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LOCAL BOUNDARY REGULARITY OF THE SZEGŐ PROJECTION AND BIHOLOMORPHIC MAPPINGS OF NON-PSEUDOCONVEX DOMAINS

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Abstract. It is shown that the Szegő projection S of a smoothly bounded domain Ω, not necessarily pseudoconvex, satisfies local regularity estimates at certain boundary points, provided that condition R holds for Ω.

It is also shown that any biholomorphic mapping $f : \Omega \to D$ between smoothly bounded domains extends smoothly near such points, provided that a weak regularity assumption holds for D.

1. Preliminaries

Throughout, Ω denotes a smoothly bounded domain in $\mathbb{C}^n$ and r a C^∞ defining function of Ω. The notation $W^s(\Omega), s \in \mathbb{R}$, stands for the Sobolev space of order s. The closure of $C_0^\infty(\Omega)$ in $W^s(\Omega), s > 0$, is denoted by $W_0^s(\Omega)$ with dual space $W^{-s}(\Omega)$. The norm of $W^{-s}(\Omega)$ is then

$$\|u\|_{-s} = \sup\{|\langle u, \phi\rangle|; \phi \in C_0^\infty(\Omega), \|\phi\|_s = 1\}, \quad u \in W^{-s}(\Omega).$$

Also the dual space $(W^s(\Omega))^*$ of $W^s(\Omega)$ is defined, and the norm of $u \in (W^s(\Omega))^*$, is

$$\|u\|_{-s}^* = \sup\{|\langle u, \phi\rangle|; \phi \in C^\infty(\bar{\Omega}), \|\phi\|_s = 1\}.$$

Certainly $(W^s(\Omega))^* \subset W^{-s}(\Omega)$, and $\|u\|_{-s} \le \|u\|_{-s}^*$ for u in $(W^s(\Omega))^*$. However, if u is harmonic then $\|u\|_{-s}^* \le C\|u\|_{-s}$ with C independent of u. Following Boas [12], the norm $\|\cdot\|_s^{(*)}$ is defined to be $\|\cdot\|_s$ if $s \ge 0$, and $\|\cdot\|_s^*$ if $s < 0$.

$W^s(\partial\Omega), s \in \mathbb{R}$, denotes the boundary Sobolev space. Any function in $W^s(\partial\Omega)$ is identified with a harmonic function in $W^{s+1/2}(\Omega)$ via the Poisson integral with equivalent norms:

$$C^{-1}\|u\|_{s+1/2} \le \|u\|_{W^s(\partial\Omega)} \le C\|u\|_{s+1/2}, \tag{1.1}$$

where C is a constant independent of u (The letter C in this paper denotes a positive constant which may vary at each of its occurrences.) There is also a local equivalence of these two norms. If ζ_1, ζ_2 are in $C_0^\infty(\mathbb{C}^n)$ and $\zeta_2 \equiv 1$ near supp ζ_1, the support of ζ_1, then

$$\begin{aligned} \|\zeta_1 u\|_s &\le C(\|\zeta_2 u\|_{W^{s-1/2}(\partial\Omega)} + \|u\|_{W^{-M}(\partial\Omega)}), \\ \|\zeta_1 u\|_{W^s(\partial\Omega)} &\le C(\|\zeta_2 u\|_{s+1/2} + \|u\|_{-M}), \end{aligned} \tag{1.2}$$

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where $M > 0$ is an arbitrary integer and C independent of u.

For each integer $t \geq 0$, let P_t be the orthogonal projection of $W^t(\Omega)$ to its closed subspace consisting of holomorphic functions. Note that $P_0 = P$ is the usual Bergman projection. If $K(w, z)$ is the Bergman kernel function, $Pu(z) = \langle u, K(\cdot, z)\rangle$ for $u \in L^2(\Omega)$. The Szegő projection S is the orthogonal projection from $L^2(\partial\Omega) = W^0(\partial\Omega)$ onto the closed subspace consisting of functions whose Poisson integrals are holomorphic in Ω. Similarly, the Szegő projection Su of u is represented by integration against the Szegő kernel $S(z, w)$:

$$Su(z) = \langle u, S(\cdot, z)\rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} S(z, w)u(w)\, d\sigma_w, \quad u \in L^2(\partial\Omega),$$

where $d\sigma_w$ is the differential surface element on $\partial\Omega$.

Definition ([9]). A domain Ω satisfies *condition* R, if the Bergman projection P of Ω maps $C^\infty(\bar{\Omega})$ into $C^\infty(\bar{\Omega})$; and Ω satisfies *local condition* R at $z_0 \in \partial\Omega$, if P maps $C^\infty(\bar{\Omega})$ to a subspace of holomorphic functions on Ω which are smoothly extendible to the boundary near z_0.

For example, smoothly bounded pseudoconvex domains of finite type in the sense of D'Angelo and also Reinhardt domains satisfy condition R (see [13],[2], and [7]). The latter need not be pseudoconvex.

Condition R was introduced by Bell and Ligocka in [9], and has proved to be extremely useful in the study of biholomorphic and proper holomorphic mappings between smoothly bounded domains. Results on regularity of the Bergman projection are often derived from the $\bar{\partial}$-Neumann theory through Kohn's formula ([18]) which relates them.

