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REALIZING HOMOLOGY BOUNDARY LINKS WITH ARBITRARY PATTERNS

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Abstract. Homology boundary links have become an increasingly important class of links, largely due to their significance in the ongoing concordance classification of links. Tim Cochran and Jerome Levine defined an algebraic object called a pattern associated to a homology boundary link which can be used to study the deviance of a homology boundary link from being a boundary link. Since a pattern is a set of m elements which normally generates the free group of rank m, any invariants which detect non-trivial patterns can be applied to the purely algebraic question of when such a set is a set of conjugates of a generating set for the free group. We will give a constructive geometric proof that all patterns are realized by some homology boundary link L^n in S^{n+2}. We shall also prove an analogous existence theorem for calibrations of $\mathbb{E}$-links, a more general and less understood class of links tha homology boundary links.

1. Introduction

A *link* of m components is a smooth, oriented, submanifold $L = \{K_1, \dots, K_m\}$ of S^{n+2} that is an ordered disjoint union of m manifolds, each of which is piecewise-linearly homeomorphic to S^n. If $m = 1$, L is usually called a *knot*. L is a *boundary link* if there exist m disjoint Seifert surfaces, i.e. oriented submanifolds $V_1, \dots, V_m$ of S^{n+2} such that $\partial V_i = K_i$ for $i = 1, \dots, m$.

Knots and links are of interest since they repeatedly arise in the classification of manifolds. An especially significant equivalence relation on links is concordance. Two links L_0 and L_1 are *concordant* if there exists a smooth, oriented submanifold of m components $C = \{C_1, \dots, C_m\}$ of $S^{n+2} \times I$ such that:

(i) C is piecewise-linearly homeomorphic to $L_0 \times I$, and
(ii) $\partial C \cap (S^{n+2} \times \{i\}) = L_i$ for $i \in \{0, 1\}$.

The classification of knot concordance groups was obtained in the mid 1960's by M. Kervaire and J. Levine [12], [14]. Among the things they prove is that the knot concordance group is trivial when n is even and is an infinitely generated group when n is odd. The techniques used in the concordance classification of knots have been found to extend in a compatible manner to the class of boundary links [2], [13], [19]. However, the extension of these ideas to links in general has been a much more difficult and less successful task. In the process, another class of links, called homology boundary links, has arisen. A *homology boundary link* of m components is a link which admits m disjoint generalized Seifert surfaces $\{Y_1, \dots, Y_m\}$ such that

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∂Y_i is homologous to K_i in the boundary of a tubular neighborhood of L. In other words, ∂Y_i may consist of many components, each of which is an oriented longitude of some K_j [20].

Homology boundary links have become an increasingly important class of links. In [6], [7], T. D. Cochran and K. E. Orr provided the first examples of homology boundary links which are not concordant to boundary links (cf. [9]). In [10], a scheme for classifying concordance classes of homology boundary links, analogous to that for boundary links, was presented. Further investigation has shown that sublinks of homology boundary links are in fact the "fundamental" class of links to use in the concordance classification of links [5], [16], [18]. Homology boundary links have also appeared in connection with the Alexander ideals of links [20] and in connection with the Andrews-Curtis conjecture [5].

An equivalent definition of homology boundary link is the following: Let L be an m-component n-dimensional link with link group $G = \pi_1(S^{n+2} - L)$. Then L is a homology boundary link if and only if there exists an epimorphism Φ from G onto $F = \langle x_1, \dots, x_m \rangle$, the free group on m letters [20]. It is known that L is a boundary link if and only if G admits an epimorphism Φ (as above) with the additional condition that for some choice of meridians $\{\mu_i\} \subseteq G$, $\{\Phi(\mu_i)\}$ is a basis for F. Cochran and Levine [5] used this set $\{\Phi(\mu_i)\}$, which they called a pattern of a link, to study the deviance of a homology boundary link from being a boundary link. Precisely, we have:

Definition 1.1. A *pattern* $P = (r_1, \dots, r_m)$ is an m-tuple of words in $F = \langle x_1, \dots, x_m \rangle$, a free group on m letters, such that P normally generates F. A homology boundary link L *admits* P *as a pattern* if there exist an epimorphism $\Phi\colon G = \pi_1(S^{n+2} - L) \to F$ and a choice of meridians $\{\mu_i\}$ of L such that $\Phi(\mu_i) = r_i$ for $i = 1, \dots, m$.

Cochran and Levine [5] proved the following theorem algebraically:

Theorem 2.6. *Given any pattern P and any positive integer n, there exists a homology boundary link L in S^{n+2} admitting P. In particular, L is a ribbon link.*

In this paper, we give a constructive geometric proof of this theorem, one which provides an actual ribbon link with the desired pattern. This completes the work of Cochran and Orr on classifying homology boundary links with specified pattern and Seifert form [8]. Further, with a slight modification of the proof, we prove an analogous existence theorem for a more general class of links, namely $\mathbb{E}$-links, which are defined and discussed in §3. The class of $\mathbb{E}$-links is essentially the class of links which are sublinks of homology boundary links. Thus, according to [5], [16], [18] this is an important class of links.

Theorem 3.4. *Given any finitely generated $\mathbb{E}$-group E, calibration $(E, \{y_1, \dots, y_m\})$, and positive integer n, there exists an $\mathbb{E}$-link L in S^{n+2} admitting the E-calibration $(E, \{y_1, \dots, y_m\})$. In particular, L is a ribbon link.*

As we will see, an $\mathbb{E}$-calibration is the analog to $\mathbb{E}$-links of a pattern.

