TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 1, January 1998, Pages 135–164
S 0002-9947(98)02032-7

DIVISOR SPACES ON PUNCTURED RIEMANN SURFACES

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ABSTRACT. In this paper, we study the topology of spaces of n-tuples of positive divisors on (punctured) Riemann surfaces which have no points in common (the *divisor spaces*). These spaces arise in connection with spaces of based holomorphic maps from Riemann surfaces to complex projective spaces. We find that there are Eilenberg-Moore type spectral sequences converging to their homology. These spectral sequences collapse at the E^2 term, and we essentially obtain complete homology calculations. We recover for instance results of F. Cohen, R. Cohen, B. Mann and J. Milgram, *The topology of rational functions and divisors of surfaces*, Acta Math. **166** (1991), 163–221. We also study the homotopy type of certain mapping spaces obtained as a suitable direct limit of the divisor spaces. These mapping spaces, first considered by G. Segal, were studied in a special case by F. Cohen, R. Cohen, B. Mann and J. Milgram, who conjectured that they split. In this paper, we show that the splitting does occur provided we invert the prime two.

0. INTRODUCTION

Let $X = M_g$ be a genus g compact oriented Riemann surface with $g \geq 0$, $M_0 = \mathbb{P}^1$ being the Riemann sphere. For the rest of this paper, we will make use of a preferred basepoint x_0 (or $*$) in M_g. Let $SP^r(X)$ denote the r-fold symmetric product of X (i.e. the space of degree r positive divisors on X). We define the subspace

$$\operatorname{Div}_{k_1,\dots,k_n}(X) \subset SP^{k_1}(X) \times \cdots \times SP^{k_n}(X) \tag{0.1}$$

to be the set of tuples of positive divisors $(D_{k_1}, \dots, D_{k_n})$ such that $D_{k_1} \cap \cdots \cap D_{k_n} = \emptyset$. In other words, $\operatorname{Div}_{k_1,\dots,k_n}(X)$ is the space of divisors D_{k_i} on X, $i = 1, \dots, n$, of degree k_i, and having no points in common. The relevance of these spaces of divisors to spaces of holomorphic maps is now explained.

First assume $g = 0$. Then by associating to every meromorphic function on $\mathbb{P}^1$ its (disjoint) sets of zeros and poles, we can identify the divisor space $\operatorname{Div}_{k,k}(S^2 - *)$ with the space of degree k based self-holomorphic maps of the Riemann sphere; that is,

$$\operatorname{Div}_{k,k}(S^2 - x_0) = \operatorname{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^1). \tag{0.2}$$

Received by the editors December 7, 1995.
1991 *Mathematics Subject Classification.* Primary 57R19; Secondary 14H55.
Key words and phrases. Riemann surfaces, symmetric products, Eilenberg-Moore spectral sequence, Whitehead product, Milgram bar construction.
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More precisely, the space $\mathrm{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^1)$ here consists of rational maps $f(z) = p(z)/q(z), z \in \mathbb{C}$, where p and q are monic polynomials of degree k (hence some people use the notation Rat_k for $\mathrm{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^1)$). We may also write $\mathrm{Rat}_k(\mathbb{P}^n)$ for $\mathrm{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^n)$ and here too we have the identification ([S])

$$\mathrm{Div}_{\underbrace{k, k, \dots, k}_{n+1}}(S^2 - x_0) = \mathrm{Rat}_k(\mathbb{P}^n). \tag{0.3}$$

When $g \geq 1$, the connection between the divisor spaces Div and spaces of holomorphic maps is less direct and is given essentially by a classical theorem of Abel. Recall that every Riemann surface M_g embeds, via the Abel-Jacobi map μ ([ACGH]), into its associated Jacobi variety $J(M_g)$, which is a complex g dimensional torus. The map μ extends additively to $SP^k(M_g), \forall k$. A classical theorem of Abel ([ACGH], chap. I) translates to the statement that the space of degree k based holomorphic maps $\mathrm{Hol}_k^*(M_g, \mathbb{P}^n)$ is the subspace of $\mathrm{Div}_{k,\dots,k}(M_g - x_0)$ consisting of $(n+1)$-tuples of divisors with the property that

$$\mu(D_1) = \mu(D_2) = \cdots = \mu(D_{n+1}).$$

Using the spaces $\mathrm{Div}(M_g - x_0)$ as intermediate constructs, Segal was able to prove an interesting stability result for spaces of holomorphic maps on Riemann surfaces. More explicitly, he showed that the natural inclusion $I : \mathrm{Hol}_k^*(M_g, \mathbb{P}^n) \hookrightarrow \mathrm{Map}_k^*(M_g, \mathbb{P}^n)$, obtained by simply forgetting the holomorphic structure, induces a homotopy equivalence through a range increasing with k. These results are greatly extended in [KM].

A systematic study of the divisor spaces was initiated in [C2M2], where the authors constructed a homotopy model whose cohomology is related to the homology of the Div spaces via Alexander-Poincaré duality. Starting with that model, we are able to construct a homology spectral sequence of the Eilenberg-Moore type, converging to the homology of the Div spaces, and then show that this spectral sequence collapses at the E^2 term for all $g \geq 0$ and for all n. This then yields our first main theorem:

Theorem 0.4. *For field coefficients $\mathbb{F}$, we have the following isomorphism*:

$$H_*(\mathrm{Div}_{\underbrace{k, \dots, k}_{n}}(M_g - *); \mathbb{F}) \cong Tor_{2nk-*,k}^{H_*(SP^\infty(M_g))}(\mathbb{F}, H_*(SP^\infty(M_g); \mathbb{F})^{\otimes n}).$$

To clarify the statement of the theorem above, we need to indicate that there is a bigraded algebra structure on the homology groups of $SP^\infty(X)$ yielding in the appropriate manner the bigrading of the Tor term above. The theorem and the details leading to it are discussed in §4. When $n > 2$, the module structure of $H_*(SP^\infty(M_g))^{\otimes n}$ over $H_*(SP^\infty(M_g))$ is trivial and so the calculations are direct. We write $\mathrm{Div}_{\underbrace{k, \dots, k}_{n}} = \mathrm{Div}_k^n$. One has for instance (§6)

Corollary 0.5. *For $n > 2$ and $g \geq 1$, the rational homology of $\mathrm{Div}_k^n(M_g - *)$ is the subset of the $(n+1)$-graded algebra*

$$\Lambda(e_{1;1}, \dots, e_{2g;1}, \dots, e_{1;n}, \dots, e_{2g;n}, E) \otimes \mathbb{Q}(h_1, \dots, h_{2g}),$$

where the grading is assigned as follows: $e_{i;r} \mapsto (1; 0, \dots, 1, \dots, 0)$, with 1 in the $r+1$ position, $1 \leq r \leq n, 1 \leq i \leq 2g$, $E \mapsto (2n-3; 1, \dots, 1)$, $h_j \mapsto (2n-2; 1, \dots, 1)$.

The multigrading is additive. In this setting, $H_*(\mathrm{Div}^n_k(M_g-*);\mathbb{Q})$ *is given by those elements of multidegree* $(*;i_1,\dots,i_n)$ *with* $i_j\le k$.

Similar results are obtained mod-p. When $g=0$, 0.4 takes a quite simple and explicit expression for all n, and one recovers the original results of [C2M2] on the homology structure of the Rat spaces (§5). In the case when $n=2$, the module structure in the Tor term of 0.4 is non-trivial and the calculations are much more tedious.

Remark 0.6. The spectral sequence that is considered in this paper and the resulting collapse are used in [KM] to study the spaces $\mathrm{Hol}^*_k(M_g,\mathbb{P}^{n-1})$ themselves. It's not coincidental that there too the homology structure does depend on whether $n=2$ or $n>2$.

We can stabilize the divisor spaces with respect to "collar" inclusions

$$\mathrm{Div}_{k_1,\dots,k_n}(M_g-*)\to\mathrm{Div}_{k_1,\dots,k_{j-1},k_j+1,k_{j+1},\dots,k_n}(M_g-*)$$

obtained by first continuously deforming $\mathrm{Div}_{k_1,\dots,k_n}(M_g-*)$ to $\mathrm{Div}_{k_1,\dots,k_n}(M_g-U)$ where U is a small neighborhood of $*$, and then adding a chosen point $x\neq *\in U$ to the jth divisor. It is now a theorem of Segal that the direct limit over these inclusions is homotopy equivalent to a component of a known (based) mapping space; i.e.

$$\lim_{k\to\infty}\mathrm{Div}^n_k(M_g-*)\simeq\mathrm{Map}^*_0(M_g,W_n(\mathbb{P}))\qquad\text{(Segal)}\tag{0.7}$$

where $W_n(\mathbb{P})$ is the n^{th} fat wedge of the infinite complex projective space $\mathbb{P}$ (or $\mathbb{P}^\infty$) and where Map_0 denotes the component of null-homotopic maps. The fat wedge $W_n(X)\subset X^n$ is the subset consisting of tuples where at least one entry is basepoint (e.g. $W_1=*\in X$ and $W_2=X\vee X$). In §7 we establish the existence of a fibration with a section

$$S^{2n-1}\to W_n(\mathbb{P}^\infty)\hookrightarrow(\mathbb{P}^\infty)^n$$

which when coupled with the mapping space fibration obtained by mapping the cofibration sequence $\bigvee S^1\to M_g\to S^2$ into $W_n\mathbb{P}$, yields the fibration

$$\Omega^2S^{2n-1}\to\mathrm{Map}^*_0(M_g,W_n\mathbb{P})\to(S^1)^{2ng}\times(\Omega S^{2n-1})^{2g}.\tag{0.8}$$

It is now not hard to see (§9) that as a result of 0.4 and 0.7 we have

Proposition 0.9. *The (cohomology) Eilenberg-Moore spectral sequence associated to*

$$\Omega^2S^{2n-1}\to\mathrm{Map}^*_0(M_g,W_n\mathbb{P})\to(\Omega W_n\mathbb{P})^{2g}$$

and converging to $H^*(\mathrm{Map}^*_0(M_g,W_n\mathbb{P});\mathbb{F})$ *collapses at the* E_2 *term.*

This leads to the determination of $H^*(\mathrm{Map}^*_0(M_g,W_n\mathbb{P});\mathbb{F})$, and the results turn out to be consistent with the conjecture of [C2M2] which states that the term $(\Omega S^{2n-1})^{2g}$ in the base of 0.8 ought to split off from the mapping space. More explicitly, and in the relevant case when $n=2$, [C2M2] states that there should be a decomposition

$$\mathrm{Map}^*(M_g,\mathbb{P}\vee\mathbb{P})\simeq(\mathbb{Z})^2\times\Omega(S^3)^{2g}\times Y_g,$$

where Y_g is the total space of a fibration $\Omega^2(S^3)\to Y_g\to(S^1)^{4g}$. The existence of such a splitting is also very much suggested by results of [BCM], who prove similar decomposition results for $\mathrm{Map}^*(M_g, S^{2n}), n \geq 1$ (see §7). It turns out, however, that there is an obstruction to such a decomposition.

In §8 we study the homotopy type of the mapping space $\mathrm{Map}^*(M_g, W_n(\mathbb{P}))$. The problem there becomes to factor the classifying map associated to 0.8 as

$$f^! : (\Omega S^{2n+1} \times (S^1)^{n+1})^{2g} \to (S^1)^{2g(n+1)} \to \Omega S^{2n+1} \hookrightarrow \Omega S^{2n+1} \times (S^1)^{(n+1)}.$$

A first look at $f^!$ shows that there are essential $\mathbb{Z}_2$ obstructions to such a factorization when $n > 2$. When $n = 2$, a close examination of the Postnikov system of $\mathbb{P}^\infty \vee \mathbb{P}^\infty$ shows that there is a non-zero obstruction to the above decomposition taking the form of a triple Whitehead product

$$[a_1, [a_1, a_2]] \in \pi_4(\mathbb{P} \vee \mathbb{P}) \cong \mathbb{Z}_2.$$

So in all cases we're up against essential $\mathbb{Z}_2$ obstructions, and we have

Proposition 0.10. *There is a splitting after inverting* 2*:*

$$\mathrm{Map}_0^*(M_g, W_n\mathbb{P}) \simeq (\Omega S^{2n-1})^{2g} \times Y_{g,n},$$

where $Y_{g,n}$ is the total space of a fibration $\Omega^2(S^{2n-1}) \to Y_{g,n} \to (S^1)^{2ng}$.

1. The Structure of Symmetric Products

Given a space X, we let $SP^n(X) = X^n/\mathcal{S}_n$ denote the n-th symmetric product of X (here $\mathcal{S}_n$ is the group on n letters acting by permuting factors). Equivalently, $SP^n(X)$ is the set of all unordered n-tuples $\langle x_1, \ldots, x_n\rangle$ of points in X.

Let $*$ be a chosen base point in X; then there are natural inclusions $SP^n(X) \hookrightarrow SP^{n+1}(X)$ which identify $\langle x_1, \ldots, x_n\rangle$ with $\langle x_1, \ldots, x_n, *\rangle$, and we get the expanding sequence of spaces

$$* \equiv SP^0(X) \subset SP^1(X) \subset \cdots \subset SP^{n-1}(X) \subset SP^n(X) \subset \cdots.$$

The direct limit over these inclusions is the infinite symmetric product $SP^\infty(X, *)$ (topologized by the weak topology relative to the union of the $SP^i(X)$.) The pairing

$$\begin{array}{ccc} SP^n(X) \times SP^m(X) & \xrightarrow{\mu} & SP^{n+m}(X), \\ \langle x_1, \ldots, x_n\rangle \times \langle y_1, \ldots, y_m\rangle & \mapsto & \langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle, \end{array}$$

turns $SP^\infty(X, *)$ into an abelian and associative monoid with pairing

$$\mu : SP^\infty(X, *) \times SP^\infty(X, *) \to SP^\infty(X, *).$$

We often write $\cdot$ (or $+$) for addition in $SP^\infty(X, *)$; that is, $\mu((x, y)) = x \cdot y = x + y = xy$ are all equivalent notations. We use the same notation for the induced Pontryagin product on $H_*(SP^\infty(X); \mathbb{A})$.

Since $SP^\infty(X, *)$ is abelian, it must be a product of Eilenberg-Mac Lane spaces, and one actually has

Theorem 1.1 (Dold-Thom)**.** *$\pi_*(SP^\infty(X, *)) \cong H_*(X; \mathbb{Z})$, and hence*

$$SP^\infty(X, *) = \prod K\left(\tilde{H}_i(X; \mathbb{Z}), i\right).$$

For example, $SP^\infty(S^n, *) \simeq K(\mathbb{Z}, n)$, $n \geq 1$.

Properties. • The finite and infinite symmetric products are covariant functors on the category of pointed topological spaces. If $f : (X, *) \to (Y, *)$ is a map of pairs, then the induced maps on the symmetric products are denoted by $SP^n f : SP^n(X) \to SP^n(Y)$ and $SP^\infty f : SP^\infty(X, *) \to SP^\infty(Y, *)$.

• $SP^n(-)$ is a homotopy functor. In particular, if cX denotes the cone on X, then $SP^n(cX)$ is contractible for all n.