Now assume that Ω is a smoothly bounded domain in $\mathbb{C}^n$, not necessarily pseudoconvex.

Definition ([20]). A point z_0 on the boundary $\partial\Omega$ of Ω is an *extreme* boundary point if there is a bounded pseudoconvex domain D in $\mathbb{C}^n$ such that

i) D contains Ω and ∂D coincides with $\partial\Omega$ near z_0;
ii) z_0 is a point of finite type of D in the sense of D'Angelo [14].

By definition, there is a neighborhood U of z_0 so that $D \cap U = \Omega \cap U$. For U sufficiently small, if f is in $C^\infty(\bar{\Omega})$ and supported in $\bar{\Omega} \cap U$, the equation $\bar{\partial}\phi = f$ is solvable in $W^s(\Omega)$ for any s with $\|\phi\|_s \leq C\|f\|_s$. This follows from the local $\bar{\partial}$-theory of pseudoconvex domains of finite type (see [13]).

For any given Ω, there exist strictly pseudoconvex domains containing Ω, with boundaries tangential to the boundary of Ω. Then it can be shown that these tangent points are extreme boundary points.

2. Background

Suppose that Ω is pseudoconvex and z_0 a boundary point of finite type. It follows from Catlin's subelliptic estimates [13] for the $\bar{\partial}$-Neumann problem that there exists a neighborhood U of z_0 so that if ζ_1 and ζ_2 are real-valued functions in $C_0^\infty(U)$ and $\zeta_2 \equiv 1$ near supp ζ_1, the Bergman projection P satisfies

$$\|\zeta_1 Pu\|_s \leq C(\|\zeta_2 u\|_s + \|u\|), \quad u \in L^2(\Omega), \quad s \geq 0. \tag{2.1}$$

And if, in addition, Ω satisfies condition R, then

$$\|\zeta_1 Pu\|_s \leq C(\|\zeta_2 u\|_s + \|u\|^*_{-N}), \quad u \in L^2(\Omega), \quad s \geq 0, \tag{2.2}$$

for any integer N (see Bell [4], or Boas [11]). If all boundary points of Ω are of finite type, thus condition R implied, it follows from (2.1) or (2.2) by simply taking $U = \mathbb{C}^n$ that $\|Pu\|_s \le C\|u\|_s$. Then it is also true that for some small $\varepsilon, 0 < \varepsilon \le \frac{1}{2}$,

$$\|\zeta_1(u - Pu)\|_s \le C(\|\zeta_2\bar{\partial}u\|_{s-\varepsilon} + \|u\|^*_{-N}). \tag{2.3}$$

More information on the above results can be obtained from [4], [10] and [11].

Boas extended (2.1) to Sobolev space projections and the Szegő projection in 1985 and 1987:

Theorem A ([10], [12]). *Let Ω be a smoothly bounded pseudoconvex domain of finite type. If ζ_1, ζ_2 are real-valued functions in $C_0^\infty(\mathbb{C}^n)$ with $\zeta_2 \equiv 1$ near* supp ζ_1, *then for $u \in W^t(\Omega)$, t a positive integer,*

$$\|\zeta_1 P_t u\|_s \le C(\|\zeta_2 u\|_s + \|u\|_t), \quad s \ge t. \tag{2.4}$$

Let Ω be a smoothly bounded pseudoconvex domain and z_0 a boundary point of finite type so that a compactness estimate holds for the $\bar{\partial}$-Neumann problem. Then for some neighborhood U of z_0, if ζ_1, ζ_2 are real-valued functions in $C_0^\infty(U)$ with $\zeta_2 \equiv 1$ near supp ζ_1 *and if $N > 0$ is an arbitrary positive integer, then for u harmonic in Ω,*

$$\|\zeta_1 Su\|_{W^s(\partial\Omega)} \le C(\|\zeta_2 u\|_{W^s(\partial\Omega)} + \|u\|_{W^{-N}(\partial\Omega)}), \quad s \ge 0. \tag{2.5}$$

Also, globally, $\|Su\|_{W^s(\partial\Omega)} \le C\|u\|_{W^s(\partial\Omega)}$.

When the domain Ω is not pseudoconvex, it is well-known that the Bergman and the Szegő projections and, more generally, the Sobolev space projections may not satisfy the global regularity estimates (see Barrett [1] and Barrett and Fornæss [3]) Nevertheless, it is shown in [20] that (2.1) can be extended to any smoothly bounded domains at extreme boundary points:

Theorem B. *For some neighborhood U of an extreme boundary point z_0, if ζ_1, ζ_2 are real-valued functions in $C_0^\infty(U)$ with $\zeta_2 \equiv 1$ near* supp ζ_1, *then the Bergman projection P of Ω satisfies*

$$\|\zeta_1 Pu\|_s \le C(\|\zeta_2 u\|_s + \|u\|), \quad s \ge 0, \tag{2.6}$$

for all $u \in L^2(\Omega)$, where the global term $\|u\|$ can be replaced by $\|u\|^*_{-N}$ for arbitrary $N > 0$ if Ω satisfies condition R.