2. Fusions of links

Given a link L^n in S^{n+2} and an arc b connecting two different components of L, i.e. b is smoothly embedded in S^{n+2} and intersects L only at its endpoints (orthogonally), choose a normal vector field ν along b which is normal to L at both

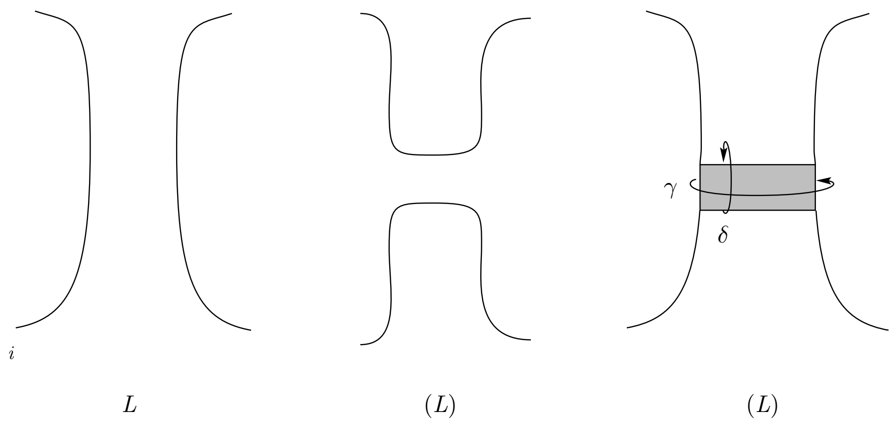


FIGURE 2.1

endpoints of b. With the proper orientation of b, one can perform the connected sum of the two components of L along b (just use the orthogonal complement of ν in a tubular neighborhood of b as the connecting tube). The resulting link $F(L)$ is a link with one less component than L and is called the *fusion* of L along the band $B = \{\nu \cup b\}$. One can perform more than one fusion to a link along a collection of bands $\{B_i\}$, thus obtaining a sequence of fusions $F_1(L), \ldots, F_k(L)$. $F(L) = F_k(L)$ is called the fusion of a link along the bands $\{B_i\}$. Of particular interest is the situation when L is a boundary link, and one performs a sequence of fusions of L along a collection of bands $\{B_i\}$, resulting in the *fusion of a boundary link* along the bands $\{B_i\}$ (cf. [3], [5]). Fusions of boundary links are of interest here because one can obtain useful information relating the link groups of L and $F(L)$.

Let L be a link in S^3, $F(L)$ its fusion along a band B, and $B(L)$ the corresponding "band link", i.e. $B(L) = L \cup B$ as shown in Figure 2.1. There is no direct way to relate $\pi_1(S^3 - L)$ and $\pi_1(S^3 - F(L))$, the link groups of L and $F(L)$, respectively. However, L and $F(L)$ both are subsets of $B(L)$. Hence, there exist inclusion maps of their complements $\phi\colon S^3 - B(L) \to S^3 - L$ and $\psi\colon S^3 - B(L) \to S^3 - F(L)$ and corresponding induced maps $\phi_*\colon \pi_1(S^3 - B(L)) \to \pi_1(S^3 - L)$ and $\psi_*\colon \pi_1(S^3 - B(L)) \to \pi_1(S^3 - F(L))$. The curves γ and δ are of particular interest here. Note that $\delta \in \ker(\phi_*)$ and $\gamma \in \ker(\psi_*)$. Observe that $S^3 - L = S^3 - B(L) \cup 2$-handle and $S^3 - F(L) = S^3 - B(L) \cup 2$-handle, so the link groups of L and $F(L)$ can be related as quotient groups of the "link" group $\pi_1(S^3 - B(L))$.

Decompose S^3 as the union of two 3-balls D^3_+ and D^3_- in such a way that D^3_- is a small 3-ball containing the bands $\{B_i\}$. Think of D^3_+ and D^3_- as the "outside" and "inside" of $S^2 = D^3_+ \cap D^3_- = \partial D^3_+ = \partial D^3_-$. Note that for every fusion band B_i, there exist curves γ_i and δ_i as described in Figure 2.1. Set:

(i) $U = D^3_+ - L$,
(ii) $V_1 = D^3_- - L$, and
(iii) $V_2 = D^3_- - F(L)$.

Applying Seifert-Van Kampen twice, one obtains the following two isomorphisms:

$$\Theta_L\colon \pi_1(S^3 - L) \to \pi_1(U)/\langle\{\delta_i\}\rangle, \tag{1}$$

$$\Theta_{F(L)}\colon \pi_1(S^3 - F(L)) \to \pi_1(U)/\langle\{\gamma_i\}\rangle. \tag{2}$$

Define an epimorphism $K\colon \pi_1(U)/\langle\{\gamma_i\}\rangle \to \pi_1(U)/\langle\{\gamma_i,\delta_i\}\rangle$ whose kernel is $\langle\{\delta_i\}\rangle$. Then

$$\Psi = \Theta_L^{-1} \circ K \circ \Theta_{F(L)} \colon \pi_1(S^3 - F(L)) \to \pi_1(S^3 - L)/\langle\{\gamma_i\}\rangle$$

is an epimorphism from the link group of $F(L)$ onto a quotient group of the link group of L. Thus, we have proved the following proposition.

Proposition 2.2. *Let $F(L)$ be a fusion of a boundary link along fusion bands $\{B_1,\dots,B_m\}$. Then there exists an epimorphism*

$$\Psi\colon \pi_1(S^3 3 - F(L)) \to \pi_1(S^3 - L)/\langle\{\gamma_i\}\rangle.$$

Remarks 2.3. (1) The proof of Proposition 2.2 shows that the epimorphism Ψ is defined independently of the order in which the bands are fused onto L.