Facts. Here are now some known properties of symmetric products that we will be using:
• $SP^\infty(X \vee Y, *) \simeq SP^\infty(X, *) \times SP^\infty(Y, *)$.
• $\pi_1(SP^n(X))$ is abelian when $n \geq 2$. 1.2
• $SP^n(S^1) \simeq S^1,\ n \geq 1$. 1.3
• $SP^n(S^2) \cong \mathbb{P}^n,\ n \geq 1$. 1.4
• There is a diffeomorphism $SP^n(\mathbb{C}) \cong \mathbb{C}^n$.

To see this last statement, choose a tuple of n points in $\mathbb{C}$, say $(v_1, v_2, \ldots, v_n)$, and associate to it the coefficients of the monic polynomial $(z-v_1)(z-v_2)\cdots(z-v_n)$. This sets up the correspondence between $SP^n(\mathbb{C})$ and $\mathbb{C}^n$, and it's easy to see that it is a diffeomorphism. A straightforward corollary of this is:

Corollary 1.5. *Let M be a smooth closed curve. Then $SP^n(M)$ is a complex n dimensional manifold.*

1.1. **The Homology of Symmetric Products.** The symmetric products exhibit interesting homological properties. For instance, it was proved by Dold [D] that for X a CW-complex, $H_*(X^n/G)$ only depends on $H_*(X)$ for any subgroup $G \subset \mathcal{S}_n$. The homology groups $H_*(SP^n(X))$, for instance, are entirely determined by the homology groups of X. Moreover, we have the following classical splitting result due to Steenrod:

Theorem 1.6 (Steenrod)**.** *For X connected and for untwisted coefficients $\mathbb{A}$, we have*

$$\begin{aligned} H_*(SP^n(X);\mathbb{A}) &= \sum_{k=1}^{n} H_*(SP^k(X), SP^{k-1}(X);\mathbb{A}) \\ &= H_*(SP^n(X), SP^{n-1}(X);\mathbb{A}) \oplus H_*(SP^{n-1}(X);\mathbb{A}). \end{aligned}$$

Remark 1.7. The splitting above induces a bigrading on $H_*(SP^\infty(X,*),\mathbb{A})$; for an element $x \in H_*(SP^\infty(X,*),\mathbb{A})$ has bidegree (i,k) iff

$$x \in H_i(SP^k(X), SP^{k-1}(X), \mathbb{A}).$$

This evidently implies that $H_*(SP^\infty(X,*),\mathbb{A})$ has the structure of a bigraded algebra. We will write $\deg(x)$ for the *homological* degree of x and $\mathrm{fil}(x)$ for its *filtration degree* k. Notice that

$$\deg(x\cdot y) = \deg(x) + \deg(y),\ \ \mathrm{fil}(x\cdot y) = \mathrm{fil}(x) + \mathrm{fil}(y).$$

Remark 1.8. For finitely generated CW-complexes, there is a standard procedure due to Milgram [M3] to determine the homology of the symmetric products. This procedure amounts to first determining the bigraded algebra structure of

$$H_*(SP^\infty A(G,n);\mathbb{A}) = H_*(K(G,n),\mathbb{A})$$

for Moore spaces, and this can be deduced from Cartan's determination of the homology of Eilenberg-Mac Lane spaces [Car].

Knowledge of the homology of symmetric products of Moore spaces can then be used to determine $H_*(SP^\infty(X,*))$ for any finitely generated CW-complex X. More precisely, given such X (arcwise connected), one can recover the homology type of X via a wedge of Moore spaces Y_i, and hence the problem reduces to calculating

$H_*(SP^\infty(\bigvee Y_i,*))$ as a bigraded algebra. But it's not hard to see that for CW-complexes X and Y (and untwisted coefficients $\mathbb{A}$) there is a bigraded algebra isomorphism

$$H_*(SP^\infty(X\vee Y,*),\mathbb{A})\cong H_*(SP^\infty(X,*),\mathbb{A})\otimes H_*(SP^\infty(Y,*),\mathbb{A}).$$

1.2. **Symmetric Products of Curves.** A genus g compact Riemann surface M_g is obtained by attaching a 2-cell, D^2, to a wedge of $2g$-circles via the commutator map. If we denote by $a_1,b_1,\cdots,a_g,b_g$ the generators of $\pi_1(M_g)$, each representing a copy of S^1 in the one skeleton $\underbrace{S^1\vee\cdots\vee S^1}_{2g}\subset M_g$, then we can write $M_g\simeq(\bigvee^{2g}S^1)\cup_{[a_1,b_1]\cdots[a_g,b_g]}D^2$.

We choose the letters $\{e_i,\ i=1,\dots,2g\}$ to label the homology generators in $H_1(M_g;\mathbb{Z})$. The boundary of the top 2-dimensional class D^2 vanishes (being a commutator), and hence D^2 generates a homology class which corresponds to the orientation class $[M_g]$ (or M for short). In homology we have that $H_*(M_g)\cong H_*(\bigvee^{2g}(S^1)\vee S^2)$ and it follows from 1.1 and 1.8 that

Lemma 1.9. *We have the following bigraded algebra isomorphism*:

$$H_*(SP^\infty(M_g,*);\mathbb{Z})\cong\Lambda(e_1)\otimes\cdots\otimes\Lambda(e_{2g})\otimes\Gamma[M]$$

where $\Gamma[M]$ *is the divided power algebra over* $\mathbb{Z}$ *generated by elements* $\gamma_i=\frac{M^i}{i!}$.

Here it is clear that $\deg(e_i)=1=\mathrm{fil}(e_i)$ and so the e_i's have bidegree $(1,1)$, while M has bidegree $(2,1)$. As a consequence of 1.6 one can check that

Lemma 1.10. $H_*(SP^n(M_g);\mathbb{Z})\subset H_*(SP^\infty(M_g,*);\mathbb{Z})$ *consists of all elements of bidegree* $(*,i), i\leq n$. *For instance* $H_*(SP^n(M),SP^{n-1}(M);\mathbb{Z})$ *has generators of the following type*:

$$e_{i_1}\cdots e_{i_r}\gamma_s,\ r+s=n.$$

Lemma 1.10 describes entirely the homology of $SP^n(M_g)$ for finite n. Notice at this point that 1.1 and then 1.3 imply that

(1.11)
$$SP^\infty(M_g,*)\simeq K(\mathbb{Z}^{2g},1)\times K(\mathbb{Z},2)\simeq(S^1)^{2g}\times\mathbb{P}\simeq SP^\infty(S^1)^{2g}\times SP^\infty(S^2).$$

We can give an explicit construction of the homotopy equivalence above as follows. First we have the obvious map $SP^\infty(\bigvee^{2g}S^1,*)=SP^\infty(S^1,*)^{2g}\overset{SP^\infty(i)}{\longrightarrow}SP^\infty(M_g,*)$ induced from the inclusion of the one skeleton $i:\bigvee^{2g}S^1\hookrightarrow M_g$ and sending the wedgepoint to basepoint $*\in M_g$. Next, we can consider the composite

$$\tau:S^1=\partial D^2\to\bigvee^{2g}S^1\to SP^2(\bigvee^{2g}S^1)\to SP^2(M_g).$$

At the level of fundamental groups, τ_* factors through a commutator f_* and since $\pi_1(SP^2(X))$ is abelian (1.2), it follows that $\tau_*([S^1])=0$. The map τ extends to a map from a new disk D'^2,

$$\tau:D'^2\to SP^2(\bigvee^{2g}S^1)\to SP^2M_g.$$

We can draw the following diagram:

$$\begin{array}{ccccc} D'^2 & & \xrightarrow{\tau} & & SP^2(\bigvee^{2g} S^1) \\ \uparrow & & & & \| \\ S^1 & \xrightarrow{f} & \bigvee^{2g} S^1 & \hookrightarrow & SP^2(\bigvee^{2g} S^1) \\ \downarrow & & & & \downarrow \\ D^2 & \to & M_g & \hookrightarrow & SP^2(M_g) \end{array}$$

which is then seen to give rise to a map

$$h : S^2 = D^2 \cup_{S^1} D'^2 \to SP^2(M_g) \tag{1.12}$$

and hence to a map $SP^\infty(h) : SP^\infty(S^2, *) = \mathbb{P}^\infty \to SP^\infty(M_g, *)$.

Lemma 1.13. *The composite*

$$\mu(SP^\infty(i) \times SP^\infty(h)) : (SP^\infty(S^1, *))^{2g} \times SP^\infty(S^2, *) \to SP^\infty(M_g, *)$$

is a homotopy equivalence (here again μ is the monoid addition in $SP^\infty(M_g, *)$).

About divided power algebras. Let A be a commutative graded algebra, and $a \in A$ an even degree element. A *divided power algebra* on a; denoted by $\Gamma(a)$, is the algebra generated by elements $\gamma_i = \gamma_i(a)$ with relations

$$\gamma_0 = 1,\ \gamma_1 = a,\ \gamma_k\gamma_h = \binom{k+h}{h}\gamma_{k+h},$$

and boundary $d\gamma_k = (da)\gamma_{k-1}, k \geq 1$. Here the degree of γ_k is determined by the fact that $\deg(\gamma_k) = k\deg(a)$. Over $\mathbb{Z}$, the generators of $\Gamma(a)$ are uniquely defined by the formula $\gamma_k = \frac{a^k}{k!}$. Over $\mathbb{Q}$, everything becomes a unit and hence $\Gamma(a) = \mathbb{Q}[a]$. With mod-$p$ coefficients, $\Gamma(a)$ splits into products of truncated polynomial algebras (see §6).

2. A Model for the Divisor Spaces

For any space X, and given integers $k_i \geq 1$, we defined (§0) the divisor space

$$\mathrm{Div}_{k_1,\dots,k_n}(X) = \{(D_{k_1}, \dots, D_{k_n}) \mid D_{k_i} \in SP^{k_i}(X),\ D_{k_1} \cap D_{k_2} \cap \cdots \cap D_{k_n} = \emptyset\}.$$

The element $D_k \in SP^k(X)$ can be represented either by an unordered k-tuple of points $\langle x_1, \dots, x_k\rangle$ (the x_j not necessarily distinct), or by a formal sum $\sum n_i x_i$ such that $\sum n_i = k$ and $x_i \neq x_j$ (when X is a curve, these are called positive divisors in the language of algebraic geometry.)

Notation. We write $\mathrm{Div}_{k_1,\dots,k_n} = \mathrm{Div}_{k_1,\dots,k_n}(M_g - *)$ and $\mathrm{Div}^n_k(M_g - *) = \mathrm{Div}_{\underbrace{k,\dots,k}_{n}}$.

Let Δ denote the diagonal multiplication

$$\left(\coprod_{j=1}^{\infty} SP^j(X)\right) \times SP^{k_1}(X) \times \cdots \times SP^{k_n}(X) \to \coprod_j SP^{k_1+j}(X) \times \cdots \times SP^{k_n+j}(X) \tag{2.1}$$

given on points by $\Delta(D, D_1, \dots, D_n) = (D_1 + D, \dots, D_n + D)$. It is clear that the divisor spaces $\mathrm{Div}_{k_1,\dots,k_n}(X)$ are included in the product $SP^{k_1}(X) \times \cdots \times SP^{k_n}(X)$ as the complement of $Im(\Delta)$; that is

Lemma 2.2. $\mathrm{Div}_{k_1,\dots,k_n} = SP^{k_1}(M_g - *)\times\cdots\times SP^{k_n}(M_g-*) - Im(\Delta)$.

Remark 2.3. The inclusion $\mathrm{Div}_{k_1,\dots,k_n}(X)\subset SP^{k_1}(X)\times\cdots\times SP^{k_n}(X)$ is an open embedding, and since for curves X the left hand side is a $k_1+\ldots+k_n$ complex manifold (Corollary 1.5), it follows that $\mathrm{Div}_{k_1,\dots,k_n}(X)$ is also a complex manifold of dimension $k_1+\ldots+k_n$.

We can define at this point the quotient space

$$\begin{aligned}TY_{k_1,\dots,k_n} &= SP^{k_1}(M_g)\times\cdots\times SP^{k_n}(M_g)\\ &\qquad / \left\{\bigcup_i SP^{k_1}(M_g)\times\cdots\times SP^{k_i-1}(M_g)\times\cdots\times SP^{k_n}(M_g)\cup Im(\Delta)\right\}\end{aligned}$$

which can be thought of as $SP^{k_1}(M_g)\times\cdots\times SP^{k_n}(M_g)/V$, where V is such that $\mathrm{Div}_{k_1,\dots,k_n} = SP^{k_1}(M_g)\times\cdots\times SP^{k_n}(M_g) - V$. One can therefore invoke Poincaré-Alexander duality to write

$$H^i(\mathrm{Div}_{k_1,\dots,k_n};\mathbb{F}) = H_{2(k_1+\cdots+k_n)-i}(TY_{k_1,\dots,k_n};\mathbb{F}). \tag{2.4}$$

The homology of the quotient space $TY_{k_1,\dots,k_n}$ is not easy to extract when the space is presented in this form. However, we can use a homotopy equivalent construction due to [C2M2] which makes the homological structure much more apparent.

Consider the *twisted* product space

$$DY^n(X) = (\underbrace{SP^\infty(X,*)\times\cdots\times SP^\infty(X,*)}_{n})\times_t SP^\infty(cX), \tag{2.5}$$

where t identifies the points

(2.6)
$$\begin{aligned}&(D_1,\cdots,D_n,\ \langle(t_1,z_1)\cdots(t_k,z_k)\rangle)\\ &\qquad \sim (D_1+z_i,\dots,D_n+z_i,\langle(t_1,z_1)\cdots(t_{i-1},z_{i-1})(t_{i+1},z_{i+1})\cdots(t_k,z_k)\rangle)\end{aligned}$$

whenever $t_i=0$ (here $D_i\in SP^{k_i}(X)$ for some $k_i\geq 1$ and $(t_j,z_j)\in cX, t_j\in[0,1]$.)

The space $DY^n(X)$ is naturally filtered as follows

$$DY_{k_1,\dots,k_n}(X) = \bigcup_{\substack{i_1+l\leq k_1\\ \vdots\\ i_n+l\leq k_n}} \left(SP^{i_1}(X)\times\cdots\times SP^{i_n}(X)\right)\times_t SP^l(cX).$$

We observe that there is a projection

$$p: DY_{k_1,\dots,k_n}\to SP^{k_1}(X)\times\cdots\times SP^{k_n}(X)/Im(\Delta)$$

given by

$$(D_{i_1},D_{i_2},\dots,D_{i_n},\langle(t_1,z_1)\cdots(t_l,z_l)\rangle)\mapsto(D_{i_1},D_{i_2},\dots,D_{i_n}).$$

It is easy to see that p is acyclic, inverse images of points being contractible sets. It follows that p induces an isomorphism in homology and combining this with 2.4 yields

Lemma 2.7. *There is an isomorphism*

$$H^i(\mathrm{Div}_{k_1,\dots,k_n}(M_g-*);\mathbb{F}) = H_{2(k_1+\cdots+k_n)-i}(DY_{k_1,\dots,k_n}/\bigcup_i DY_{k_1,\dots,k_i-1,\dots,k_n};\mathbb{F}).$$

It is the quotient spaces $DY_{k_1,\dots,k_n}/\bigcup_i DY_{k_1,\dots,k_i-1,\dots,k_n}$ that we analyze in this paper.

