TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 1, January 1998, Pages 43–53
S 0002-9947(98)01556-6

DECOMPOSITION THEOREMS AND APPROXIMATION BY A "FLOATING" SYSTEM OF EXPONENTIALS

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Abstract. The main problem considered in this paper is the approximation of a trigonometric polynomial by a trigonometric polynomial with a prescribed *number* of harmonics. The method proposed here gives an opportunity to consider approximation in different spaces, among them the space of continuous functions, the space of functions with uniformly convergent Fourier series, and the space of continuous analytic functions. Applications are given to approximation of the Sobolev classes by trigonometric polynomials with prescribed number of harmonics, and to the widths of the Sobolev classes.

This work supplements investigations by Maiorov, Makovoz and the author where similar results were given in the integral metric.

Introduction

Let X be a Banach space of functions defined on the cube $\mathbb{T}^n = (-\pi, \pi]^n$ and 2π-periodic in each variable. For $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}^n$, let $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, $(\mathbf{x}, \mathbf{k}) = k_1 x_1 + ... + k_n x_n$, and let $T(\theta_m; \mathbf{x}) = \sum_{\mathbf{k} \in \theta_m} c_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}$ be a trigonometric polynomial with the spectrum from the set θ_m of finite cardinality m.

Let

$$e_m(f; X) \overset{def}{=} \inf_{\theta_m} \inf_{T(\theta_m)} \|f - T(\theta_m)\|_X$$

be the degree of best approximation of f by polynomials with m harmonics. The quantity $e_m(f; L_2)$ originally appeared in a paper by S. B. Stechkin [St], who used it in the criterion for absolute convergence of orthogonal series. This characteristic has become popular after Ismagilov's excellent paper [Is], in which he found nontrivial estimates for $e_m(|x|; L_\infty)$ and gave interesting and important applications to the widths of Sobolev classes. The Ismagilov method was developed in a series of papers by V. Maiorov (see, for example, [Mr1], [Mr2], [Mr3]). The probabilistic method for constructing approximating polynomials was proposed by Y. Makovoz [Mk] and the author [Be1].

This paper is organized as follows. First we obtain an estimate in the space $C(\mathbb{T})$ of continuous functions of one variable and describe the method in all details. In the subsequent sections the method is used in other situations: the space U of uniformly convergent Fourier series, the space C_A of continuous analytic functions,

Received by the editors March 13, 1995.
1991 *Mathematics Subject Classification.* Primary 42A61.
Key words and phrases. Approximation, width.
This research was supported by the Israeli Ministry of Science and the Arts through the Ma'agara program for absorption of immigrant mathematicians at the Technion, Israel Institute of Technology.

and the space $C(\mathbb{T}^n)$ of continuous functions of n real variables. The proofs in these sections are less detailed, and often only necessary differences are pointed out.

Applications are given in the concluding section. As usual C_s denotes an absolute constant.

Approximation in the space $C(\mathbb{T})$

Let T_N be a trigonometric polynomial of degree at most N, namely,

$$T_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}.$$

Theorem 1. *For every $2 \le p < \infty$ and $1 \le M \le 2N$ there exists a trigonometric polynomial $T(\theta_M; x)$ with number of harmonics at most M, such that*

$$\|T_N(x) - T(\theta_M; x)\|_\infty \le C_1(\frac{N}{M} \log(1 + \frac{N}{M}))^{1/p} \|T_N\|_p, \tag{1.1}$$

and the spectrum θ_M is in the segment $[-2N, 2N]$.

The proof is based on the following lemmas.

Lemma 1. *Let $V_{2N}(f; x)$ be the de la Vallée-Poussin means of the function f. Then*

$$\|V_{2N}(f; x)\|_\infty \le C_2 \|f\|_\infty. \tag{1.2}$$

This lemma is well known. Its proof may be found, for example, in [Z]. We note only that $V_{2N}(f)$ is a trigonometric polynomial of degree $\le 2N$ and for every trigonometric polynomial T_N of degree $\le N$

$$V_{2N}(T_N; x) \equiv T_N(x).$$

We denote by $\| \cdot \|_\psi$ the norm of the Orlicz space generated by the function $\psi(u) = e^{u^2} - 1$ (see, for example, [Z]).