(2) One can easily generalize Proposition 2.2 to links L^n of higher dimensions. The proof can be altered in the obvious way so that γ_i is a circle and δ_i is an n-sphere. Thus, Ψ will be an isomorphism since its kernel is $\langle\delta_i\rangle$ (which is trivial in π_1).

(3) We can further analyze the curves γ_i as follows. If the fusion band B_i fuses together components K_j and K_k of L and if x_{ij} and x_{ik} are meridional elements of the link group of L for K_j and K_k, respectively, then $\gamma_i = \bar{x}_{ij}\eta_i x_{ik}\bar{\eta}_i$ for $\eta_i \in \pi_1(S^3 - L)$ and where $\bar{x}_i = x_i^{-1}$ (see Figure 2.4).

This leads to the following useful corollary (cf. [3]).

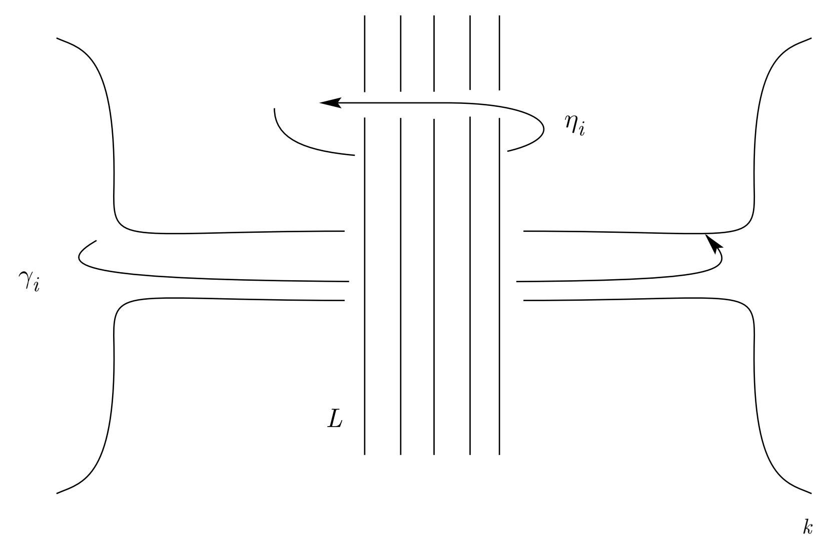


FIGURE 2.4

Corollary 2.5. *Let $L = \{K_1,\dots,K_m\}$ be an m-component trivial link, i.e. $\pi_1(S^3 - L) = \langle x_1,\dots,x_m\rangle$ where x_j is a meridional generator of K_j. Let $F(L)$ be a fusion of L along fusion bands $\{B_1,\dots,B_k\}$, where B_i fuses together components K_j and K_k. Then there exists an epimorphism*

$$\Psi\colon \pi_1(S^3 - F(L)) \to \langle x_1,\dots,x_m | x_{ij} = \eta_i x_{ik}\bar{\eta}_i\rangle$$

where $\eta_i \in \pi_1(S^3 - L)$.

We are now ready to present a constructive geometric proof of Cochran and Levine's theorem (cf. [9]).

Theorem 2.6. *Given any pattern P and any positive integer n, there exists a homology boundary link L in S^{n+2} admitting P. In particular, L is a ribbon link.*

Proof. Let $F = \langle x_1, \ldots, x_m\rangle$ be a free group and suppose $P = (r_1, \ldots, r_m)$ is a pattern in F. Since the map $\mathrm{Aut}(F) \to \mathrm{Aut}(F/[F,F])$ is onto, assume without loss of generality that $x_i = r_i \mod [F,F]$ for all i. Since P is a pattern, the elements of P normally generate F, so each generator x_i may be expressed as a word w_i such that

$$w_i = \prod_{j=1}^{\lambda_i} \eta_{ij} r_{\nu(i,j)}^{\varepsilon_{ij}} \bar{\eta}_{ij}$$

where $r_{\nu(i,j)} \in \{r_1, \ldots, r_m\}$, η_{ij} is a word $\prod_{k=1}^{\lambda_{ij}} x_{\sigma(i,j,k)}^{\varepsilon_{ijk}}$ in $\{x_i\}$, and ε_{ij}, $\varepsilon_{ijk} \in \{\pm 1\}$. By representing each element of the given basis of F as a product of conjugates of pattern elements, we have achieved a decomposition which will appear in the construction upon application of Corollary 2.5.

Begin the construction by drawing m circles one below another, as shown in Figure 2.7; call these circles β_i. Next to β_i, draw λ_i concentric circles; call these circles α_{ij} for $1 \le i \le m$, $1 \le j \le \lambda_{ij}$. Orient the β_i's all clockwise and orient α_{ij} clockwise or counterclockwise depending on whether the exponent ε_{ij} of $r_{\nu(i,j)}$ in w_i is $+1$ or -1. Call the result L_0.