2.1. **Homotopy invariance.** Here we show that the topology of the space DY doesn't depend on the choice of the diagonal approximation Δ. Choose a map $\Delta' : SP^\infty(X) \to SP^\infty(X)^n$ homotopic to Δ and define the corresponding space $DY'(X)$ obtained from $(SP^\infty(X,*))^n \times SP^\infty(cX)$ via the identification

$$\begin{aligned}&(D_1,\cdots,D_n,\langle(t_1,z_1)\cdots(t_k,m_k)\rangle)\\ &\qquad\sim(\nu(\Delta'(z_i),(D_1,\cdots,D_n)),\langle(t_1,z_1)\cdots(t_{i-1},z_{i-1})(t_{i+1},z_{i+1})\cdots(t_k,m_k)\rangle)\end{aligned}$$

whenever $t_i = 0$. Here ν is the componentwise symmetric product multiplication.

Lemma 2.8. $DY'(M_g) \simeq DY(M_g)$.

Proof. Denote by $AG(X)$ the free abelian group on points of X or equivalently the group completion of $SP^\infty(X)$. Points of $AG(X)$ have the form $*$ or

$$\{x_1\cdots x_r, y_1^{-1}\cdots y_s^{-1} \mid * \neq x_i, y_j, x_i \neq y_j\}.$$

It is known ([DT]) that for connected CW complexes, the inclusion $SP^\infty(X) \hookrightarrow AG(X)$ is a homotopy equivalence.

For simplicity of notation, write $G = AG(M_g)$. The diagonal Δ (resp. Δ') extends in the obvious way to a map $G \to G^n$ and it induces an action $\delta : G \times G^n \to G^n$ (resp. δ') as described previously. We can then consider the associated "completed" model

$$\hat{DY}(M_g) = G^n \times_G AG(cM_g)$$

where G acts on G^n via δ. Similary we can construct $\hat{DY'}(M_g)$ associated to δ'. It is easy to see that the new model $\hat{DY}(M_g)$ is homotopy equivalent to $DY(M_g)$. This follows by considering the map of quasifiberings

$$\begin{array}{ccc} SP^\infty(M_g)^n & \to & G^n \\ \downarrow & & \downarrow \\ DY & \to & \hat{DY} \\ \downarrow & & \downarrow \\ SP^\infty(\Sigma M_g) & \to & AG(\Sigma M_g) \end{array}$$

where the top and bottom maps are homotopy equivalences (similarly $\hat{DY'}(M_g) \simeq DY'(M_g)$).

Since Δ is a diagonal approximation, we have a homotopy $G \times G^n \times I \xrightarrow{\phi} G^n$, where if we write $\phi(g,x,t) = g_t(x)$, the map g_0 corresponds to componentwise multiplication $G \times G^n \to G^n$ and $g_1 = \delta$. The inclusion $i : (G^n \times \{0\}) \times AG(cM_g) \hookrightarrow (G^n \times I) \times AG(cM_g)$ is a G-map (here the action of G on the right hand side is given by $g((x,t),w) = ((g_t(x),t),gw))$, and it is clearly a homotopy equivalence. Since G acts freely on $AG(cM_g)$, i is a map of free G-spaces, and it is then a theorem of equivariant homotopy [Br] that i is actually a homotopy equivalence through G-maps. This then descends to an equivalence of quotients, and we have

$$\hat{D}Y(M_g) = (G^n \times \{1\}) \times_G AG(cM_g) \simeq (G^n \times \{0\}) \times_G AG(cM_g).$$

The same argument shows that $\hat{D}Y'(M_g)$ is homotopy equivalent to the right-hand, side and the lemma is proved. $\square$

3. Stabilization

As indicated in the introduction, there are homotopy inclusions

$$\mathrm{Div}^n_k = \mathrm{Div}_{k,\dots,k}(M_g - *) \hookrightarrow \mathrm{Div}^n_{k+1}$$

defined as follows (and more generally, there are inclusions

$$\mathrm{Div}_{k_1,\dots,k_i,\dots,k_n} \hookrightarrow \mathrm{Div}_{k_1,\dots,k_i+1,\dots,k_n}$$

that raise any degree): Choose a sequence of concentric neighborhoods $\{U_k\}, k \geq 1$, $U_{k+1} \subset U_k$, around the basepoint $x_0 \in M_g$. For each neighborhood U_k pick an n-tuple of *distinct* points $(x^k_1, \dots, x^k_n) \in U_k - U_{k+1}$. The map

$$\mathrm{Div}^n_k(M_g - U_k) \to \mathrm{Div}^n_{k+1}(M_g - U_{k+1}) \tag{3.1}$$

given by sending a configuration $(D_1, \dots, D_n)$ to $(D_1 + x^k_1, \dots, D_n + x^k_n)$ is a closed embedding, and it extends to an open embedding

$$e : \mathrm{Div}^n_k(M_g - U_k) \times (U_k - \bar{U}_{k+1}) \to \mathrm{Div}^n_{k+1}(M_g - U_{k+1}).$$

It is not hard to see that $\mathrm{Div}^n_k(M_g - U_k) \cong \mathrm{Div}^n_k$, and so we regard Div^n_k as a codimension $2n$ (real) submanifold of Div^n_{k+1}.

Notation. The direct limit of Div^n_k over the embeddings e is denoted by $\mathrm{Div}^n(M_g - *)$. Consider now the following diagram:

$$\begin{array}{ccc}
H_*(\mathrm{Div}^n_k) & \stackrel{e_*}{\to} & H_*(\mathrm{Div}^n_{k+1}) \\
\downarrow\cong & & \downarrow\cong \\
H^{2kn-*}(\mathrm{Div}^n_k, \partial\,\mathrm{Div}^n_k) & \stackrel{f}{\to} & H^{2(k+1)n-*}(\mathrm{Div}^n_{k+1}, \partial\,\mathrm{Div}^n_{k+1}) \\
\downarrow\cong & & \downarrow\cong \\
H^{2kn-*}(TY^n_k) & \stackrel{f}{\to} & H^{2(k+1)n-*}(TY^n_{k+1}).
\end{array}$$

Lemma 3.2. *The map f corresponds to cupping with $a_1 \cup a_2 \cup \dots \cup a_n$.*

Proof. Let V be a tubular neighborhood of Div^n_k in Div^n_{k+1}. We have that Div^n_k is a (complex) codimension n submanifold of Div^n_{k+1} and we can identify V with the normal disc bundle to $\mathrm{Div}^n_k \subset \mathrm{Div}^n_{k+1}$. Denote by η the entire normal bundle and let $M(\eta) = V/\partial V$ be the corresponding Thom space.

Note that we can compactify Div^n_j by adding a boundary term $\partial\,\mathrm{Div}^n_j$ (corresponding to its complement in $SP^j(M_g)^n$). Poincaré duality and the Thom isomorphism interlock in the following diagram of isomorphisms:

$$\begin{array}{ccc}
H^*(\mathrm{Div}^n_k, \partial\,\mathrm{Div}^n_k) & \stackrel{\cup U}{\longrightarrow} & H^{*+2n}(M(\eta), M(\eta_{|\partial})) \\
\downarrow\cong & & \downarrow\cong \\
H_{2kn-*}(\mathrm{Div}^n_k) & \stackrel{=}{\to} & H_{2kn-*}(\mathrm{Div}^n_k)
\end{array} \tag{3.3}$$

where U is the Thom class and $M(\eta_{|\partial})$ is the Thom space of the normal bundle η restricted to $\partial\,\mathrm{Div}^n_k$. Note also that there is an (excision) isomorphism

$$H^*(M(\eta)) = H^*(V, \partial V) \stackrel{\cong}{\to} H^*(\mathrm{Div}^n_{k+1}, \mathrm{Div}^n_{k+1} - \mathrm{Div}^n_k)$$

which then yields a map

$$H^*(M(\eta), M(\eta_{|\partial})) \to H^*(\mathrm{Div}^n_{k+1}, \partial\,\mathrm{Div}^n_{k+1}). \tag{3.4}$$

This is now enough to give a description of the map f, for we have that the Thom isomorphism (given by the top map in 3.3) combines with 3.4 to yield

$$f : H^*(\mathrm{Div}^n_k, \partial\,\mathrm{Div}^n_k) \stackrel{\cong}{\to} H^{*+2n}(M(\eta), M(\eta_{|\partial})) \to H^{*+2n}(\mathrm{Div}^n_{k+1}, \partial\,\mathrm{Div}^n_{k+1}).$$

Write the tubular neighborhood V of Div_k^n as

$$V = \mathrm{Div}_k^n \times V_{x_1^k} \times \cdots \times V_{x_n^k},$$

where $V_{x_i^k}$ is a small disc around x_i^k. The Thom class is by definition the orientation class $U \in H^{2n}(V_{x_1^k} \times \cdots \times V_{x_n^k}, \partial(V_{x_1^k} \times \cdots \times V_{x_n^k}))$. Since $H^2(V_{x_i^k}, \partial V_{x_i^k})$ is generated by a_i, it is now clear that

$$\begin{aligned} U = a_1 \dots a_n \in H^2(V_{x_1^k}, \partial V_{x_1^k}) \otimes \quad & \cdots \quad \otimes H^2(V_{x_n^k}, \partial V_{x_n^k}) \\ & \hookrightarrow \quad H^{2n}\left(V_{x_1^k} \times \cdots \times V_{x_n^k}, \partial(V_{x_n^k} \times \cdots \times V_{x_n^k})\right) \end{aligned}$$

and the proof is complete. □

Corollary 3.5 (Segal)**.** *The "collar" inclusions* $H_*(\mathrm{Div}_k^n) \xrightarrow{e_*} H_*(\mathrm{Div}_{k+1}^n)$ *are injections.*

4. The Homology of Divisor Spaces

In this section, we prove our main result (0.4 in the introduction and 4.14 below). So we start with the model (§2)

$$DY^n(M_g) = (SP^\infty(M_g, *))^n \times_t SP^\infty(cM_g, *),$$

where t is the diagonal twisting described in 2.5. For simplicity, we will write M for M_g and $SP^\infty(M)$ for $SP^\infty(M_g, *)$.

We fix a diagonal approximation

$$\Delta_* : C_*(SP^\infty(M)) \to C_*(SP^\infty(M))^{\otimes^n}.$$

This induces an action of $C_*(SP^\infty(M))$ on $C_*(SP^\infty(M))^{\otimes^n}$. On the other hand, the inclusion

$$M \hookrightarrow cM,\ x \mapsto (0, x)$$

induces an action of $C_*(SP^\infty(M))$ on $C_*(SP^\infty(cM))$. Using these actions, a chain complex for DY is given by

$$C_*(SP^\infty(M))^{\otimes n} \otimes_{C_*(SP^\infty(M))} C_*(SP^\infty(cM)) \tag{4.1}$$

and by Lemma 2.8, any other choice of Δ_* yields chain homotopic complexes.

At this point we need to describe the module structure of $C_*(SP^\infty(cM))$ over $C_*(SP^\infty(M))$, and for that purpose we need to review some constructions.

4.1. **Milgram's Bar Construction.** Infinite symmetric products provide models for topological bar constructions and classifying spaces, as was observed by Milgram [M1]–[M2].

Let X be an associative topological monoid $\mu : X \times X \to X$ with μ cellular. We assume that μ has a unit *. Let σ^n be the n-simplex which we parametrize as follows:

$$\sigma^n = \{(t_1, t_2, \cdots, t_n) |\ 0 \le t_1 \le \cdots \le t_n \le 1\}.$$

The (acyclic) Milgram's bar construction on X is the space

$$E_T(X) = \coprod_{i=1}^{\infty} X \times \sigma^i \times X^i / \sim$$

with identifications $\sim$ given as follows:

$$\begin{array}{rl}
(i) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n) \\
 & \quad \sim (x_0,t_1,\cdots,\hat{t}_j,\cdots,t_n,x_1,\cdots,\hat{x}_j,x_jx_{j+1},\cdots,x_n) \text{ if } t_j=t_{j+1}, \\
(ii) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n) \sim (x_0x_1,t_2,\cdots,t_n,x_2,\cdots,x_n) \text{ if } t_1=0, \\
(iii) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n) \sim (x_0,t_1,\cdots,t_{n-1},x_1,\cdots,x_{n-1}) \text{ if } t_n=1, \\
(iv) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n) \\
 & \quad \sim (x_0,t_1,\cdots,\hat{t}_j,\cdots,t_n,x_1,\cdots,\hat{x}_j,\cdots,x_n) \text{ if } x_j=*.
\end{array}$$

Clearly, X acts freely on $E_T(X)$ by multiplying on the left and it turns out that $E_T(X)$ is contractible [M1]. This implies that the quotient space $B_T(X) = E_T(X)/X$ is a classifying space for X. The space $B_T(X)$ is also referred to as the *topological bar construction* on X (and we will sometimes write B_X for $B_T(X)$.)

Let $E(A)$ and $B(A)$ denote respectively the *acyclic* and the *reduced* algebraic bar constructions on A. In our case, A will be a differential graded, or *DG*, algebra. Suppose now that X is an abelian H-space; then $C_*(X)$, the chain complex for X, is a DG algebra. There is a correspondence $\lambda : C_*(B_T(X)) \to BC_*(X)$ given on generators by

$$\lambda(\sigma^n \times e_1 \times \cdots \times e_n) = |e_1|\cdots|e_n|.$$

Both $C_*(B_T(X))$ and $BC_*(X)$ are bigraded and it can be checked that λ is a differential bigraded algebra homomorphism. Actually, more is true:

Theorem 4.2 (Milgram)**.** *There is an isomorphism of differential bigraded algebras (dba);*

$$C_*(B_T(X)) \cong B(C_*(X)),\ C_*(E_T(X)) \cong E(C_*(X)).$$

Remark 4.3. We mentioned earlier that there is an interesting connection between infinite symmetric products and the classifying space construction above. Indeed, one can order points

$$\langle (t_1,z_1),\ldots,(t_n,z_n)\rangle \in SP^n(\Sigma X)$$

according to the ascending order of the t_i's. However there is an ambiguity whenever $t_i = t_j$, in which case we identify $\langle (t_i,z_i),(t_i,z_j)\rangle$ with $\langle t_i, \langle z_iz_j\rangle\rangle$, where $\langle z_iz_j\rangle$ is the product in $SP^\infty(X)$. Of course when $t_i = 0$ or $t_i = 1$ we get the basepoint identification (in the suspension). It then follows that when elements of $SP^\infty(\Sigma X)$ are represented in the *normal form* $(t_1 \le t_2 \le \cdots \le t_n, x_1,\ldots,x_n)$, the following homeomorphism becomes apparent:

$$SP^\infty(\Sigma X) = B_T(SP^\infty(X)).$$

Corollary 4.4 (Milgram)**.** $C_*(SP^\infty(\Sigma X,*)) \cong_{dba} B\left(C_*(SP^\infty(X,*))\right).$

A similar statement holds for cX; that is, $SP^\infty(cX) \cong E_T(SP^\infty(X))$ and there is a dba (i.e. differential bigraded algebra) isomorphism

$$C_*(SP^\infty(cX,*)) \cong_{dba} E\left(C_*(SP^\infty(X,*))\right).$$

It then follows that

$$C_*(SP^\infty(cM)) \cong EC_*(SP^\infty(M))$$

as modules over $C_*(SP^\infty(M))$. Combining this with 4.1 gives

Lemma 4.5. $H_*(DY^n(M);\mathbb{A}) \cong Tor^{C_*(SP^\infty(M,*))}(\mathbb{A}, C_*(SP^\infty(M,*))^{\otimes n}).$

4.2. **The Collapse.** The total space for $Tor^{C_*(SP^\infty(M))}(\mathbb{A}, C_*(SP^\infty(M))^{\otimes n})$ is given by

$$C_*(SP^\infty(M))^{\otimes n} \otimes_{C_*(SP^\infty(M))} E(C_*(SP^\infty(M))). \tag{4.6}$$

We will write $\otimes_{\Delta_*}$ instead of $\otimes_{C_*(SP^\infty(M))}$ for shorthand. Of course, filtering 4.6 by the number of bar degrees yields the classical homology Eilenberg-Moore spectral sequence.