Lemma 2. *For every trigonometric polynomial $T_N(x)$ and for every $M \le 2N$ there exists a trigonometric polynomial $T(\theta_M; x)$, with the number of harmonics at most M, such that*

$$\|T_N(x) - T(\theta_M; x)\|_\psi \le C_3(\frac{N}{M})^{1/2} \|T_N\|_2, \tag{1.3}$$

where the spectrum $\theta_M \subset [-N, N]$.

In the space L_q $(2 < q < \infty)$ such an inequality was proved in [Be2], [Mk]. The proof proposed in [Bo] is based on Khinchin's inequality and can be used in this situation with minimal changes. For the sake of completeness we shall give the proof later.

Lemma 3. *For each $\lambda > 0$, every polynomial $T_N \in L_\psi$ can be represented as a sum of two polynomials $T_N = T^1_{2N} + T^2_{2N}$, each of degree $\le 2N$, such that*

$$\|T^1_{2N}\|_\infty \le C_4 \lambda \|T_N\|_\psi, \qquad \|T^2_{2N}\|_\infty \le C_5 e^{-\lambda^2} \|T_N\|_\psi. \tag{1.4}$$

For the proof, it is sufficient to truncate the polynomial T_N at level λ and apply the de la Vallée-Poussin operator to both sides of the equality (see Lemma 1).

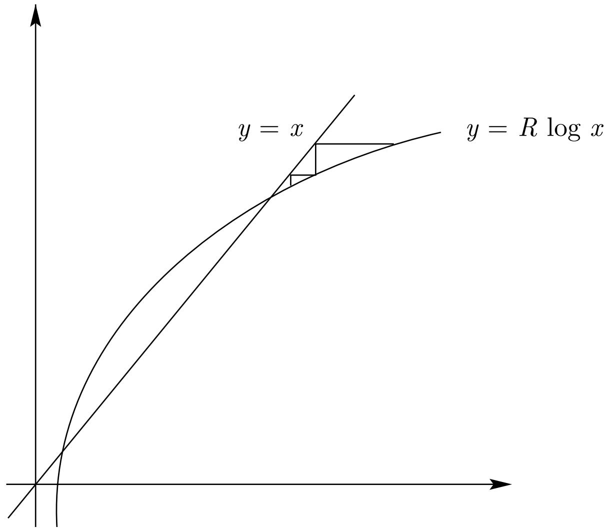

FIGURE 1

Lemma 4. *For every trigonometric polynomial T_N and for every $M \leq 2N$ there exists a trigonometric polynomial $T(\theta_M; x)$, with the cardinality of the spectrum $\leq M$, such that*

$$\|T_N(x) - T(\theta_M; x)\|_\infty \leq C_6(\frac{N}{M}\log N)^{1/2}\|T_N\|_2. \tag{1.5}$$

This can be derived from the original proof given in [Be1], as well as from the corresponding statement for the space L_q $(2 < q < \infty)$ (see [Be2], [Mk], [Bo]), if the order of growth $C_q \sim \sqrt{q}$ in Khinchin's inequality is taken into account.

Proof of Theorem 1. First let $p = 2$. Denote $N/M = R$ and consider the iterated sequence

$$a_1(N, M) = R\log N, \quad a_2(N, M) = R\log a_1(N, M), ...,$$

$$a_{k+1}(N, M) = R\log a_k(N, M), ...$$

The sequence $\{a_k(N, M)\}$ converges to x_0, the largest root of the equation $x = R\log x$. This can be easily seen from Figure 1.

The x_0 can be estimated by $R\log R < x_0 < 2R\log R$. We will prove that there is a number α such that for every k and $1 \leq M \leq \alpha N$ there exists a trigonometric polynomial $T(\theta_M^k)$ such that

$$\|T_N - T(\theta_M^k)\|_\infty \leq C_6\sqrt{a_k(N, M)}\|T_N\|_2. \tag{1.6}$$

Then after a finite number of steps we will obtain the desired estimate.