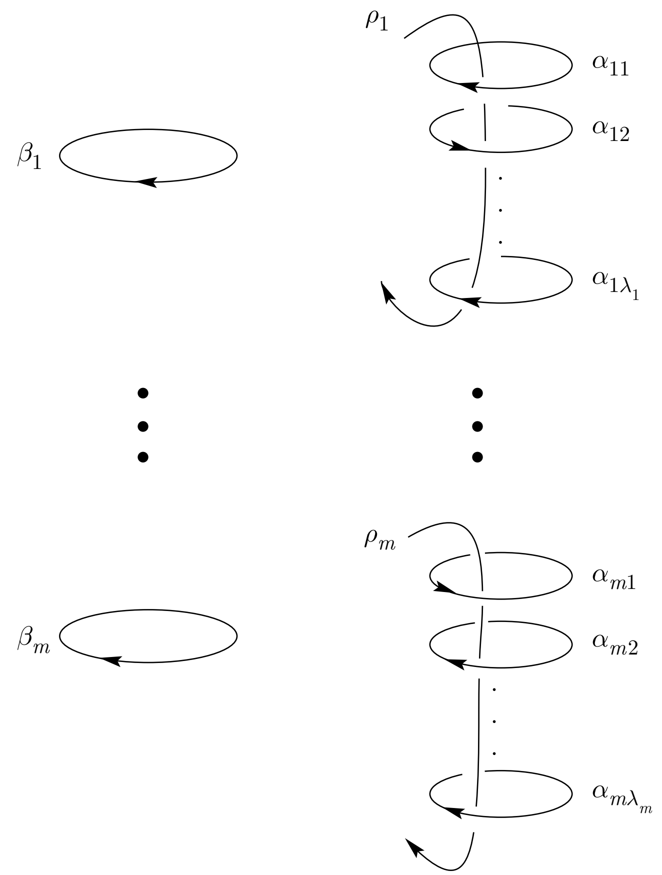
Note that $G_0 = \pi_1(S^3 - L_0)$ is a free group on $m + \sum_{i=1}^m \lambda_i$ letters. Let b_i and a_{ij} be simple meridional generators in G_0 for β_i and α_{ij}, respectively, oriented using the right-hand rule. Then $G_0 \approx \langle b_i, a_{ij}\rangle$. Now we will add fusion bands B_{ij} to L_0 to obtain the desired link L. Fuse a band B_{ij} to L_0 starting at α_{ij} and connecting to $\beta_{\nu(i,j)}$. Pass the band B_{ij} through the center of the $\sigma(i,j,k)$th set of concentric circles for every $x_{\sigma(i,j,k)}$ in η_{ij}. B_{ij} should pass top to bottom if the exponent ε_{ijk} of $x_{\sigma(i,j,k)}$ is $+1$ and bottom to top if the exponent is -1. If the circle α_{ij} is oriented clockwise, B_{ij} must be given a half-twist before it is attached to β_j to preserve orientation; if α_{ij} is oriented counterclockwise, no twists are necessary. Fuse such a band B_{ij} to L_0 for $1 \le i \le m$ and $1 \le j \le \lambda_i$ in the manner described above to obtain the desired link L; i.e. L will be the fusion of L_0 along the bands $\{B_{ij}\}$.

To show that L admits P as a pattern, we must exhibit an epimorphism $\Phi\colon \pi_1(S^3 - L) \to F$ such that for some choice of meridians $\{\mu_i\}$ of L, $\Phi(\mu_i) = r_i$ for $i = 1, \ldots, m$ (L_0 has $m + \sum_{i=1}^m \lambda_i$ components and $\sum_{i=1}^m \lambda_i$ bands were fused to L_0, so L is an m-component link). From the construction, one observes that each component β_i of L_0 lies in a different component of L (so we can consider L similarly ordered). Choose a set of meridians $\{\mu_i\}$ of L such that each μ_i is a simple meridian of the ith component of L equivalent to b_i. The curve γ_{ij} corresponding to the band B_{ij}, by construction, is expressed as:

$$a_{ij} = \prod_{k=1}^{\lambda_{ij}} \left[\left(\prod_{t=1}^{\lambda_{ijk}} a_{\sigma(i,j,k)t}^{\varepsilon_{\sigma(i,j,k)t}}\right)\right]^{\varepsilon_{ijk}} b_{\nu(i,j)} \overline{\prod_{k=1}^{\lambda_{ij}} \left[\left(\prod_{t=1}^{\lambda_{ijk}} a_{\sigma(i,j,k)t}^{\varepsilon_{\sigma(i,j,k)t}}\right)\right]^{\varepsilon_{ijk}}}$$

in $\pi_1(S^3 - L_0)$. By Proposition 2.2, there exists an epimorphism

$$\Psi\colon \pi_1(S^3 - L) \to \pi_1(S^3 - L_0)/\langle \gamma_{ij}\rangle,$$



L_0

FIGURE 2.7

where by Corollary 2.5

$$\pi_1(S^3 - L_0)/\langle \gamma_{ij} \rangle$$
$$= \left\langle b_i, a_{ij} | a_{ij} = \prod_{k=1}^{\lambda_{ij}} \left[\left(\prod_{t=1}^{\lambda_{ijk}} a_{\sigma(i,j,k)t}^{\varepsilon_{\sigma(i,j,k)t}} \right) \right]^{\varepsilon_{ijk}} b_{\nu(i,j)} \overline{\prod_{k=1}^{\lambda_{ij}} \left[\left(\prod_{t=1}^{\lambda_{ijk}} a_{\sigma(i,j,k)t}^{\varepsilon_{\sigma(i,j,k)t}} \right) \right]^{\varepsilon_{ijk}}} \right\rangle.$$

Using Tietze transformations on the group $\pi_1(S^3 - L_0)/\langle \gamma_{ij} \rangle$, add generators A_i and relations $A_i = \prod_{j=1}^{\lambda_i} a_{ij}^{\varepsilon_{ij}}$. Then recall Figure 2.7 and observe that the curve ρ_i is represented in $\pi_1(S^3 - L_0)/\langle \gamma_{ij} \rangle$ by $\prod_{j=1}^{\lambda_i} a_{ij}^{\varepsilon_{ij}}$. So we have obtained the following presentation:

$$\pi_1(S^3 - L_0)/\langle \gamma_{ij} \rangle = \left\langle b_i, a_{ij}, A_i | a_{ij} = \prod_{k=1}^{\lambda_{ij}} A_{\sigma(i,j,k)}^{\varepsilon_{ijk}} b_{\nu(i,j)} \overline{\prod_{k=1}^{\lambda_{ij}} A_{\sigma(i,j,k)}^{\varepsilon_{ijk}}}, A_i = \prod_{j=1}^{\lambda_i} a_{ij}^{\varepsilon_{ij}} \right\rangle.$$