Proposition 4.7. *There is an embedding*

$$e : H_*(SP^\infty(M_g); \mathbb{A}) \hookrightarrow C_*(SP^\infty(M_g), \mathbb{A})$$

inducing an isomorphism

$$Tor^{H_*(SP^\infty(M_g))}(\mathbb{A}, H_*(SP^\infty(M_g))^{\otimes n}) \cong Tor^{C_*(SP^\infty(M_g))}(\mathbb{A}, C_*(SP^\infty(M_g))^{\otimes n}).$$

From the Cartan-Moore comparison theorem ([McCl], corollary 7.6.), 4.7 would follow if e is compatible with the module structures; that is if e commutes with the diagonal action

$$\begin{array}{ccc} H_*(SP^\infty(M)) & \xrightarrow{e} & C_*(SP^\infty(M)) \\ \downarrow{\scriptstyle \Delta_*} & & \downarrow{\scriptstyle \Delta_*} \\ H_*(SP^\infty(M))^{\otimes n} & \xrightarrow{\otimes^n e} & C_*(SP^\infty(M))^{\otimes n}. \end{array}$$

So first we record the existence of e as a separate lemma.

Lemma 4.8. *A chain complex for $SP^\infty(M_g)$ can be chosen so that there is an embedding $e : H_*(SP^\infty(M_g); \mathbb{Z}) \hookrightarrow C_*(SP^\infty(M_g), \mathbb{Z})$.*

Proof. We go back to the standard representation of M_g as $M_g \simeq (\bigvee^{2g} S^1) \cup_{[-]} D^2$, where D^2 is the top 2-cell attached via the commutator map $[-]$ to a bouquet of $2g$ one dimensional S^1's. The i-th copy of S^1 in $\bigvee^{2g} S^1$ represents a one dimensional cell e_i attached trivially to basepoint. If $*$ denotes the product in $SP^\infty(M)$, then we see dirctly that $e_{i_1} * e_{i_2} * \ldots * e_{i_n}, i_j \neq i_k$, and $e_i * D^2$ give genuine cells in $SP^\infty(M)$ (which can be thought of as the cross product cells). We also know that $SP^n(D^2), n \geq 1$, are cells of dimension $2n$ (Lemma 1.5). We can consider then the complex $\mathcal{C}$ generated by the different products $e_{i_1} * e_{i_2} * \ldots * e_{i_n} * SP^l(D^2)$ with $i_j \neq i_k$. Since $\partial e_i = 0,\ \partial D^2 = 0$, we notice that $\partial\left(e_i * SP^n(D^2)\right) = 0$ and hence elements of $\mathcal{C}$ represent homology classes. As such, it is clear that $n!(SP^n(D^2)) = [M]$, for this is simply equivalent to the statement that the projection map $M^n \to SP^n(M)$ has degree $n!$. It then follows that the $SP^l(D^2)$'s generate a divided power algebra in $\mathcal{C}$. From Lemma 1.8 we see that $\mathcal{C} \cong H_*(SP^\infty(M))$, and the embedding of the homology into the chain complex is constructed. ☐

Proof of 4.7. The diagonal approximation $\Delta : M \to M^n$ can be extended multiplicatively (on each component) to a map $\Delta^\infty : SP^\infty(M) \to (SP^\infty(M))^n$, and clearly Δ^∞ is homotopic to the diagonal on $SP^\infty(M)$. We have the following commuting diagram:

$$\begin{array}{ccccc} SP^\infty(M) \times SP^\infty(M) & \xrightarrow{*} & SP^\infty(M) & \xrightarrow{\Delta^\infty} & SP^\infty(M)^n \\ \downarrow{\scriptstyle \Delta^\infty \times \Delta^\infty} & & & & \downarrow{\scriptstyle * \times *} \\ SP^\infty(M)^n \times SP^\infty(M)^n & & \xrightarrow{\text{shuffle}} & & SP^\infty(M)^n \times SP^\infty(M)^n \end{array}$$

where as before shuff $((x_1, \dots, x_n), (y_1, \dots y_n)) = ((x_1, y_1), \dots, (x_n, y_n))$. Note that shuff_* is cellular. Now Δ^∞ is already cellular on M by construction, and the diagram above shows that Δ^∞_* must be cellular on $\mathcal{C}$. By standard considerations, Δ^∞ extends to a cellular map on all of $SP^\infty(M)$. The embedding $e: \mathcal{C} \hookrightarrow C_*(SP^\infty(M))$ does commute with the diagonal action, and since $\mathcal{C} = H_*(SP^\infty(M_g)$ the proposition follows by Cartan-Moore. □

4.3. **The Main Result.** Write $\mathcal{A} = H_*(SP^\infty(M))$. We now put a multigrading on $Tor^{\mathcal{A}}(\mathbb{A}, \mathcal{A}^{\otimes n})$ and derive Theorem 0.4. First, by Steenrod's splitting 1.6, the total space for $Tor^{\mathcal{A}}(\mathbb{A}, \mathcal{A}^n)$ takes the form

$$Tot = \bigoplus H_*(T_{k_1,\dots,k_n}; \mathbb{A}) \otimes_{\mathcal{A}} E(\mathcal{A}), \tag{4.9}$$

where $T_{k_1,k_2,\dots,k_n} = SP^{k_1}(M) \times \cdots \times SP^{k_n}(M)/V$ with

$$V = \bigcup_{1 \le i \le n} SP^{k_1}(M) \times \cdots SP^{k_i - 1} \times \cdots \times SP^{k_n}(M).$$

Since the algebra $\mathcal{A}$ is bigraded, so is $E(\mathcal{A})$. More precisely, let $a_i \in \mathcal{A}$ have bidegree $(\deg(a_i), \mathrm{fil}(a_i))$ as in 1.7; then

$$\text{bidegree } (a_0|a_1|a_2|\dots|a_n|) = \left(n + \sum_{i=0}^{n} \deg(a_i), \sum \mathrm{fil}(a_i)\right),$$

which means that the homological degree of $a_0|a_1|a_2|\dots|a_n| \in E(\mathcal{A})$ is given by $\sum \deg(a_i) + n$, while the filtration degree is simply $\sum \mathrm{fil}(a_i)$ (this is not the bar degree). Define $E_{m,k}(\mathcal{A})$ to consist of all elements of bidegree $(m,k) \in E(\mathcal{A})$.

The boundary ∂ in Tot is described as follows:

$$\partial(|a_1|a_2|\dots|a_n|) = \Delta_*(a_1) \otimes |a_2|\dots|a_n| + \partial_B(|a_1|\dots|a_n|), \tag{4.10}$$

where ∂_B is the reduced bar differential which in this case is given by

$$\partial_B(|a_1|\dots|a_n|) = (-1)^i \sum_{1}^{n-1} |a_1|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_n|.$$

By definition we have that $H_*(Tot, \partial) = Tor^{\mathcal{A}}(\mathbb{A}, \mathcal{A}^n)$ for untwisted coefficients $\mathbb{A}$. Note that $\Delta_* : H_*(SP^r(M)) \to H_*(SP^r(M))^{\otimes n}$ preserves the filtration degree r and hence the total space Tot splits as a sum of subchain complexes:

$$\bigoplus_{l_i \le k_i} Tot_{l_1,\dots,l_n} = \bigoplus_{l_i = r_i + j \le k_i} H_*(T_{r_1,\dots,r_n}; \mathbb{A}) \otimes_{\mathcal{A}} E_{*,j}(\mathcal{A}).$$

The homology $H_*\left(\bigoplus_{l_i \le k_i} Tot_{k_1,\dots,k_n}\right)$ coincides with $H_*(DY_{k_1,\dots,k_n}, \mathbb{A})$, and this is a direct summand of $H_*(DY, \mathbb{A})$. On the other hand, and by construction, $H_*(TY_{k_1,\dots,k_n})$ is a quotient of $H_*(DY_{k_1,\dots,k_n})$, which only sees elements of the exact filtration $(k_1, \dots, k_n)$. When passing to this quotient, the diagonal term Δ_* gets *reduced* according to

(4.11)

$$\Delta_*^{\mathrm{red}} : H_*(SP^r(M)) \xrightarrow{\Delta_*} (H_*(SP^r(M)))^{\otimes n} \xrightarrow{q} H_*(SP^r(M), SP^{r-1}(M))^{\otimes n}.$$

We write

$${}^2Tor^{H_*(SP^\infty(M))}(\mathbb{A}, H_*(SP^\infty(M))^{\otimes n})$$

for the new tor term, where the action is understood to be reduced. This new action induces a new boundary ${}^2\partial$ as in 4.10, and in this case

$$ {}^2Tot_{k_1,\dots,k_n} = \bigoplus_{r_i+j=k_i} H_*(T_{r_1,\dots,r_n};\mathbb{A}) \otimes_{\Delta_*} E_{*,j}(\mathcal{A}) \tag{4.12}$$

is a *subcomplex* of $(Tot, {}^2\partial)$. One then has

(4.13)

$$H_*(TY_{k_1,\dots,k_n};\mathbb{A}) \cong \ 2Tor^{H_*(SP^\infty(M))}_{*,k_1,\dots,k_n}(\mathbb{A}, H_*(SP^\infty(M))^{\otimes n}) = H_*\left({}^2Tot_{k_1,\dots,k_n}\right),$$

and these represent the homology classes of the exact filtration $(k_1,\dots,k_n)$. When $k_i = k$, we shorten the notation $(*;k,\dots,k)$ to a bidegree notation $(*;k)$ for simplicity. From now on, we drop the superscript ${}^2Tor$ and write Tor with the understanding that the module action of $H_*(SP^\infty(M))$ on the tensor product is reduced (cf. 4.11). The preceding discussion then yields

Theorem 4.14. *For field coefficients* $\mathbb{F}$*, we have the following isomorphism*:

$$H_*(\mathrm{Div}_{k,\dots,k}(M_g - *);\mathbb{F}) \cong Tor^{H_*(SP^\infty(M_g))}_{2nk-*;k}(\mathbb{F}, H_*(SP^\infty(M))^{\otimes n})$$

with the module structure induced from $\Delta_*^{\mathrm{red}} : H_*(SP^\infty(M)) \to H_*(SP^\infty(M)^n)$ *in 4.11.*

Proof. Apply the duality $H_*(TY_{k,\dots,k};\mathbb{F}) \cong H^{2nk-*}(\mathrm{Div}^n_k(M - *);\mathbb{F})$. □

Remark 4.15. Note that $H_*(\mathrm{Div}_{k,\dots,k}(M_g - *);\mathbb{F})$ must vanish beyond the middle dimension $* > nk$, this being a pecularity of Stein spaces.

4.4. **An Alternate Description.** One could have filtered the space

$$TY^n = \bigcup DY_{k_1,\dots,k_n}/\cup_j\, DY_{k_1,\dots,k_j-1,\dots,k_n}$$

not by the number of bars but as follows. Write

$$TY^n \;=\; \bigcup TY_{k_1,\dots,k_n} = \bigcup_{r_i+l=k_i} T_{r_1,\dots,r_n} \times_t (SP^l(cM_g)/SP^{l-1}(cM_g))$$

with filtration pieces

$$\mathcal{F}^j = \bigcup_{\substack{l\le j\\ r_i+l=k_i}} (T_{r_1,\dots,r_n}) \otimes_t (SP^l(cM_g)/SP^{l-1}(cM_g)).$$

The same arguments as in §4.3 can now be expressed in the following form

Proposition 4.16. *There exists a spectral sequence converging to* $H_*(TY_{k_1,\dots,k_n})$ *with* E^1 *term*

$$E^1 = \coprod_{r_i+j=k_i} H_*(T_{r_1,\dots,r_n};\mathbb{F}) \otimes H_*(SP^j(\Sigma M_g), SP^{j-1}(\Sigma M_g), \mathbb{F}).$$

The spectral sequence collapses at E^1 *for* $n > 2$.

Proof. It can easily be checked in light of §4.3 that

$$d_1(c_* \otimes |a_1|\dots|a_l|) = c_*\Delta_*^{\mathrm{red}}(a_1) \otimes |a_2|\dots|a_l|$$

where $a_i \in H_*(SP^\infty M)$ and $|a_1|\dots|a_l| \in H_*(SP^\infty(\Sigma M)) = H_*(B_T(SP^\infty(M))) = B(H_*(SP^\infty M))$. This yields the first part of the proposition (cf. 4.12). That the spectral sequence collapses when $n > 2$ is a corollary of the fact that Δ_*^{red} vanishes in this case (see Lemma 6.2). □

Remark 4.17. When $n = 2$ we have $d_1(|M|) \neq 0$, and there are higher differentials d_{p^i} described in Remark 6.16.