For $k = 1$ the estimate (1.6) is given in Lemma 4. Assume that (1.6) holds for k for all N; we now prove it for $k + 1$. According to Lemma 2, there exists a polynomial $T(\theta_M; x)$ such that

$$\|T_N(x) - T(\theta_M; x)\|_\psi \leq C_3(\frac{N}{M})^{1/2}\|T_N\|_2.$$

Applying Lemma 3, we find the decomposition

$$T_N(x) - T(\theta_M; x) = T^1_{2N}(x) + T^2_{2N}(x),$$

for a parameter λ yet to be prescribed. Using our assumption for k, we approximate the polynomial T^2_{2N} by a polynomial $T(\theta^k_M; x)$ in the norm L_∞. Then

$$\|T_N - T(\theta_M) - T(\theta^k_M)\|_\infty \le \|T^1_{2N}\|_\infty + \|T^2_{2N} - T(\theta^k_M)\|_\infty$$

$$\le C_4\lambda\|T_N - T(\theta_M)\|_\psi + C_6\sqrt{a_k(2N,M)}\|T^2_{2N}\|_2.$$

From (1.3) and (1.4) we obtain

$$\|T_N - T(\theta_M) - T(\theta^k_M)\|_\infty$$

$$\le \sqrt{\frac{N}{2M}}\|T_N\|_2(\sqrt{2}C_4C_3\lambda + C_3C_5C_6\sqrt{2a_k(2N,M)}e^{-\lambda^2}).$$

Set $\lambda = \sqrt{1/2\log a_k(N,2M)}$. Then the right-hand side can be rewritten in the form

(1.7)

$$(\frac{N}{2M}\log a_k(N,2M))^{1/2}\|T_N\|_2(C_4C_3 + C_3C_5C_6\frac{\sqrt{2a_k(2N,M)}}{\sqrt{a_k(N,2M)}\sqrt{\log a_k(N,2M)}}).$$

As $\frac{N}{2M}\log a_k(N,2M) = a_{k+1}(N,2M)$, we need only prove that the expression in parentheses does not exceed an absolute constant C_6. To this end we show that for every k

$$\frac{2a_k(2N,M)}{a_k(N,2M)} \le 4\log 4(1 + \frac{1}{\log N/2M} + ... + \frac{1}{(\log N/2M)^{k-1}}). \tag{1.8}$$

For $k = 1$

$$\frac{a_1(2N,M)}{a_1(N,2M)} = \frac{4\log 2N}{\log N} \le 4\log 4.$$

Assume that (1.8) holds for $k-1$. Then, according the definition for k

$$\frac{a_k(2N,M)}{a_k(N,2M)} = \frac{4\log a_{k-1}(2N,M)}{\log a_{k-1}(N,2M)} = 4(1 + \frac{1}{\log a_{k-1}(N,2M)}\log\frac{a_{k-1}(2N,M)}{a_{k-1}(N,2M)})$$

$$\le 4(1 + \frac{1}{\log N/2M}\log(4\log 4(1 + \frac{1}{\log N/2M} + ... + \frac{1}{(\log N/2M)^{k-2}})$$

$$\le 4\log 4(1 + \frac{1}{\log N/2M} + ... + \frac{1}{(\log N/2M)^{k-1}}).$$

Therefore if $N \ge 2Me^2$, then there exists an absolute constant C_7 such that $\frac{2a_k(2N,M)}{a_k(N,2M)} \le C_7$. We can take $C_6 > C_2C_3$ and bound $M < \alpha N$ in such a way that the expression in the parentheses in (1.7) does not exceed C_6.

An odd M is treated similarly.

The theorem is proved for all M and N such that $M \le \alpha N$.

For M satisfying $\alpha N < M \le 2N$ the conclusion of the theorem follows from the above.

The proof of the theorem in the case $2 < p < \infty$ is based on the following decomposition lemma, analogous to Lemma 3.

Lemma 5. *Let a trigonometric polynomial T_N and $\lambda > 0$ be given. Then T_N can be represented as the sum of two polynomials T^1_{2N} and T^2_{2N}, $T_N = T^1_{2N} + T^2_{2N}$, such that*

$$\|T^1_{2N}\|_\infty \leq C_8 \lambda^{1/p} \|T_N\|_p, \quad \|T^2_{2N}\|_2 \leq C_9 \lambda^{1/p-1/2} \|T_N\|_p. \tag{1.9}$$

Now we can prove the theorem for $2 < p < \infty$. By the above, there exists a polynomial $T(\theta_M; x)$ such that

$$\|T^2_{2N} - T(\theta_M; x)\|_\infty \leq C_6 \sqrt{\frac{N}{M} \log(\frac{N}{M} + 1)} \|T^2_{2N}\|_2.$$

Therefore by (1.9) we have

$$\|T_N - T(\theta_M)\|_\infty \leq C_{10} [\lambda^{1/p} + \lambda^{1/p-1/2} (N/M \log{(N/M+1)})^{1/2}] \|T_N\|_p.$$

Putting $\lambda = N/M \log{(N/M + 1)}$, we obtain the desired result. □

Remark 1. As one can see from the proof, the interval where the spectrum of the approximating polynomial is located is doubled on every step. But the actual support of the spectrum is contained in the segment $[-2N, 2N]$. This follows from Lemma 1.