Define a map $\Psi_0\colon \pi_1(S^3 - L_0)/\langle\gamma_{ij}\rangle \to F$ where

$$b_i \to r_i, \quad a_{ij} \to \eta_{ij} r_{\nu(i,j)} \bar{\eta}_{ij}, \quad \text{and} \quad A_i \to x_i.$$

Ψ_0 is well defined since the image of relations

$$a_{ij} = \prod_{k=1}^{\lambda_{ij}} A_{\sigma(i,j,k)}^{\varepsilon_{ijk}} b_{\nu(i,j)} \overline{\prod_{k=1}^{\lambda_{ij}} A_{\sigma(i,j,k)}^{\varepsilon_{ijk}}}$$

are trivial in F, and the relations $A_i = \prod_{j=1}^{\lambda_i} a_{ij}^{\varepsilon_{ij}}$ become $x_i = w_i$. Ψ_0 is onto since $\{\Psi_0(A_i)\}$ is a basis for F. Define $\Phi\colon \pi_1(S^3 - L) \to F$ by $\Phi = \Psi_0 \circ \Psi$. One checks that $\Phi(\mu_i) = \Psi_0(b_i) = r_i$. Thus L is a homology boundary link admitting P as a pattern. One further observes that L is a ribbon link by construction. ◇

Remarks 2.8. (1) The link L obtained in the proof is not unique. The proof only requires that fusion bands be added which do not intersect one another or the link L_0 (except where they are properly attached). The fusion band may be twisted and knotted in any fashion as long as these necessary conditions are maintained!

(2) The proof is the same for the construction of higher dimensional links (cf. Proposition 2.2, Remarks 2.3).

We close this section with two examples. The first is a relatively simple one, which provides a homology boundary link which is not concordant to a boundary link. The second example uses a pattern which is only slightly more complicated than the first example. However, the resulting link is much more complicated!

Example 2.9. Consider the free group $F = \langle x_1, x_2\rangle$ and pattern $P_1 = (r_1, r_2) = (x_1[x_1, x_2], x_2)$. Cochran and Orr constructed a link admitting this pattern which was the first (and still simplest known) example of a homology boundary link which is not concordant to a boundary link [6], [7]. Since P_1 is a pattern, P_1 normally generates F, so x_i is expressible as a word w_i where

$$w_i = \prod_{j=1}^{\lambda_i} \eta_{ij} r_{\nu(i,j)}^{\varepsilon_{ij}} \bar{\eta}_{ij}.$$

For P_1, we have

$$\begin{aligned}
&\lambda_1 = 3, \quad \lambda_2 = 1,\\
&r_{\nu(1,1)} = r_1, \quad r_{\nu(1,2)} = r_{\nu(2,1)} = r_2, \quad r_{\nu(1,3)} = \bar{r}_2,\\
&\eta_{11} = \eta_{12} = \bar{x}_1, \quad \eta_{13} = \eta_{21} = 1.
\end{aligned}$$

Applying the construction in Theorem 2.6, one obtains a homology boundary link L_1, pictured in Figure 2.10. One may also observe that L_1 is equivalent to the initial link used by Cochran and Orr, shown in Figure 2.11.

