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DOUBLE WALSH SERIES WITH COEFFICIENTS OF BOUNDED VARIATION OF HIGHER ORDER

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ABSTRACT. Let $D_j^k(x)$ denote the Cesàro sums of order k of the Walsh functions. The estimates of $D_j^k(x)$ given by Fine back in 1949 are extended to the case $k > 2$. As a corollary, the following properties are established for the rectangular partial sums of those double Walsh series whose coefficients satisfy conditions of bounded variation of order $(p, 0)$, $(0, p)$, and (p, p) for some $p \geq 1$: (a) regular convergence; (b) uniform convergence; (c) L^r-integrability and L^r-metric convergence for $0 < r < 1/p$; and (d) Parseval's formula. Extensions to those with coefficients of generalized bounded variation are also derived.

0. INTRODUCTION

Let $I \equiv [0, 1)$. Denote by $\{\omega_n(t)\}$ the Paley-Walsh orthonormal system defined on I. Consider the double Walsh series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\omega_j(x)\omega_k(y) \qquad (x, y \in I), \tag{0.1}$$

where $\{c_{jk} : j, k \geq 0\}$ satisfies the following conditions for some positive integer p:

$$c_{jk} \longrightarrow 0 \qquad \text{as} \quad \max\{j, k\} \to \infty, \tag{0.2}$$

$$\lim_{k\to\infty} \sum_{j=0}^{\infty} |\Delta_{p0}c_{jk}| = 0, \tag{0.3}$$

$$\lim_{j\to\infty} \sum_{k=0}^{\infty} |\Delta_{0p}c_{jk}| = 0, \tag{0.4}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{pp}c_{jk}| < \infty. \tag{0.5}$$

The finite-order differences $\Delta_{pq}c_{jk}$ are defined by

$$\begin{aligned}
&\Delta_{00}c_{jk} = c_{jk};\\
&\Delta_{pq}c_{jk} = \Delta_{p-1,q}c_{jk} - \Delta_{p-1,q}c_{j+1,k} \qquad (p \geq 1);\\
&\Delta_{pq}c_{jk} = \Delta_{p,q-1}c_{jk} - \Delta_{p,q-1}c_{j,k+1} \qquad (q \geq 1).
\end{aligned}$$

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We have

$$\Delta_{pq}c_{jk} = \sum_{s=0}^{p}\sum_{t=0}^{q}(-1)^{s+t}\binom{p}{s}\binom{q}{t}c_{j+s,k+t}.$$

Conditions (0.3)–(0.5) are known as conditions of bounded variation of order $(p,0)$, $(0,p)$, and (p,p), respectively. For $p=1$, conditions (0.3) and (0.4) are excessive, because they can be derived from (0.2) and (0.5). Obviously, conditions (0.3)–(0.5) generalize the concept of monotone sequences.

For $m,n\geq 0$, the rectangular partial sums $s_{mn}(x,y)$ and the first arithmetic (or Cesàro) means $\sigma_{mn}(x,y)$ of series (0.1) are defined as

$$s_{mn}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} c_{jk}\omega_j(x)\omega_k(y),$$

$$\sigma_{mn}(x,y) = \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n} s_{jk}(x,y).$$

If the c_{jk} are the Walsh-Fourier coefficients of some $f\in L^1(I^2)$, we shall write $s_{mn}(f;x,y)$ and $\sigma_{mn}(f;x,y)$ instead of $s_{mn}(x,y)$ and $\sigma_{mn}(x,y)$, respectively.

Let $\Omega\subseteq I^2$. As in [CH], we say that s_{mn} *converges uniformly on* Ω *to* f *in the unrestricted sense* (or briefly, converges uniformly on Ω to f) if s_{mn} converges uniformly on Ω to f as $\min\{m,n\}\to\infty$. In contrast, s_{mn} is said to *converge uniformly on* Ω *to* f *in the restricted sense* if for all $0<a<b<\infty$,

$$\lim_{\substack{a\leq m/n\leq b\\ m,n\to\infty}} s_{mn}(x,y) = f(x,y) \qquad \text{uniformly on} \quad \Omega. \tag{0.6}$$

We also say that $s_{mn}(x_0,y_0)$ converges unrestrictedly (or restrictedly) to $f(x_0,y_0)$ if s_{mn} converges uniformly on Ω to f in the unrestricted (or restricted) sense, where $\Omega=\{(x_0,y_0)\}$. The above definitions will apply to other sequences of functions, such as σ_{mn}. Conventionally we say that series (0.1) has the mentioned property whenever s_{mn} does. If series (0.1) converges unrestrictedly to $f(x,y)$, the row series $\sum_{j=0}^{\infty} c_{jk}\omega_j(x)\omega_k(y)$ converges for each fixed k, and the column series $\sum_{k=0}^{\infty} c_{jk}\omega_j(x)\omega_k(y)$ converges for each fixed j, then we shall say that series (0.1) *converges regularly* to $f(x,y)$, (*cf.* [H]). Set

$$\|f\|_r \equiv \left(\int_0^1\int_0^1 |f(x,y)|^r\,dxdy\right)^{1/r}.$$

Note that $\|\cdot\|_r^r$ defines a metric for $0<r<1$, and $\|\cdot\|_r$ is a norm for $r\geq 1$.

In this paper, we are concerned with the following convergence problems for suitable r:

(0.7) where $s_{mn}(x,y)$ converges uniformly to $f(x,y)$,
(0.8) where $s_{mn}(x,y)$ converges regularly to $f(x,y)$,
(0.9) whether $f\in L^r(I^2)$ and $\|s_{mn}-f\|_r\to 0$ as $\min\{m,n\}\to\infty$.

We also investigate the validity of the following Parseval's formula for suitable ϕ and $\Omega_{\epsilon\delta}$:

$$\lim_{\epsilon,\delta\downarrow 0}\iint_{\Omega_{\epsilon\delta}} f(x,y)\phi(x,y)\,dxdy = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\,\hat{\phi}_\Omega^*(j,k), \tag{0.10}$$

where $\{\Omega_{\epsilon\delta} : 0 < \epsilon, \delta < 1\}$ is a decreasing family of subsets of I^2 and

$$\hat{\phi}^*_\Omega(j,k) \equiv \lim_{\epsilon,\delta\downarrow 0} \iint_{\Omega_{\epsilon\delta}} \phi(x,y)\omega_j(x)\omega_k(y)\,dxdy.$$

These problems were investigated by many mathematicians, including Balašov [B], Fine [F], Móricz [M1], Móricz-Schipp [MS1], Rubinštein [R], Šneider [S], Yano [Y] for the one-dimensional case, and Chen [C2], [C3], [C4], [C5], Móricz [M2], [M3], Móricz-Schipp [MS2], Móricz-Schipp-Wade [MSW1] for higher dimensions. They were thoroughly discussed in [C5] for those series of type (0.1) whose coefficients c_{jk} satisfy conditions of bounded variation of order $(p,0), (0,p)$, and (p,p) with the weight $(\overline{jk})^{p-1}$, where $\bar{\xi} \equiv \max\{\xi, 1\}$. The purpose of this paper is to relax these weight conditions to (0.2)–(0.5) or to more generalized conditions. Details on these are stated below. The first main result reads as follows.

Theorem 0.1. *Assume that conditions* (0.2)–(0.5) *are satisfied for some* $p \geq 1$. *Then series* (0.1) *converges regularly to some measurable function* $f(x,y)$ *for all* $x, y \in I \setminus E_p$, *and the convergence is uniform on any compact subset* Ω *of* $(I \setminus E_p)^2$, *where* E_p *is a suitable countable set of dyadic rationals. Moreover, the following statements are true.*

(i) *For all* $0 < r < 1/p$, *we have* $f \in L^r(I^2)$ *and* $\|s_{mn} - f\|_r \to 0$ *as* $\min\{m,n\} \to \infty$.

(ii) *Let* $\{\Omega_{\epsilon\delta} : 0 < \epsilon, \delta < 1\}$ *be a decreasing family of compact subsets of* $(I \setminus E_p)^2$. *Assume that* $\phi : [0,1] \times [0,1] \mapsto \mathbb{C}$ *is measurable and locally bounded in* $(0,1] \times (0,1]$, $\hat{\phi}^*_\Omega(j,k)$ *exists for all* (j,k), *and the condition*

$$\sup_{\substack{j,k\geq 0\\ 0<\epsilon,\delta<1}} \left| \iint_{\Omega_{\epsilon\delta}} \phi(x,y) D^p_j(x) D^p_k(y)\,dxdy \right| < \infty \tag{0.11}$$

is satisfied. Then formula (0.10) *holds.*

The set E_p and the Cesàro sums $D^k_j(x)$ will be defined in §1. We say that $\{\Omega_{\epsilon\delta} : 0 < \epsilon, \delta < 1\}$ is a *decreasing family* of subsets of I^2 if whenever $\epsilon_1 \leq \epsilon_2$ and $\delta_1 \leq \delta_2$, we have $\Omega_{\epsilon_1\delta_1} \supseteq \Omega_{\epsilon_2\delta_2}$. For Ω and $\Omega_{\epsilon\delta}$, a wider class is allowed in a more general setting. This will be proved in §2. In [C4], [C5], the Parseval's formula involved there corresponds to the case $\Omega_{\epsilon\delta} = [\epsilon, 1) \times [\delta, 1)$. Theorem 0.1 and its extension generalize [C5, Theorems 2.1 & 4.1], [F, Theorem X], [M2, Theorems 1 & 2], [R], and [S]. For regular convergence and mean convergence, the cases $p = 1$ and 2 of Theorem 0.1 were proved in [M2]. Those proofs were based on suitable estimates for the Walsh-Dirichlet kernels $D^1_j(x)$ and the Walsh-Fejér kernels $K_j(x)$. These estimates were given by Fine in [F]. The obstacle to extending the results of Móricz [M2] to $p > 2$ is that an analogous estimate for $D^k_j(x)$ with $k > 2$ has not been found yet. We shall set up such a result in §1.