The main result of §3 is Theorem 3.1, which says that under the assumption of condition R, estimate (2.3) holds at extreme boundary points. Theorem 3.13 extends Boas' local estimates (2.4) and (2.5) to arbitrary domains at extreme boundary points, with the assumption of condition R. The proof uses Theorem 3.1 as an essential tool. If the domain is actually pseudoconvex, the condition in Theorem 3.13 is loosened in comparison to Boas' theorems, since condition R can be inferred from the compactness of the $\bar{\partial}$-Neumann problem or from the finite type of the domain. There is, however, a loss of control on the derivative of the global term in the estimate for the Szegő projection.

Now let $f : \Omega \to D$ be a biholomorphic mapping between smoothly bounded domains. Suppose one of the domains is pseudoconvex (as a consequence, so is the other). A theorem of Bell [5] says that if Ω satisfies local condition R at a boundary point z_0, then f extends smoothly to the boundary near z_0. There are also theorems about boundary extendibility of the mapping with domains not being assumed pseudoconvex (see Bell [6], Lempert [19], Forstnerič and Rosay [17]).

In §4 the local boundary behavior of biholomorphic mappings near an extreme boundary point is studied. Theorem 4.1 states that if f is biholomorphic then f is smoothly extendible to the boundary near any extreme boundary point, assuming that for a fixed point w in the target domain D the Bergman kernel $K(\cdot, w)$ of D is in $L^{2+\varepsilon}(D)$ for an arbitrarily small $\varepsilon > 0$.

3. The Szegő projection and the Sobolev space projections

Following [4], if the global term of an estimate is of the form $\|\cdot\|$ or, $\|\cdot\|^*_{-N}$, the estimate will be referred as "weak pseudolocal estimate" or "strong pseudolocal estimate" respectively.

The following theorem, together with Theorem B, shows that the Bergman projection behaves similarly in a non-pseudoconvex domain at extreme boundary points as if in a pseudoconvex domain at boundary points of finite type.

Theorem 3.1. *Let Ω be any smoothly bounded domain in $\mathbb{C}^n$ satisfying condition R, and z_0 an extreme boundary point. Then the orthogonal projection $I - P$ admits a strong pseudo-local estimate at z_0. Namely, there is a neighborhood U of z_0 and a number $\varepsilon, 0 < \varepsilon \le \frac{1}{2}$, so that if ζ_1, ζ_2 are real-valued functions in $C_0^\infty(U)$ with $\zeta_2 \equiv 1$ near* supp ζ_1*, then*

$$\|\zeta_1(u - Pu)\|_s \le C(\|\zeta_2\bar\partial u\|_{s-\varepsilon} + \|u\|^*_{-N}), \quad s \ge 0, \tag{3.2}$$

for all u in $L^2(\Omega)$.

Proof. It suffices to show the estimates for $u \in C^\infty(\bar\Omega)$, since an approximation argument implies the general case (see Proposition 3.2 of [20]). Let D be a smoothly bounded pseudoconvex domain of finite type contained in Ω with the property that ∂D coincides with $\partial\Omega$ near z_0. The existence of such a domain is shown, for example, in [4]. Let P_D be the Bergman projection associated to D. Choose a neighborhood U of z_0 so that $\Omega \cap U = D \cap U$ and (2.6) holds.

Let $\eta_j, j = 1, 2, 3$, be real-valued functions in $C_0^\infty(U)$ so that $\zeta_2 \equiv 1$ near supp η_3, $\eta_{j+1} \equiv 1$ near supp $\eta_j, j = 1, 2$, and $\eta_1 \equiv 1$ near supp ζ_1. Set $v = \eta_2 u$. Then $\zeta_1 v = \zeta_1 u$ and

$$\|\zeta_1(u - Pu)\|_s \le \|\zeta_1(v - Pv)\|_s + \|\zeta_1 P(v - u)\|_s. \tag{3.3}$$

Applying (2.6) to the last term gives

$$\begin{aligned}\|\zeta_1 P(v - u)\|_s \le& C(\|\eta_1(v - u)\|_s + \|v - u\|^*_{-N} \\ =& C\|(1 - \eta_2)u\|^*_{-N} \le C\|u\|^*_{-N}.\end{aligned} \tag{3.4}$$

For the first term on the right side of (3.3), since v is supported in D, by the triangle inequality

$$\|\zeta_1(v - Pv)\|_s \le \|\zeta_1(v - P_D v)\|_s + \|\zeta_1(P_D v - Pv)\|_s. \tag{3.5}$$

On the support of η_1, $\bar\partial v = \bar\partial u$. Since D is pseudoconvex of finite type, (2.3) yields that

$$\begin{aligned}\|\zeta_1(v - P_D v)\|_s \le& C(\|\eta_1\bar\partial v\|_{s-\varepsilon} + \|v\|^*_{W^{-N}(D)}) \\ \le& C(\|\eta_1\bar\partial u\|_{s-\varepsilon} + \|v\|^*_{W^{-N}(D)}) \\ \le& C(\|\zeta_2\bar\partial u\|_{s-\varepsilon} + \|u\|^*_{-N}),\end{aligned} \tag{3.6}$$

where the inequality $\|v\|^*_{W^{-N}(D)} \leq C\|u\|^*_{-N}$ is used, whose validity is clear since v is supported away from $\Omega \setminus D$. Estimates (3.3)–(3.6) imply

$$\|\zeta_1(u - Pu)\|_s \leq C(\|\zeta_2\bar{\partial}u\|_{s-\varepsilon} + \|u\|^*_{-N}) + \|\zeta_1(P_D v - Pv)\|_s. \tag{3.7}$$