5. The Rat Spaces

When $g = 0$, we have the homeomorphism described in the introduction:

$$\mathrm{Div}_k^{n+1}(S^2 - *) = \mathrm{Hol}_k^*(S^2, \mathbb{P}^n) = \mathrm{Rat}_k(\mathbb{P}^n).$$

Applying Theorem 4.14, we see that the module structure of $H_*(SP^\infty(S^2)) = \Gamma(a)$ on $H_*(SP^\infty(S^2))^n$ is trivial, for in this case $\Delta_*(a = [S^2]) = a \otimes 1 + 1 \otimes a$ and hence by 4.11, $\Delta_*^{\mathrm{red}}(a) = 0$. It follows that

$$Tor^{H_*(SP^\infty(S^2))}(\mathbb{F}, H_*(SP^\infty(S^2))^{n+1}) = \Gamma(a_1, \dots, a_{n+1}) \otimes Tor^{\Gamma(a)}(\mathbb{F}, \mathbb{F}).$$

We observe that we have an identification

$$Tor^{\Gamma(a)}(\mathbb{F}, \mathbb{F}) = H_*(SP^\infty(S^3), \mathbb{F}) = \coprod_i H_*(SP^i(S^3), SP^{i-1}(S^3), \mathbb{F})$$

and can then write

(5.1)
$$H_*(TY_k^{n+1}(S^2); \mathbb{F}) \cong \coprod_j \gamma_j(a_1 \dots a_{n+1}) H_{*-2(n+1)j}(SP^{k-j}(S^3), SP^{k-j-1}(S^3); \mathbb{F}).$$

We can consider the inclusion $\mathrm{Rat}(\mathbb{P}^1) \hookrightarrow \Omega^2 S^2$. The space $\Omega_0^2 S^2 \simeq \Omega^2 S^3$ stably splits (Snaith) as an infinite bouquet

$$\Omega^2 \Sigma^2 S^1 \simeq_s \bigvee_0^\infty D_k,$$

where $D_k = F(\mathbb{C}, k) \wedge_{\mathcal{S}_k} S^{(k)}$ are the building blocks of the May-Milgram model for S^1 (here $S^{(k)}$ denotes the k-fold smash of S^1 with itself). It is known [BCM] that there is a duality isomorphism

$$H_*(D_k, \mathbb{F}) \cong H^{4k-*}(SP^k(S^3), SP^{k-1}(S^3), \mathbb{F}).$$

The identity 5.1 and the duality $H_{2k(n+1)-*}(TY_k^{n+1}, \mathbb{F}) \cong \tilde{H}^*(\mathrm{Rat}_k(\mathbb{P}^n), \mathbb{F})$ combine to yield

$$\begin{aligned} H_*(\mathrm{Rat}_k(\mathbb{P}^n), \mathbb{F}) &= H^{2k(n+1)-*}(TY_k^{n+1}, \mathbb{F}) \\ &= \bigoplus_j H^{2(n+1)(k-j)-*}(SP^{k-j}(S^3), SP^{k-j-1}(S^3); \mathbb{F}) \\ &= \bigoplus H_{4(k-j)-2(n+1)(k-j)+*}(D_{k-j}; \mathbb{F}) \\ &= \bigoplus_j H_{*-(2n-2)(k-j)}(D_{k-j}; \mathbb{F}). \end{aligned}$$

Proposition 5.2 ([C2M2])**.** $H_*(\mathrm{Rat}_k(\mathbb{P}^n), \mathbb{F}) \cong H_*(\bigvee_{j=1}^k \Sigma^{(2n-2)j} D_j, \mathbb{F})$.

Corollary 5.3 (Segal)**.** *Let* $\mathrm{Rat}_\infty(\mathbb{P}^n)$ *be the direct limit induced from the system of collar inclusions* $\mathrm{Rat}_k(\mathbb{P}^n) \to \mathrm{Rat}_{k+1}(\mathbb{P}^n)$*. Then*

$$H_*(\mathrm{Rat}_\infty(\mathbb{P}^n), \mathbb{Z}) \cong H_*(\Omega_0^2 S^{2n+1}, \mathbb{Z}).$$

Remark 5.4. Cohen and Shimamoto show that the isomorphism in 5.2 is induced from an actual homotopy equivalence $\mathrm{Rat}_k(\mathbb{P}^n) \simeq C_k(\mathbb{C}, S^{2n-1})$ whenever $n > 1$. We refer to [CS] for a definition of the labelled configuration space $C_k(\mathbb{C}, S^{2n-1})$ and its relation with $\Omega^2 S^{2n-1}$.

6. The Positive Genus Case

In this section we determine the full structure of $Tor^{\mathcal{A}}(\mathbb{F}, \mathcal{A}^{\otimes n})$ for $g \geq 1$, and for both rational and $\mathbb{Z}_p$ coefficients. We start by making explicit the action of $\mathcal{A}$ on $\mathcal{A}^{\otimes n}$. The algebra $\mathcal{A} = H_*(SP^\infty(M_g))$ acts on $\mathcal{A}^{\otimes n}$ via the prescription

$$x \cdot (c_1 \otimes \cdots \otimes c_n) = \nu_* \left(\Delta_*^{\mathrm{red}}(x) \otimes (c_1 \otimes \cdots \otimes c_n) \right),$$

where ν is the componentwise symmetric product multiplication and where Δ_*^{red} is as in 4.11. Now recall that $\mathcal{A}$ has generators the 1-dimensional classes e_i, $1 \leq i \leq 2g$, which are primitive, as well as the top orientation class $[M]$.

Notation. We denote by $e_{i;r}$ the element $1 \otimes \cdots \otimes e_i \otimes \cdots \otimes 1$, where e_i is in the r^{th} position, $1 \leq r \leq n$. By $e_{i;r}e_{j;s}$ for $r < s$ we then mean $1 \otimes \cdots \otimes e_i \otimes \cdots \otimes e_j \otimes \cdots \otimes 1$.

Lemma 6.1. $\Delta_*([M_g]) = \sum_r [M_g]_r + \sum_{j=1}^{g} \sum_{r<s} (e_{2j;r}e_{2j-1;s} - e_{2j-1;r}e_{2j;s})$.

Proof. There is a natural collapse map from M_g to a wedge of g tori $T_1 \vee \cdots \vee T_g$ inducing an isomorphism in H_1 and such that the image of $[M]$ is $\sum [T_i]$. We have that $T_i = S^1 \times S^1$ and $H_*(T_i) = \Lambda(e_{2i-1}, e_{2i})$. It is easy to see that

$$\Delta_*[T_i] = \sum [T_i]_r + \sum_{r<s} (e_{2i;r}e_{2i-1;s} - e_{2i-1;r}e_{2i;s})$$

and hence by adding these up the lemma follows. □

Lemma 6.2. $\Delta_*^{\mathrm{red}}(e_i) = 0$, $\forall 1 \leq i \leq 2g, n \geq 2$. *On the other hand,*

$$\Delta_*^{\mathrm{red}}([M]) = \begin{cases} \sum_{j=1}^{g} (e_{2j} \otimes e_{2j-1} - e_{2j-1} \otimes e_{2j}), & \text{if } n = 2, \\ 0, & \text{if } n > 2. \end{cases}$$

Proof. Note that $e_{2j-1} \otimes e_{2j} - e_{2j} \otimes e_{2j-1} \in H_1(M,*)^{\otimes 2} \subset H_2(SP^\infty(M)^2)$ (here $n = 2$) and this is non-trivial in the image of Δ_*^{red}. The rest is a direct consequence of 4.11. □

Corollary 6.3. *Suppose $n > 2$. Then*

$$\begin{aligned} Tor^{\mathcal{A}}(\mathbb{F}, \mathcal{A}^{\otimes n}) &\cong \mathcal{A}^{\otimes n} \otimes Tor^{\mathcal{A}}(\mathbb{F}, \mathbb{F}) \\ &\cong \mathcal{A}^{\otimes n} \otimes H_*(SP^\infty(\Sigma M_g, *); \mathbb{F}) \\ &\cong \mathcal{A}^{\otimes n} \otimes_i Tor^{\Lambda(e_i)}(\mathbb{F}, \mathbb{F}) \otimes Tor^{\Gamma([M])}(\mathbb{F}, \mathbb{F}). \end{aligned}$$

Proof. Since both $\Delta_*^{\mathrm{red}}(e_i)$ and $\Delta_*^{\mathrm{red}}([M])$ vanish for $n > 2$, it follows that Δ_*^{red} vanishes on the generators of $\mathcal{A}$ and hence induces a trivial action on $\mathcal{A}^{\otimes n}$ whenever $n > 2$. This gives the first isomorphism. The last two identities are a consequence of the embedding $H_*(SP^\infty(M)) \hookrightarrow C_*(SP^\infty(X))$ (4.7) and of Cartan-Moore; i.e.

$$\begin{aligned} H_*(SP^\infty(\Sigma X); \mathbb{F}) &= Tor^{H_*(SP^\infty(M))}(\mathbb{F}, \mathbb{F}) = Tor^{\Lambda(e_1, \cdots, e_{2g}) \otimes \Gamma([M])}(\mathbb{F}, \mathbb{F}) \\ &= \bigotimes_i Tor^{\Lambda(e_i)}(\mathbb{F}, \mathbb{F}) \otimes Tor^{\Gamma([M])}(\mathbb{F}, \mathbb{F}). \quad \square \end{aligned}$$

We now describe $H_*(SP^n(\Sigma M); \mathbb{F})$ for $\mathbb{F} = \mathbb{Q}$ and $\mathbb{F} = \mathbb{Z}_p$.

6.1. **The Homology of $SP^\infty(\Sigma M_g) = B_{SP^\infty(M_g)}$.** The acyclic bar construction for $\Lambda(e)$ over $\mathbb{Z}$ gives rise to a minimal resolution which is generated at each level $B_i(\Lambda(e))$ by elements of the form $|e|\cdots|e|$ or $e|e|\cdots|e|$ (# of bars is i) and has boundary $\partial|e|e|\cdots|e| = e|e|\cdots|e|$. The generators $|e|\cdots|e|$ generate a divided power algebra (under the shuffle product), and it is readily seen that

$$Tor_{*,*}^{\Lambda(e)}(\mathbb{Z},\mathbb{Z}) = \Gamma(|e|).$$

The case of divided power algebras is harder. When $\mathbb{F} = \mathbb{Q}$, divided power algebras turn into polynomial algebras, and so in this case $Tor^{\Gamma(a)}(\mathbb{Q},\mathbb{Q}) = \Lambda(|a|)$, implying that

$$H_*(SP^\infty(\Sigma M);\mathbb{Q}) = \mathbb{Q}(|e_1|,\dots,|e_{2g}|)\otimes\Lambda([M]). \tag{6.4}$$

When $\mathbb{F} = \mathbb{Z}_p$, we see that $a^p = p!\gamma_p = 0$. Similarly, $\gamma_{p^i}^p$ is also zero. This shows that each γ_{p^i} generates a *truncated polynomial algebra*

$$P_T(a,p) = \mathbb{Z}_p[a,a^2,\cdots,a^{p-1}]/a^p = 0.$$

Lemma 6.5 (Cartan)**.** *$\{\gamma_{p^i}, i\geq 0\}$ generate $\Gamma(a)$ as an algebra over $\mathbb{Z}_p$ and*

$$\Gamma(a)\otimes\mathbb{Z}_p \cong P_T(a,p)\otimes P_T(\gamma_p,p)\otimes\cdots\otimes P_T(\gamma_{p^i},p)\otimes\cdots.$$

We assume in what follows that a has even degree (for our purpose $a = [M]$). One can construct a minimal resolution for $P_T(a,p)$ (over $\mathbb{Z}_p$) which is generated by elements

$$\{|a^{p-1}|a|\cdots|a^{p-1}|a|, |a||a^{p-1}|a|\cdots|a^{p-1}|a|\}$$

with boundary

$$\partial\,\underbrace{|a^{p-1}|a|\cdots|a^{p-1}|a|}_{i} = a^{p-1}\,\underbrace{|a^{p-1}|a|\cdots|a^{p-1}|a|}_{i-1}.$$

As an algebra under the shuffle product, it is checked that

Lemma 6.6.

$$Tor^{P_T(a,p)}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\Lambda(|a|)\otimes\Gamma(|a^{p-1}|a|),\qquad p>2.$$

When $p=2$, then $P_T(a,2) = \Lambda(a)$ and $Tor^{P_T(a,2)}(\mathbb{Z}_2,\mathbb{Z}_2)\cong\Gamma[|a|]$.

Remark 6.7. The element $|a|a^{p-1}|$ in the bar construction is known as the *transpotence* of a. It can be checked that $\beta(|\gamma_{p^{i+1}}|) = |\gamma_{p^i}^{p-1}|\gamma_{p^i}|$, where β is the mod-p Bockstein.

Remark 6.8. The generators $|e_i|$, $|M|$, all represent homology classes in $H_*(\Sigma M)\subset H_*(SP^\infty(\Sigma M))$ and this explains why they are referred to as *suspension classes*. All generators in $H_*(SP^\infty(\Sigma M))$ are assigned a bidegree as in Remark 1.7 and we find that

Generator	Bigrading
$\lvert e_i\rvert$	$(2;1)$
$\lvert M\rvert$	$(3;1)$
$\lvert\gamma_{p^i}\rvert$	$(2p^i+1;p^i)$.

Bidegrees are additive; for example the bidegree of $|\gamma_{p^i}|\gamma_{p^j}|$ is $(2(p^i+p^j)+2;p^i+p^j)$. If we let $h_{2p^i+1,p^i} = |\gamma_{p^i}|$, we can then write

$$Tor^{\Gamma(a)}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\Lambda(|a|,\dots,h_{2p^i+1,p^i},\dots)\otimes\Gamma(\beta h_{2p+2,p},\dots,\beta h_{2p^i+2,p^i},\dots)$$

and this describes the algebra $H_*(SP^\infty(S^3);\mathbb{Z}_p) = H_*(K(\mathbb{Z},3);\mathbb{Z}_p)$. Generally we have the following.

Lemma 6.9. *The homology $H_*(SP^n(\Sigma M_g);\mathbb{F})$ is given by those classes of bidegree $(*;i)$ with $i \leq n$, in*

$$\Gamma(|e_1|,\dots,,|e_{2g}|)\otimes Tor^{\Gamma(a)}(\mathbb{F},\mathbb{F}).$$

6.2. **Homology Calculations, $n>2$.** The preceding discussion as well as Corollary 6.3 show that for $n>2$

(6.10)
$$\begin{aligned} Tor^{\mathcal{A}}(\mathbb{F},\mathcal{A}^{\otimes n}) = \Gamma(a_1,\dots,a_n)\otimes\Gamma(|e_1|,\dots,|e_{2g}|)\otimes\Lambda(e_1,\dots,e_{2ng})\\ \otimes Tor^{\Gamma([M])}(\mathbb{F},\mathbb{F}), \end{aligned}$$

the terms of $(n+1)$-grading $(*,k\dots,k)$ making up all of $H_{2nk-*}(\mathrm{Div}_k(M-*;\mathbb{F})$.

Proposition 6.11. *Assume $n>2$ and $g\geq 1$ and consider the algebra*

$$\Lambda(f_{1;1},\dots,f_{2ng;1},\dots,f_{1;n},\dots,f_{2g;n})\otimes\Lambda(E)\otimes\mathbb{Q}(h_1,\dots,h_{2g}).$$

This algebra is $(n+1)$-graded according to $f_{i;r}\mapsto(1;0,\dots,1,\dots,0)$ with 1 in the $r+1$ position, $1\leq r\leq n$, $E\mapsto(2n-3;1,\dots,1)$ and $h_j\mapsto(2n-2;1,\dots,1)$. The multigrading is additive. The homology groups $H_*(\mathrm{Div}_k(M_g-*);\mathbb{Q})$ are now given by those elements of multidegree $(*;i_1,\dots,i_n)$ with $i_j\leq k$.