Remark 2. If T_N is the Dirichlet kernel

$$T_N(x) = \sum_{k=-N}^{N} e^{ikx}, \tag{1.10}$$

then estimate (1.1) can be derived immediately from [G1] or [Sp]. A more accurate estimate of the approximation of kernel (1.10) is proved in [Mr2].

Remark 3. This method of approximation can be used in every space where statements similar to Lemmas 1-4 are valid.

Proof of Lemma 2. Suppose that $\|T_N\|_2 = 1$, fix $N/M \sim 2^k$, and write

$$\begin{aligned}
\sum_{j=-N}^{N} c_j e^{ijx} &= \sum_{j=-N}^{N} c_j \epsilon^1_j e^{ijx} + \sum_{j=-N}^{N} c_j (1-\epsilon^1_j) e^{ijx} \\
&= \sum_{j=-N}^{N} c_j \epsilon^1_j e^{ijx} + \sum_{j=-N}^{N} c_j (1-\epsilon^1_j)\epsilon^2_j e^{ijx} + \sum_{j=-N}^{N} c_j (1-\epsilon^1_j)(1-\epsilon^2_j) e^{ijx} \\
&= \sum_{j=-N}^{N} c_j \epsilon^1_j e^{ijx} + \sum_{j=-N}^{N} c_j (1-\epsilon^1_j)\epsilon^2_j e^{ijx} + ... \\
&\quad + \sum_{j=-N}^{N} c_j (1-\epsilon^1_j)...(1-\epsilon^{k-1}_j)\epsilon^k_j e^{ijx} \\
&\quad + \sum_{j=-N}^{N} c_j (1-\epsilon^1_j)...(1-\epsilon^k_j) e^{ijx},
\end{aligned} \tag{1.11}$$

where $\{\epsilon^m_j\}_{-N \leq j \leq N, 1 \leq m \leq k}$ are ± 1, the signs to be specified. Denoting (1.11) by $\phi(\epsilon, x)$, we deduce from the definition of norm in the Orlicz space [Kr] and Khinchin's inequality that

$$\int \|\phi(\epsilon,x)\|_{\psi(dx)}d\epsilon = \int \inf_{t>0}\frac{1}{t}(1+\int_{\mathbb{T}}\psi(t\phi(\epsilon,x))dx)d\epsilon^1...d\epsilon^m$$

$$\le \int d\epsilon^1...d\epsilon^{m-1}\inf_{t>0}\frac{1}{t}(\int_{\mathbb{T}}\int \psi(t\phi(\epsilon,x))d\epsilon^m dx)$$

$$\le C_{11}\sum_{m\le k}\int (\sum_{j=-N}^{N}|c_j|^2|(1-\epsilon_j^1)|^2...|(1-\epsilon_j^{m-1})|^2)^{1/2}d\epsilon^1...d\epsilon^{m-1}.$$

The Hölder inequality and integration with respect to $\epsilon^1,...,\epsilon^{m-1}$ give

$$\int \|\phi(\epsilon,x)\|_{\psi(dx)}d\epsilon \le C_{12}2^{\frac{k}{2}}, \tag{1.12}$$

and if we let

$$A_\epsilon = \{j\in[-N,N] \mid \epsilon_j^1 = ... = \epsilon_j^k = -1\},$$

then

$$|A_\epsilon| = \frac{1}{2^k}\sum_{j=-N}^{N}(1-\epsilon_j^1)...(1-\epsilon_j^k),$$

$$\int |A_\epsilon|d\epsilon = \frac{N}{2^k}\sim M. \tag{1.13}$$

Now (1.12) and (1.13) allow us to choose the signs ϵ_j^m so that

$$T(\theta_M;x) = \sum_j (1-\epsilon_j^1)...(1-\epsilon_j^k)e^{ijx}$$

satisfies the condition

$$\|T_N(x) - T(\theta_M;x)\|_\psi \le C_{13}(\frac{N}{M})^{1/2}.$$

Using the homogeneity of the norm, we conclude the proof of the lemma. □

Approximation in the norm of uniform convergence

Let U be the Banach space of continuous functions defined on $\mathbb{T}$ with the norm $\|f\|_U = \sup_k \|S_k(f)\|_\infty$, where $S_k(f)$, $k=1,2,...$, are the partial sums of the Fourier series of f.