For the last term in (3.7), by extending $P_D v$ to be defined on $\Omega \setminus D$ via zero extension, then $P_D v \in L^2(\Omega)$ and $PP_D v = Pv$. Therefore if $w = \zeta_2 P_D v$, then

$$\begin{aligned} \|\zeta_1(P_D v - Pv)\|_s &= \|\zeta_1(P_D v - PP_D v)\|_s \\ &\leq \|\zeta_1(w - Pw)\|_s + \|\zeta_1 P(P_D v - w)\|_s. \end{aligned} \tag{3.8}$$

As before, from (2.6) it follows that for some arbitrary integer $N > 0$,

$$\begin{aligned} \|\zeta_1 P(P_D v - w)\|_s \leq & C(\|\eta_1(P_D v - w)\|_s + \|P_D v - w\|^*_{-N}) \\ = & \|P_D v - w\|^*_{-N} \leq C\|P_D v\|^*_{-N}. \end{aligned} \tag{3.9}$$

The function $P_D v$ vanishes on $\Omega \setminus D$; thus

$$\|P_D v\|^*_{-N} \leq \|P_D v\|^*_{W^{-N}(D)} \leq C\|v\|^*_{W^{-N}(D)} \leq C\|u\|^*_{-N}. \tag{3.10}$$

Summarizing (3.7)–(3.10) gives

$$\|\zeta_1(u - Pu)\|_s \leq C(\|\zeta_2\bar{\partial}u\|_{s-\varepsilon} + \|u\|^*_{-N}) + \|\zeta_1(w - Pw)\|_s. \tag{3.11}$$

The function $\bar{\partial}w = (\bar{\partial}\zeta_2)P_D v$ is in $C^\infty(\bar{\Omega})$ since v is in $C^\infty(\bar{D})$ and since $\bar{\partial}\zeta_2$ is supported in $\bar{\Omega} \cap U$. Thus there is a function $\psi \in W^s(\Omega)$ solving the equation $\bar{\partial}\psi = \bar{\partial}w$ and satisfying

$$\|\psi\|_s \leq C\|\bar{\partial}w\|_s = C\|(\bar{\partial}\zeta_2)P_D v\|_s.$$

Recall that $v = \eta_2 u$, and since the intersection of supp $\bar{\partial}\zeta_2$ and supp η_2 is empty, (2.2) implies that

$$\begin{aligned} \|(\bar{\partial}\zeta_2)P_D(\eta_2 u)\|_s \leq & C(\|(1 - \eta_3)\eta_2 u\|_s + \|\eta_2 u\|^*_{W^{-N}(D)}) \\ \leq & \|\eta_2 u\|^*_{W^{-N}(D)} \leq C\|u\|^*_{-N}. \end{aligned}$$

So $\|\psi\|_s$ is bounded by a constant times $\|u\|^*_{-N}$. The function $\psi - w$ is holomorphic in Ω. So $P\psi - Pw = \psi - w$. It follows from (2.6) that

$$\begin{aligned} \|\zeta_1(w - Pw)\|_s &\leq C(\|\zeta_1\psi\|_s + \|\zeta_1 P\psi\|_s) \\ &\leq C(\|\zeta_1\psi\|_s + \|\zeta_1\psi\|_s + \|\psi\|) \\ &\leq C\|\psi\|_s \leq C\|u\|^*_{-N}. \end{aligned} \tag{3.12}$$

Now (3.2) follows by combining (3.11) and (3.12). □

The estimate (3.2) implies that when a square-integrable function is decomposed into the sum of the holomorphic part and its orthogonal complement, locally at an extreme boundary point the holomorphic part is more closely related to the function. The next theorem gives local boundary regularity of the Szegő projection and of all the Sobolev space projections at extreme boundary points. In its proof (3.2) is applied in a key step.

Theorem 3.13. *Under the same hypotheses of Theorem 3.1, the Szegő projection S and all Sobolev space projections P_t, where $t > 0$ is an integer, satisfy a weak pseudo-local estimate at any extreme boundary point z_0. Namely, there is a neighborhood U of z_0 so that if ζ_1, ζ_2 are real-valued functions in $C_0^\infty(U)$ with $\zeta_2 \equiv 1$ near* supp *ζ_1, then*

i) *For any $s \geq 0$, if $u \in L^2(\partial\Omega)$, then*

$$\|\zeta_1 Su\|_{W^s(\partial\Omega)} \leq C(\|\zeta_2 u\|_{W^s(\partial\Omega)} + \|u\|_{L^2(\partial\Omega)}). \tag{3.14}$$

ii) *For any $s \geq t$, if $u \in W^t(\Omega)$, then*

$$\|\zeta_1 P_t u\|_s \leq C(\|\zeta_2 u\|_s + \|u\|_t).$$

The proofs of i) and ii) are similar, using the methods developed by Boas in [10] and [12]. Besides relying on Theorem 3.1, the proofs also employ a local approximation argument, which is different from Boas' proofs. Only the proof of i) will be given.