Obviously, condition (0.5) implies any of the following conditions:

$$\lim_{\lambda\downarrow 1} \limsup_{n\to\infty} \sum_{j=0}^{\infty} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] + 1 - k}{[\lambda n] - n} |\Delta_{pp} c_{jk}| = 0, \tag{0.12}$$

$$\lim_{\lambda\downarrow 1} \limsup_{m\to\infty} \sum_{j=m+1}^{[\lambda m]} \sum_{k=0}^{\infty} \frac{[\lambda m] + 1 - j}{[\lambda m] - m} |\Delta_{pp} c_{jk}| = 0. \tag{0.13}$$

Conditions (0.12) and (0.13) have appeared in many places. They were originally taken into consideration in the development of pointwise convergence of double trigonometric series, (*cf.* [C1], [CH], [CMW]). The second main result of this paper is the following.

Theorem 0.2. *Assume that conditions* (0.2)–(0.4) *and* (0.12)–(0.13) *are satisfied for some* $p \geq 1$. *Then the following statements are true.*

(i) *Let* Ω *be a compact subset of* $(I \setminus E_p)^2$. *If* $\sigma_{mn}(x,y)$ *converges uniformly on* Ω *to* $f(x,y)$, *then so does* $s_{mn}(x,y)$.
(ii) *If* $\|\sigma_{mn} - f\|_r \to 0$ *unrestrictedly for some* r *with* $0 < r < 1/p$, *then* $\|s_{mn} - f\|_r \to 0$ *as* $\min\{m,n\} \to \infty$.

Theorem 0.2 generalizes [M3, Theorems 1 & 1′]. The particular case $\Omega = \{(x_0, y_0)\}$ of Theorem 0.2 (i) reduces to a pointwise convergence property. Let $C_W(I^2)$ be the collection of all uniformly W-continuous functions $f : I^2 \mapsto \mathbb{R}$. Then $C_W(I^2)$ is the uniform closure of the double Walsh polynomials (*cf.* [SSW, pp. 156-158]). As proved in [M3, Lemma 4], $\sigma_{mn}(f; x, y)$ converges uniformly on I^2 to $f(x,y)$ for all $f \in C_W(I^2)$. Given $n \in \mathbb{N}$ and $x \in I$, let $J_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. Set

$$f^{\#}(x,y) = \sup_{n \in \mathbb{N}} \frac{1}{|J_n(x)|} \left| \int_{J_n(x)} f(t,y)\,dt \right| \qquad (f \in L^1(I^2)).$$

Recently, Weisz [W1] extended a result of Móricz-Schipp-Wade [MSW2] from $f \in L^1 \log^+ L^1(I^2)$ to $f^{\#} \in L^1(I^2)$. He proved that if $f^{\#} \in L^1(I^2)$, then

$$\sigma_{mn}(f; x, y) \longrightarrow f(x,y) \quad a.e. \qquad \text{as} \quad \min\{m,n\} \to \infty.$$

As for $f \in L^1(I^2)$, the Riemann-Lebesgue lemma ensures (0.2). By the Hölder inequality and the two-dimensional extension of [M], we find that

$$\|\sigma_{mn} - f\|_r \leq \|\sigma_{mn} - f\|_1 \longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty,$$

where $0 < r < 1/p$. Using these, we obtain the following result, which generalizes [M3, Theorems 2 & 2′, Corollaries 1, 2 & 2′].

Corollary 0.3. *Let* c_{jk} *be the Walsh-Fourier coefficients of* $f \in L^1(I^2)$. *Assume that conditions* (0.3)–(0.4) *and* (0.12)–(0.13) *are satisfied for some* $p \geq 1$. *Then the following statements remain true.*

(i) *If* $f^{\#} \in L^1(I^2)$, *then* $s_{mn}(f; x, y) \to f(x,y)$ *a.e. as* $\min\{m,n\} \to \infty$.
(ii) *If* $f \in C_W(I^2)$, *then* $s_{mn}(f; x, y)$ *converges uniformly on any compact subset* Ω *of* $(I \setminus E_p)^2$ *to* $f(x,y)$.
(iii) *If* $f \in L^1(I^2)$, *then for all* $0 < r < 1/p$, $\|s_{mn}(f) - f\|_r \to 0$ *as* $\min\{m,n\} \to \infty$.

It is clear that Theorem 0.2 and Corollary 0.3 will apply to the following case:

$c_{jk} = a_j b_k \,(j, k \geq 0)$, $a_j = 0$ except perhaps for a finite number of j, and $\{b_k : k \geq 0\}$ is a null sequence satisfying the property stated below for some positive integer p:

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} ([\lambda n] - k + 1)|\Delta_p b_k| = 0.$$

The differences $\Delta_p b_k$ are defined in a way similar to those for $\Delta_{pq}c_{jk}$. A special example of $\{a_j\}$ is as follows: $a_0 = 1$ and $a_j = 0$ for all $j > 0$. This particular case of Corollary 0.3 generalizes [M1, Theorems 1 & 2].

The definition of restricted convergence given in (0.6) means that $s_{mn}(x, y)$ converges uniformly on Ω to $f(x, y)$ as both m and n tend to infinity in such a way that $a \le m/n \le b$. For restricted convergence, we need the following concept of restricted limit superior, introduced in [CH]:

$$\limsup_{\substack{a\le m/n\le b\\ m,n\to\infty}} d_{mn} \equiv \inf_{\substack{a\le m/n\le b\\ m,n\ge 1}} \Big(\sup_{\substack{a\le j/k\le b\\ j\ge m,k\ge n}} d_{jk}\Big) = \lim_{m\to\infty} \Big(\sup_{\substack{a\le j/k\le b\\ j\ge m,k\ge m}} d_{jk}\Big),$$

where $\{d_{jk} : j, k \ge 0\}$ is a double sequence of extended real numbers. In the sequel, we drop "$m, n \to \infty$" under the sign "lim sup". Instead of (0.3)–(0.4) and (0.12)–(0.13), we consider the following weaker conditions:

$$\limsup_{a\le m/k\le b} \sum_{j=0}^{m} |\Delta_{p0}c_{jk}| = 0, \tag{0.14}$$

$$\limsup_{a\le j/n\le b} \sum_{k=0}^{n} |\Delta_{0p}c_{jk}| = 0, \tag{0.15}$$

$$\lim_{\lambda\downarrow 1} \limsup_{a\le m/n\le b} \sum_{j=0}^{m} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] + 1 - k}{[\lambda n] - n} |\Delta_{pp}c_{jk}| = 0, \tag{0.16}$$

$$\lim_{\lambda\downarrow 1} \limsup_{a\le m/n\le b} \sum_{j=m+1}^{[\lambda m]} \sum_{k=0}^{n} \frac{[\lambda m] + 1 - j}{[\lambda m] - m} |\Delta_{pp}c_{jk}| = 0, \tag{0.17}$$

where $0 < a < b < \infty$. We have

Theorem 0.4. *Assume that conditions* (0.2) *and* (0.14)–(0.17) *are satisfied for some $p \ge 1$ and for all $0 < a < b < \infty$. Then the following statements hold.*

(i) *Let Ω be a compact subset of $(I \setminus E_p)^2$. If $\sigma_{mn}(x, y)$ converges uniformly on Ω in the restricted sense to $f(x, y)$, then so does $s_{mn}(x, y)$.*

(ii) *If $\|\sigma_{mn} - f\|_r \to 0$ restrictedly for some r with $0 < r < 1/p$, then $\|s_{mn} - f\|_r$ converges restrictedly to* 0.

Recently, Weisz [W2] proved that $\sigma_{mn}(f; x, y)$ converges almost everywhere to $f(x, y)$ in the restricted sense for any $f \in L^1(I^2)$. As explained in the paragraphs before Corollary 0.3, the following is a consequence of Theorem 0.4, which generalizes Corollary 0.3 for restricted convergence.

Corollary 0.5. *Let c_{jk} be the Walsh-Fourier coefficients of $f \in L^1(I^2)$. Assume that conditions* (0.14)–(0.17) *are satisfied for some $p \ge 1$ and for all $0 < a < b < \infty$. Then the following statements remain true.*

(i) *If $f \in L^1(I^2)$, then $s_{mn}(f; x, y) \to f(x, y)$ restrictedly for almost all $(x, y) \in I^2$.*

(ii) *If $f \in C_W(I^2)$, then $s_{mn}(f; x, y)$ converges uniformly on Ω in the restricted sense to $f(x, y)$ for all compact subsets Ω of $(I \setminus E_p)^2$.*

(iii) *If $f \in L^1(I^2)$, then for all $0 < r < 1/p$, $\|s_{mn}(f) - f\|_r$ converges restrictedly to* 0.

Throughout this paper C and C_p denote constants, which are not necessarily the same at each occurrence.

1. Magnitude of the Cesàro sums $D_j^k(x)$

Let $D_j^0(x) = \omega_j(x)$ and $D_j^k(x)$ denote the Cesàro sums of order k of the sequence $\{D_j^0(x)\}$ (*cf.* [Z] for this terminology). Then

$$D_j^k(x) = \sum_{u=0}^{j} D_u^{k-1}(x) \qquad (j \geq 0; k \geq 1).$$

For the sake of convenience, we also define $D_j^k(x) = 0$ for $j < 0$ or $k < 0$. The number $D_j^1(x)$ is known as the Walsh-Dirichlet kernel of order j, and $K_j(x) \equiv D_j^2(x)/(j+1)$ is called the jth Walsh-Fejér kernel. In [F, §6-§7], Fine established the following expansions and estimates.