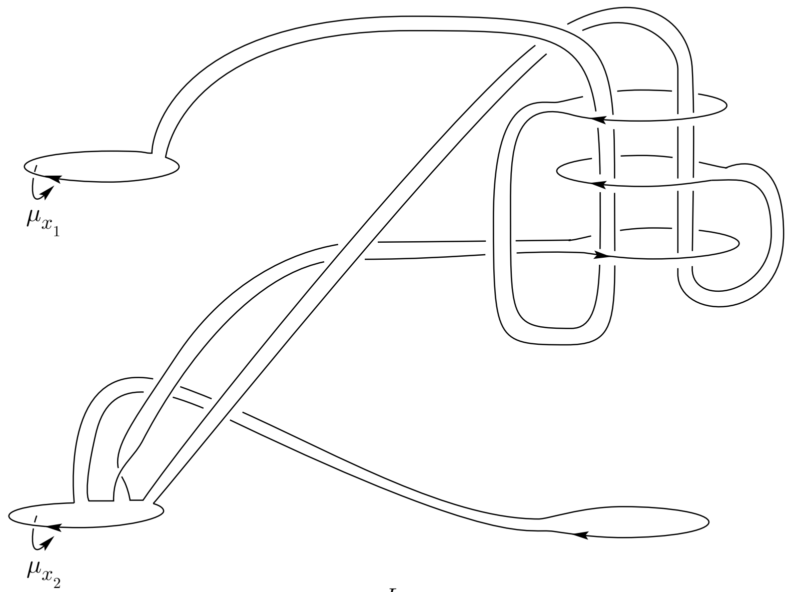


FIGURE 2.10

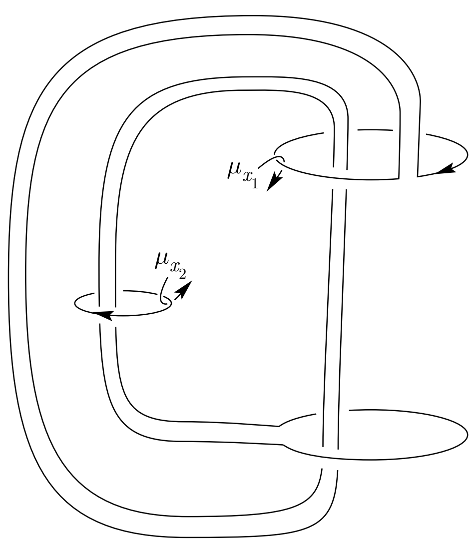
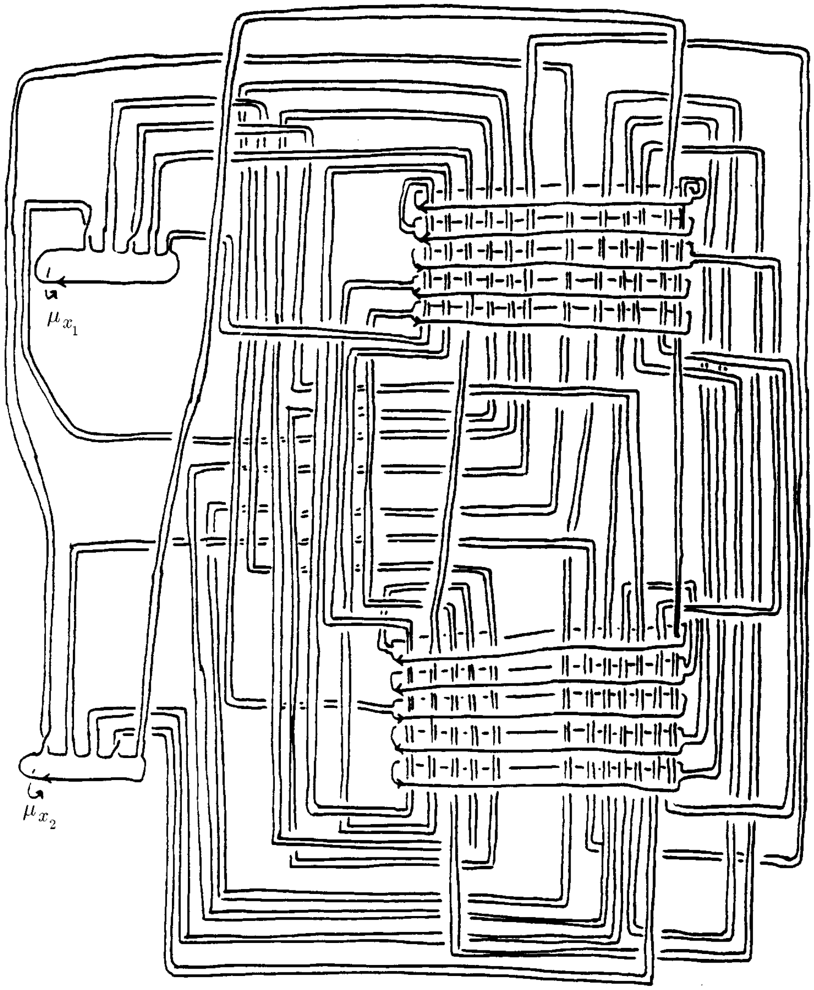


FIGURE 2.11

Example 2.12. Consider the free group $F = \langle x_1, x_2 \rangle$ and pattern $P_2 = (r_1, r_2) = ([x_2, \bar{x}_1]x_1, [x_1, \bar{x}_2]x_2)$. Note that no element of P_2 is conjugate to any generator of F, so any homology boundary link admitting P cannot be a boundary link [5]. Nor can L be the strong fusion of a boundary link [10]. Since P_2 is a pattern, P_2



L_2

FIGURE 2.13

normally generates F, so x_i is expressible as a word w_i where

$$w_i = \prod_{j=1}^{\lambda_i} \eta_{ij} r^{\varepsilon_{ij}}_{\nu(i,j)} \bar{\eta}_{ij}.$$

For the pattern P_2 given above we have the following:

$$\lambda_1 = \lambda_2 = 5,$$
$$r_{\nu(1,2)} = r_{\nu(1,4)} = r_{\nu(2,1)} = r_1, \quad r_{\nu(1,5)} = r_{\nu(2,3)} = \bar{r}_1,$$
$$r_{\nu(1,1)} = r_{\nu(2,2)} = r_{\nu(2,4)} = r_2, \quad r_{\nu(1,3)} = r_{\nu(2,5)} = \bar{r}_2,$$
$$\eta_{11} = \eta_{25} = x_1 x_2 \bar{x}_1, \quad \eta_{12} = \eta_{23} = x_1, \quad \eta_{13} = \eta_{22} = x_2,$$
$$\eta_{14} = [x_2, x_1], \quad \eta_{15} = \eta_{21} = x_2 x_1 \bar{x}_2, \quad \eta_{24} = [x_1, x_2].$$

So applying the construction in Theorem 2.6 one obtains the following homology boundary link L_2, as shown in Figure 2.13 which admits P as a pattern, via some epimorphism $\Phi\colon \pi_1(S^3 - L) \to F$ where $\Phi(\mu_i) = x_i$ for $i = 1, 2$. Note that L_2 is the fusion of a 12-component trivial link along $10(= \sum \lambda_i)$ fusion bands [3], [4].

3. Generalizations to $\mathbb{E}$-links

With some small modifications, we can extend Theorem 2.6 to a more general class of links, namely $\mathbb{E}$-links [3], [16].

Definition 3.1. A group E is an $\mathbb{E}$-group if there exists a (not necessarily finite) 2-complex K such that:

(i) $\pi_1(K) = E$,
(ii) $H_1(K)$ is torsion-free, and
(iii) $H_2(K) = 0$.

If K is a finite complex, then E is said to be a *finite* $\mathbb{E}$-group.

All free groups are necessarily $\mathbb{E}$-groups. $\mathbb{E}$-groups are a more general class of groups than free groups. For example, any group E whose deficiency is equal to $\mathrm{rank}(H_1(E))$ is also an $\mathbb{E}$-group [3], [16].