Proof. The space $W^s(\partial\Omega)$ can be identified with $W^{s+1/2}(\Omega)$ by harmonic extension with equivalent norms. Thus it suffices to show, by (1.1) and (1.2), that

$$\|\zeta_1 Su\|_{s+1/2} \leq C(\|\zeta_2 u\|_{s+1/2} + \|u\|_{1/2})$$

for $u \in W^{1/2}(\Omega)$ and harmonic in Ω.

For $u \in L^2(\Omega)$, let $Tu = \int_{\partial\Omega} K(\cdot, w)u(w)d\sigma_w$. The integral is well-defined since $K(z, \cdot) \in C^\infty(\bar{\Omega})$ for fixed $z \in \Omega$, which follows from condition R on Ω. By Stokes' theorem,

$$Tu = \int_\Omega K(\cdot, w)\Delta(ru)(w)\, dV_w,$$

where $\Delta = 4\sum \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ is the usual Laplacian operator and r the normalized defining function, i.e., the gradient of r is equal to 1 on $\partial\Omega$. It is clear that $TSu = Tu$ by Fubini's theorem.

Claim. a) For any $s \geq 0$, $M \geq 0$ and any $u \in C^\infty(\bar{\Omega})$ and harmonic in Ω,

$$\|\zeta_1 Tu\|_{s-1/2} \leq C(\|\zeta_2 u\|_{s+1/2} + \|u\|_{-M}). \tag{3.15}$$

b) For any s and $h \in W^{1/2}(\Omega)$ and holomorphic in Ω,

$$\|\zeta_1 h\|_{s+1/2} \leq C(\|\zeta_2 Th\|_{s-1/2} + \|h\|_{-M}). \tag{3.16}$$

The theorem is now a consequence of the claim. In fact, it is again sufficient to prove (3.14) for $u \in C^\infty(\bar{\Omega})$ and harmonic in Ω. From the claim, for such a u, if η is a smooth cut-off function so that $\zeta_2 \equiv 1$ near supp η and $\eta \equiv 1$ near supp ζ_1, then

$$\begin{aligned}\|\zeta_1 Su\|_{s+1/2} \leq& C(\|\eta TSu\|_{s-1/2} + \|Su\|_{1/2}) \\ \leq& C(\|\eta Tu\|_{s-1/2} + \|u\|_{1/2}) \\ \leq& C(\|\zeta_2 u\|_{s+1/2} + \|u\|_{1/2}).\end{aligned}$$

Thus the proof of the theorem will be completed when the claim is established.

Proof of the Claim. Fix a sequence of smooth functions $\eta_0, \eta_1, \cdots$, such that $\eta_0 = \zeta_1$ and $\eta_{j+1} \equiv 1$ near supp η_j, $j = 0, 1, \cdots$, and $\zeta_2 \equiv 1$ on supp η_j for all j. For any $\phi \in C_0^\infty(\Omega)$ and u a harmonic function in $C^\infty(\bar{\Omega})$,

$$\begin{aligned}\langle \zeta_1 Tu, \phi\rangle =& \langle \Delta(ru), P(\zeta_1\phi)\rangle \\ =& \langle \eta_1 \Delta(ru), \eta_2 P(\zeta_1\phi)\rangle + \langle \Delta(ru), (1-\eta_1)P(\zeta_1\phi)\rangle.\end{aligned}$$

Hence,

$$|\langle \zeta_1 Tu, \phi\rangle| \leq |\langle \eta_1 \Delta(ru), \eta_2 P(\zeta_1\phi)\rangle| + |\langle \Delta(ru), (1-\eta_1)P(\zeta_1\phi)\rangle|. \tag{3.17}$$

Since u is harmonic, $\Delta(ru)$ involves its derivatives of at most first order. From the estimate dual to (2.6) it follows that

$$
\begin{aligned}
|\langle \eta_1 \Delta(ru), P(\zeta_1\phi)\rangle \le & C\|\eta_1\Delta(ru)\|_{s-1/2}\|\eta_2 P(\zeta_1\phi)\|^*_{-s+1/2} \\
\le & C(\|\eta_2 u\|_{s+1/2} + \|u\|_{-M})\|\phi\|^{(*)}_{-s+1/2}.
\end{aligned} \tag{3.18}
$$

For the last term in (3.17), since Ω satisfies condition R and the support of $(1-\eta_1)$ does not meet the support of ζ_1,

$$
\begin{aligned}
|\langle \Delta(ru), (1-\eta_1)P(\zeta_1\phi)\rangle| \le & C\|\Delta(ru)\|^*_{-N-1}\|(1-\eta_1)P(\zeta_1\phi)\|^{(*)}_{N+1} \\
\le & C\|u\|_{-N}\|\phi\|^{(*)}_{-t+1/2}.
\end{aligned} \tag{3.19}
$$

Combining (3.17)–(3.19) and taking the supremum over all $\phi \in C_0^\infty(\Omega)$ satisfying $\|\phi\|^{(*)}_{-t+1/2} = 1$ gives (3.15).