Lemma A. *Let $j = p \cdot 2^n + q$ with $p \geq 0$ and $0 \leq q < 2^n$. Then for all x, we have*

$$D_j^1(x) = D_{p-1}^1(2^n x) D_{2^n-1}^1(x) + \omega_p(2^n x) D_q^1(x).$$

Lemma B. *Let $n \geq 1$. Then for all j and for $x \in (2^{-n}, 2^{-n+1})$, we have*

$$|D_j^1(x)| < \frac{2}{x},$$
$$|D_j^2(x)| < \frac{4}{x(x-2^{-n})} + \frac{4}{x^2}.$$

In [C5], the first author used the estimate for $|D_j^1(x)|$ to derive the inequality

$$|D_j^k(x)| \leq 2^{k(k+1)/2} \min\{(\bar{j})^k, (\bar{j})^{k-1} x^{-1}\}, \tag{1.1}$$

where $x \in I, j \geq 0$, and $k \geq 1$. The purpose of this section is to get an estimate like the second one given in Lemma B, which is better than (1.1) for use. To do so, we first extend Lemma A from $k = 1$ to the general k. Define the numbers A_j^k by the recursive formulas

$$A_j^0 \equiv 1, \quad A_j^k \equiv A_0^{k-1} + A_1^{k-1} + \cdots + A_j^{k-1} \quad (j \geq 0; k \geq 1).$$

As shown in [Z, p.77], we have

$$A_j^k = \binom{j+k}{j} = \frac{(j+k)(j+k-1)\cdots(j+1)}{k!} \simeq \frac{j^k}{k!}.$$

Moreover,

$$A_j^k \leq (k+1)(\bar{j})^k \qquad (j, k \geq 0). \tag{1.2}$$

Let $\lambda_{1\beta}^k(\xi, q) \equiv A_q^{k-\beta}$ for $1 \leq \beta \leq k$, and 0 otherwise. For $\alpha \geq 2$, set

$$\lambda_{\alpha\beta}^k(\xi, q) \equiv \sum_{\substack{s_1, s_2, \cdots, s_{\alpha-1} \geq 1; t \geq 0 \\ s_1 + s_2 + \cdots + s_{\alpha-1} + t + \beta = k}} A_{\xi-1}^{s_1} A_{\xi-1}^{s_2} \cdots A_{\xi-1}^{s_{\alpha-1}} A_q^t.$$

Then $\lambda^k_{\alpha\beta}(\xi, q) = 0$ for $\beta > k$, $\lambda^k_{kk}(\xi, q) = 0$ for $k \geq 2$, and $\lambda^k_{\alpha\beta}(\xi, q) = 0$ for $\alpha + \beta > k + 1$. An elementary calculation gives

$$\lambda^{k+1}_{1,k+1}(2^n, q) = 1;$$

$$\lambda^{k+1}_{1\beta}(2^n, q) = \sum_{t=0}^{q} \lambda^k_{1\beta}(2^n, t) \qquad (1 \leq \beta \leq k);$$

$$\lambda^{k+1}_{\alpha\beta}(2^n, q) = \sum_{t=0}^{2^n-1} \lambda^k_{\alpha-1,\beta}(2^n, t) + \sum_{t=0}^{q} \lambda^k_{\alpha\beta}(2^n, t) \qquad (\alpha \geq 2;\ \text{all}\, \beta);$$

$$\lambda^{k+1}_{k+1,\beta}(2^n, q) = \sum_{t=0}^{2^n-1} \lambda^k_{k\beta}(2^n, t) \qquad (1 \leq \beta \leq k;\ \text{all}\, q).$$

Based on these, the following extension of Lemma A can easily be derived.

Lemma 1.1. *Let $j = p \cdot 2^n + q$ with $p \geq 0$ and $0 \leq q < 2^n$. Then for all $k \geq 1$ and for all $x \in I$, we have*

$$D^k_j(x) = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \lambda^k_{\alpha\beta}(2^n, q) D^{\alpha}_{p-\alpha}(2^n x) D^{\beta}_{2^n-1}(x) + \omega_p(2^n x) D^k_q(x).$$

Proof of Lemma 1.1. The proof will be carried out by induction on k. Lemma A guarantees the case $k = 1$. We have

$$D^{k+1}_j(x) = \sum_{r=0}^{p-1} \sum_{t=0}^{2^n-1} D^k_{r\cdot 2^n+t}(x) + \sum_{t=0}^{q} D^k_{p\cdot 2^n+t}(x).$$

If this lemma holds for k, then for all $0 \leq r \leq p-1$ and $0 \leq t \leq 2^n - 1$,

$$D^k_{r\cdot 2^n+t}(x) = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \lambda^k_{\alpha\beta}(2^n, t) D^{\alpha}_{r-\alpha}(2^n x) D^{\beta}_{2^n-1}(x) + \omega_r(2^n x) D^k_t(x). \tag{1.3}$$

Moreover,

$$D^k_{p\cdot 2^n+t}(x) = \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \lambda^k_{\alpha\beta}(2^n, t) D^{\alpha}_{p-\alpha}(2^n x) D^{\beta}_{2^n-1}(x) + \omega_p(2^n x) D^k_t(x). \tag{1.4}$$

Summing (1.3) and (1.4) with respect to r and t results in

$$\begin{aligned} D^{k+1}_j(x) &= \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \left\{ \sum_{t=0}^{2^n-1} \lambda^k_{\alpha\beta}(2^n, t) \right\} \left\{ \sum_{r=0}^{p-1} D^{\alpha}_{r-\alpha}(2^n x) \right\} D^{\beta}_{2^n-1}(x) \\ &\quad + \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \left\{ \sum_{t=0}^{q} \lambda^k_{\alpha\beta}(2^n, t) \right\} D^{\alpha}_{p-\alpha}(2^n x) D^{\beta}_{2^n-1}(x) \\ &\quad + \left\{ \sum_{r=0}^{p-1} \omega_r(2^n x) \right\} \left\{ \sum_{t=0}^{2^n-1} D^k_t(x) \right\} + \omega_p(2^n x) \left\{ \sum_{t=0}^{q} D^k_t(x) \right\} \\ &= \sum_{\alpha=1}^{k+1} \sum_{\beta=1}^{k+1} \lambda^{k+1}_{\alpha\beta}(2^n, q) D^{\alpha}_{p-\alpha}(2^n x) D^{\beta}_{2^n-1}(x) + \omega_p(2^n x) D^{k+1}_q(x). \end{aligned}$$

Thus, the desired result follows by induction on k. $\square$

To extend Lemma B, we introduce the sequence $\{Q_k(x) : k \geq 0\}$ of rational functions defined on $[0,1)$ below. Set

$$\bigoplus_{s=\alpha}^{\beta} n_s \equiv 2^{-(\sum_{\alpha \leq s \leq \beta} n_s)} \qquad (\alpha \leq \beta).$$

Let $I(n_1, \cdots, n_k)$ denote the open interval

$$\left(\sum_{j=1}^{k} \bigoplus_{s=j}^{k} n_s, \quad \sum_{j=1}^{k} \bigoplus_{s=j}^{k} n_s + \bigoplus_{s=1}^{k} n_s \right).$$

Then the interval $I(n_1, \cdots, n_k)$ can be obtained in the following way. Divide the open interval $(0,1)$ into the open subintervals $I(1), I(2), I(3), \cdots$, with the partition points $1/2, 1/4, 1/8, \cdots$, where

$$I(1) = (1/2, 1), \qquad I(2) = (1/4, 1/2), \qquad I(3) = (1/8, 1/4), \cdots .$$

Take the interval $I(n_k)$. Then $I(n_k) = (2^{-n_k}, 2^{-n_k+1})$. Divide the interval $I(n_k)$ into the open subintervals $I(1, n_k), I(2, n_k), I(3, n_k), \cdots$, with the partition points $2^{-n_k} + 2^{-n_k-1}, 2^{-n_k} + 2^{-n_k-2}, 2^{-n_k} + 2^{-n_k-3}, \cdots$, where

$$\begin{aligned} I(1, n_k) &= (2^{-n_k} + 2^{-n_k-1},\ 2^{-n_k+1}), \\ I(2, n_k) &= (2^{-n_k} + 2^{-n_k-2},\ 2^{-n_k} + 2^{-n_k-1}), \\ I(3, n_k) &= (2^{-n_k} + 2^{-n_k-3},\ 2^{-n_k} + 2^{-n_k-2}), \end{aligned}$$

and so on. We have $|I(1, n_k)| = |I(n_k)|/2, |I(2, n_k)| = |I(n_k)|/4, |I(3, n_k)| = |I(n_k)|/8, \cdots$, where $|\cdot|$ denotes the length of the interval. Continue this process for $I(n_{k-1}, n_k), I(n_{k-2}, n_{k-1}, n_k), \cdots$. After k steps, we get $I(n_1, \cdots, n_k)$. Set $E_0 = \phi$ and $E_1 = \{0\}$. For $k \geq 2$, let E_k be the set consisting of 0 and those partition points to get all $I(n_1, \cdots, n_{k-1})$, where $n_1, \cdots, n_{k-1} \geq 1$. Then E_k is a countable subset of I and

$$I \setminus E_k = \bigcup I(n_1, \ldots, n_{k-1}),$$

where the union is disjoint and runs over all positive integers $n_1, \cdots, n_{k-1}$. Moreover, the following properties can easily be derived:

(1.5) $I(n_1, \cdots, n_k) \subset (0,1)$ for all $n_1, \cdots, n_k \geq 1$;
(1.6) $I(n_1, \cdots, n_k) \subset I(n_2, \cdots, n_k)$;
(1.7) $2^{n_k} I(n_1, \cdots, n_k) - 1 = I(n_1, \cdots, n_{k-1})$;
(1.8) $I(n_1, \cdots, n_k) \cap I(n_1^*, \cdots, n_k^*) = \phi$ if $(n_1, \cdots, n_k) \neq (n_1^*, \cdots, n_k^*)$.

Set $Q_0(x) = 1$ and $Q_1(x) = 2/x$. For $k \geq 2$, define

$$Q_k(x) = \begin{cases} 0 & \text{if} \quad x \in E_k, \\ \rho_k (x - \sum_{j=1}^{k-1} \oplus_{s=j}^{k-1} n_s)^{-k} & \text{if} \quad x \in I(n_1, \cdots, n_{k-1}), \end{cases}$$

where $n_1, \cdots, n_{k-1}$ run over all possibilities of positive integers. The numbers ρ_k are defined by the recursive formulas: $\rho_0 = 1, \rho_1 = 2$, and for $k \geq 2$,

$$\rho_k = 2^k \left(1 + \sum_{\substack{\alpha+\beta \leq k+1 \\ \alpha < k, \beta \geq 2}} 2^{-\alpha} (k - \beta + 2)^{2\alpha} \rho_\alpha \right).$$

By (1.6) and (1.7), we easily prove that

$$Q_\alpha(x) \le \rho_\alpha \left(\frac{Q_k(x)}{\rho_k} \right)^{\alpha/k} \qquad (1 \le \alpha \le k-1), \tag{1.9}$$

$$Q_\alpha(2^n x - 1) \le 2^{-n\alpha} \rho_\alpha \left(\frac{Q_k(x)}{\rho_k} \right)^{\alpha/k} \qquad (1 \le \alpha \le k-1), \tag{1.10}$$

where $k \ge 2$, $x \in I(n_1, \cdots, n_{k-1})$, and $n = n_{k-1}$. Moreover, we have

Lemma 1.2. *Let $k \ge 0$ and $0 < r < 1/k$. Then*

$$\left(\int_0^1 |Q_k(x)|^r \, dx \right)^{1/r} = (1-kr)^{-1/r} (2^{1-kr} - 1)^{(1-k)/r} \rho_k.$$

