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FACTORISATION IN NEST ALGEBRAS. II

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Abstract. The main result of this paper is Theorem 5, which provides a necessary and sufficient condition on a positive operator A for the existence of an operator B in the nest algebra $AlgN$ of a nest N satisfying $A = BB^*$ (resp. $A = B^*B$). In Section 3 we give a new proof of a result of Power concerning outer factorisation of operators. We also show that a positive operator A has the property that there exists for every nest N an operator B_N in $AlgN$ satisfying $A = B_N B_N^*$ (resp. $A = B_N^* B_N$) if and only if A is a Fredholm operator. In Section 4 we show that for a given operator A in $B(H)$ there exists an operator B in $AlgN$ satisfying $AA^* = BB^*$ if and only if the range $r(A)$ of A is equal to the range of some operator in $AlgN$. We also determine the algebraic structure of the set of ranges of operators in $AlgN$. Let $F_r(N)$ be the set of positive operators A for which there exists an operator B in $AlgN$ satisfying $A = BB^*$. In Section 5 we obtain information about this set. In particular we discuss the following question: Assume A and B are positive operators such that $A \leq B$ and A belongs to $F_r(N)$. Which further conditions permit us to conclude that B belongs to $F_r(N)$?

1. Introduction and preliminaries

Let H be a separable Hilbert space. A nest N on H is a totally ordered set of closed subspaces of H containing $\{0\}$ and H which is closed under intersection and closed span. The associated nest algebra $AlgN$ is the set of bounded operators A on H leaving each member of N invariant. The problem of factorisation of operators with respect to a nest N consists in writing a positive operator A in the form BB^* (or B^*B) with B in $AlgN$. The factorisation of a positive invertible finite matrix A as B^*B with B and its inverse in upper triangular form is known as the Cholesky decomposition. In [13] Gohberg and Krein obtain factorisations for operators which are compact perturbations of the identity with respect to arbitrary nests. Larson [15] studied factorisations of positive invertible operators A in the form B^*B with B invertible in $AlgN$. He showed that such a factorisation exists for every positive invertible operator if and only if the nest is countable. These results are concerned with factorisations of invertible or essentially invertible operators. Arveson [4] has introduced the concept of the outer operator in analogy with the outer functions in Hardy spaces. He has given a necessary and sufficient condition on a positive operator A for the existence of a factorisation $A = B^*B$ with B outer in $AlgN$, with respect to nests of a certain order type. Shields [23] obtained a factorisation for any positive trace class operator in the case of a nest of order type $\mathbb{N}$. In [20] Power, making a constructive approach, proved that every positive operator

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A has a factorisation $A = B^*B$ with B outer in $AlgN$ if and only if the nest N is well-ordered. Factorisation problems for other types of operator algebras which are related to nest algebras are also studied in [3],[17],[19]. Factorisation theory of operators is closely related to the theory of factorisation of positive operator-valued functions on the unit circle. We refer the reader to [18] for a survey of results and bibliography.

In this work we give a necessary and sufficient condition on a positive operator A for the existence of an operator B in the nest algebra $AlgN$ of a nest N satisfying $A = BB^*$ (resp. $A = B^*B$). This result, which we prove in Section 2, holds for an arbitrary positive operator A and for any nest N. If the nest N is of order type $\mathbb{Z}$ the condition we give for the factorisation $A = B^*B$ is the same as the one given by Arveson in Theorem 3.3. of [4]. However in the general case the condition has a more elaborate form. The main idea in order to obtain the factorisation $A = BB^*$ is to consider the biggest projection Q in N which satisfies $Q = Q_-$ and study the behaviour of the operator A "near" Q. In Section 3 we use the technique developed in Section 2 to obtain a new proof of the above mentioned result of Power. We also show that a positive operator A has the property that there exists for every nest N an operator B_N in $AlgN$ satisfying $A = B_N B_N^*$ (resp. $A = B_N^* B_N$) if and only if A is a Fredholm operator. In Section 4 we show that for a given operator A in $B(H)$ there exists an operator B in $AlgN$ satisfying $AA^* = BB^*$ if and only if the range $r(A)$ of A is equal to the range of some operator in $AlgN$ (Theorem 13). A consequence of this is that if A and C are positive operators in $B(H)$ with the same range, then there exists an operator B in $AlgN$ satisfying $A = BB^*$ if and only if there exists an operator D in $AlgN$ satisfying $C = DD^*$. Theorem 13 motivates the study of the set of ranges of operators in $AlgN$, which we denote by $OR(N)$. In the rest of Section 4 we caracterise the nests N for which the set $OR(N)$ is a meet semi-lattice (resp. a join semi-lattice). In Section 5 we consider a nest N satisfying $I = I_-$ and study the set of positive operators A for which there exists an operator B in $AlgN$ satisfying $A = BB^*$. This set is denoted by $F_r(N)$. We show that if A is in $F_r(N)$ then A^λ is in $F_r(N)$ for every positive number λ with $0 < \lambda \leq 1$, and show by an example that this is not true if $\lambda > 1$. We also show that if A and C are in $F_r(N)$ then $A + C$ is in $F_r(N)$. A simple criterion is given which permits one to decide if an operator A with closed range belongs to $F_r(N)$. We close Section 5 with Theorem 29, which provides a decomposition of a positive operator A into a "factorable" and a "completely non-factorable" part with respect to a nest N satisfying $I = I_-$. An analogous decomposition has been obtained in [21] and [1] for special cases.