Theorem 2. *Let $2<p<\infty$, $1\le M\le 2N$, and let $T_N(x)$ be a trigonometric polynomial of degree N. Then there exists a trigonometric polynomial with a number of harmonics $\le M$, such that*

$$\|T_N - T(\theta_M)\|_U \le C_1(\frac{N}{M})^{1/p}(\log(\frac{N}{M}+1))^{3/p}\|T_N\|_p, \tag{2.1}$$

and the spectrum θ_M is contained in the segment $[-N,N]$.

In order to use the method of Theorem 1, we need the following results.

Lemma 6. *Let $\lambda>1$. Then every polynomial T_N can be represented as a sum of two polynomials T_N^1, T_N^2, of degree at most N, such that*

$$\|T_N^1\|_U \le C_2\lambda\|T_N\|_\psi, \qquad \|T_N^2\|_2 \le C_3\lambda^{1/3}e^{-\lambda^{2/3}}\|T_N\|_\psi.$$

Proof. By Lemma 3, T_N can be represented in the form $T_N = T^1_{2N} + T^2_{2N}$, where

$$\|T^1_{2N}\|_\infty \le C_4 t \|T_N\|_\psi, \quad \|T^2_{2N}\|_2 \le C_5 e^{-t^2} \|T_N\|_\psi.$$

By [Ki], the function T^1_{2N} may be decomposed into $T^1_{2N} = g + h$, where

$$\|g\|_U \le C_6 \mu \|T^1_{2N}\|_\infty, \quad \|h\|_2 \le C_7 e^{-\mu} \|T^1_{2N}\|_\infty. \tag{2.2}$$

Therefore $T_N = g + T^2_{2N} + h$, where

$$\|g\| \le C_8 t\mu \|T_N\|_\psi, \quad \|T^2_{2N} + h\|_2 \le C_9 (e^{-t^2} + te^{-\mu}) \|T_N\|_\psi.$$

We put now $t = \lambda^{1/3}$ and $\mu = \lambda^{2/3}$. By the uniform boundedness of the operators of partial sums in the space U, the lemma follows. □

Lemma 7. *Let $1 \le M \le 2N$, and let T_N be a trigonometric polynomial of degree $\le N$. Then there exists a trigonometric polynomial $T(\theta_M)$, with a number of harmonics $\le M$, such that*

$$\|T_N - T(\theta_M)\|_U \le C_{10} (\frac{N}{M} \log N)^{1/2} \|T_N\|_2.$$

The proof is analogous to that of Lemma 4.

Now the proof of Theorem 2 follows as Theorem 1. We consider first the case $p = 2$. The iterative sequence ($R = N/M$)

$$a_1 = R \log N, \quad a_2 = R \log^3 a_1, \ ... \ a_{k+1} = R \log^3 a_k, ...$$

converges to x_0 the largest root of the equation $x = R \log^3 x$ which can be estimated by $R \log^3 R < x_0 < 2R \log^3 R$. After a finite number of steps, the desired result is achieved.

Let $2 < p < \infty$. The following result can be proved by the same method as in [Ki2]

Lemma 8. *Let $\lambda > 0$ and $2 < p < \infty$. Then every polynomial T_N can be represented as a sum of two polynomials T^1_N and T^2_N, of degree at most N, such that*

$$\|T^1_N\|_U \le C_{11} \lambda^{1/q} \|T_N\|_q, \qquad \|T^2_N\|_2 \le C_{12} \lambda^{1/q - 1/2} \|T_N\|_q.$$

Using this lemma and the already-proved result for $p = 2$, we obtain the assertion of the theorem.

Remark 4. Theorem 2 gives a better estimate than Lemma 7 only for M close to N.

Remark 5. The essential part of the proof is the decomposition obtained in Lemma 6, the proof of which is based on Carleson's famous theorem about convergence almost everywhere of the Fourier series for functions in L_2. It would be interesting to give another proof independent of this deep result.

Approximation of analytic polynomials in the space H^∞

Let us consider an analytic polynomial, i.e.

$$T_N(x) = \sum_{k=0}^{N} c_k e^{ikx}, \tag{3.1}$$

and estimate its best approximation by analytic polynomials with M harmonics. Note that we cannot use Theorem 1 because the polynomial in Theorem 1 is not analytic. It happens that for an analytic polynomial T_N, the approximating polynomial in Theorem 2 is analytic. However Theorem 2 yields a worse estimate of the approximation.