Proof of Lemma 1.2. The cases $k = 0$ and $k = 1$ are trivial. Assume that $k \ge 2$. We have $1 - kr > 0$, and so the definition of $Q_k(x)$ gives

$$\left(\int_{I(n_1, \dots, n_{k-1})} |Q_k(x)|^r \, dx \right)^{1/r} = 2^{-(\sum_{1 \le s < k} n_s)(1-kr)/r} (1-kr)^{-1/r} \rho_k.$$

Therefore,

$$\begin{aligned}
&\left(\int_0^1 |Q_k(x)|^r \, dx \right)^{1/r} \\
&= \left\{ \sum_{n_1=1}^{\infty} \cdots \sum_{n_{k-1}=1}^{\infty} \int_{I(n_1, \cdots, n_{k-1})} |Q_k(x)|^r \, dx \right\}^{1/r} \\
&= \left\{ \sum_{n_1=1}^{\infty} \cdots \sum_{n_{k-1}=1}^{\infty} 2^{-(\sum_{1 \le s < k} n_s)(1-kr)} \right\}^{1/r} (1-kr)^{-1/r} \rho_k \\
&= (1-kr)^{-1/r} \rho_k \left\{ \sum_{n_1=1}^{\infty} 2^{-n_1(1-kr)} \right\}^{1/r} \cdots \left\{ \sum_{n_{k-1}=1}^{\infty} 2^{-n_{k-1}(1-kr)} \right\}^{1/r} \\
&= (1-kr)^{-1/r} (2^{1-kr} - 1)^{(1-k)/r} \rho_k.
\end{aligned}$$

□

Theorem 1.3. *Let $k \ge 0$. Then for all $j \ge 0$ and all $x \in I \setminus E_k$, we have*

$$|D_j^k(x)| \le Q_k(x), \tag{1.11}$$

and so the inequality (1.11) *holds for almost all $x \in I$. Moreover, for $0 < r < 1/k$ and for all j,*

$$\left(\int_0^1 |D_j^k(x)|^r \, dx \right)^{1/r} \le (1-kr)^{-1/r} (2^{1-kr} - 1)^{(1-k)/r} \rho_k. \tag{1.12}$$

Proof of Theorem 1.3. We first prove (1.11). The case $k = 0$ follows from the fact that $|D_j^0(x)| = |\omega_j(x)| = 1$. For general k, we use induction on k. Lemma B ensures the case $k = 1$. Assume that (1.11) holds for any positive integer $\le k - 1$, where $k \ge 2$. For $x \in I \setminus E_k$, we have $x \in I(n_1, \cdots, n_{k-1})$ for some positive integers

$n_1, \cdots, n_{k-1}$. Write $j = p \cdot 2^n + q$ with $p \ge 0$ and $0 \le q < 2^n$, where $n = n_{k-1}$. By Lemma 1.1, we obtain

$$\begin{aligned} |D_j^k(x)| &\le \left| \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} \lambda_{\alpha\beta}^k(2^n, q) D_{p-\alpha}^{\alpha}(2^n x) D_{2^n-1}^{\beta}(x) \right| + |\omega_p(2^n x) D_q^k(x)| \\ &= J_1 + J_2, \text{say.} \end{aligned}$$

We have $x \in (2^{-n}, 2^{-n+1})$. Paley's lemma says $D_{2^n-1}^1(x) = 0$. On the other hand, $\lambda_{\alpha\beta}^k(2^n, q) = 0$ for all $\alpha + \beta > k + 1$. Hence,

$$J_1 \le \sum_{\substack{\alpha+\beta \le k+1 \\ \alpha < k, \beta \ge 2}} |\lambda_{\alpha\beta}^k(2^n, q)| \cdot |D_{p-\alpha}^{\alpha}(2^n x)| \cdot |D_{2^n-1}^{\beta}(x)|. \tag{1.13}$$

Let $\alpha < k, \beta \ge 2$, and $\alpha + \beta \le k + 1$. By (1.2), we get

$$\begin{aligned} |\lambda_{\alpha\beta}^k(2^n, q)| &\le \Bigg\{ \sum_{\substack{s_1, \cdots, s_\alpha \ge 0 \\ s_1 + \cdots + s_\alpha = k - \beta}} (s_1 + 1) \cdots (s_\alpha + 1) \Bigg\} 2^{n(k-\beta)} \\ &\le 2^{-\alpha}(k - \beta + 2)^{2\alpha} 2^{n(k-\beta)} \\ &\le 2^{k-\alpha-\beta}(k - \beta + 2)^{2\alpha} x^{\beta-k}. \end{aligned} \tag{1.14}$$

We have

$$\begin{aligned} |D_{2^n-1}^{\beta}(x)| &\le 2^n \left(\sup_{0 \le j < 2^n} |D_j^{\beta-1}(x)| \right) \le 2^{2n} \left(\sup_{0 \le j < 2^n} |D_j^{\beta-2}(x)| \right) \\ &\le \cdots \le 2^{n\beta} \le 2^{\beta} x^{-\beta}. \end{aligned} \tag{1.15}$$

The induction hypothesis, (1.7), and (1.10) together imply

$$\begin{aligned} |D_{p-\alpha}^{\alpha}(2^n x)| &= |D_{p-\alpha}^{\alpha}(2^n x - 1)| \le Q_\alpha(2^n x - 1) \\ &\le 2^{-n\alpha} \rho_\alpha \left(\frac{Q_k(x)}{\rho_k} \right)^{\alpha/k} \le x^{\alpha} \rho_\alpha \left(\frac{Q_k(x)}{\rho_k} \right)^{\alpha/k}. \end{aligned} \tag{1.16}$$

It is clear that

$$x^{\alpha-k} \left(\frac{Q_k(x)}{\rho_k} \right)^{\alpha/k} \le \frac{Q_k(x)}{\rho_k},$$

so (1.13)–(1.16) together yield

$$\begin{aligned} J_1 &\le 2^k \left\{ \sum_{\substack{\alpha+\beta \le k+1 \\ \alpha < k, \beta \ge 2}} 2^{-\alpha}(k - \beta + 2)^{2\alpha} \rho_\alpha x^{\alpha-k} \left(\frac{Q_k(x)}{\rho_k} \right)^{\alpha/k} \right\} \\ &\le \left(\frac{2^k}{\rho_k}\right) \Bigg(\sum_{\substack{\alpha+\beta \le k+1 \\ \alpha < k, \beta \ge 2}} 2^{-\alpha}(k - \beta + 2)^{2\alpha} \rho_\alpha \Bigg) Q_k(x). \end{aligned}$$

The same argument as (1.15) also implies

$$J_2 \le (q + 1) \left(\sup_{0 \le j \le q} |D_j^{k-1}(x)| \right) \le \cdots \le (q + 1)^k \le 2^k x^{-k} \le \frac{2^k Q_k(x)}{\rho_k}.$$

From here, we get

$$|D_j^k(x)| \le (\frac{2^k}{\rho_k})\Bigg(1 + \sum_{\substack{\alpha+\beta\le k+1\\ \alpha<k,\beta\ge 2}} 2^{-\alpha}(k-\beta+2)^{2\alpha}\rho_\alpha\Bigg)Q_k(x) = Q_k(x),$$

which concludes (1.11). Combining (1.11) with Lemma 1.2 results in (1.12). This finishes the proof. $\square$

2. Generalizations of main results and proofs

Let Ω be a subset of $(I \setminus E_p)^2$. Denote by $d_j(\Omega)$ the shortest distance from $\Omega_j \cap I(n_1, \cdots, n_{p-1})$ to the left endpoint of $I(n_1, \cdots, n_{p-1})$ for all $n_1, \cdots, n_{p-1} \ge 1$, where Ω_1 and Ω_2 are the projections of Ω to the x-axis and to the y-axis, respectively. Set $d(\Omega) = \min\{d_1(\Omega), d_2(\Omega)\}$. Notice that for $p = 1$, $d_j(\Omega)$ is defined as the distance from Ω_j to 0. Obviously, if Ω is compact, then Ω_1 and Ω_2 are compact subsets of $I \setminus E_p$. For $p = 1$, we have $d(\Omega) > 0$. As for $p \ge 2$, $\{I(n_1, \cdots, n_{p-1}) : n_1, \cdots, n_{p-1} \ge 1\}$ is an open cover of Ω_1 and of Ω_2, so a finite number of $I(n_1, \cdots, n_{p-1})$ will cover both Ω_1 and Ω_2. For this case, we also have $d(\Omega) > 0$. Hence, the following result generalizes Theorem 0.1. For $\Omega_{\epsilon\delta} = [\epsilon, 1) \times [\delta, 1)$, it extends [C5, Theorem 4.1].

Theorem 2.1. *Assume that conditions* (0.2)–(0.5) *are satisfied for some* $p \ge 1$. *Then series* (0.1) *converges regularly to some measurable function* $f(x, y)$ *for all* $x, y \in I \setminus E_p$, *and the convergence is uniform on any subset* Ω *of* $(I \setminus E_p)^2$ *with* $d(\Omega) > 0$. *Moreover, the following statements are true.*

(i) *For all* $0 < r < 1/p$, *we have* $f \in L^r(I^2)$ *and* $\|s_{mn} - f\|_r \to 0$ *as* $\min\{m, n\} \to \infty$.

