neq 2} b_{k2} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & -\sum_{k \neq n} b_{kn} \end{pmatrix}.$$

Then

$$\det B = p_1 p_2 \cdots p_n + \left(\sum_{1 \leq i_1 < i_2 < \dots < i_l \leq n-1} c_{i_1, \dots, i_l} p_{i_1} p_{i_2} \cdots p_{i_l} \right) + \det B' \\ = p_1 p_2 \cdots p_n + \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq n-1} c_{i_1, \dots, i_l} p_{i_1} p_{i_2} \cdots p_{i_l},$$

a polynomial in $p_1, \dots, p_n$, where $c_{i_1, \dots, i_l}$ is the principal minor of B' obtained by deleting the i_j^{th} row and column of B' , for $j = 1, 2, \dots, l$. We only need to show that $c_{i_1, \dots, i_l} \geq 0$, for all $l = 1, 2, \dots, n-1$. However, this follows from the induction hypothesis, since B' as well as all its principal submatrices satisfy the hypotheses of Lemma 3.5. $\square$

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