As a general rule (with an exception in Theorem 10) we prove our results for the factorisation BB^* and present the results concerning the factorisation B^*B as corollaries. A reason for this choice is that it makes Theorem 13 appear more elegant.

Some of the results of this work (Theorem 5, Theorem 11 and Proposition 28) generalise previous results that we have obtained in [2] in the particular case of a continuous nest.

Throughout this work H is a separable Hilbert space. The inner product on H will be denoted by $\langle \, , \, \rangle$. By a subspace of H we mean a subset of H which is closed under addition of vectors and scalar multiplication. If W is a subspace of H, $W^\perp$ is the subspace of H consisting of the vectors orthogonal to each vector in W. If V is a subset of H, $[V]$ is the linear span of V. If ξ is a vector in H, $[\xi]$ is the linear span of ξ. If $\{V_n\}_{n=1}^{\infty}$ is a sequence of closed mutually orthogonal subspaces of H, we denote by $\sum_{n=1}^{\infty} \oplus V_n$ the closure of their linear span. We will denote by $B(H)$ the space of all bounded operators from H into itself. Let x, y be in H. The rank-one operator $x \otimes y$ is the operator on H defined by: $z \to \langle z, x\rangle y$. If A is in $B(H)$ we denote by $r(A)$ the range of A and by $cokerA$ the orthogonal complement of the kernel of A. If P is an orthogonal projection in $B(H)$, $P^{\perp}$ is the (orthogonal) projection $I - P$.

Let N be a nest on H. The nest $N^{\perp}$ is defined to be $\{P^{\perp} : P \in N\}$. If P is in N we will denote by the same symbol the orthogonal projection on the subspace P. When the subspace H (resp. $\{0\}$) is considered as an element of the nest N it will be denoted by I (resp. 0). If E is a projection commuting with the elements of N, EN is the nest in the Hilbert space EH defined by $EN = \{EP \colon P \in N\}$. We will say that a vector x in H is N-proper if there exists a projection P in N, $P \neq I$, such that $Px = x$. The set of N-proper vectors will be denoted by PrN. Given an element P of N, we define P_- to be $\overline{[\bigcup_{L\in N, L<P} L]}$ and P_+ to be $\bigcap_{L\in N, L>P} L$. We define 0_- to be 0 and I_+ to be I. The nest N is continuous if $P = P_-$ for every P in N. The associated nest algebra $AlgN$ is the set of operators A in $B(H)$ satisfying $PAP = AP$ for every P in N. For a general discussion of nest algebras the reader is refered to [6].

We will say that a positive operator A in $B(H)$ admits a right factorisation (resp. a left factorisation) with respect to N if there exists an operator B in $AlgN$ such that $A = BB^*$ (resp. $A = B^*B$). We will denote by $F_r(N)$ (resp. $F_l(N)$) the set of positive operators in $B(H)$ which admit a right factorisation (resp. a left factorisation) with respect to N.

2. The factorisation theorem

Throughout this section the letter N will denote a nest on H, and Q will be the element of N defined by $Q = \overline{\bigcup_{P\in N, P=P_-} P}$. Then it is easy to see that $Q = Q_-$ and that for every P in N, $P > Q$, we have $P \neq P_-$.

Lemma 1. *Let R be in N. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of elements of N such that $P_0 = 0$, $P_{n+1} > P_n, P_n \neq R$ for each n, and P_n converges strongly to R. Then there exists a sequence $\{M_n\}_{n=1}^{\infty}$ of closed mutually orthogonal infinite dimensional subspaces of H, such that $M_n \subset R \ominus P_n$ for every n.*

Proof. Take for each n a vector e_n in $P_{n+1} \ominus P_n$ of norm 1. Take a sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint infinite subsets of $\mathbb{N}$. Put $M_n = \overline{[e_m : m \in A_n, m \geq n]}$. $\square$

Lemma 2. *Let A be an operator in $B(H)$. Let R be in N. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of elements of N such that $P_0 = 0$, $P_{n+1} > P_n, P_n \neq R$ for each n, and P_n converges strongly to R. Then*

$$\sum_{n=1}^{\infty} \oplus (A^{-1}(P_n) \ominus A^{-1}(P_{n-1})) = \overline{\bigcup_{P<R} A^{-1}(P) \cap cokerA}.$$

Proof. For each n the subspace $A^{-1}(P_n) \ominus A^{-1}(P_{n-1})$ is contained in $A^{-1}(P_n) \cap cokerA$, which is in $\overline{\bigcup_{P<R} A^{-1}(P) \cap cokerA}$.

Let P be in N, $P < R$. There exists m such that $P < P_m$. We have $A^{-1}(P) \cap coker A \subset A^{-1}(P_m) \cap coker A$. But

$$A^{-1}(P_m) \cap coker A = \sum_{n=1}^{m} \oplus (A^{-1}(P_n) \ominus A^{-1}(P_{n-1})).$$

We conclude that $A^{-1}(P) \cap coker A$ is contained in $\sum_{n=1}^{\infty} \oplus (A^{-1}(P_n) \ominus A^{-1}(P_{n-1}))$. □

Lemma 3. *Let A be an operator in $B(H)$. Then we have:*

a) $(\sum_{P>Q} \oplus (A^{-1}(P) \ominus A^{-1}(P_-))) \oplus A^{-1}(Q) = H$.

b) $(\sum_{P>Q} \oplus (A^{-1}(P) \ominus A^{-1}(P_-))) \oplus (A^{-1}(Q) \cap coker A) = coker A$.