(ii) *Let* $\{\Omega_{\epsilon\delta} : 0 < \epsilon, \delta < 1\}$ *be a decreasing family of subsets of* $(I \setminus E_p)^2$ *with* $d(\Omega_{\epsilon\delta}) > 0$ *for all* $0 < \epsilon, \delta < 1$. *Assume that* $\phi : [0, 1] \times [0, 1] \mapsto \mathbb{C}$ *is measurable and locally bounded in* $(0, 1] \times (0, 1]$, $\hat{\phi}^*_\Omega(j, k)$ *exists for all* (j, k), *and* (0.11) *is satisfied. Then formula* (0.10) *holds.*

Proof of Theorem 2.1. The proof is essentially same as that given in [C5]. We first prove that (0.3) and (0.5) together imply

$$(0.3') \qquad \sum_{j=0}^{\infty} |\Delta_{p0} c_{jk}| < \infty \quad \text{for all} \quad k.$$

Condition (0.3) ensures the existence of a positive integer N so that

$$(0.3'') \qquad \sum_{j=0}^{\infty} |\Delta_{p0} c_{jk}| < \infty \quad \text{for all} \quad k > N.$$

Let $M \equiv [\alpha_{jk}]$ be the $(N + 1) \times (N + 1)$ matrix defined by

$$\alpha_{jk} = \begin{cases} (-1)^{k-j}\binom{p}{k-j} & \text{if} \quad 0 \le j \le k \le \min\{N, j + p\}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\det M = 1$, there exist $\alpha_0, \alpha_1, \cdots, \alpha_N$, depending on p and N only, such that

$$[\alpha_0, \alpha_1, \cdots, \alpha_N]M = [1, 0, \cdots, 0].$$

This guarantees the existence of $\beta_1, \cdots, \beta_p$, depending on p and N only, such that for all $j, k \geq 0$,

$$\begin{aligned}\Delta_{p0}c_{jk} =& \alpha_0\Delta_{pp}c_{jk} + \alpha_1\Delta_{pp}c_{j,k+1} + \cdots + \alpha_N\Delta_{pp}c_{j,k+N} \\ &+ \beta_1\Delta_{p0}c_{j,k+N+1} + \cdots + \beta_p\Delta_{p0}c_{j,k+N+p}.\end{aligned}$$

Thus, by (0.3″) and (0.5), we obtain

$$\sum_{j=0}^{\infty}|\Delta_{p0}c_{jk}| \leq \sum_{s=0}^{N}|\alpha_s|(\sum_{j=0}^{\infty}|\Delta_{pp}c_{j,k+s}|) + \sum_{t=1}^{p}|\beta_t|(\sum_{j=0}^{\infty}|\Delta_{p0}c_{j,k+N+t}|) < \infty.$$

This verifies (0.3′). Similarly, (0.4) and (0.5) together imply (0.4′):

$$\sum_{k=0}^{\infty}|\Delta_{0p}c_{jk}| < \infty \quad \text{for all} \quad j. \tag{0.4′}$$

The summation by parts yields

$$\begin{aligned} s_{mn}(x,y) =& \sum_{j=0}^{m}\sum_{k=0}^{n}(\Delta_{pp}c_{jk})D_j^p(x)D_k^p(y) \\ &+ \sum_{t=0}^{p-1}\sum_{j=0}^{m}(\Delta_{pt}c_{j,n+1})D_j^p(x)D_n^{t+1}(y) \\ &+ \sum_{s=0}^{p-1}\sum_{k=0}^{n}(\Delta_{sp}c_{m+1,k})D_m^{s+1}(x)D_k^p(y) \\ &+ \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}(\Delta_{st}c_{m+1,n+1})D_m^{s+1}(x)D_n^{t+1}(y). \end{aligned} \tag{2.1}$$

Assume that $x, y \in I \setminus E_p$. Then (1.6) implies that $x, y \in I \setminus E_k$ for all $0 \leq k \leq p$. By Theorem 1.3, we get

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\big|(\Delta_{pp}c_{jk})D_j^p(x)D_k^p(y)\big| \leq \left(\sum_{j=0}^{m}\sum_{k=0}^{n}|\Delta_{pp}c_{jk}|\right)Q_p(x)Q_p(y) \tag{2.2}$$

and

$$\begin{aligned} &\sum_{t=0}^{p-1}\sum_{j=0}^{m}\big|(\Delta_{pt}c_{j,n+1})D_j^p(x)D_n^{t+1}(y)\big| \\ &\quad\leq \sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\left(\sum_{j=0}^{m}|\Delta_{p0}c_{j,n+v+1}|\right)Q_p(x)Q_{t+1}(y) \\ &\quad\leq \Big(\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\Delta_{p0}c_{jk}|\Big)Q_p(x)\Big(\sum_{t=0}^{p-1}2^tQ_{t+1}(y)\Big). \end{aligned} \tag{2.3}$$

Similarly, we have

$$
\begin{aligned}
&\sum_{s=0}^{p-1}\sum_{k=0}^{n}\left|(\Delta_{sp}c_{m+1,k})D_m^{s+1}(x)D_k^p(y)\right| \\
&\qquad \le \Big(\sup_{m<j\le m+p}\sum_{k=0}^{n}|\Delta_{0p}c_{jk}|\Big)\Big(\sum_{s=0}^{p-1}2^sQ_{s+1}(x)\Big)Q_p(y)
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
&\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\left|(\Delta_{st}c_{m+1,n+1})D_m^{s+1}(x)D_n^{t+1}(y)\right| \\
&\qquad \le \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}\left|\Delta_{00}c_{m+u+1,n+v+1}\right| \\
&\qquad\qquad \times Q_{s+1}(x)Q_{t+1}(y) \\
&\qquad \le \Big(\sup_{j>m,k>n}|c_{jk}|\Big)\Big(\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}2^{s+t}Q_{s+1}(x)Q_{t+1}(y)\Big).
\end{aligned}
\tag{2.5}
$$

Putting (0.2)–(0.5) and (2.1)–(2.5) together, we infer that $s_{mn}(x,y)$ converges unrestrictedly to some measurable function $f(x,y)$ for $x,y\in I\setminus E_p$. Summation by parts gives

$$
\begin{aligned}
\sum_{j=0}^{m}c_{jk}\omega_j(x)\omega_k(y) &= \sum_{j=0}^{m}(\Delta_{p0}c_{jk})D_j^p(x)\omega_k(y) \\
&\quad + \sum_{s=0}^{p-1}(\Delta_{s0}c_{m+1,k})D_m^{s+1}(x)\omega_k(y).
\end{aligned}
$$

A similar argument says that $\sum_{j=0}^{\infty}c_{jk}\omega_j(x)\omega_k(y)$ converges for each fixed k. The same conclusion also holds for each column series. Thus, series (0.1) converges regularly to $f(x,y)$ for $x,y\in I\setminus E_p$. Let Ω be any subset of $(I\setminus E_p)^2$ with $d(\Omega)>0$. Denote by Ω_1 and Ω_2 the projections of Ω to the x-axis and to the y-axis, respectively. By (1.6),

$$
|Q_k(x)|\le \rho_k d(\Omega)^{-k}<\infty \qquad (0\le k\le p\,;\,x\in\Omega_1\cup\Omega_2).
$$

By (0.2)–(0.5) and (2.1)–(2.5), we confirm that (0.1) converges uniformly on Ω to $f(x,y)$. Indeed, the same argument also verifies that

$$
f(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\Delta_{pp}c_{jk})D_j^p(x)D_k^p(y) \qquad \text{uniformly on} \quad \Omega. \tag{2.6}
$$

Let $0<r<1/p$. Then Lemma 1.2 tells us that

$$
\mu_k^r \equiv \int_0^1 |Q_k(x)|^r\,dx<\infty \qquad (0\le k\le p). \tag{2.7}
$$

By (0.5), (2.2), and (2.6), we infer that

$$
\int_0^1\int_0^1 |f(x,y)|^r\,dxdy \le \Big\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{pp}c_{jk}|\Big\}^r(\mu_p^r)^2<\infty.
$$

Set $\Lambda_{mn} \equiv \{(j,k) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) : j > m \text{ or } k > n\}$. By (0.2)–(0.5) and (2.1)–(2.6), we obtain

$$\begin{aligned}
&\|s_{mn} - f\|_r^r \\
&\le \left(\sum_{\Lambda_{mn}} |\Delta_{pp} c_{jk}|\right)^r (\mu_p^r)^2 + \left(\sup_{k>n} \sum_{j=0}^{m} |\Delta_{p0} c_{jk}|\right)^r (\sum_{t=0}^{p-1} 2^{tr} \mu_p^r \mu_{t+1}^r) \\
&\quad + \left(\sup_{j>m} \sum_{k=0}^{n} |\Delta_{0p} c_{jk}|\right)^r (\sum_{s=0}^{p-1} 2^{sr} \mu_{s+1}^r \mu_p^r) \\
&\quad + \left(\sup_{j>m, k>n} |c_{jk}|\right)^r (\sum_{s=0}^{p-1} \sum_{t=0}^{p-1} 2^{(s+t)r} \mu_{s+1}^r \mu_{t+1}^r) \\
&\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty.
\end{aligned}$$

It remains to prove (ii). Set

$$\Phi_{jk}^{st}(\epsilon,\delta) \equiv \iint_{\Omega_{\epsilon\delta}} \phi(x,y) D_j^s(x) D_k^t(y)\, dxdy.$$

Then condition (0.11) is same as

$$\sup_{\substack{j,k \ge 0 \\ 0<\epsilon,\delta<1}} |\Phi_{jk}^{pp}(\epsilon,\delta)| < \infty. \tag{2.8}$$

For $s,t \ge 1$, we have

$$\Phi_{jk}^{st}(\epsilon,\delta) = \sum_{u=0}^{j} \Phi_{uk}^{s-1,t}(\epsilon,\delta) = \sum_{v=0}^{k} \Phi_{jv}^{s,t-1}(\epsilon,\delta) = \sum_{u=0}^{j} \sum_{v=0}^{k} \Phi_{uv}^{s-1,t-1}(\epsilon,\delta),$$

and so (0.11) is equivalent to the existence of constant C such that

$$\sup_{\substack{j,k \ge 0 \\ 0<\epsilon,\delta<1}} |\Phi_{jk}^{st}(\epsilon,\delta)| \le C < \infty \qquad \text{for all} \quad 0 \le s,t \le p. \tag{2.9}$$

We have $\hat{\phi}_\Omega^*(j,k) = \lim_{\epsilon,\delta\downarrow 0} \Phi_{jk}^{00}(\epsilon,\delta)$, and $\hat{\phi}_\Omega^*(j,k)$ exists for all (j,k). Therefore, the limit $\zeta_{jk}^{st} \equiv \lim_{\epsilon,\delta\downarrow 0} \Phi_{jk}^{st}(\epsilon,\delta)$ exists for all s,t,j,k, and (2.9) implies that

$$|\zeta_{jk}^{st}| \le C \qquad (j,k \ge 0; \quad 0 \le s,t \le p). \tag{2.10}$$