The proof of (3.16) is by induction. Let $\varepsilon > 0$ be as in Theorem 3.1. Assume that $\|\eta_{j+3}h\|_{s-\varepsilon+1/2} < +\infty$ and (3.16) holds with $\eta_{j+3}, \eta_{j+6}, s-\varepsilon$ in place of ζ_1, ζ_2, s. It will be shown that $\|\eta_j h\|_{s+1/2} < +\infty$ and (3.16) holds with η_j, η_{j+3} in place of ζ_1, ζ_2.

Again let $\phi \in C_0^\infty(\Omega)$. Choose a sequence $u_1, u_2, \cdots$ in $C^\infty(\bar\Omega)$ such that $u_j \to h$ in $L^2(\Omega)$ and $\eta_{j+4}u_j \to \eta_{j+4}h$ in $W^{s-\varepsilon+1/2}(\Omega)$. Let $h_j = Pu_j$. Then $h_j \in C^\infty(\bar\Omega)$, and (2.6) implies that $\eta_{j+3}h_j \to \eta_{j+3}h$ in $W^{s-\varepsilon+1/2}(\Omega)$. Now

$$
\begin{aligned}
\langle \eta_{j+1}Th, \phi\rangle = & \langle h, P(\eta_{j+1}\phi)\rangle_{\partial\Omega} \\
= & \langle \eta_{j+3}h, P(\eta_{j+1}\phi)\rangle_{\partial\Omega} + \langle h, (1-\eta_{j+3})P(\eta_{j+1}\phi)\rangle_{\partial\Omega} \\
= & \lim \langle \eta_{j+3}h_k, P(\eta_{j+1}\phi)\rangle_{\partial\Omega} + \langle h, (1-\eta_{j+3})P(\eta_{j+1}\phi)\rangle_{\partial\Omega}.
\end{aligned} \tag{3.20}
$$

The last term can be estimated by using (2.6) as follows:

$$
\begin{aligned}
|\langle h, (1-\eta_{j+3})P(\eta_{j+1}\phi)\rangle_{\partial\Omega}| = & C|\langle \Delta(rh), (1-\eta_{j+3})P(\eta_{j+1}\phi)\rangle| \\
\le & C\|\Delta(rh)\|^*_{-N-1}\|(1-\eta_{j+3})P(\eta_{j+1}\phi)\|_{N+1} \\
\le & C\|h\|_{-N}\|\phi\|^{(*)}_{-s+1/2}.
\end{aligned} \tag{3.21}
$$

Write the term whose limit is taken in (3.20) as

$$
\begin{aligned}
\langle \eta_{j+3}h_k, P(\eta_{j+1}\phi)\rangle_{\partial\Omega} = & \langle \Delta(r\eta_{j+3}h_k), P(\eta_{j+1}\phi)\rangle \\
= & \langle \eta_{j+1}P\Delta(r\eta_{j+3}h_k), \phi\rangle \\
= & \langle \eta_{j+1}\Delta(r\eta_{j+3}h_k), \phi\rangle - \langle \eta_{j+1}(\Delta(r\eta_{j+3}h_k) - P\Delta(r\eta_{j+3}h_k)), \phi\rangle.
\end{aligned} \tag{3.22}
$$

Now, because of Theorem 3.1 and because $[\bar\partial, \Delta r]$ is of order 1 when applied to h_k,

$$
\begin{aligned}
\|\eta_{j+1}(\Delta(r\eta_{j+3}h_k) - & P\Delta(r\eta_{j+3}h_k))\|_{s-1/2} \\
& \le C(\|\eta_{j+2}\bar\partial\Delta(rh_k)\|_{s-\varepsilon-1/2} + \|\Delta(r\eta_{j+3}h_k)\|^*_{-N-1}) \\
& \le C(\|\eta_{j+2}[\bar\partial, \Delta r]h_k\|_{s-\varepsilon-1/2} + \|h_k\|_{-N}) \\
& \le C(\|\eta_{j+3}h_k\|_{s-\varepsilon+1/2} + \|h_k\|_{-N}).
\end{aligned} \tag{3.23}
$$

Hence the absolute value of the last inner product in (3.22) is bounded by a constant times

$$
(\|\eta_{j+3}h_k\|_{s-\varepsilon+1/2} + \|h_k\|_{-N})\|\phi\|^{(*)}_{-s+1/2}.
$$

Passing to the limit in (3.22) and (3.23) and combining (3.20)–(3.23), we get

$$
|\langle \eta_{j+1}\Delta(rh), \phi\rangle| \le C|\langle \eta_{j+1}Th, \phi\rangle| + C(\|\eta_{j+3}h\|_{s-\varepsilon+1/2} + \|h\|_{-N})\|\phi\|^{(*)}_{-s+1/2}.
$$

Taking the supremum over all $\phi \in C_0^\infty(\Omega)$ with $\|\phi\|^{(*)}_{-s+1/2} = 1$ yields

$$\|\eta_{j+1}\Delta(rh)\|_{s-1/2} \le C(\|\eta_{j+1}Th\|_{s-1/2} + \|\eta_{j+3}h\|_{s-\varepsilon+1/2} + \|h\|_{-N}). \tag{3.24}$$

Since h is holomorphic, it follows from Lemma B.8 of [10] that

$$\|\eta_j h\|_{s+1/2} \le C(\|\eta_{j+1}\Delta(rh)\|_{s-1/2} + \|h\|_{-N}). \tag{3.25}$$

Finally, (3.24) and (3.25), together with a reduction procedure, imply (3.16). □

4. Biholomorphic mappings of non-pseudoconvex domains

The proof of the following theorem follows mainly from the transformation formulae of the Bergman projection and kernel.