Proof. With $\mathbb{Q}$ coefficients 6.10 takes the form

$$\mathbb{Q}(a_1,\dots,a_n)\otimes\mathbb{Q}(|e_1|,\dots,|e_{2g}|)\otimes\Lambda(e_1,\dots,e_{2ng})\otimes\Lambda(|M|).$$

It is now a matter of counting the multidegree $(2nk-*;k,\dots,k)$ elements. The degree one generators are represented by $a_1^k\dots e_{i;r}a_j^{k-1}\dots a_n^k$ in $H_{2nk-1}(TY_k)$, $1\leq i\leq 2g$ and $1\leq r\leq n$, and to them correspond the $f_{i;r}\in H_1(\mathrm{Div}_k(M-*);\mathbb{Q})$. Similarly, $|M|$ is Poincaré dual to an element E of the right filtration and of homology degree $2n-3$, whereas the h_j's are dual to the $|e_j|$'s. Here $\mathbb{Q}(a_1,\dots,a_n)$ serves as a "calibrating" factor, and the calculation follows. $\square$

When $\mathbb{F}=\mathbb{Z}_p$, p odd, we can facilitate the counting by dualizing $Tor^{\mathcal{A}}(\mathbb{Z}_p,\mathcal{A}^{\otimes n})$. Divided power algebras turn into polynomial algebras and we get the total space

$$\begin{aligned} \mathbb{Z}_p(a_1,\dots,a_n)\otimes\Lambda(e_1,\dots,e_{2ng})\otimes\mathbb{Z}_p\left(|e_1|,\dots,|e_{2g}|\right)\\ \otimes\Lambda(|M|,|\gamma_p|,\dots,|\gamma_{p^i}|\dots)\otimes\mathbb{Z}_p\left(|M^{p-1}|M|,\dots,|\gamma_{p^i}^{p-1}|\gamma_{p^i}|,\dots\right). \end{aligned}$$

Here we ought to write $e_i^*,|e_i|^*,[M]^*$ for the classes above, but for simplicity we leave that out. Our calibration procedure leads generators $e_{i;r}$ as well as

$$\begin{array}{cccc} & h_i & E_i & H_i \\ (n+1)\text{-degree} & (2(n-1);1,\dots,1) & \left(2(n-1)p^i-1;p^i,\dots,p^i\right) & \left(2(n-1)p^i;p^i,\dots,p^i\right). \end{array}$$

Lemma 6.12. *Assume $n>2$ and p odd; then $H_*(\mathrm{Div}_k^n(M_g-*);\mathbb{Z}_p)$ is given by those classes in*

$$\begin{aligned} \bigotimes_{1\leq r\leq n}\Lambda(e_{1;r},\dots,e_{2g;r})\otimes\mathbb{Z}_p(h_1,\dots,h_{2g})\\ \otimes\Lambda(E_1,\dots,E_j,\dots)\otimes\mathbb{Z}_p(H_1,\dots,H_j,\dots) \end{aligned}$$

of multidegree $(*;i_1,\dots,i_n)$ with $i_j\leq k$.

6.3. **Homology Calculations, $n = 2$.** As pointed out in 6.3, the action of $\mathcal{A}$ on $\mathcal{A}^{\otimes 2}$ is not trivial. The Tor term $Tor^{\mathcal{A}}(\mathbb{F}, \mathcal{A} \otimes \mathcal{A})$ takes the form

$$\Gamma(a_1, a_2) \otimes \Gamma(|e_1|, \dots, |e_{2g}|) \otimes Tor^{\Gamma([M])}(\mathbb{F}, \Lambda(e_1, \dots, e_{4g})), \tag{6.13}$$

and the calculation boils down to understanding the term on the far right. The boundary here in the total space $\Lambda(e_1, \dots, e_{4g}) \otimes E(\Gamma[M])$ is generated by $\partial(|M|) = \sum_{i=1}^{g} e_{2i-1}e_{2i} - e_{2i}e_{2i-1}$ (cf. Lemma 6.2). By reordering the e_i's and renaming, we can rewrite it as $\partial(|M|) = \sum_{i=1}^{2g} e_{2i-1}e_{2i}$. Assume for now that $\mathbb{F} = \mathbb{Q}$. We can rewrite 6.13 as follows:

$$H_*(\mathcal{W}_g; \mathbb{Q}) \otimes \mathbb{Q}(h_1, \cdots, h_{2g}) \otimes \mathbb{Q}(a_1, a_2),$$

where $\mathcal{W}_g$ is the complex

$$\Lambda(e_1, \cdots, e_{4g}) \otimes \Lambda|M| \xrightarrow{\partial} \Lambda(e_1, \cdots, e_{4g}),\ \partial(|M|) = e_1e_2 + e_3e_4 + \cdots + e_{4g-1}e_{4g}.$$

By taking Poincaré duals we get

Lemma 6.14. *Let $\bar{\mathcal{W}}_g$ denote the complex*

$$\Lambda(e_1, \dots, e_{4g}, f) \to \Lambda(e_1, \dots, e_{4g}),\ \delta f = e_1e_2 + \cdots + e_{4g-1}e_{4g}.$$

Then

$$H_*(\mathrm{Div}^2_k(M_g - *); \mathbb{Q}) \subset H_*(\bar{\mathcal{W}}_g, \mathbb{Q}) \otimes \mathbb{Q}(h_1, \dots, h_{2g})$$

consists of elements with tridegrees $(*; i, j), i, j \leq k$, where tridegrees are assigned as follows: $e_{\text{odd}} \mapsto (1; 1, 0), e_{\text{even}} \mapsto (1; 0, 1), f \mapsto (1; 1, 1), h_i \mapsto (2; 1)$.

The complex $\bar{\mathcal{W}}_g$ has been studied in both [BC] and [BCM], and its Betti numbers have been completely determined. It is shown there for instance that the map

$$\bigcup(e_1e_2 + \cdots + e_{4g-1}e_{4g}) : \Lambda(e_1, \cdots, e_{4g}) \to \Lambda(e_1, \cdots, e_{4g})$$

is injective in degrees $\leq 2g$ and surjective in degrees $\geq 2g$. Moreover, if $\nu(i, g)$ denotes the rank of $H_i(\bar{\mathcal{W}}_g; \mathbb{Q})$, then we have

Lemma 6.15 ([BCM])**.** *$\nu(i, g) = 0$ for $i > 4g + 1$, and otherwise*

$$\nu(i, g) = \begin{cases} \binom{4g}{i} - \binom{4g}{i-2} & \text{for } i \leq 2g, \\ \binom{4g}{i-1} - \binom{4g}{i+1} & \text{for } 2g < i \leq 4g + 1. \end{cases}$$

Remark 6.16. When $\mathbb{F} = \mathbb{Z}_p$, the boundary terms take the form

$$\begin{aligned} \partial(|\gamma_{p^i}|) &= \frac{1}{p^i}\left(\sum_1^g e_{2i-1}e_{2i}\right)^{p^i}, \\ \partial(|\gamma_{p^i}^{p-1}|\gamma_{p^i}|) &= \left[\frac{1}{p^i}\left(\sum_1^g e_{2i-1}e_{2i}\right)^{(p-1)p^i}\right]|\gamma_{p^i}|. \end{aligned}$$

These last differentials correspond to the Kudo differential in the Serre spectral sequence associated to the quasi-fibration $SP^\infty(M)^n \to DY \to SP^\infty(\Sigma M)$. They also describe the d_{p^i} in §4.4.

Example 6.17. As an example, we carry out the calculation for $T = M_1$, a genus 1 surface, $n = 2$ and $p = 2$. The complex at hand can be written as

$$\Lambda(e_1, e_2, e_3, e_4) \otimes \mathbb{Z}_2(|e_1|, |e_2|, |T|, |\gamma_2|, \dots, |\gamma_{2^i}| \dots)$$

on generators with tridegrees: $e_1, e_3 \mapsto (1;1,0), e_2, e_4 \mapsto (1;0,1), |e_i| \mapsto (2;1,1)$, $|T| \mapsto (1;1,1)$ and $|\gamma_{2^i}| \mapsto (2^{i+1}-1; 2^i, 2^i)$. The coboundary is given by

$$\delta(|T|) = e_1e_2 + e_3e_4,\ \delta(|\gamma_{2^i}|) = \frac{1}{2^i}(e_1e_2 + e_3e_4)^{2^i}.$$

This implies that $\delta(|\gamma_2|) = e_1e_2e_3e_4$ and $\delta(|\gamma_{2^i}|) = 0$ for $i \geq 2$. We're interested in all elements of tridegree $(*, i, j), i, j \leq k$. For example, we have

Lemma 6.18. *The Poincaré series for $H_*(\mathrm{Div}^2_k(T - *); \mathbb{F}_2)$, $k = 1, 2$, are given by*

$$P(x) = 1 + 4x + 5x^2\ (k = 1),\ P(x) = 1 + 4x + 7x^2 + 9x^3 + 6x^4\ (k = 2).$$

7. Homotopy Constructions

We start with a fibration sequence due to Segal, which along with the scanning map, also first constructed in [S], constitutes the main tool in setting up the correspondence between divisor spaces and mapping spaces. We denote by $\mathbb{P}^n$ the nth complex projective space, and we use $\mathbb{P}$ and $\mathbb{P}^\infty$ interchangeably for the infinite complex space.

7.1. **Fat-Wedge Fibrations.** Let $W_n\mathbb{P}$ denote the n^{th} *fat wedge* of $\mathbb{P}^\infty$; that is, $W_n\mathbb{P}$ is the subset of $(\mathbb{P}^\infty)^n$ consisting of all n-tuples with at least one entry equal to the basepoint in $\mathbb{P}^\infty$ (we sometimes write W_n for $W_n\mathbb{P}$). Of course $W_1 = \{x_0\}$ and $W_2 = \mathbb{P}^\infty \vee \mathbb{P}^\infty$. One has the following

Lemma 7.1 (Segal)**.** *There is a fibration sequence*

$$(S^1)^n \to \mathbb{P}^n \to W_{n+1}\mathbb{P} \xrightarrow{\theta} (\mathbb{P}^\infty)^n = B(S^1)^n.$$

The projection θ is described in [S],§2. When $n = 1$, this is the folding map (i.e. the restriction of $\theta : \mathbb{P}\times\mathbb{P} \to \mathbb{P}, \theta(a, b) = a \cdot b^{-1}$). In this particular case, the fiber of θ is the total space of the fibration over $\mathbb{P}\vee\mathbb{P}$ induced from the path/loop fibration $S^1 \to S^\infty \to \mathbb{P}$ and this is seen to be $S^\infty \cup_{S^1} S^\infty \simeq \Sigma S^1 = S^2$.

Lemma 7.2. *There is a fibration*

$$S^{2n+1} \to W_{n+1}\mathbb{P} \hookrightarrow (\mathbb{P}^\infty)^{n+1}.$$

Proof. The proof proceeds by induction. In the case $n = 1$, we have the inclusion $W_2 = \mathbb{P}\vee\mathbb{P} \hookrightarrow \mathbb{P}\times\mathbb{P}$. A general result of Ganea states that the homotopy fiber of the inclusion $X\vee Y \hookrightarrow X\times Y$ is $\Omega(X)*\Omega(Y) \simeq \Sigma(\Omega(X)\wedge\Omega(Y))$ (here $*$ denotes the join product.) In our case, the fiber then becomes $\Sigma(\Omega\mathbb{P}\wedge\Omega\mathbb{P})) \simeq \Sigma(S^1\wedge S^1) = S^3$.

For $n > 1$, the fiber of $W_{n+1} \hookrightarrow (\mathbb{P}^\infty)^{n+1}$ is given as the total space of the pull-back of the path-loop fibration $(S^1)^{n+1} \to P \to (\mathbb{P}^\infty)^{n+1}$. Write W_{n+1} as the double mapping cylinder

$$W_n \times \mathbb{P} \leftarrow 2W_n \times * \to (\mathbb{P}^\infty)^n \times *.$$

The fiber of $W_n \times \mathbb{P} \to (\mathbb{P}^\infty)^{n+1}$ is S^{2n-1} by the induction hypothesis, while the fiber of $(\mathbb{P}^\infty)^n \to (\mathbb{P}^\infty)^{n+1}$ is $\Omega\mathbb{P} = S^1$. It follows that the fiber of $W_n \times * \simeq W_n \times S^\infty \to (\mathbb{P}^\infty)^{n+1} = (\mathbb{P}^n) \times \mathbb{P}$ is $S^{2n-1} \times S^1$, and hence the homotopy fiber $W_{n+1}\mathbb{P} \to (\mathbb{P}^\infty)^{n+1}$ can be written as the mapping cylinder $S^{2n-1} \times S^1 \times [0, 1]/\sim$ with S^1 collapsed at one end and S^{2n-1} collapsed at the other. But this is no other than $S^{2n-1} * S^1 = S^{2n+1}$, and the proof is complete. ∎

Corollary 7.3. *We have the following commutative diagram of fibrations (here h denotes the Hopf map and Δ_{n+1} is the diagonal inclusion):*

$$\begin{tikzcd}
S^{2n+1} \arrow[r, "="] \arrow[d, "h"] & S^{2n+1} \arrow[r] \arrow[d, "G"] & * \arrow[d] \\
\mathbb{P}^n \arrow[r] \arrow[d] & W_{n+1} \arrow[r, "\theta"] \arrow[d] & (\mathbb{P}^\infty)^n \arrow[d, "="] \\
\mathbb{P}^\infty \arrow[r, "\Delta_{n+1}"] & (\mathbb{P}^\infty)^{n+1} \arrow[r] & (\mathbb{P}^\infty)^n
\end{tikzcd}$$

Remark 7.4. Looping 7.2 yields a principal fibration $\Omega(i) : \Omega(S^{2n-1}) \to \Omega(W_n) \to \Omega(\mathbb{P}^\infty)^n$ which admits a cross section obtained as follows. Let s_i be the inclusion of $\mathbb{P}^\infty$ into W_n as the i-th factor. Then the composition

$$s : (\Omega\mathbb{P}^\infty)^n \xrightarrow{(\Omega s_1 \times \cdots \times \Omega s_n)} (\Omega W_n)^n \xrightarrow{\ *\ } \Omega W_n$$

provides the desired section of 7.2 (here $*$ is loop multiplication). Naturally this implies that

$$\Omega W_n \simeq (\Omega\mathbb{P}^\infty)^n \times \Omega(S^{2n-1}) \simeq (S^1)^n \times \Omega(S^{2n-1}).$$

This splitting is not an H-space splitting (in the case $n = 2$ for instance, the right hand side is abelian while $\Omega(\mathbb{P} \vee \mathbb{P})$ is not). The inclusion $\Omega S^{2n-1} \hookrightarrow \Omega W_n$ is, however, loop-sum preserving.