Since $d(\Omega_{\epsilon\delta}) > 0$, the set $\Omega_{\epsilon\delta}$ is contained in some compact subset of $(0,1] \times (0,1]$. We have assumed that ϕ is locally bounded in $(0,1] \times (0,1]$, so it follows from (2.6), with $\Omega_{\epsilon\delta}$ in place of Ω, that as $\min\{m,n\} \to \infty$,

$$\sum_{j=0}^{m} \sum_{k=0}^{n} (\Delta_{pp} c_{jk}) \Phi_{jk}^{pp}(\epsilon,\delta) \longrightarrow \iint_{\Omega_{\epsilon\delta}} f(x,y)\phi(x,y)\, dxdy. \tag{2.11}$$

Putting (0.5) and (2.9)–(2.11) together, we infer that

$$\lim_{\epsilon,\delta\downarrow 0} \iint_{\Omega_{\epsilon\delta}} f(x,y)\phi(x,y)\, dxdy = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (\Delta_{pp} c_{jk}) \zeta_{jk}^{pp}.$$

The above limit of the double integral exists, and the double series on the right is absolutely convergent. For $m, n \geq 0$, we have

$$\lambda_{mn} \equiv \sum_{j=0}^{m} \sum_{k=0}^{n} c_{jk}\, \hat{\phi}^*_\Omega(j,k) = \lim_{\epsilon,\delta \downarrow 0} \iint_{\Omega_{\epsilon\delta}} s_{mn}(x,y)\phi(x,y)\, dxdy.$$

It follows from (2.1) that

$$\begin{aligned}\lambda_{mn} =& \sum_{j=0}^{m} \sum_{k=0}^{n} (\Delta_{pp} c_{jk})\, \zeta_{jk}^{pp} + \sum_{t=0}^{p-1} \sum_{j=0}^{m} (\Delta_{pt} c_{j,n+1})\, \zeta_{jn}^{p,t+1} \\ &+ \sum_{s=0}^{p-1} \sum_{k=0}^{n} (\Delta_{sp} c_{m+1,k})\, \zeta_{mk}^{s+1,p} + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} (\Delta_{st} c_{m+1,n+1})\, \zeta_{mn}^{s+1,t+1}.\end{aligned}$$

As mentioned before, the series $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (\Delta_{pp} c_{jk})\, \zeta_{jk}^{pp}$ converges absolutely. On the other hand, (0.2)–(0.4) and (2.10) imply

$$\begin{aligned}\sum_{t=0}^{p-1} \sum_{j=0}^{m} |\Delta_{pt} c_{j,n+1}||\zeta_{jn}^{p,t+1}| &\leq C \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{j=0}^{m} |\Delta_{p0} c_{j,n+v+1}| \\ &\leq C_p \Big(\sup_{k>n} \sum_{j=0}^{m} |\Delta_{p0} c_{jk}| \Big) \\ &\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty,\end{aligned}$$

$$\begin{aligned}\sum_{s=0}^{p-1} \sum_{k=0}^{n} |\Delta_{sp} c_{m+1,k}||\zeta_{mk}^{s+1,p}| &\leq C_p \Big(\sup_{j>m} \sum_{k=0}^{n} |\Delta_{0p} c_{jk}| \Big) \\ &\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty,\end{aligned}$$

and

$$\begin{aligned}\sum_{s=0}^{p-1} \sum_{t=0}^{p-1} |\Delta_{st} c_{m+1,n+1}||\zeta_{mn}^{s+1,t+1}| &\leq C_p \Big(\sup_{j>m, k>n} |c_{jk}| \Big) \\ &\longrightarrow 0 \qquad \text{as} \quad \min\{m,n\} \to \infty.\end{aligned}$$

Hence, as $\min\{m,n\} \to \infty$, λ_{mn} tends to $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (\Delta_{pp} c_{jk}) \zeta_{jk}^{pp}$. This gives (0.10), and the proof is complete. □

With the help of Theorem 1.3, we get

$$\sup_{\substack{j,k \geq 0 \\ 0<\epsilon,\delta<1}} \left| \iint_{\Omega_{\epsilon\delta}} \phi(x,y) D_j^p(x) D_k^p(y)\, dxdy \right| \leq \|\phi(x,y) Q_p(x) Q_p(y)\|_1.$$

Hence, if $\phi(x,y)Q_p(x)Q_p(y) \in L^1(I^2)$, then (0.11) holds. Moreover, from the inequality $|\phi(x,y)|\rho_p^2 \leq |\phi(x,y)Q_p(x)Q_p(y)|$, we find that $\phi \in L^1(I^2)$. Using these facts, we obtain the following extension of [C5, Corollary 4.2].

Corollary 2.2. *Assume that conditions* (0.2)–(0.5) *are satisfied for some* $p \geq 1$. *If* $\phi : [0,1] \times [0,1] \mapsto \mathbb{C}$ *is locally bounded in* $(0,1] \times (0,1]$ *and* $\phi(x,y)Q_p(x)Q_p(y) \in$

$L^1(I^2)$, then $f(x,y)\phi(x,y) \in L^1(I^2)$ and

$$\int_0^1 \int_0^1 f(x,y)\phi(x,y)\,dxdy = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\,\hat{\phi}(j,k),$$

where f is the limit function of the series (0.1).

Proof of Corollary 2.2. It follows from (2.6) and Theorem 1.3 that

$$|f(x,y)\phi(x,y)| \le \Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_{pp}c_{jk}|\Big)|\phi(x,y)Q_p(x)Q_p(y)| \qquad (x,y \in I \setminus E_p).$$

Since $\phi(x,y)Q_p(x)Q_p(y) \in L^1(I^2)$, we know that $f(x,y)\phi(x,y) \in L^1(I^2)$. Choose a decreasing family $\{\Omega_{\epsilon\epsilon} : 0 < \epsilon < 1\}$ of compact subsets of $(I \setminus E_p)^2$ with

$$\bigcup_{0<\epsilon<1} \Omega_{\epsilon\epsilon} = (I \setminus E_p)^2.$$

Then, by applying the Lebesgue dominated convergence theorem, we find that $\hat{\phi}^*_{\Omega}(j,k) = \hat{\phi}(j,k)$ for all $j,k \ge 0$. Therefore, Theorem 2.1 gives us

$$\begin{aligned}\int_0^1 \int_0^1 f(x,y)\phi(x,y)\,dxdy &= \lim_{\epsilon\downarrow 0} \iint_{\Omega_{\epsilon\epsilon}} f(x,y)\phi(x,y)\,dxdy \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk}\,\hat{\phi}(j,k),\end{aligned}$$

which is the desired result. □

To prove Theorem 0.2, we introduce the following three sums for $\lambda > 1$:

$$\begin{aligned}\Sigma^{\lambda}_{10}(m,n;x,y) &\equiv \sum_{j=m+1}^{[\lambda m]}\sum_{k=0}^{n} \frac{[\lambda m]+1-j}{[\lambda m]-m} c_{jk}\omega_j(x)\omega_k(y),\\ \Sigma^{\lambda}_{01}(m,n;x,y) &\equiv \sum_{j=0}^{m}\sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]+1-k}{[\lambda n]-n} c_{jk}\omega_j(x)\omega_k(y),\\ \Sigma^{\lambda}_{11}(m,n;x,y) &\equiv \sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} \frac{[\lambda m]+1-j}{[\lambda m]-m}\cdot\frac{[\lambda n]+1-k}{[\lambda n]-n} c_{jk}\omega_j(x)\omega_k(y).\end{aligned}$$

They involve those $c_{jk}\omega_j(x)\omega_k(y)$ with (j,k) lying between the two rectangles $[0,\lambda m] \times [0,\lambda n]$ and $[0,m] \times [0,n]$. The coefficients corresponding to the terms $c_{jk}\omega_j(x)\omega_k(y)$ have absolute value not greater than 1. As indicated in [CMW, p. 639], we have

$$\begin{aligned}s_{mn} - \sigma_{mn} =& \frac{[\lambda m]+1}{[\lambda m]-m}(\sigma_{[\lambda m],n} - \sigma_{mn}) + \frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{m,[\lambda n]} - \sigma_{mn})\\ &+ \frac{[\lambda m]+1}{[\lambda m]-m}\cdot\frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda m],[\lambda n]} - \sigma_{[\lambda m],n} - \sigma_{m,[\lambda n]} + \sigma_{mn})\\ &- \Sigma^{\lambda}_{01}(m,n;x,y) - \Sigma^{\lambda}_{10}(m,n;x,y) - \Sigma^{\lambda}_{11}(m,n;x,y)\end{aligned} \tag{2.12}$$

and

$$\Sigma_{11}^{\lambda}(m,n;x,y) = \frac{1}{[\lambda m]-m}\sum_{u=m+1}^{[\lambda m]}(\Sigma_{01}^{\lambda}(u,n;x,y)-\Sigma_{01}^{\lambda}(m,n;x,y)) \tag{2.13}$$

$$= \frac{1}{[\lambda n]-n}\sum_{v=n+1}^{[\lambda n]}(\Sigma_{10}^{\lambda}(m,v;x,y)-\Sigma_{10}^{\lambda}(m,n;x,y)).$$

Obviously, (2.13) implies

$$|\Sigma_{11}^{\lambda}(m,n;x,y)| \le \begin{cases} 2\sup_{m\le u\le[\lambda m]}|\Sigma_{01}^{\lambda}(u,n;x,y)|, \\ 2\sup_{n\le v\le[\lambda n]}|\Sigma_{10}^{\lambda}(m,v;x,y)|. \end{cases} \tag{2.14}$$

By (2.12) and (2.14), we easily prove

Theorem 2.3. *(i) Let $\Omega \subseteq I^2$. Assume that* (2.15)–(2.16) *are satisfied:*

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in\Omega}|\Sigma_{01}^{\lambda}(m,n;x,y)|\right) = 0, \tag{2.15}$$

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in\Omega}|\Sigma_{10}^{\lambda}(m,n;x,y)|\right) = 0. \tag{2.16}$$

If $\sigma_{mn}(x,y)$ converges uniformly on Ω to $f(x,y)$, then so does $s_{mn}(x,y)$.

(ii) Assume that (2.17)–(2.18) *hold for some $r \ge 1$:*

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\|\Sigma_{01}^{\lambda}(m,n;x,y)\|_r = 0, \tag{2.17}$$

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\|\Sigma_{10}^{\lambda}(m,n;x,y)\|_r = 0. \tag{2.18}$$

If $\|\sigma_{mn}-f\|_r \to 0$ unrestrictedly, then $\|s_{mn}-f\|_r \to 0$ as $\min\{m,n\}\to\infty$.