Theorem 4.1. *Let Ω, D be smoothly bounded domains in $\mathbb{C}^n$ and z_0 an extreme boundary point of Ω. Suppose that for some open subset O of D and any $w \in O$ there exists an $\varepsilon > 0$ such that the Berman kernel $K(\cdot, w)$ of D is in $L^{2+\varepsilon}(D)$. Then any biholomorphic mapping $f : \Omega \to D$ extends smoothly to the boundary near z_0.*

Also, it follows easily from the proof of the main theorem in [9] that if D satisfies condition R and Ω satisfies local condition R at a boundary point z_0, then a biholomorphic mapping $f : \Omega \to D$ extends smoothly to the boundary near z_0.

Proof. Let F be the inverse of f. Assume that $u = \det[f']$ and $U = \det[F']$ are the Jacobian determinants of f and F. Let P_j and $K_j(\cdot,\cdot)$, $j = 1, 2$, be the Bergman projections and kernels of Ω and D respectively. They obey the following transformation rules:

$$u\,(P_2\phi) \circ f = P_1(u\,\phi \circ f),$$
$$K_1(z, F(w))\overline{U(w)} = u(z)K_2(f(z), w).$$

For any $h \in C^\infty(\bar{D})$, holomorphic in D, let ψ be in $C^\infty(\bar{D})$, vanish to infinite order at all boundary points of D, and satisfy $P_2\psi = h$. The existence of such a ψ is shown, for example, in [21]. Then $u\,h \circ f = u\,(P_2\psi) \circ f = P_1(u\,\psi \circ f)$.

Since z_0 is an extreme boundary point, it is shown by Diederich and Fornæss in [16] that there exists a smooth function ϕ on Ω such that i) locally ϕ defines Ω near z_0; ii) Ω is contained in $\{z; \phi(z) < 0\}$; and iii) the function $-(-\phi)^\delta$ is plurisubharmonic in Ω for some $\delta > 0$.

For w in D, let $\rho(w) = -(-\phi)^\delta(F(w))$. Then $\rho(w)$ is plurisubharmonic in D. By the Hopf lemma, there is a positive constant C such that $\rho(w) \le -Cd(w, \partial D)$. If U is some small neighborhood of z_0, it follows from i) above that

$$z \in \Omega \cap U \Longrightarrow d(f(z), \partial D)^{1/\delta} \le C\, d(z, \partial\Omega). \tag{4.2}$$

Let ζ_1, ζ_2 be as in (2.6). The transformation formula and (2.6) yield that for any $s \ge 0$

$$\|\zeta_1(u\,h \circ f)\|_s = \|\zeta_1 P_1(u\,\psi \circ f)\|_s \le C(\|\zeta_2(u\,\psi \circ f)\|_s + \|u\,\psi \circ f\|). \tag{4.3}$$

The function $u\,\psi \circ f$ is square integrable. So the last term of the above estimate is bounded. Since $\zeta_2(u\,\psi\circ f)$ is supported in U, for any l and any index α the inequality

$\left|\frac{\partial^\alpha \psi}{\partial z^\alpha}(w)\right| \leq C_{l,\alpha} d(w, \partial D)^l$ holds for some constant $C_{l,\alpha}$. Letting $l = 2(n+1+s)/\delta$, from (5.2) we get

$$\begin{aligned}
\|\zeta_2(u\,\psi \circ f)\|_s \leq & C\left[\int_{\Omega_1} d(z, \partial\Omega_1)^{-2(n+1+s)} d(f(z), \partial\Omega_2)^l \, dV_z\right]^{1/2} \\
\leq & C\left[\int_{\Omega_1} d(z, \partial\Omega_1)^{-2(n+1+s)} d(f(z), \partial\Omega_2)^{\frac{2(n+1+s)}{\delta}} \, dV_z\right]^{1/2} \\
= & C < +\infty.
\end{aligned} \tag{4.4}$$

The integer s is arbitrary, so (5.3) and (5.4) give that $\zeta_1(u\,h \circ f)$ is in $C^\infty(\bar{\Omega})$. Taking $h = 1$, we see that u is smooth near z_0.

Claim. The Jacobian determinant u vanishes to at most finite order at z_0.

The proof of the main theorem in [8] or [15] shows that if the claim is proved then a division theorem implies the smooth extension of f near z_0.