Proof. a) It is clear that the sum is orthogonal. Let y be in H. Assume that y is orthogonal to $\sum_{P>Q} \oplus (A^{-1}(P) \ominus A^{-1}(P_-)) \oplus A^{-1}(Q)$. Let $R = inf\{P \in N : PAy = Ay\}$. We have $RAy = Ay$. If $R > Q, y$ is orthogonal to $A^{-1}(R) \ominus A^{-1}(R_-)$; hence y is in $A^{-1}(R_-)$. We conclude that Ay is in R_-, which is contrary to the definition of R. Therefore $R \leq Q$. But then y is in $A^{-1}(Q)$ and is orthogonal to $A^{-1}(Q)$; hence $y = 0$.

b) follows from a) and the fact that $(\sum_{P>Q} \oplus (A^{-1}(P) \ominus A^{-1}(P_-)))$ is contained in $coker A$. □

Lemma 4. *Let L be in N and $\{L_n\}_{n=0}^{\infty}$ be a sequence of elements of N such that $L_{n+1} < L_n$, $L_n \neq L$ for each n and L_n converges strongly to L. Let M be a closed subspace of H contained in $L^\perp$. Assume that there exists m in $\mathbb{N}$ such that $dim(((L_n)_-)^\perp \ominus (((L_n)_-)^\perp \cap M)) \leq m$ for each n. Then $dim\left(L^\perp \ominus M\right) \leq m$.*

Proof. Assume $dim\left(L^\perp \ominus M\right) > m$. Then there exist $m+1$ linearly independent vectors $x_1, x_2, ..., x_{m+1}$ in $L^\perp \ominus M$. For each n the vectors $((L_n)_-)^\perp x_1$, $((L_n)_-)^\perp x_2$, ...,$((L_n)_-)^\perp x_{m+1}$ are orthogonal to $((L_n)_-)^\perp \cap M)$ and so their Grammian is 0. The Grammians of the vectors $((L_n)_-)^\perp x_1$, $((L_n)_-)^\perp x_2$, ...,$((L_n)_-)^\perp x_{m+1}$ converge to the Grammian of the vectors $x_1, x_2, ..., x_{m+1}$. Hence the vectors $x_1, x_2, ..., x_{m+1}$ are linearly dependent. □

Let A be an operator in $B(H)$. We set

$$n(A) = dim(A^{-1}(Q) \ominus \overline{\bigcup_{P<Q} A^{-1}(P)})$$

if $Q \neq \{0\}$; $n(A) = 0$ if $Q = 0$.

Let P be in N, $P > Q$. We set $n_P(A) = +\infty$ if $dim(P \ominus P_-) = +\infty$, and

$$n_P(A) = dim(P \ominus P_-) - dim(A^{-1}(P) \ominus A^{-1}(P_-))$$

if $dim(P \ominus P_-) < +\infty$. Note that $n_P(A) \geq 0$.

Theorem 5. *Let A be an operator in $B(H)$. The following are equivalent:*

a) There exists an operator B in $AlgN$ such that $AA^* = BB^*$.

b) $\sum_{P>Q} n_P(A) \geq n(A)$.

Proof. Assume b) holds. Consider for $P > Q$ a partial isometry V_P with domain contained in $P \ominus P_-$ and range $A^{-1}(P) \ominus A^{-1}(P_-)$. If $dim(P \ominus P_-) = +\infty$, we choose V_P in such a way that $dim((P \ominus P_-) \ominus domV_P) = +\infty$. Put $V_1 = \sum_{P>Q} V_p$. Then V_1 is a partial isometry with range $\sum_{P>Q} \oplus(A^{-1}(P) \ominus A^{-1}(P_-))$. We set $E_P = (P \ominus P_-) \ominus domV_P$.

Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of elements of N such that $P_0 = 0$, $P_{n+1} > P_n, P_n \neq Q$ for each n, and P_n converges strongly to Q. It follows from Lemma 1 that there exists a sequence $\{M_n\}_{n=1}^{\infty}$ of closed mutually orthogonal infinite dimensional subspaces of H, such that $M_n \subset Q \ominus P_n$ for every n. Consider for $n \geq 1$ a partial isometry W_n with domain contained in M_n and range $A^{-1}(P_n) \ominus A^{-1}(P_{n-1})$. Put $V_2 = \sum_{n=1}^{\infty} W_n$. Then V_2 is a partial isometry, and it follows from Lemma 2 that its range is $\overline{\bigcup_{P<Q} A^{-1}(P) \cap cokerA}$.

Put $E = \sum_{P>Q} \oplus E_P$. Let V_3 be apartial isometry with domain contained in E and range $A^{-1}(Q) \ominus \overline{\bigcup_{P<Q} A^{-1}(P)}$. Such an isometry exists, because

$$dimE = \sum_{P>Q} n_P(A) \geq dim(A^{-1}(Q) \ominus \overline{\bigcup_{P<Q} A^{-1}(P)})$$

by hypothesis.