Lemma 7.5. *Consider 7.2; $S^{2n-1} \xrightarrow{G} W_n \hookrightarrow (\mathbb{P}^\infty)^n$, and let a_i denote the homotopy class of the ith inclusion $S^2 = \mathbb{P}^1 \hookrightarrow 1^{i-1} \times \mathbb{P}^1 \times 1^{n-i} \hookrightarrow (\mathbb{P}^\infty)^n$. Then G is an iterated Whitehead product*

$$G = [\cdots[[a_1, a_2], a_3], \ldots], a_n]\,.$$

Remark 7.6. One can apply the functor $\mathrm{Map}^*(M_g, -)$ to 7.3 and obtain a new diagram of fibrations. It is not hard to see that $\mathrm{Map}^*_c(M_g, \mathbb{P}^\infty) \simeq (S^1)^{2g}$, where Map^*_c is any component of Map^*. Indeed, since the attaching map of the two disc in M_g maps into $\bigvee^{2g} S^1$ as a commutator, it follows that its suspension is null. This implies that

$$\begin{aligned}
\mathrm{Map}^*(M_g, \mathbb{P}^\infty) &\simeq \mathrm{Map}^*(\Sigma M_g, K(\mathbb{Z}, 3)) \simeq \mathrm{Map}^*(S^3 \vee \bigvee^{2g} S^2, K(\mathbb{Z}, 3)) \\
&\simeq \mathbb{Z} \times \coprod^{2g} \Omega^2(K(\mathbb{Z}, 3)) \simeq \mathbb{Z} \times (S^1)^{2g}.
\end{aligned}$$

One therefore gets the diagram

$$\begin{tikzcd}
\mathrm{Map}^*(M_g, S^{2n+1}) \arrow[r, "="] \arrow[d, "h"] & \mathrm{Map}^*(M_g, S^{2n+1}) \arrow[r] \arrow[d, "G"] & * \arrow[d] \\
\mathrm{Map}^*(M_g, \mathbb{P}^n) \arrow[r] \arrow[d] & \mathrm{Map}^*(M_g, W_{n+1}) \arrow[r] \arrow[d] & (S^1)^{2ng} \arrow[d, "="] \\
(S^1)^{2g} \arrow[r] & (S^1)^{2(n+1)g} \arrow[r] & (S^1)^{2ng}
\end{tikzcd}$$

When $n = 1$, we know that

$$\mathrm{Map}^*(M_g, S^3) \simeq (\Omega^2 S^3) \times (\Omega S^3)^{2g}$$

and that $\mathrm{Map}^*(M_g, S^2)$ splits as $(\Omega S^3)^{2g} \times X_g$ for some total space $\Omega^2(S^3) \to X_g \to (S^1)^{2g}$ (see [BCM],§11; or [C2M2],§7). This however still is not enough to conclude any splitting for $\mathrm{Map}^*(M_g, \mathbb{P}^\infty \vee \mathbb{P}^\infty)$ (see §8.2).

The Samelson Product. This is standard [C] but we review it briefly. Given a loop space $\Omega(X)$, we denote by $S : \Omega(X) \times \Omega(X) \to \Omega(X)$ the commutator map $S(f, g) = f{*}g{*}f^{-1}{*}g^{-1}$. The map S is null homotopic when either f or g is constant at the basepoint, and hence it descends to a map $S : \Omega(X) \wedge \Omega(X) \to \Omega(X)$. The Samelson product

$$\langle , \rangle : \pi_p(\Omega(X)) \otimes \pi_q(\Omega(X)) \to \pi_{p+q}(\Omega(X))$$

is defined to be the composite $S^p \wedge S^q \xrightarrow{\alpha\wedge\beta} \Omega(X) \wedge \Omega(X) \xrightarrow{S} \Omega(X)$.

Theorem 7.7 (Samelson)**.** *Consider the suspension $E : X \to \Omega\Sigma X$ and the induced map $ad : X \wedge X \to \Omega\Sigma X$ given by $S \circ (E \wedge E)$. Then if x and y are primitive, we have*

$$ad_*(x \otimes y) = x \otimes y - (-)^{|x||y|} y \otimes x \in H_*(\Omega\Sigma X) \cong T(H_*(X)).$$

7.2. **Segal's Scanning Map.** We can now describe the map

$$S : \mathrm{Div}^n_k(M_g - *) \to \mathrm{Map}^*_{\vec{0}}(M_g, W_n\mathbb{P}) \tag{7.8}$$

where $\mathrm{Map}^*_{\vec{0}}(M_g, W_n\mathbb{P})$ refers to the subspace of based, null-homotopic maps (or equivalently-based maps of multidegree $\vec{0} = (0, \dots, 0)$) in $\mathrm{Map}_{\vec{0}}(M_g, W_n) \subset \mathrm{Map}_{\vec{0}}(M_g, (\mathbb{P}^\infty)^n)$.

Fix $r > 0$ (r small) and let $D_r(x) \subset M_g$ be the disc of radius r around the point $x \in M_g$. Since $M_g - *$ is parallelizable, one can canonically identify the pair $(\bar{D}(x), \partial\bar{D}(x))$ with (S^2, ∞), where the north pole ∞ is chosen to be the basepoint in S^2. To a given positive divisor $D \in SP^r(M_g)$ and to any $x \in M_g$, we can associate the divisor $D^x \in SP^\infty(S^2, \infty) = \mathbb{P}$ made out of points of $D \cap D_r(x)$ and extended out by basepoints; i.e.

$$D^x = \langle D \cup D_r(x), \infty, \dots \rangle.$$

Let $(D_1, \dots, D_n) \in \mathrm{Div}^n_k(M_g - *)$; then one defines

$$S : M_g \to \mathrm{Div}^n(S^2, \infty),\ x \mapsto (D^x_1, \dots, D^x_n) \tag{7.9}$$

where here $\mathrm{Div}^n(S^2, \infty) \subset (\mathbb{P}^\infty)^n$ consists of all n-tuples of divisors whose supports do not have a point in common (here the support of $D = \sum n_i z_i \in SP^\infty(S^2, \infty)$ is the set of $z_i \neq \infty$). One should probably point out the important difference in topology between $\mathrm{Div}^n(S^2, \infty)$ and $\mathrm{Div}^n(S^2 - \infty)$.

As was observed in [S], we can let Q_ϵ be the open subset of $\mathrm{Div}^n(S^2, \infty)$ consisting of n-tuples of divisors such that (at least) one such divisor, say D_i, is disjoint from the closed disk of radius ϵ about the origin (south pole). Then radial expansion defines a deformation retraction of Q_ϵ into W_n (more precisely in this case, the ith component of $(\mathbb{P}^\infty)^n$ gets retracted to ∞). This shows that $Q_\epsilon \simeq W_n$, and since $\mathrm{Div}^n(S^2, \infty) = \bigcup_{\epsilon > 0} Q_\epsilon$ we get

Lemma 7.10. $\mathrm{Div}^n(S^2, \infty) \simeq W_n\mathbb{P}$.

It is clear that 7.9 has multidegree $(k,\dots,k)$, and hence when combined with 7.10 it yields a map $S:\mathrm{Div}^n_k(M_g-*)\to\mathrm{Map}^*_{(k,\dots,k)}(M_g,W_n\mathbb{P})$. Since all components of the mapping space are homotopy equivalent, we obtain the map 7.8. Note that the stabilization process of §3 yields a (homotopy) commutative diagram

$$\begin{array}{ccc} \mathrm{Div}^n_k(M_g-*) & \to & \mathrm{Map}^*_{\vec{k}}(M_g,W_n\mathbb{P}) \\ \downarrow & & \downarrow \\ \mathrm{Div}^n_{k+1}(M_g-*) & \to & \mathrm{Map}^*_{\vec{k+1}}(M_g,W_n\mathbb{P}), \end{array}$$

and in the direct limit we obtain

Theorem 7.11 (Segal)**.** $S:\mathrm{Div}^n(M_g-*)\to\mathrm{Map}^*_0(M_g,W_n\mathbb{P})$ *is a homotopy equivalence.*

8. The Splitting

Recall that associated to M_g, we have the cofibration sequence

$$S^1\xrightarrow{f}\bigvee_1^{2g}S^1\longrightarrow M_g\longrightarrow S^2\longrightarrow\bigvee_1^{2g}S^2, \tag{8.1}$$

with f given as a product of commutators

$$[x_1,x_2]\cdots[x_{2g-1},x_{2g}]\in\pi_1(\vee_1^{2g}S^1).$$

Applying the functor $\mathrm{Map}^*(-,X)$ to 8.1 yields the fibration sequence

$$\Omega^2X\to\mathrm{Map}^*(M_g,X)\to(\Omega X)^{2g}\xrightarrow{f^!}\Omega X.$$

The map $f^!$ classifies $\Omega^2X\to\mathrm{Map}^*(M_g,X)\to(\Omega X)^{2g}$, and since $f^!=[x_1,x_2]^!\cdots[x_{2g-1},x_{2g}]^!$ is described in terms of commutators, it is natural to suspect that obstructions to the nullity of $f^!$ lie in various Whitehead products. This is indeed the case.

Write $X=W_n$. To analyze $f^!$, it is enough to consider one commutator at a time, say $[x_1,x_2]^!$. This we write as composition

$$(\Omega W_n)^2\xrightarrow{\ \Delta\ }\big((\Omega W_n)^2\big)^2\xrightarrow{id^2\times\chi^2}\big((\Omega W_n)^2\big)^2\longrightarrow(\Omega W_n)^4\xrightarrow{\ *^4\ }\Omega W_n,$$

where χ is the inverse map with respect to the loop sum, $\chi(f)(t)=f(1-t)=f^{-1}(t)$.

In §7 we saw that we had a map $\Omega W_n\xrightarrow{\pi}(S^1)^n$ (and a splitting $\Omega W_n\simeq\Omega S^{2n-1}\times(S^1)^n$). The composite

$$(\Omega W_n)^2\xrightarrow{[x_1,x_2]^!}\Omega W_n\xrightarrow{\ \pi\ }(S^1)^n$$

is a commutator in an abelian group and hence it is trivial. It follows that $\pi f^!$ is also homotopy trivial. and hence $f^!$ factors (up to homotopy):

$$f^!:(\Omega S^{2n-1}\times(S^1)^n)^{2g}\to\Omega S^{2n-1}\hookrightarrow\Omega W_n.$$

Question. Does $f^!$ factor further through $(S^1)^{2ng}$ as

$$f^!:(\Omega S^{2n-1}\times(S^1)^n)^{2g}\to(S^1)^{2ng}\to\Omega S^{2n-1}\hookrightarrow\Omega W_n. \tag{8.2}$$

In studying $\mathrm{Div}^2_k(M_g-*)$, [C2M2] only needed to consider the case $n=2$, and the question above was conjectured to be true.

We can analyze the obstruction to factoring $f^!$ as in 8.2 as follows. Start with

$$\big(\Omega S^{2n-1}\times(S^1)\big)^2\xrightarrow{(\Omega G*e)^2}\Omega(W_n)^2\xrightarrow{[x_1,x_2]^!}\Omega(W_n),$$

where $e : S^1 \to W_n$ is any one of the Ωs_i described in 7.5. Letting e_1 and e_2 (resp. ΩG_1 and ΩG_2) be the maps of S^1 (resp. ΩS^{2n-1}) into the first and second copies of ΩW_n, one can write (up to sign)

$$\begin{aligned}&[x_1, x_2]^1 ((e_1 * \Omega G_1) \times (e_2 * \Omega G_2))\\ &\quad \mapsto e_1 * \Omega G_1 * e_2 * \Omega G_2 * \chi(\Omega G_1) * \chi(e_1) * \chi(\Omega G_2) * \chi(e_2).\end{aligned}$$

Suppose the image of ΩG and the image of e commute. Then we can rewrite the above as follows:

$$\begin{aligned}(e_1, e_2, \Omega G_1, \Omega G_2) &\mapsto (e_1 * e_2 * [\Omega G_1 * \Omega G_2 * \chi(\Omega G_1) * \chi(\Omega G_2)] * \chi(e_1) * \chi(e_2))\\ &= e_1 * e_2 * \{\Omega G_1, \Omega G_2\} * e_1^{-1} * e_2^{-1}.\end{aligned}$$

If we suppose further that ΩG_1 and ΩG_2 commute in ΩW_n, then $f^!$ would factor as desired through $(S^1)^{2gn}$. This then shows that the desired factorization 8.2 happens under the following conditions:

- ΩG and e commute in ΩW_n,
- ΩS^{2n-1} is homotopy abelian.

The second condition is true after inverting 2. Indeed, an odd sphere is an H-space after inverting 2, at which point the loop space becomes abelian. To address the validity of the first condition, we restrict our attention to the commutator

$$\Omega S^{2n-1} \times S^1 \xrightarrow{\{\Omega G, e\}} \Omega(W_n \mathbb{P}).$$

Observe that $\Omega S^{2n-1} = \Omega\Sigma(S^{2n-2}) \simeq J(S^{2n-2})$ where $J(S^{2n-2})$ is the James construction on S^{2n-2} corresponding to the free monoid generated by points of S^{2n-2}. The commutator map can therefore be reduced to $S^{2n-2} \times S^1 \xrightarrow{\{\Omega G, e\}} \Omega(W_n\mathbb{P})$ and from there one can use the correspondence between the Samelson and Whitehead products [C] to write

$$ad\{\Omega G, e\} = [G, a] \in \pi_{2n}(W_n\mathbb{P}) = \pi_{2n} S^{2n-2} = \mathbb{Z}_2, \tag{8.3}$$

where $a = ade : \Sigma S^1 = S^2 \hookrightarrow \mathbb{P} \hookrightarrow W_n$.

In either case, then, it follows that the obstructions to factoring $f^! : (W_n\mathbb{P})^{2g} \to \Omega S^{2n-1} \times (S^1)^n$ through $(S^1)^n$ are $\mathbb{Z}_2$ obstructions. We have proved the following.

Proposition 8.4. *The following splits after inverting* 2*:*

$$\mathrm{Map}^*(M_g, W_n\mathbb{P}) \simeq (\mathbb{Z})^n \times \Omega(S^{2n-1})^{2g} \times Y_{g,n},$$

where $Y_{g,n}$ *is the total space of a (principal) fibering* $\Omega^2(S^{2n-1}) \to Y_{g,n} \to (S^1)^{2gn}$.

8.1. **The obstruction when** $n = 2$**.** When $n > 2, n \neq 4, 8$, it is clear that 8.4 is best possible. However when $n = 2$, one can hope to relax the localization condition there, for in this case ΩS^3 is homotopy abelian (S^3 being a group) and the first obstruction discussed earlier is not essential. We show, however, that the second obstruction 8.3 is.

Let G be as in Lemma 7.5. We know that the homotopy class of G is represented by $[a_1, a_2]$ and hence $[G, a_1]$ corresponds to $[[a_1, a_2], a_1]$. We show that this triple Whitehead product generates $\pi_4(\mathbb{P} \vee \mathbb{P}) \cong \mathbb{Z}_2$.