Definition 3.2. Suppose E is an $\mathbb{E}$-group with $\mathrm{rank}(H_1(E)) = m$. An *E-calibration* is a pair $(E, \{y_1, \ldots, y_m\})$ where the $\{y_1, \ldots, y_m\}$ normally generates E.

An E-calibration is the analog of a pattern for $\mathbb{E}$-groups. Note than an E-calibration can only exist for $\mathbb{E}$-groups which are normally generated by m elements where $m = \mathrm{rank}(H_1(E))$.

Definition 3.3. A link L is said to be an $\mathbb{E}$-link if it admits an E-calibration for some $\mathbb{E}$-group E.

In [3], it is shown that every sublink of a homology boundary link is an $\mathbb{E}$-link. With a small additional assumption (it is still unknown whether this assumption is necessary), Levine showed that every finite $\mathbb{E}$-link is a sublink of a homology boundary link [16], [18].

Given a finitely generated $\mathbb{E}$-group E normally generated by elements $\{y_1, \ldots, y_m\}$, where $m = \mathrm{rank}(H_1(E))$, one can construct a second $\mathbb{E}$-group E' of rank m such that:

(i) there exists an epimorphism $\phi\colon E' \to E$,
(ii) E' is normally generated by y'_i where $\phi(y'_i) = y_i$, and
(iii) E' has the following presentation:

$$E' = \left\langle x'_1, \ldots, x'_{m'}, y'_1, \ldots, y'_m | x'_i = \prod_{j=1}^{\lambda_i} \eta_{ij} y'^{\varepsilon_{ij}}_{\nu(i,j)} \bar{\eta}_{ij} \right\rangle,$$

where $\{x'_i\}$, $1 \le i \le m'$, generate $E'(m \le m')$.

Construct E' as follows. Suppose E is generated by $\{x'_1, \ldots, x'_{m'}\}$. Then the free group on $\{x'_i\} \cup \{y'_i\}$ maps onto E by $\phi(x'_i) = x_i$, $\phi(y'_i) = y_i$. Since $\{y_i\}$ normally generate E, this map factors through the group E' above. Consider a 2-complex K' consisting of the wedge product of m 1-cells with m' 2-cells attached giving the presentation listed above. By construction, K' satisfies Definition 3.1, so E' is a finite $\mathbb{E}$-group.

In order to generalize Theorem 2.6, we would like the collection $\{y_i\}$ to play the role that the pattern did in the case of homology boundary links. Note that it is sufficient to realize $(E', \{y_i'\})$ as an E'-calibration for L since we can then map $(E', \{y_i'\})$ onto $(E, \{y_i\})$ via ϕ to obtain an E-calibration for L. The immediate obstruction is that the group E' (and E) onto which the calibration maps is not necessarily a free group and, worse yet, may not even be generated by the same number of elements as in the normal generating set $\{y_i\}$, i.e. $m' \geq m$. This obstruction is overcome by beginning with a trivial link of more components and fusing more bands to obtain the desired $\mathbb{E}$-link L.

Theorem 3.4. *Given any finitely generated $\mathbb{E}$-group E, calibration $(E, \{y_1, \ldots, y_m\})$, and positive integer n, there exists an $\mathbb{E}$-link L in S^{n+2} admitting the E-calibration $(E, \{y_1, \ldots, y_m\})$. In particular, L is a ribbon link.*

Proof. Let $\{y_1, \ldots, y_m\}$ be a normal generating set for E. By the remarks preceding the statement of the theorem, we may assume that E has presentation:

$$E = \left\langle x_1, \ldots, x_{m'}, y_1, \ldots, y_m | x_i = \prod_{j=1}^{\lambda_i} \eta_{ij} y_{\nu(i,j)}^{\varepsilon_{ij}} \bar{\eta}_{ij} \right\rangle.$$

Follow Theorem 2.6 and begin the construction by drawing m circles one below another; orient all m circles counterclockwise and call these circles β_k. Next to the β_k's draw m' sets of λ_i concentric circles; call these circles α_{ij}. Orient the circles β_k and α_{ij} as in Theorem 2.6 for $1 \leq k \leq m$, $1 \leq i \leq m'$, and $1 \leq j \leq \lambda_i$. For simplicity, let $R_{ij} = \eta_{ij} y_{\nu(i,j)}^{\varepsilon_{ij}} \bar{\eta}_{ij}$; that is, let R_{ij} be the jth conjugate in the relation expressing x_i as a product of conjugates in the presentation for E. Orient α_{ij} clockwise if the exponent ε_{ij} of $y_{\nu(i,j)}$ in R_{ij} is $+1$ and counterclockwise if -1. Call the resulting link L_0 (see Figure 3.5).

Note that L_0 is a trivial link of $m + \sum_{i=1}^{m'} \lambda_i$ components. Following the construction in Theorem 2.6, add $\sum_{i=1}^{m'} \lambda_i$ fusion bands to L_0 which connect α_{ij} to $\beta_{\nu(i,j)}$ to obtain the desired $\mathbb{E}$-link L. One then proceeds exactly as in Theorem 2.6 to obtain an E-calibration Ψ for L, completing the proof. $\diamond$

As was the case with the proof of Theorem 2.6, the proof of this theorem can be interpreted in the obvious way to produce $\mathbb{E}$-links of any dimension. Lastly, we close with an example illustrating the theorem.