Here the limit superior of a double sequence $\{d_{jk} : j,k \ge 0\}$ of extended real numbers is defined as

$$\limsup_{m,n\to\infty} d_{mn} \equiv \inf_{m,n\ge 1}(\sup_{j\ge m,k\ge n} d_{jk}) = \lim_{m\to\infty}(\sup_{j\ge m,k\ge m} d_{jk}).$$

We shall use (i) of Theorem 2.3 to establish (i) of Theorem 0.2. For (ii) of Theorem 0.2, the range of r is different from that in (ii) of Theorem 2.3, so we shall prove it by a different method. It is unknown whether the conclusion (ii) of Theorem 2.3 holds for $0 < r < 1$.

Proof of Theorem 2.3. With the help of (2.14), we find that (2.15) implies

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in\Omega}|\Sigma_{11}^{\lambda}(m,n;x,y)|\right) = 0. \tag{2.19}$$

Assume that $\sigma_{mn}(x,y)$ converges uniformly on Ω to $f(x,y)$. Then by (2.12), we get

$$
\begin{aligned}
&\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in\Omega}|s_{mn}(x,y)-\sigma_{mn}(x,y)|\right)\\
&\quad\le \limsup_{m,n\to\infty}\left(\sup_{(x,y)\in\Omega}|\Sigma_{01}^{\lambda}(m,n;x,y)|\right)+\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in\Omega}|\Sigma_{10}^{\lambda}(m,n;x,y)|\right)\\
&\quad\quad+\limsup_{m,n\to\infty}\left(\sup_{(x,y)\in\Omega}|\Sigma_{11}^{\lambda}(m,n;x,y)|\right),
\end{aligned}
$$

where $\lambda > 1$. The quantity on the left-hand side is independent of λ. Thus, the same inequality will remain true after taking "$\lambda \downarrow 1$". This indicates that (i) follows from (2.15)–(2.16) and (2.19). For (ii), by (2.13), we obtain

$$
\begin{aligned}
&\|\Sigma_{11}^{\lambda}(m,n;x,y)\|_r\\
&\qquad\le \frac{1}{[\lambda m]-m}\sum_{u=m+1}^{[\lambda m]}\left(\|\Sigma_{01}^{\lambda}(u,n;x,y)\|_r+\|\Sigma_{01}^{\lambda}(m,n;x,y)\|_r\right)\\
&\qquad\le 2\big(\sup_{m\le u\le[\lambda m]}\|\Sigma_{01}^{\lambda}(u,n;x,y)\|_r\big).
\end{aligned}
$$

Thus, (2.17) implies

$$
\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\|\Sigma_{11}^{\lambda}(m,n;x,y)\|_r=0.
$$

To replace $\sup_{(x,y)\in\Omega}$ by $\|\cdot\|_r$, we find that the preceding proof also verifies (ii). This completes the proof. □

As explained before, the following generalizes Theorem 0.2.

Theorem 2.4. *Assume that conditions* (0.2)–(0.4) *and* (0.12)–(0.13) *are satisfied for some* $p \ge 1$. *Then the following statements are true.*

(i) *Let* Ω *be any subset of* $(I\setminus E_p)^2$ *with* $d(\Omega) > 0$. *If* $\sigma_{mn}(x,y)$ *converges uniformly on* Ω *to* $f(x,y)$, *then so does* $s_{mn}(x,y)$.

(ii) *If* $\|\sigma_{mn}-f\|_r \to 0$ *unrestrictedly for some* r *with* $0 < r < 1/p$, *then* $\|s_{mn}-f\|_r \to 0$ *as* $\min\{m,n\}\to\infty$.

Proof of Theorem 2.4. We show (i) first. Let Ω be any subset of $(I\setminus E_p)^2$ with $d(\Omega) > 0$. Denote by Ω_1 and Ω_2 the projections of Ω to the x-axis and to the y-axis, respectively. Then $\Omega_j \subseteq I\setminus E_p$ for $j=1,2$. By (1.6) and Theorem 1.3,

$$
|D_j^k(x)|\le|Q_k(x)|\le\rho_k d(\Omega)^{-k}<\infty\qquad(0\le k\le p\,;\,x\in\Omega_1\cup\Omega_2).
$$

For $(x, y) \in \Omega$, summation by parts yields

$$\begin{aligned}
&|\Sigma_{01}^{\lambda}(m,n;x,y)| \\
&\le \sum_{j=0}^{m} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]+1-k}{[\lambda n]-n} |\Delta_{pp} c_{jk}| Q_p(x) Q_p(y) \\
&\quad + \frac{1}{[\lambda n]-n} \sum_{t=0}^{p-1} \sum_{j=0}^{m} \sum_{k=n+1}^{[\lambda n]} |\Delta_{pt} c_{j,k+1}| Q_p(x) Q_{t+1}(y) \\
&\quad + \sum_{t=0}^{p-1} \sum_{j=0}^{m} |\Delta_{pt} c_{j,n+1}| Q_p(x) Q_{t+1}(y) \\
&\quad + \sum_{s=0}^{p-1} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]+1-k}{[\lambda n]-n} |\Delta_{sp} c_{m+1,k}| Q_{s+1}(x) Q_p(y) \\
&\quad + \frac{1}{[\lambda n]-n} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{k=n+1}^{[\lambda n]} |\Delta_{st} c_{m+1,k+1}| Q_{s+1}(x) Q_{t+1}(y) \\
&\quad + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} |\Delta_{st} c_{m+1,n+1}| Q_{s+1}(x) Q_{t+1}(y) \\
&= I_1^{\lambda}(m,n;x,y) + I_2^{\lambda}(m,n;x,y) + I_3(m,n;x,y) \\
&\quad + I_4^{\lambda}(m,n;x,y) + I_5^{\lambda}(m,n;x,y) + I_6(m,n;x,y), \text{ say.}
\end{aligned} \tag{2.20}$$

We have

$$I_1^{\lambda}(m,n;x,y) = \left(\sum_{j=0}^{m} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]+1-k}{[\lambda n]-n} |\Delta_{pp} c_{jk}| \right) Q_p(x) Q_p(y)$$

and

$$\begin{aligned}
I_2^{\lambda}(m,n;x,y) &\le \sup_{n<k\le[\lambda n]} \sum_{t=0}^{p-1} \sum_{j=0}^{m} |\Delta_{pt} c_{j,k+1}| Q_p(x) Q_{t+1}(y) \\
&\le \sup_{n<k\le[\lambda n]} \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \Big(\sum_{j=0}^{m} |\Delta_{p0} c_{j,k+v+1}|\Big) Q_p(x) Q_{t+1}(y) \\
&\le \left(\sup_{n<k\le[\lambda n]+p} \sum_{j=0}^{m} |\Delta_{p0} c_{jk}| \right) Q_p(x) \Big(\sum_{t=0}^{p-1} 2^t Q_{t+1}(y)\Big).
\end{aligned}$$

Similarly, we also have

$$I_3(m,n;x,y) \le \Big(\sup_{n<k\le n+p} \sum_{j=0}^{m} |\Delta_{p0} c_{jk}| \Big) Q_p(x) \Big(\sum_{t=0}^{p-1} 2^t Q_{t+1}(y)\Big),$$

$$I_4^{\lambda}(m,n;x,y) \le \Big(\sup_{m<j\le m+p} \sum_{k=n+1}^{[\lambda n]} |\Delta_{0p} c_{jk}| \Big) \Big(\sum_{s=0}^{p-1} 2^s Q_{s+1}(x)\Big) Q_p(y),$$

$$\begin{aligned}
I_5^\lambda(m,n;x,y) &\le \sup_{n<k\le[\lambda n]} \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} |\Delta_{st}c_{m+1,k+1}| Q_{s+1}(x)Q_{t+1}(y) \\
&\le (\sup_{j>m;k>n} |c_{jk}|)(\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} 2^{s+t}Q_{s+1}(x)Q_{t+1}(y)), \\
I_6(m,n;x,y) &\le (\sup_{j>m;k>n} |c_{jk}|)(\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} 2^{s+t}Q_{s+1}(x)Q_{t+1}(y)).
\end{aligned}$$

Combining these with (0.2)–(0.4) and (0.12) results in (2.15). Replacing (0.12) by (0.13), we get (2.16). Thus, (i) follows from (i) of Theorem 2.3.

It remains to show (ii). Assume that $\|\sigma_{mn} - f\|_r$ converges unrestrictedly to 0 for some r with $0 < r < 1/p$. We have

$$\|s_{mn} - f\|_r^r \le \|s_{mn} - \sigma_{mn}\|_r^r + \|\sigma_{mn} - f\|_r^r,$$

and so it suffices to show that $\|s_{mn} - \sigma_{mn}\|_r^r \to 0$ as $\min\{m,n\} \to \infty$. By (2.12),

$$\begin{aligned}
&\|s_{mn} - \sigma_{mn}\|_r^r \\
&\le (\frac{[\lambda m]+1}{[\lambda m]-m})^r \|\sigma_{[\lambda m],n} - \sigma_{mn}\|_r^r + (\frac{[\lambda n]+1}{[\lambda n]-n})^r \|\sigma_{m,[\lambda n]} - \sigma_{mn}\|_r^r \\
&\quad + (\frac{[\lambda m]+1}{[\lambda m]-m})^r (\frac{[\lambda n]+1}{[\lambda n]-n})^r \|\sigma_{[\lambda m],[\lambda n]} - \sigma_{[\lambda m],n} - \sigma_{m,[\lambda n]} + \sigma_{mn}\|_r^r \\
&\quad + \|\Sigma_{01}^\lambda(m,n;x,y)\|_r^r + \|\Sigma_{10}^\lambda(m,n;x,y)\|_r^r + \|\Sigma_{11}^\lambda(m,n;x,y)\|_r^r \\
&= J_1^\lambda(m,n) + J_2^\lambda(m,n) + J_3^\lambda(m,n) \\
&\quad + J_4^\lambda(m,n) + J_5^\lambda(m,n) + J_6^\lambda(m,n), \text{ say.}
\end{aligned}$$