We set $V = V_1 + V_2 + V_3$. Then V is a partial isometry with range

$$(\sum_{P>Q} \oplus(A^{-1}(P) \ominus A^{-1}(P_-))) \oplus (\overline{\bigcup_{P<Q} A^{-1}(P) \cap cokerA}) \oplus (A^{-1}(Q) \ominus \overline{\bigcup_{P<Q} A^{-1}(P)}).$$

Since

$$A^{-1}(Q) \ominus \overline{\bigcup_{P<Q} A^{-1}(P)} = ((A^{-1}(Q) \cap cokerA) \ominus (\overline{\bigcup_{P<Q} A^{-1}(P) \cap cokerA}),$$

the range of V is $(\sum_{P>Q} \oplus(A^{-1}(P) \ominus A^{-1}(P_-))) \oplus ((A^{-1}(Q) \cap cokerA)$, which by Lemma 3 is equal to $cokerA$. We have $A = AVV^*$. We show that AV is in $AlgN$. Let R be in N and x be in R. We show that AVx is in R. i) Assume $R < Q$. If $R \leq P_1$, $AVx = 0$. If $R > P_1$, there exists $m \geq 1$ such that $P_m < R \leq P_{m+1}$. Then $AVx = AV_2x = A(\sum_{n=1}^{m} W_n)x$, which is contained in

$$A(\sum_{n=1}^{m} \oplus(A^{-1}(P_n) \ominus A^{-1}(P_{n-1}))).$$

But $A(\sum_{n=1}^{m} \oplus(A^{-1}(P_n) \ominus A^{-1}(P_{n-1})))$ is contained in P_m. Hence AVx is in R. ii) Assume $R = Q$. We have $AVx = A(V_2 + V_3)x$. But $r(V_2 + V_3)$ is contained in $A^{-1}(Q)$. We conclude that AVx is in Q. iii) Assume $R > Q$. Since $r(V_2 + V_3)$ is contained in $A^{-1}(Q)$ we see that $A(V_2 + V_3)x$ is in Q. We have $AV_1x = A(\sum_{Q<P\leq R} V_p)x$, which is contained in $A(\sum_{Q<P\leq R} \oplus(A^{-1}(P) \ominus A^{-1}(P_-)))$. But $A(\sum_{Q<P\leq R} \oplus(A^{-1}(P) \ominus A^{-1}(P_-)))$ is contained in R. Hence AVx is in R.

Put $B = AV$. Then $BB^* = AVV^*A^* = AA^*$ and B is in $AlgN$.

Assume a) holds. It follows from polar decomposition that there exists a partial isometry U with domain $\overline{cokerA}$ and range $\overline{cokerB}$ such that $A = BU$. Put $D = \overline{(\bigcup_{P<Q} A^{-1}(P) \cap cokerA)}$ and $M = ((A^{-1}(Q) \cap cokerA) \ominus \overline{(\bigcup_{P<Q} A^{-1}(P) \cap cokerA)}$. We show that UM is contained in $Q^{\perp}$. Take m in M and P in N, $P < Q$. Since $r(A) = r(B)$ we have $BPUm = Ax_P$ for some x_P in $cokerA$. Since $BPUm$ is in P, x_P is in $A^{-1}(P) \cap cokerA$. We have $BPUm = BUx_P$, and so $PUm - Ux_P$ is in $KerB$. We have $PUm = PUm - Ux_P + Ux_P$, which belongs to $KerB \oplus UD$. We conclude that $QUm = \lim_{P \in N, P \neq Q, P \to Q} PUm$ is in $KerB \oplus UD$. Since UM is orthogonal to $KerB \oplus UD$, we see that QUm is orthogonal to Um. Therefore Um is in $Q^{\perp}$.

We set $K_P = A^{-1}(P) \ominus A^{-1}(P_-)$ for P in N, $P > Q$. We show that UK_P is contained in $(P_-)^{\perp}$. Let x be in K_P. Since $r(A) = r(B)$ we have $B(P_-)Ux = Ay$ for some y in $A^{-1}(P_-) \cap cokerA$. But then $B(P_-)Ux = BUy$, and so $(P_-)Ux - Uy$ is in $KerB$. We have

$$(P_-)Ux = (P_-)Ux - Uy + Uy,$$

which is in $KerB \oplus U(A^{-1}(P_-) \cap cokerA)$. Since Ux is orthogonal to $KerB \oplus U(A^{-1}(P_-) \cap cokerA)$ we see that Ux is orthogonal to $(P_-)Ux$. Hence $(P_-)Ux = 0$.

Put $K = \sum_{P>Q} \oplus K_P$. Since UM is contained in $Q^{\perp}$ and is orthogonal to UK, in order to prove b) it is enough to prove that $dim(Q^{\perp} \ominus UK) \leq \sum_{P>Q} n_P(A)$. Put $\Omega = \{P \in N,\ P \geq Q\}$, $\Pi = \{P \in \Omega : dim((P_-)^{\perp} \ominus (\sum_{R \geq P} \oplus UK_R)) \leq \sum_{R \geq P} n_R(A)\}$, $\Sigma = \{P \in \Omega,\ P \notin \Pi\}$. It suffices to show that Σ is empty.