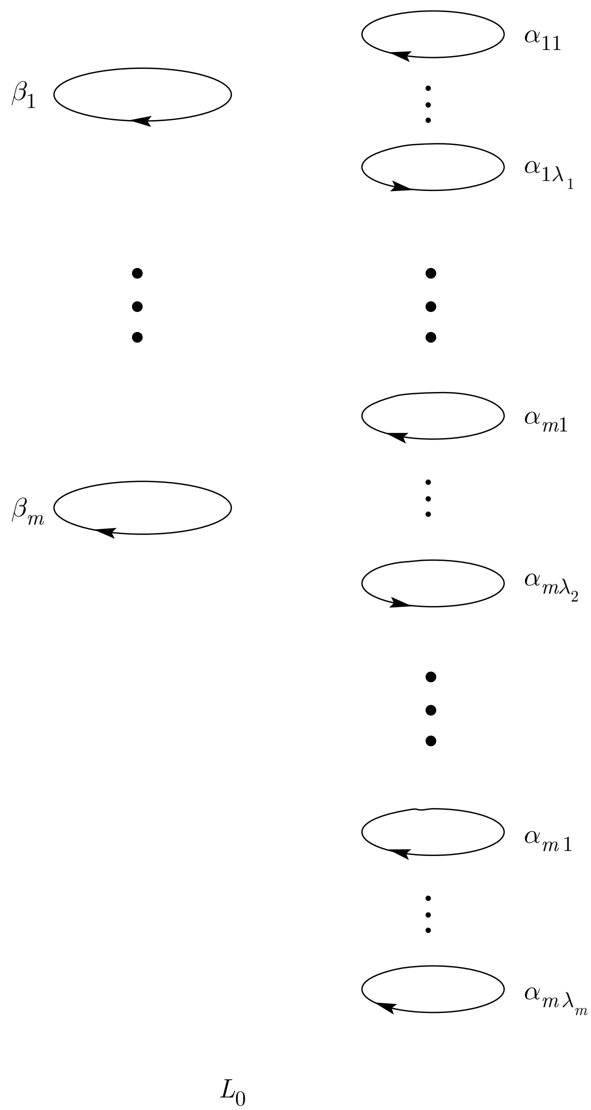


FIGURE 3.5

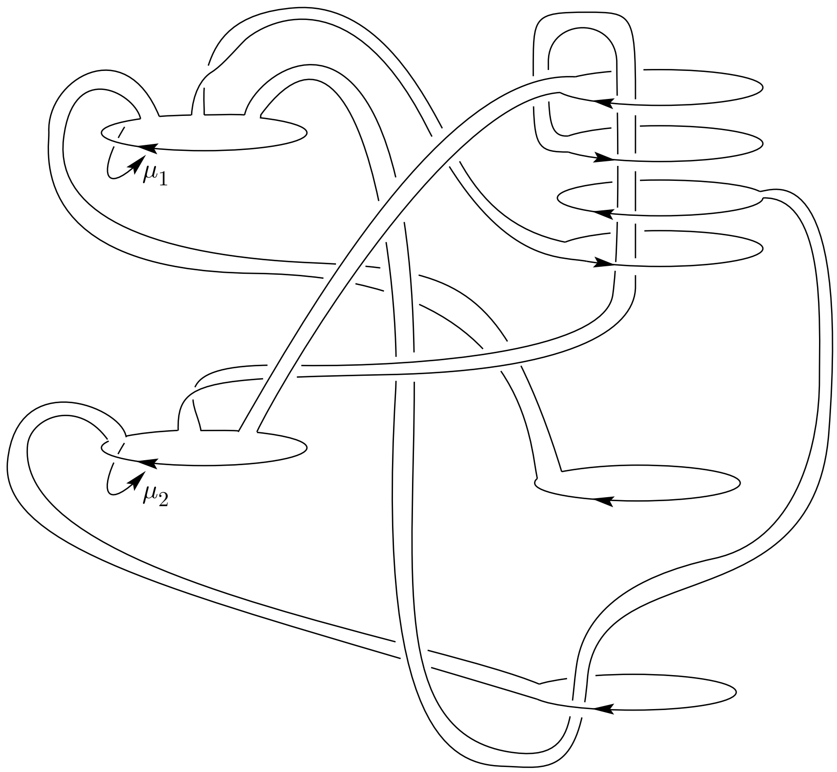
Example 3.6. Consider the $\mathbb{E}$-group with presentation

$$E = \langle x_1, x_2, x_3 | \bar{x}_1[x_3, x_1][x_3, x_2] \rangle.$$

Using $\{x_2, x_3\}$ as a normal generating set for E, we may equivalently present E by

$$\langle x_1, x_2, x_3, y_1, y_2 | x_1 = y_2 x_1 \bar{y}_2 \bar{x}_1 x_3 y_1 \bar{x}_3 \bar{y}_1, x_2 = y_1, x_3 = y_2 \rangle.$$

G. Baumslaug [1] has shown that E is parafree but not free, and therefore cannot map onto a free group of rank 2. Hence, any $\mathbb{E}$-link L calibrating E cannot be a homology boundary link; however L will be a sublink of a homology boundary link [10]. Following the construction outlined in Theorem 3.4, a link L calibrating E is shown in Figure 3.7. Observe that the meridians μ_1 and μ_2 labeled in the diagram



L

FIGURE 3.7

are the meridians which the E-calibration will map to the normal generating set $\{y_1 = x_2, y_2 = x_3\}$ in E.

$$\lambda_1 = 4, \quad \lambda_2 = \lambda_3 = 1,$$
$$r_{\nu(1,1)} = y_2, \quad r_{\nu(1,2)} = \bar{y}_2, \quad r_{\nu(1,3)} = y_1, \quad r_{\nu(1,4)} = \bar{y}_1,$$
$$r_{\nu(2,1)} = y_1, \quad r_{\nu(3,1)} = y_2,$$
$$\eta_{12} = x_1, \quad \eta_{13} = x_3, \quad \eta_{11} = \eta_{14} = \eta_{21} = \eta_{31} = 1.$$

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