The hypothesis on σ_{mn} guarantees that $|J_k^\lambda(m,n)| \longrightarrow 0$ as $\min\{m,n\} \longrightarrow \infty$, where $\lambda > 1$ and $k = 1,2,3$. Notice that (2.20) holds for $x,y \in I \setminus E_p$, (see Theorem 1.3). For $|J_4^\lambda(m,n)|$, we have

$$\begin{aligned}
|J_4^\lambda(m,n)| &\le \|I_1^\lambda(m,n;x,y)\|_r^r + \|I_2^\lambda(m,n;x,y)\|_r^r + \|I_3(m,n;x,y)\|_r^r \\
&\quad + \|I_4^\lambda(m,n;x,y)\|_r^r + \|I_5^\lambda(m,n;x,y)\|_r^r + \|I_6(m,n;x,y)\|_r^r \\
&\le \left(\sum_{j=0}^{m}\sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]+1-k}{[\lambda n]-n} |\Delta_{pp}c_{jk}|\right)^r (\mu_p^r)^2 \\
&\quad + 2\left(\sup_{n<k\le[\lambda n]+p} \sum_{j=0}^{m} |\Delta_{p0}c_{jk}|\right)^r \mu_p^r (\sum_{t=0}^{p-1} 2^{tr}\mu_{t+1}^r) \\
&\quad + \left(\sup_{m<j\le m+p} \sum_{k=n+1}^{[\lambda n]} |\Delta_{0p}c_{jk}|\right)^r (\sum_{s=0}^{p-1} 2^{sr}\mu_{s+1}^r)\mu_p^r \\
&\quad + 2\left(\sup_{j>m;k>n} |c_{jk}|\right)^r (\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} 2^{(s+t)r}\mu_{s+1}^r\mu_{t+1}^r),
\end{aligned}$$

where μ_k^r is defined by (2.7). Lemma 1.2 tells us that $\mu_k^r < \infty$ for all $0 \le k \le p$. By (0.2)–(0.4) and (0.12), we conclude that

$$\lim_{\lambda \downarrow 1} \limsup_{m,n\to\infty} |J_4^\lambda(m,n)| = 0.$$

Similarly, conditions (0.2)–(0.4) and (0.13) will imply

$$\lim_{\lambda \downarrow 1} \limsup_{m,n\to\infty} |J_5^\lambda(m,n)| = 0.$$

If we substitute u for m in (2.20), the inequality (2.14) gives us

$$\begin{aligned}
|J_6^\lambda(m,n)| \le{}& 2^r \left(\sup_{m \le u \le [\lambda m]} \sum_{j=0}^{u} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] + 1 - k}{[\lambda n] - n} |\Delta_{pp} c_{jk}| \right)^r (\mu_p^r)^2 \\
&+ 2^{r+1} \left(\sup_{\substack{m \le u \le [\lambda m] \\ n < k \le [\lambda n] + p}} \sum_{j=0}^{u} |\Delta_{p0} c_{jk}| \right)^r \mu_p^r (\sum_{t=0}^{p-1} 2^{tr} \mu_{t+1}^r) \\
&+ 2^r \left(\sup_{m < j \le [\lambda m] + p} \sum_{k=n+1}^{[\lambda n]} |\Delta_{0p} c_{jk}| \right)^r (\sum_{s=0}^{p-1} 2^{sr} \mu_{s+1}^r) \mu_p^r \\
&+ 2^{r+1} \left(\sup_{j > m; k > n} |c_{jk}| \right)^r (\sum_{s=0}^{p-1} \sum_{t=0}^{p-1} 2^{(s+t)r} \mu_{s+1}^r \mu_{t+1}^r).
\end{aligned}$$

By (0.2)–(0.4) and (0.12), we infer that

$$\lim_{\lambda \downarrow 1} \limsup_{m,n\to\infty} |J_6^\lambda(m,n)| = 0.$$

Therefore, $\|s_{mn} - \sigma_{mn}\|_r \longrightarrow 0$ as $\min\{m,n\} \longrightarrow \infty$, and the desired result follows. □

Let $0 < a < b < \infty$. For $a \le m/n \le b$, we have $a/(p+\lambda) \le u/v \le (p+\lambda)b$ for all $m \le u \le [\lambda m] + p$ and for all $n \le v \le [\lambda n] + p$, where $\lambda > 1$. Following the proofs of Theorems 2.3 and 2.4, we can easily extend Theorem 0.4 in the following way.

Theorem 2.5. *(i) Let $\Omega \subseteq I^2$. Assume that for all $0 < a < b < \infty$ the conditions*

$$\lim_{\lambda \downarrow 1} \limsup_{a \le m/n \le b} \left(\sup_{(x,y)\in\Omega} |\Sigma_{01}^\lambda(m,n;x,y)| \right) = 0, \tag{2.21}$$

$$\lim_{\lambda \downarrow 1} \limsup_{a \le m/n \le b} \left(\sup_{(x,y)\in\Omega} |\Sigma_{10}^\lambda(m,n;x,y)| \right) = 0 \tag{2.22}$$

are satisfied. If $\sigma_{mn}(x,y)$ converges uniformly on Ω in the restricted sense to $f(x,y)$, then so does $s_{mn}(x,y)$.

(ii) Suppose there exists some $r \ge 1$ such that for all $0 < a < b < \infty$

$$\lim_{\lambda \downarrow 1} \limsup_{a \le m/n \le b} \|\Sigma_{01}^\lambda(m,n;x,y)\|_r = 0, \tag{2.23}$$

$$\lim_{\lambda \downarrow 1} \limsup_{a \le m/n \le b} \|\Sigma_{10}^\lambda(m,n;x,y)\|_r = 0. \tag{2.24}$$

If $\|\sigma_{mn} - f\|_r \to 0$ restrictedly, then $\|s_{mn} - f\|_r$ converges restrictedly to 0.

Theorem 2.6. *Assume that conditions* (0.2) *and* (0.14)–(0.17) *are satisfied for some* $p \geq 1$ *and for all* $0 < a < b < \infty$. *Then the following statements hold.*

(i) *Let* Ω *be any subset of* $(I \setminus E_p)^2$ *with* $d(\Omega) > 0$. *If* $\sigma_{mn}(x, y)$ *converges uniformly on* Ω *in the restricted sense to* $f(x, y)$, *then so does* $s_{mn}(x, y)$.

(ii) *If* $\|\sigma_{mn} - f\|_r \to 0$ *restrictedly for some* r *with* $0 < r < 1/p$, *then* $\|s_{mn} - f\|_r$ *converges restrictedly to* 0.

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References

[B] L. A. Balašov, *Series with respect to the Walsh system with monotone coefficients*, Sibirsk. Math. Zh. **12** (1971), 25-39; English transl. in Siberian Math. J. **12** (1971). MR **44:**1982

[C1] C.-P. Chen, *Pointwise convergence of trigonometric series*, J. Austral. Math. Soc. Ser. A **43** (1987), 291-300. MR **88m:**42011

[C2] ———, *Integrability and L-convergence of multiple trigonometric series*, Bull. Austral. Math. Soc. **49** (1994), 333-339. MR **95b:**42009

[C3] ———, *Weighted integrability and L^1-convergence of multiple trigonometric series*, Studia Math. **108** (1994), 177-190. MR **95b:**42010

[C4] ———, *Integrability of multiple Walsh series and Parseval's formula*, Analysis Math. **22** (1996), 99–112. CMP 1996#16

[C5] ———, *Integrability, mean convergence, and Parseval's formula for double Walsh series*, preprint.

[CH] C.-P. Chen and P.-H. Hsieh, *Pointwise convergence of double trigonometric series*, J. Math. Anal. Appl. **172** (1993), 582-599. MR **94d:**42013

[CMW] C.-P. Chen, F. Móricz, and H.-C. Wu, *Pointwise convergence of multiple trigonometric series*, J. Math. Anal. Appl. **185** (1994), 629-646. MR **95k:**42020

[F] N, J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372-414. MR **11:**352b

[H] G. H. Hardy, *On the convergence of certain multiple series*, Proc. Cambridge Phil. Soc. **19** (1916-1919), 86-95.

[M] G. Morgenthaler, *Walsh-Fourier series*, Trans. Amer. Math. Soc. **84** (1957), 472-507. MR **19:**956d

[M1] F. Móricz, *Walsh-Fourier series with coefficients of generalized bounded variation*, J. Austral. Math. Soc. Ser. A **47** (1989), 458-465. MR **91b:**42045

[M2] ———, *Double Walsh series with coefficients of bounded variation*, Z. Anal. Anwendungen **10** (1991), 3-10. MR **93c:**42026

[M3] ———, *Pointwise convergence of double Walsh series*, Analysis **12** (1992), 121-137. MR **93e:**42022

[MS1] F. Móricz and F. Schipp, *On the integrability and L^1-convergence of Walsh series*, J. Math. Anal. Appl. **146** (1990), 99-109. MR **91b:**42047

[MS2] ———, *On the integrability and L^1-convergence of double Walsh series*, Acta. Math. Hung. **57** (1991), 371-380. MR **92m:**42033

[MSW1] F. Móricz, F. Schipp, and W. R. Wade, *On the integrability of double Walsh series with special coefficients*, Michigan Math. J. **37** (1990), 191-201. MR **91d:**42028

[MSW2] ———, *Cesàro summability of double Walsh-Fourier series*, Trans. Amer. Math. Soc. **329** (1992), 131-140. MR **92j:**42028

[R] A. I. Rubinštein, *The A-integral and series with respect to a Walsh system*, Uspekhi. Mat. Nauk **18** (1963), no. 3, 191-197. (in Russian). MR **27:**4017

[S] A. A. Šneider, *On series with respect to Walsh functions with monotone coefficients*, Izv. Akad. Nauk. SSSR, Ser. Mat. **12** (1948), 179-192 (in Russian) MR **10:**34d

[SSW] F. Schipp, P. Simon, and W. R. Wade, *Walsh series, An Introduction to Dyadic Harmonic Analysis*, IOP Publishing Ltd, Akadémiai Kiadó, Budapest, 1990. MR **92g:**42001

[W1] F. Weisz, *Cesàro summability of two-parameter Walsh-Fourier series*, J. Approx. Theory **88** (1997), 168–192. CMP 1997#7

[W2] ———, *Cesàro summability of two-dimensional Walsh-Fourier series*, Trans. Amer. Math. Soc. **348** (1996), 2169–2181. MR **96i:**42004

[Y] S. Yano, *On Walsh-Fourier series*, Tôhoku Math. J. **3** (1951), 223-242. MR **13:**550a

[Z] A. Zygmund, *Trigonometric Series* (2nd ed.), Vol. 1, Cambridge Univ. Press, Cambridge, 1959. MR **21:**6498

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