Assume that $\Sigma \neq \emptyset$. Note that if P is in Ω and is different from Q then $P \neq P_-$. It follows that every non-empty subset of Ω has a maximum. Let S be the maximum of Σ.

i) Suppose first that $S = L_-$ for some L in Π. Let π be the canonical projection from $(S_-)^{\perp}$ onto $(S_-)^{\perp}/\,(\sum_{P \geq L} \oplus UK_P)$. Since $ker\pi$ is contained in $(\sum_{P \geq S} \oplus UK_P)$, we see that

$$dim((S_-)^{\perp} \ominus (\sum_{P \geq S} \oplus UK_P)) = dim\pi((S_-)^{\perp}) - dim\pi(\sum_{P \geq S} \oplus UK_P).$$

We have

$$dim\pi((S_-)^{\perp}) = dim\pi(S \ominus S_-) + dim\pi((L_-)^{\perp})$$

and

$$dim\pi(\sum_{P \geq S} \oplus UK_P) = dimUK_S.$$

So

$$\begin{aligned} dim((S_-)^{\perp} \ominus (\sum_{P \geq S} \oplus UK_P)) &= dim\pi(S \ominus S_-) + dim\pi((L_-)^{\perp}) - dimUK_S \\ &= dim(S \ominus S_-) + dim\pi((L_-)^{\perp}) - dimUK_S. \end{aligned}$$

But

$$dim\pi((L_-)^{\perp}) = dim((L_-)^{\perp} \ominus (\sum_{P \geq L} \oplus UK_P))$$

and

$$dim\pi((L_-)^\perp) \le \sum_{P\ge L} n_P(A)$$

since L is in Π. We conclude that

$$dim((S_-)^\perp \ominus (\sum_{P\ge S} \oplus UK_P)) \le \sum_{P\ge L} n_P(A) + dim(S\ominus S_-) - dimUK_S = \sum_{P\ge S} n_R(A).$$

Hence S is in Π, which is contrary to our assumption.

ii) Suppose that $S \ne L_-$ for every L in Π. Then there exists a sequence $\{L_n\}_{n=0}^\infty$ of elements of Π such that $L_{n+1} < L_n, L_n \ne S$ for each n and L_n converges strongly to S. There exist finitely many $P > Q$ such that $n_P(A) \ne 0$. Put $m = \sum_{P>S} n_P(A)$. Then for each n we have

$$dim(((L_n)_-)^\perp \ominus (((L_n)_-)^\perp \cap (\sum_{P>S} \oplus UK_P)))$$

$$\le dim(((L_n)_-)^\perp \ominus (\sum_{P\ge P_n} \oplus UK_P)) \le m.$$

It follows then from Lemma 4 that $dim(S^\perp \ominus (\sum_{P>S} \oplus UK_P)) \le m$. Let π be the canonical projection from $(S_-)^\perp$ onto $(S_-)^\perp / (\sum_{P>S} \oplus UK_P)$. Proceeding as in (i), we see that

$$dim((S_-)^\perp \ominus (\sum_{P\ge S} \oplus UK_P)) \le \sum_{P\ge S} n_P(A).$$

Hence S is in Π, which is contrary to our assumption.

We conclude that Σ is empty. □

If the nest N has the property $I = I_-$, condition b) of Theorem 5 says that $A^{-1}(PrN)$ is dense in H . In this particular case this condition is essentially the same as the density condition given in Theorem 3.1 in [1] in a different but related context.

Corollary 6. *Let A be a positive operator in $B(H)$. Then A admits a right factorisation with respect to N if and only if* $\sum_{P>Q} n_P(A^{\frac{1}{2}}) \ge n(A^{\frac{1}{2}})$.

Let $R = \overline{\bigcap_{P\in N, P=P_+} P}$. Then it is easy to see that $R = R_+$ and that for every P in N, $P < R$, we have $P \ne P_+$. Let A be an operator in $B(H)$. We set

$$m(A) = dim(\bigcap_{P>R} \overline{r(AP)} \ominus \overline{r(AR)})$$

if $R \ne H$; $m(A) = 0$ if $R = I$.

Let P be in N, $P < R$. We set: $m_P(A) = +\infty$, if $dim(P_+ \ominus P) = +\infty$, and

$$m_P(A) = dim(P_+ \ominus P) - dim(\overline{r(AP_+)} \ominus \overline{r(AP)})$$

otherwise. Note that $m_P(A) \ge 0$.

Corollary 7. *Let A be an operator in $B(H)$. The following are equivalent:*

a) There exists an operator B in $AlgN$ such that $A^*A = B^*B$.

b) $\sum_{P<R} m_P(A) \ge m(A)$.

One should note that if the nest N is of order type $\mathbb{Z}$ condition b) of Corollary 7 says that $\bigcap\limits_{P>0} \overline{r(AP)} = \{0\}$. In this particular case this condition is the same as the one given by Arveson in Theorem 3.3. of [4]. A condition of the same type has been given by Lowdenslager in Theorem 1 of [16] in the context of factorisation of operator functions.

Corollary 8. *Let A be a positive operator in $B(H)$. Then A admits a left factorisation with respect to N if and only if* $\sum\limits_{P<R} m_P(A^{\frac{1}{2}}) \geq m(A^{\frac{1}{2}})$.

3. Outer factorisation and universally factorable operators

Let N be a nest on H. An operator A in $AlgN$ is called outer if its range projection commutes with N and $r(BP)$ is dense in $r(B) \cap P$ for every P in N. Outer operators were introduced by Arveson in [4] in analogy with outer functions in Hardy spaces. Theorem 3.3 in [4] gives a necessary and sufficient condition on a positive operator X for the existence of an outer operator A in $AlgN$ satisfying $X = A^*A$ under the assumption that the nest N is of a certain order type. In [20] Power proves that for every positive operator X in $B(H)$ there exists an outer operator A in $AlgN$ satisfying $X = A^*A$ if and only if the nest N is well ordered. In what follows we give a proof of the result of Power based on the ideas of Section 2. Note that a nest N is well ordered if and only if $P \neq P_+$ for every P in N, $P \neq I$.