Theorem 3. *Let T_N be an analytic trigonometric polynomial, and let $2 \le p < \infty$ and $1 \le M \le N$. Then there exists an analytic trigonometric polynomial $T(\theta_M; x)$, with a number of harmonics $\le M$, such that*

$$\|T_N - T(\theta_M)\|_\infty \le C_1\Big(\frac{N}{M}\log\Big(\frac{N}{M}+1\Big)\Big)^{1/p}\|T_N\|_p \tag{3.2}$$

and $\theta_M \subset [0, 2N]$.

We begin with the case $p = 2$. The main lemma to be proved is the decomposition lemma.

Lemma 9. *Let $\lambda > 0$ be given. Every analytic polynomial can be represented as the sum of two analytic polynomials, $T_N = T^1_{2N} + T^2_{2N}$, of degree $\le 2N$, such that*

$$\|T^1_{2N}\|_\infty \le C_2\lambda\|T_N\|_\psi, \quad \|T^2_{2N}\|_2 \le C_3 e^{-\lambda^2}\|T_N\|_\psi.$$

Proof. We follow [Ki1] and assume without loss of generality that $\|T_N\|_\psi = 1$. We truncate T_N by multiplying it by an analytic function $\tau(x) = [\alpha + iH(\alpha)]^{-1}$, where H is the Hilbert transform and $\alpha(x) = \max(1, \frac{|T_N|}{\lambda})$. It is obvious that $|\tau(x)| \le 1/\alpha(x)$; therefore $|T_N(x)\tau(x)| \le \lambda$. From the definition of the norm in the Orlicz space and the Tchebychev inequality we have

$$\mu\{x : |T_N(x)| > \lambda\} \le e^{-\lambda^2/2}. \tag{3.3}$$

Denote $E_\lambda = \{x : \ |T_N| > \lambda\}$. Then

$$\int_{\mathbb{T}} |(1-\tau(x))T_N(x)|^2 dx \le C_4\Big(\int_{\mathbb{T}} |T_N|^2 \frac{|1-\alpha|^2}{\alpha^2}dx + \int_{\mathbb{T}} |T_N|^2 \frac{H(1-\alpha)|^2}{\alpha^2}dx\Big). \tag{3.4}$$

Because $\alpha(x) = 1$ if $|T_N| \le \lambda$, the first term in (3.4) can be rewritten in the form $\int_{E_\lambda} ||T_N| - \lambda|^2 dx$, which by virtue of (3.3) can in turn be estimated by

$$\int_\lambda^\infty |u-\lambda| e^{-u^2/2} du \le C_5 e^{\lambda^2/2}.$$

For the second term we have

$$\int_{\mathbb{T}} |T_N|^2 \frac{|H(1-\alpha)|^2}{\alpha^2}dx \le \lambda^2 \int_{\mathbb{T}} |H(1-\alpha)|^2 dx.$$

Since the Hilbert transform is bounded in L_2, it follows that

$$\int_{\mathbb{T}} |H(1-\alpha)|^2 dx \le \int_{\mathbb{T}} (1-\alpha)^2 dx = \int_{|T_N|>\lambda} (1 - |T_N|/\lambda)^2 dx,$$

which we have already estimated. Now $T_N = \tau T_N + (1-\tau)T_N \equiv f_1 + f_2$, where f_1 and f_2 are analytic functions. We apply the de la Vallée-Poussin operator and Lemma 1, and the proof follows.

Now the theorem for $p = 2$ can be proved by the method of Theorem 1. We omit the details.

Since for $2 < p < \infty$ every analytic polynomial can be decomposed as $T_N = T^1_{2N} + T^2_{2N}$ with $\|T^1_{2N}\|_\infty \le C_6\lambda^{1/p}\|T_N\|_p$, $\quad \|T^2_{2N}\|_2 \le C_7\lambda^{1/p-1/2}\|T_N\|_p$ (see [Ki2]), it is sufficient to apply the statement for $p = 2$ and choose an optimal λ. $\square$

Applications

1. Best approximation by trigonometric polynomials of functions from the Sobolev classes. We consider $W_p^{r,\alpha}$ $(1 \le p < \infty)$, the class of functions which have the integral representation

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(x-u) F_{r,\alpha}(u) du, \tag{4.1}$$

where

$$F_{r,\alpha}(u) = \sum_{k=1}^{\infty} \frac{1}{k^r} cos(ku - \pi\alpha/2), \quad 0 \le \alpha < 2,$$

is a Bernoulli kernel and $\|\phi\|_p \le 1$. We consider

$$e_m(W_p^{2,\alpha}; X) = \sup_{f \in W_p^{r,\alpha}} e_m(f; X),$$

the error of the best approximation in the Banach space X of the class $W_p^{r,\alpha}$ by trigonometric polynomials with a prescribed number of harmonics.