We start by considering the first few stages of the Postnikov decomposition for $X = \mathbb{P} \vee \mathbb{P}$. Notice that

$$\pi_1(X) = 0,\ \pi_2(X) \cong \mathbb{Z} \times \mathbb{Z},\ \pi_3(X) \cong \mathbb{Z} \text{ and } \pi_4(X) \cong \mathbb{Z}_2.$$

and hence

$$\begin{array}{ccccc} K(\mathbb{Z}_2,4) & \rightarrow & X_4 & & \\ & & \downarrow & & \\ K(\mathbb{Z},3) & \xrightarrow{i} & X_3 & \xrightarrow{k^5} & K(\mathbb{Z}_2,5) \\ & & \downarrow & & \\ \mathbb{P}\vee\mathbb{P} & \xrightarrow{f_2} & \mathbb{P}\times\mathbb{P} & \xrightarrow{k^4} & K(\mathbb{Z},4), \end{array}$$

where f_2 is the inclusion $\mathbb{P}\vee\mathbb{P}\hookrightarrow\mathbb{P}\times\mathbb{P}$. The fiber of f_2 is S^3, and so

$$\tau(\iota_3)=k^4,\ \ \iota_3\in\mathbb{Z}\cong H^3(S^3,\pi_3(X)),$$

where τ is the transgression. Since $H^*(\mathbb{P}\times\mathbb{P})\cong\mathbb{Z}[a_1,a_2]$, with a_1 and a_2 being the dual cohomology classes to the 2-dimensional generators corresponding to the inclusions $S^2\hookrightarrow\mathbb{P}\hookrightarrow\mathbb{P}\times\mathbb{P}$, and since

$$H^*(\mathbb{P}\vee\mathbb{P})\cong\mathbb{Z}[a_1,a_2]/(a_1a_2),$$

it follows that the class a_1a_2 must be hit by the transgression and hence $k^4=a_1a_2$.

Lemma 8.5. *Let γ be the class in $H^5(X_3,\mathbb{Z}_2)$ that restricts to $Sq^2(\iota_3)$, $\iota_3\in H^3(K(\mathbb{Z},3),\mathbb{Z}_2)$. Then γ is non-zero, and $k^5=\gamma$.*

Proof. Since d_4 corresponds to the transgression in this case, we have $d_4(\iota_3)=\tau(\iota_3)=a_1a_2$. Recall that

$$H^*(K(\mathbb{Z},3),\mathbb{Z}_2)=\mathbb{F}_2[\iota_3,Sq^2(\iota_3),(\iota_3)^2,Sq^4Sq^2(\iota_3),\dots,Sq^{2^i}\cdots Sq^4Sq^2(\iota_3),\dots].$$

A quick inspection of the E_4 quadrant shows that the d_4 differential vanishes on all homology generators in the fiber but ι_3. Since $d_4(\iota_3)=a_1a_2$, and since the classes a_1 and a_2 survive (and their powers), it follows that

$$E_5=H^*(\mathbb{P}\times\mathbb{P})\otimes\mathbb{F}_2[Sq^2(\iota_3),\iota_3^2,Sq^4Sq^2(\iota_3),\dots].$$

Since ι_3 transgresses, so does $Sq^2(\iota_3)$. We then have

$$d_6(Sq^2(\iota_3))=Sq^2(d_4(\iota_3))=Sq^2(a_1a_2).$$

But $Sq^2(a_1a_2)$ is already hit by d_4, as the following application of the Cartan formula (with $Sq^1(a_i)=0$ in $\mathbb{P}$) shows:

$$Sq^2(a_1a_2)=Sq^2(a_1)a_2+a_1Sq^2(a_2)=(a_1+a_2)a_1a_2=d_4(a_1+a_2)\iota_3.$$

It follows that $d_6(Sq^2(\iota_3))=0$ and that $Sq^2(\iota_3)$ survives to E_∞. Since it is the only class in $H^5(X_3,\mathbb{Z}_2)$, it must be the image of the transgression $\tau(H^4(K(\mathbb{Z}_2,4),\mathbb{Z}_2))$ in the next stage of the Postnikov tower. This proves the lemma. $\square$

Now consider the pulback diagram

$$\begin{array}{ccc} K(\mathbb{Z},3) & \xrightarrow{=} & K(\mathbb{Z},3) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ E & \xrightarrow{j} & X_3 \\ \downarrow & & \downarrow \\ \mathbb{P}^\infty & \xrightarrow{i_1} & \mathbb{P}^\infty\times\mathbb{P}^\infty \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle a_1a_2} \\ K(\mathbb{Z},4) & \xrightarrow{=} & K(\mathbb{Z},4) \end{array}$$

where i_1 is the inclusion of $\mathbb{P}$ into the first factor. The pullback of the k-invariant a_1a_2 under i_1 is trivial (since $i_1^*(a_2) = 0$). The induced total space is then $E = K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$

Lemma 8.6. $j^*(\gamma) = Sq^2(\iota_3) + \iota_3 \cup a_1$.

Proof. In the fibration

$$K(\mathbb{Z}, 3) \xrightarrow{i} X_3 \xrightarrow{p} \mathbb{P} \times \mathbb{P},$$

$\tau(Sq^2(\iota_3)) = a_1a_2(a_1 + a_2)$ means that there is a class $\beta \in \mathcal{C}^5(X_3)$ (where $(\mathcal{C}, \delta)$ is a cochain complex) such that $i^*([\beta]) = Sq^2(\iota_3)$ (this is the only time that we differentiate between a cochain x and its cohomology class $[x]$) and that

$$\delta(\beta) = p^*(a_1a_2(a_1 + a_2)) = p^*(a_1a_2) \cup p^*(a_1 + a_2).$$

The cochain β is chosen modulo $p^*(\mathbb{P} \vee \mathbb{P})$. Now, the pullback $i^*(j^*(\beta))$ must correspond to $Sq^2(\iota_3)$ and hence $j^*(\beta) - Sq^2(\iota_3) \in ker(i^*) = \{0, i_3 \cup a_1\}$. By the choice of β modulo $p^*(\mathbb{P} \vee \mathbb{P})$, we must then have that $j^*(\beta) = Sq^2(\iota_3)$.

On the other hand, since ι_3 transgresses to a_1a_2 it follows also that $p^*(a_1a_2) = \delta(\iota_3)$. (Here we think of ι_3 as some cochain in X_3 mapping onto $\iota_3 \in H^3(K(\mathbb{Z}, 3))$ under the epimorphism $i^* : \mathcal{C}^3(X_3) \to \mathcal{C}^3(K(\mathbb{Z}, 3))$.) It then follows that

$$\delta(\beta) = \delta(\iota_3) \cup (a_1 + a_2) = \delta(\iota_3(a_1 + a_2))$$

and hence that $\delta(\beta + \iota_3(a_1 + a_2)) = 0$. Moreover $i^*(\beta + \iota_3(a_1 + a_2)) = Sq^2(\iota_3)$. This shows that we can choose $\gamma \in H^5(X_3)$ to be equal to $\beta + \iota_3(a_1 + a_2)$. Therefore

$$j^*(\gamma) = j^*(\beta) + j^*(\iota_3(a_1 + a_2)) = Sq^2(\iota_3) + \iota_3 \cup a_1,$$

and the lemma follows. $\square$

Lemma 8.7. $[[a_1, a_2], a_1]$ *generates* $\pi_4(\mathbb{P} \vee \mathbb{P}) \cong \mathbb{Z}_2$.

Proof. Consider the composite map

$$S^3 \times S^2 \xrightarrow{f=\iota_3\times\iota_2} E = K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2) \xrightarrow{j} X_3.$$

The pullback of γ via $f \circ j$ is

$$f^*(j^*(\gamma)) = f^*(Sq^2(\iota_3)) + f^*(\iota_3 \cup a_1) = f^*(\iota_3) \cup f^*(a_1) = \kappa_3 \cup \kappa_2,$$

where κ_i is the generator of $H^i(S^i)$. It follows that the composite $S^3 \times S^2 \to X_3 \to K(\mathbb{Z}_2, 5)$ is not zero and hence $S^3 \times S^2$ does not lift to the next stage, X_4, of the Postnikov resolution, since the latter $K(\mathbb{Z}_2, 4) \to 1.5X_4 \to 1.5X_3$ has k-invariant γ.

The fact that $S^3 \times S^2$ doesn't lift to X_4 implies that the class of the Whitehead product $[\kappa_3, \kappa_2] \in \pi_3(S^3 \vee S^2)$ has image the non-zero generator in $\pi_4(E_2) = \pi_4(\mathbb{P} \vee \mathbb{P})$ (here κ_i also denotes the generating class in $\pi_i(S^i)$). The image of κ_2 is a_1 by construction, while the image of $\kappa_1 \in \pi_3(S^3)$ is the Whitehead product $[a_1, a_2] \in \pi_3(\mathbb{P} \vee \mathbb{P})$ according to the diagram

$$\begin{array}{ccc} S^3 \to K(\mathbb{Z},3) \to & X_3 \\ \downarrow & \downarrow \\ \mathbb{P} \vee \mathbb{P} \longrightarrow & \mathbb{P} \times \mathbb{P}. \end{array}$$

This concludes the proof. Note that the non-zero generator $[[a_1, a_2], a_1]$ must correspond to $[a_1, a_2] \circ \eta$, where η is the Hopf map $S^4 \to S^3$. $\square$

9. Relation with Mapping Spaces

In this section, we prove the following easy consequence of our previous study of the divisor spaces.

Proposition 9.1. *The Eilenberg-Moore spectral sequence associated to the fibration*

$$\Omega^2 S^{2n-1} \to \mathrm{Map}_0^*(M_g, W_n\mathbb{P}) \to (S^1)^{2ng} \times (\Omega S^{2n-1})^{2g}$$

collapses at $E_2 = Tor_{H^*(\Omega S^{2n-1})}(\mathbb{F}, H^*((S^1)^{2ng} \times (\Omega S^{2n-1})^{2g}))$.

Proof. Consider first the case $n = 2$. In this case, the classifying map being null homotopic on ΩS^3, it follows that the action of $H^*(\Omega S^3)$ on $H^*(\Omega S^3)^{2g}$ is trivial and that

$$(*) \qquad E_2 = H^*(\Omega S^3)^{2g} \otimes Tor_{H^*(\Omega S^3)}(\mathbb{F}, H^*(S^1)^{4g}).$$

We write $H^*(S^1)^{4g} = \Lambda(e_1, \dots, e_{4g})$ and $H^*(\Omega S^3) = \Gamma(a)$. We should point out that in the Eilenberg-Moore spectral sequence the bar degrees are subtracted from the total degree of resolution elements rather than added (compare with §1). For instance, in this case $\deg|a| = 2 - 1 = 1$. To understand the module structure of $\Gamma(a)$ on $\Lambda(e_1, \dots, e_{4g})$, we need to know first about the ring structure of $H_*(\Omega(\mathbb{P} \vee \mathbb{P}))$.

Lemma 9.2 ([C2M2]). *Let* $e_1, e_2 \in H_1(\Omega(\mathbb{P} \vee \mathbb{P}))$ *be the generators corresponding to the inclusions of* $\mathbb{P}$ *into the first and second factors of* $\mathbb{P}\vee\mathbb{P}$ *(respectively), and let* a *represent the class in the Hurewicz image of the generating sphere in* $H_2(\Omega(\mathbb{P}\vee\mathbb{P}); \mathbb{Z})$ *coming from* $\pi_2(\Omega S^3)$. *Then if* $T(\ )$ *denotes the tensor algebra, we have*

$$H_*(\Omega(\mathbb{P} \vee \mathbb{P}); \mathbb{Z}) \cong T(e_1, e_2, a)/(e_1^2 = e_2^2 = 0, e_1e_2 + e_2e_1 = a).$$

Proof. Since $e_1, e_2, \langle e_1, e_2\rangle$ and $[e_1, e_2]$ are maps of spheres, we will use the same notation for the maps and the corresponding spherical classes they generate. That $e_1^2 = e_2^2 = 0$ follows trivially from the homology of S^1. Since the inclusion $G : S^3 \to \mathbb{P} \vee \mathbb{P}$ is given by $G = [\Sigma e_1, \Sigma e_2]$ (Lemma 8.5), it then follows that $a = adG = \langle ad\Sigma e_1, ad\Sigma e_2\rangle = \langle e_1, e_2\rangle : S^2 \to \Omega S^3 \to \Omega(\mathbb{P} \vee \mathbb{P})$. By Samelson's theorem 7.6, we must have that $\langle e_1, e_2\rangle = e_1e_2 + e_2e_1 = a$, as desired. ☐

Going back to the proof of 9.1, we can look at the effect of the commutator $[x_1, x_2]^!$ at the level of homology on $H_*(S^1 \times S^1)$. We have

$$\begin{aligned} [x_1, x_2]^!(e_{ij} \otimes e_{kl}) &= (*) \times (id^2 \times \chi^2)\Delta_*(e_{ij} \otimes e_{kl}) \\ &= (*)(e_{ij}e_{kl} \otimes 1 + e_{ij} \otimes e_{kl} - e_{kl} \otimes e_{ij} + 1 \otimes e_{ij}e_{kl}) \end{aligned}$$

and using the relations in 9.2 above, we see that $[x_1, x_2]^!(e_{ij} \otimes e_{kl}) = 0$ when $i = k$ or $j = l$, and that

$$\begin{aligned} [x_1, x_2]^!(e_{11} \otimes e_{22}) &= [x_1, x_2]^!(e_{12} \otimes e_{21}) \\ &= e_1 \otimes e_2 + e_2 \otimes e_1 = a \in \mathbb{Z}[a] = H_*(\Omega S^3) \end{aligned}$$

This then defines the map $f^!$ completely. In cohomology, it follows that $f^*(a) = \sum_1^{2g} e_{2i+1}e_{2i}$, which implies that the action of a on $\Lambda(e_1, e_2, \dots, e_{4g})$ is given by multiplication with $\sum_1^{2g} e_{2i+1}e_{2i}$. In this case $(*)$ takes the form

$$(9.3) \qquad \Gamma(h_1, \dots, h_{2g}) \otimes Tor_{\Gamma[a]}(\mathbb{F}, \Lambda(e_1, \dots, e_{4g})).$$

This already makes up for the homology $H_*(\mathrm{Div}^2(M_g - *);\mathbb{F})$ (cf. §6.2) and hence in light of Segal's homotopy equivalence 7.10, this must give the entire homology of $\mathrm{Map}_0^*(M_g, \mathbb{P}\vee\mathbb{P})$ and $E_2 = E_\infty$.

The case $n > 2$ is simpler, for $f^* : H^*(\Omega(W_n\mathbb{P})) \to H^*\big((\Omega W_n\mathbb{P})^{2g}\big)$ is trivial and hence the E_2 term (*) takes the form

$$E_2 = H^*((\Omega S^{2n-1})^{2g};\mathbb{F}) \otimes H^*((S^1)^{2ng};\mathbb{F}) \otimes H^*(\Omega^2 S^{2n-1};\mathbb{F}). \tag{9.4}$$

Here too results of §6 show that 9.4 accounts for all classes in $H^*(\mathrm{Map}_0^*(M_g, W_n);\mathbb{F})$, and the Eileberg-Moore spectral sequence must then collapse at E^2. This completes the proof. □

Acknowledgements

This work is part of the author's doctoral dissertation written at Stanford University under the supervision of Professor R.J. Milgram. The author is much indebted to his advisor for suggesting the problem and for sharing ideas without which this work would not have been possible. The author would also like to thank G. Carlsson and R.L. Cohen for helpful conversations. The final version of this paper was written at the Centre de Recherches Mathematiques of the Université of of Montréal, and the author thanks both the center and Professor J. Hurtubise for their support.

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