Lemma 9. *Let N be a well-ordered nest on H and A be an operator in $B(H)$. Let P_0 be in N. Then*

$$\sum_{P\in N, P<P_0} \oplus(\overline{r(AP_+)} \ominus \overline{r(AP)}) = \overline{r(AP_0)}.$$

Proof. It is clear that $\sum\limits_{P\in N, P<P_0} \oplus(\overline{r(AP_+)} \ominus \overline{r(AP)})$ is contained in $\overline{r(AP_0)}$. Let x be in $\overline{r(AP_0)}$, $x \neq 0$, and assume that x is orthogonal to

$$\sum_{P\in N, P<P_0} \oplus(\overline{r(AP_+)} \ominus \overline{r(AP)}).$$

Put $S = sup\{L \in N : x \text{ is orthogonal to } \overline{r(AL)}\}$. Then since $x \neq 0$ we get $S < P_0$. Since x is orthogonal to $(\overline{r(AS_+)} \ominus \overline{r(AS)})$ we conclude that x is orthogonal to $\overline{r(AS_+)}$, which is contrary to the definition of S. Therefore

$$\sum_{P\in N, P<P_0} \oplus(\overline{r(AP_+)} \ominus \overline{r(AP)}) = \overline{r(AP_0)}. \quad \square$$

Theorem 10. *a) Let N be a well-ordered nest on H. Let X be a positive operator in $B(H)$. Then there exists an outer operator B in $AlgN$ such that $X = B^*B$. Moreover, B belongs to the von Neumann algebra generated by X and the nest N.*

b) Let N be a nest on H. Assume that for every positive operator X in $B(H)$ there exists an outer operator B in $AlgN$ such that $X = B^*B$. Then N is well-ordered.

Proof. a) Put $A = X^{\frac{1}{2}}$. Let P be in N. We denote by M_P the orthogonal projection on $\overline{r(AP_+)} \ominus \overline{r(AP)}$. We set $A_P = M_P A(P_+ - P)$. We show that $r(A_P)$ contains

$r(AP_+) \cap r(AP)^\perp$. Let y be in $r(AP_+) \cap r(AP)^\perp$. Then there exists x in P_+ such that $y = Ax$. We have $y = A(P_+ - P)x + APx$, and therefore

$$M_P y = M_P A(P_+ - P)x + M_P APx.$$

Now since APx is contained in $r(AP)$ we have $M_P APx = 0$. Hence $y = M_P y = M_P A(P_+ - P)x$ and y is in $r(A_P)$. Let $V_P\,|A_P|$ be the polar decomposition of A_P. Then V_P is a partial isometry with domain contained in $P_+ \ominus P$ and range $\overline{r(A_P)}$ which is equal to $\overline{r(AP_+)} \ominus \overline{r(AP)}$. Put $V = \sum_{P \in N} V_P$. We have

$$VV^* = \sum_{P \in N} \oplus(\overline{r(AP_+)} \ominus \overline{r(AP)}),$$

which is equal to $\overline{r(A)}$ by Lemma 9. Therefore $A = VV^*A$. We set $B = V^*A$. Then $X = B^*B$. Since M_P lies in the von Neumann algebra generated by X and the nest N, the same holds for the operators V_P and hence also for V. We conclude that B belongs to the von Neumann algebra generated by X and the nest N. To finish the proof we have to show that B is outer and lies in $AlgN$. The range of $(V_P)^*A$ is contained in $P_+ \ominus P$, and hence the range projection of $(V_P)^*A$ commutes with N for every P. It follows that the range projection of B commutes with N. Let P_0 be in N. We will show now that $r(BP_0)$ is dense in $r(B) \cap P_0$. Let y be in $r(B) \cap P_0$. Then $y = Bx$ for some x in H. We have $y = V^*Ax$. Then $Vy = Ax$ and Ax is in VP_0, which is equal to

$$\sum_{P \in N, P < P_0} \oplus(\overline{r(AP_+)} \ominus \overline{r(AP)}).$$

By Lemma 9, $\sum_{P \in N, P < P_0} \oplus(\overline{r(AP_+)} \ominus \overline{r(AP)})$ is equal to $\overline{r(AP_0)}$. It follows that V^*Ax is in $V^*\overline{r(AP_0)}$. But $V^*\overline{r(AP_0)}$ is contained in $\overline{r(V^*AP_0)} = \overline{r(BP_0)}$. Therefore y is in $\overline{r(BP_0)}$ and $r(B) \cap P_0$ is contained in $\overline{r(BP_0)}$. We conclude that B is an outer operator in $AlgN$.

b) Assume that N is not well-ordered. Then there exists P_0 in N such that $P_0 = (P_0)_+$. Let ξ be a unit vector in H such that $(P_0)\xi = 0$ and $P\xi \neq 0$ for every P in N, $P > P_0$. Put $X = \xi \otimes \xi$. Assume there exists an outer operator B in $AlgN$ such that at $X = B^*B$. The operator B must be a rank one operator. So there exist vectors x,y in H such that $B = x \otimes y$. We have $\xi \otimes \xi = \langle y,y\rangle x \otimes x$, and hence x is a multiple of ξ. It follows from the characterisation of the rank-one operators in $AlgN$ given in [22] that y belongs to P_0. But now we have $r(BP_0) = \{0\}$ and $r(B) \cap P_0 = [y]$. We conclude that B cannot be outer. □

Now we are going to characterise the positive operators A in $B(H)$ which admit a right factorisation (resp. a left factorisation) with respect to any nest N. We use some results from Fredholm theory which may be found in [5]. As in Section 2, if N is a nest we denote by Q the element of N defined by $Q = \overline{\bigcup_{P \in N, P = P_-} P}$.