Fix a $w \in O \setminus f(\Gamma)$ with $K_2(\cdot, w) \in L^{2+\varepsilon}(D)$ for some ε. Let t be a large number so that $\frac{2}{t} < \varepsilon$, and set $g_w(z) = u(z)^t K_2(f(z), w)^{t+1}$. The function $g_w(z)$ is in $L^{\frac{1}{t}}(\Omega)$. Indeed, by the Cauchy-Schwarz inequality

$$\begin{aligned}
\int_\Omega |g_w(z)|^{\frac{1}{t}} \, dV_z = & \int_\Omega |u(z) K_2(f(z), w)^{\frac{t+1}{t}}| \, dV_z \\
= & \int_D |K_2(\lambda, w)^{\frac{t+1}{t}} U(\lambda)| \, dV_\lambda \\
\leq & C\left(\int_D |U(\lambda)|^2 \, dV_\lambda\right)^{\frac{1}{2}} \left(\int_D |K_2(\lambda, w)|^{2+\frac{2}{t}} \, dV_\lambda\right)^{\frac{1}{2}},
\end{aligned} \tag{4.5}$$

so it is therefore bounded. Since $g_w(z)$ is holomorphic, $|g_w(z)| \leq C\, d(z, \partial\Omega)^{-(n+1)t}$. In view of the transformation formula of the Bergman kernel the following equality holds:

$$u(z) g_w(z) = \left(\overline{U(w)} K_1(z, F(w))\right)^{t+1}. \tag{4.6}$$

Since Ω satisfies local condition R at z_0, the function $K_1(z_0, F(w))$ for $w \in D \setminus f(\Gamma)$ does not vanish identically. Therefore, for some positive constants C and some $w \in O$, after possibly shrinking U,

$$\overline{U(w)} K_1(z, F(w)) \geq C^{-1},$$

for all $z \in \Omega \cap U$. It follows from (4.6) that for all z close to z_0,

$$C\,|u(z)|\, d(z, \partial\Omega)^{-(n+1)t} \geq |u(z) g_w(z)| \geq C^{-1} > 0.$$

Hence $|u(z)| \geq C^{-1}\, d(z, \partial\Omega)^{(n+1)t}$, which implies that u vanishes to at most finite order at z_0. Thus the proof of the claim is complete, and so is the proof of the theorem. $\square$

References

1. D. Barrett, *Irregularity of the Bergman projection on a smooth bounded domain in* $\mathbb{C}^2$, Ann. Math. **119** (1984), 431–436. MR **85e:**32030
2. ———, *Regularity of the Bergman projection on domains with transverse symmetries*, Math. Ann. **258** (1982), 441–446. MR **83i:**32032
3. D. Barrett and J. Fornæss, *Uniform approximation of holomorphic functions on bounded Hartogs domains in* $\mathbb{C}^2$, Math. Z. **191** (1986), 61–72. MR **82e:**32022

4. S. Bell, *Differentiability of the Bergman kernel and pseudo-local estimates*, Math. Z. **192** (1986), 467–472. MR **87i:**32034
5. ———, *Local boundary behavior of proper holomorphic mappings*, Proc. Sympos. Pure Math., vol. 41, A.M.S., Providence, 1984, pp. 1–7. MR **85j:**32043
6. ———, *Boundary behavior of proper holomorphic mappings between non-pseudoconvex domains*, Am. J. of Math. **106** (1984), 639–643. MR **86a:**32054
7. S. Bell and H. Boas, *Regularity of the Bergman projection in weakly pseudoconvex domains*, Math. Ann. **257** (1981), 23–30. MR **83b:**32021
8. S. Bell and D. Catlin, *Boundary regularity of proper holomorphic mappings*, Duke Math. J. **49** (1982), 385–396. MR **84b:**32037a
9. S. Bell and E. Ligocka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. **57** (1980), 283–289. MR **81i:**32017
10. H. Boas, *The Szegő projection: Sobolev estimates in regular domains*, Trans. Amer. Math. Soc. **300** (1987), 109–132. MR **88d:**32030
11. ———, *Extension of Kerzman's theorem on differentiability of the Bergman kernel function*, Indiana Univ. Math. Journal **36** (1987), 495–499. MR **88j:**32028
12. ———, *Sobolev space projections in strictly pseudoconvex domains*, Trans. Amer. Math. Soc. **288** (1985), 227–240. MR **86g:**32041
13. D. Catlin, *Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains*, Ann. Math. **126** (1987), 131–191. MR **88i:**32025
14. J. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. Math. **115** (1982), 615–637. MR **84a:**32027
15. K. Diederich and J. E. Fornæss, *Boundary regularity of proper holomorphic mappings*, Invent. Math. **67** (1982), 363–384. MR **84b:**32037b
16. ———, *Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129–141. MR **55:**10728
17. F. Forstnerič and J-P. Rosay, *Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. **279** (1987), 239–252. MR **89c:**32070
18. J. J. Kohn, *A survey of the $\bar{\partial}$-Neumann problem*, Proc. Sympos. Pure Math. 41, A.M.S., Providence, 1984, pp. 137–145. MR **85e:**32023
19. L. Lempert, *On the boundary behavior of holomorphic mappings*, Contributions to Several Complex Variables (A. Howard and P.-M. Wong, editors), Vieweg, Braunschweig, 1986, pp. 193–215. MR **87m:**32058
20. P. Ma, *Local boundary regularity of the Bergman projection in non-pseudoconvex domains*, Ill. J. of Math. **37** (1993), 49–68. MR **93k:**32048
21. E. Straube, *Harmonic and analytic functions admitting a distribution boundary value*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 559–586. MR **87c:**31006

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