Theorem 4. *Let $1 \le p \le \infty$ and $r > 1/p$. Then*

$$e_m(W_p^{r,\alpha}; U) \asymp m^{-\min(r, r-1/p+1/2)}.$$

The lower estimate follows immediately from [Be1], since

$$e_m(W_p^{r,\alpha}; U) \ge e_m(W_p^{r,\alpha}; L_2).$$

The upper estimate can be proved in the usual way (see for example [Be2]).

The exact order of $e_m(W_p^{r,\alpha}; L_q)$, $2 < q < \infty$, was found earlier in [Be2].

Corollary. *Let $1 \le p \le \infty$ and $r > 1/p$. Then*

$$e_m(W_p^{r,\alpha}; L_\infty) \asymp m^{-\min(r, r-1/p+1/2)}.$$

2. Best approximation of analytic functions in the unit disk $|z| \le 1$. In our approach to this problem we consider the subspace $W_{A,p}^{r,\alpha} \subset W_p^{r,\alpha}$ of functions of the form (4.1) with Fourier series containing only exponentials with positive indices.

Theorem 5. *Let $1 \le p \le \infty$ and $r > 1/p$. Then*

$$e_m(W_{A,p}^{r,\alpha}; H^\infty) \asymp m^{-\min(r, r-1/p+1/2)}.$$

The proof is based on Theorem 3.

3. Best approximation and the estimate of trigonometric widths for the classes of functions of several variables. Let f be a real function on $\mathbb{T}^n = [-\pi, \pi]^n$, 2π-periodic in each variable, and let r be an integer. Let W_p^r be the class of functions f with

$$\sum_{r_1 + r_2 + \dots + r_n = r} \|D_{x_1}^{r_1} D_{x_2}^{r_2} \dots D_{x_n}^{r_n} f\|_p \le 1.$$

We shall estimate the trigonometric widths

$$d_m^T(W_p^r; L_q) \overset{def}{=} \inf_{\theta_m} \sup_{f \in W_p^r} \inf_{T(\theta_m)} \|f - T(\theta_m)\|_q,$$

where the infimum is taken over all subsets of cardinality m in $\mathbb{Z}^n$. It is obvious that the trigonometric m-width is no less than the Kolmogorov m-width, namely,

$$d_m(W_p^r; L_q) \stackrel{def}{=} \inf_{L_m} \sup_{f\in W_p^r} \inf_{u\in L_m} \|f-u\|_q, \tag{4.2}$$

where the infimum is taken over all subspaces L_m of dimension m.

Another obvious but useful inequality is

$$e_m(W_p^r; L_q) \le d_m^T(W_p^r; L_q).$$

Theorem 6. *Let $1 \le p \le 2$, $p' = p/p-1$, $2 < q \le p' \le \infty$, and $r > n$. Then*

$$d_m^T(W_p^r; L_q) \asymp m^{-(r/n-1/p+1/2)}.$$

Due to estimates for the Kolmogorov widths ([Mr3], [Ho]), the lower estimate is a corollary of them. The upper estimate strengthens the corresponding estimate for the Kolmogorov widths. The proof of the upper estimate is based on the extension of the estimate of Theorem 1 to the case of several variables.

Remark 6. In this connection it would be interesting to find an extension of Theorem 4 to the case of several variables. The main open question is a corresponding decomposition lemma.

Acknowledgments.

I would like to express my deep gratitude to Yuly Makovoz and Dany Leviatan for their helpful comments on this paper. I also thank the referee, who pointed out to me how Lemma 8 and therefore Theorem 2 could be strengthened.

Note. After this paper was submitted the author learned that a result analogous to Theorem 1 was proved by another method by R. DeVore and V. Temlyakov. Their proof is mainly based on Gluskin's finite-dimensional result [G1].

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