Theorem 11. *Let A be an operator in $B(H)$.*

a) There exists for every nest N an operator B_N in $AlgN$ satisfying $AA^* = B_N B_N^*$ if and only if A is a right Fredholm operator.

b) There exists for every nest N an operator B_N in $AlgN$ satisfying $A^*A = B_N^* B_N$ if and only if A is a left Fredholm operator.

Proof. a) Assume that A is a right Fredholm operator. Then $r(A)$ is closed and of co-finite dimension in H. Then $r(A) \cap Q$ is of co-finite dimension in Q. It follows from [2, Prop. 4] that $\bigcup_{P<Q} P \cap r(A)$ is dense in $r(A) \cap Q$. Since the restriction of A onto $coker A$ is an isomorphism from $coker A$ onto $r(A)$, we see that the set $A^{-1}(\bigcup_{P<Q} P \cap r(A)) \cap coker A$ is dense in $A^{-1}(Q \cap r(A)) \cap coker A$. We have

$$A^{-1}(\bigcup_{P<Q} P \cap r(A)) \cap coker A = A^{-1}(\bigcup_{P<Q} P) \cap coker A$$

and

$$A^{-1}(Q \cap r(A)) \cap coker A = A^{-1}(Q) \cap coker A.$$

Since

$$A^{-1}(Q) \ominus \overline{A^{-1}(\bigcup_{P<Q} P)} = (A^{-1}(Q) \cap coker A) \ominus \overline{(A^{-1}(\bigcup_{P<Q} P) \cap coker A)},$$

we conclude that $n(A) = 0$. It then follows from Theorem 5 that for every nest N there exists an operator B_N in $AlgN$ satisfying $AA^* = B_N B_N^*$.

Assume that for every nest N there exists an operator B_N in $AlgN$ satisfying $AA^* = B_N B_N^*$. It follows from [2, Th. 15] that A is a right Fredholm operator.

b) Assume that A is a left Fredholm operator. Then A^* is a right Fredholm operator. Let N be a nest. It follows from a) that there exists an operator $C_{N^\perp}$ in $AlgN^\perp$ satisfying $A^*A = C_{N^\perp} C_{N^\perp}^*$. Put $B_N = C_{N^\perp}^*$. Then B_N is in $AlgN$ and satisfies $A^*A = B_N^* B_N$.

Assume that for every nest N there exists an operator B_N in $AlgN$ satisfying $A^*A = B_N^* B_N$. It follows then from [2, Th. 15] that A is a left Fredholm operator. □

Corollary 12. *Let A be a positive operator in $B(H)$. Then A admits a right factorisation (a left factorisation) with respect to every nest N if and only if A is a Fredholm operator.*

Proof. It follows from Theorem 11 that A admits a right factorisation with respect to every nest N if and only if $A^{\frac{1}{2}}$ is a right Fredholm operator. Since $A^{\frac{1}{2}}$ is selfadjoint, $A^{\frac{1}{2}}$ is a right Fredholm operator if and only if it is a Fredholm operator. But $A^{\frac{1}{2}}$ is a Fredholm operator if and only if A is a Fredholm operator.

The other assertion is proved in the same way. □

4. Factorisation and ranges of operators

Let N be a nest on H. We set $OR(N) = \{W : W = r(X) \text{ for some } X \text{ in } AlgN\}$. Let A be an operator in $B(H)$. Assume that there exists an operator B in $AlgN$ such that $AA^* = BB^*$. Then $r(A) = r(B)$ by [10], and so $r(A)$ is in $OR(N)$. In Theorem 13 below we show that this condition is also sufficient for the existence of an operator B in $AlgN$ satisfying $AA^* = BB^*$. As in Section 2, we denote by Q the element of N defined by $Q = \overline{\bigcup_{P \in N, P = P_-} P}$.

Theorem 13. *Let N be a nest on H. Let A be an operator in $B(H)$. The following are equivalent:*

a) There exists an operator B in $AlgN$ such that $AA^* = BB^*$.

b) $r(A)$ is in $OR(N)$.

Proof. It is clear that a) implies b).

b) implies a): Let C be an operator in $AlgN$ such that $r(A) = r(C)$. The operator C clearly satisfies condition b) of Theorem 5. We prove that so does A. For this it suffices to show that $n(A) = n(C)$ and $n_P(A) = n_P(C)$ for every $P > Q$ such that $dim(P \ominus P_-) < +\infty$. It follows from [10] that there exists operators X and Y in $B(H)$ such that i) $cokerX = cokerA$, $r(X) = cokerC$ and $A = CX$, ii) $cokerY = cokerC$, $r(Y) = cokerA$ and $C = AY$. Then it is easy to see that $XYx = x$ for every x in $cokerC$ and $YXx = x$ for every x in $cokerA$.

We prove first that $n_P(A) = n_P(C)$ for every $P > Q$ such that $dim(P \ominus P_-) < +\infty$. Let P be an element of N such that $P > Q$ and $dim(P \ominus P_-) < +\infty$, and π be the canonical projection from $C^{-1}(P)$ onto $C^{-1}(P)/C^{-1}(P_-)$. Let x be in $A^{-1}(P) \ominus A^{-1}(P_-)$. Then it is easy to see that Xx is in $C^{-1}(P)$. We are going to show that the linear map from $A^{-1}(P) \ominus A^{-1}(P_-)$ to $C^{-1}(P)/C^{-1}(P_-)$ defined by $x \to \pi(Xx)$ is injective. In fact, if $\pi(Xx) = 0$, then Xx belongs to $C^{-1}(P_-)$ and $Ax = CXx$ belongs to P_-. This implies that x is in $A^{-1}(P_-)$, and hence it is 0. It follows that $dim(A^{-1}(P) \ominus A^{-1}(P_-)) \leq dim(C^{-1}(P) \ominus C^{-1}(P_-))$ and hence $n_P(C) \leq n_P(A)$. A similar argument proves that $n_P(A) \leq n_P(C)$.We conclude that $n_P(C) = n_P(A)$ for every P in N such that $P > Q$ and $dim(P \ominus P_-) < +\infty$.

We show now that $n(A) = n(C)$. Let π be the canonical projection from $C^{-1}(Q)$ onto $C^{-1}(Q)/\overline{C^{-1}(PrQN)}$. Let x be in $A^{-1}(Q) \ominus \overline{A^{-1}(PrQN)}$. Then it is easy to see that Xx is in $C^{-1}(Q)$. We are going to show that the linear map from $A^{-1}(Q) \ominus \overline{A^{-1}(PrQN)}$ to $C^{-1}(Q)/\overline{C^{-1}(PrQN)}$ defined by $x \to \pi(Xx)$ is injective. Assume $\pi(Xx) = 0$. Then Xx is in $\overline{C^{-1}(PrQN)}$, and consequently there exists a sequence w_n in $C^{-1}(PrQN)$ converging to Xx. Therefore the sequence Yw_n converges to YXx, which is equal to x. Since w_n is in $C^{-1}(PrQN)$, Cw_n is in $Pr(QN)$. Since $AYw_n = Cw_n$, we conclude that Yw_n is in $A^{-1}(PrQN)$. Therefore x, being the limit of Yw_n, is in $\overline{A^{-1}(PrQN)}$. Hence $x = 0$. It follows that

$$dim(A^{-1}(Q) \ominus \overline{A^{-1}(PrQN)}) \leq dim(C^{-1}(Q) \ominus \overline{C^{-1}(PrQN)}).$$

A similar argument shows that

$$dim(C^{-1}(Q) \ominus \overline{C^{-1}(PrQN)}) \leq dim(A^{-1}(Q) \ominus \overline{A^{-1}(PrQN)}).$$

We conclude that $n(A) = n(C)$.

It follows now from Theorem 5 that there exists an operator B in $AlgN$ such that $AA^* = BB^*$. □

Corollary 14. *Let N be a nest on H. Let A and C be operators in $B(H)$. We assume that $r(A) = r(C)$. Then there exists an operator B in $AlgN$ such that $AA^* = BB^*$ if and only if there exists an operator D in $AlgN$ such that $CC^* = DD^*$.*

Proof. This follows from Theorem 13. □

Corollary 15. *Let N be a nest on H. Let A and C be positive operators in $B(H)$. We assume that $r(A) = r(C)$. Then:*

a) The operator A is in $F_r(N)$ if and only if the operator C is in $F_r(N)$.

b) The operator A is in $F_l(N)$ if and only if the operator C is in $F_l(N)$.

Proof. a) Theorem 1 in [10] implies that there exist positive numbers λ and μ such that $A^2 \leq \lambda C^2 \leq \mu A^2$. It follows then from [14, Prop. 4.2.8.] that $A \leq \lambda^{\frac{1}{2}} C \leq \mu^{\frac{1}{2}} A$.

Hence, using again Theorem 1 of [10], we obtain $r(A^{\frac{1}{2}}) = r(C^{\frac{1}{2}})$. Now the assertion follows from Corollary 14.

b) follows from a) and the fact that $F_l(N) = F_r(N^{\perp})$. ☐

In the rest of this section we study the set $OR(N)$. The importance of this set emerges from Theorem 13.

Let $\mathcal{S}$ be a set of subspaces of H containing $\{0\}$ and H. We say that $\mathcal{S}$ is a join semi-lattice if whenever V and W are in $\mathcal{S}$, $V + W$ is in $\mathcal{S}$. We say that $\mathcal{S}$ is a meet semi-lattice if whenever V and W are in $\mathcal{S}$, $V \cap W$ is in $\mathcal{S}$. If $\mathcal{S}$ is a join semi-lattice and a meet semi-lattice we say that it is a lattice. When N is the trivial nest consisting of the subspaces $\{0\}$ and H, the set $OR(N)$ is the set of ranges of operators in $B(H)$. In this case it was shown by Dixmier that $OR(N)$ is a lattice [8]. A proof of this result may also be found in [11] or [12]. In what follows we characterise the nests N for which $OR(N)$ is a join or a meet semi-lattice.

Proposition 16. *Let N be a nest and W a linear subspace of H. The following are equivalent:*

a) The subspace W is in $OR(N)$.

b) There exists an operator A in $B(H)$ with $r(A) = W$ which satisfies condition b) of Theorem 5.

c) Every operator A in $B(H)$ with $r(A) = W$ satisfies condition b) of Theorem 5.

Proof. It follows from Theorem 11 and Theorem 5 that a) implies c).

It is clear that c) implies b).

b) implies a) by Theorem 5 and Theorem 13. ☐

Corollary 17. *Let N be a nest. We assume that one of the following holds:*

a) $Q = 0$.

b) There exist P in N, $P > Q$, such that $dim(P \ominus P_{-}) = +\infty$.

Then $OR(N)$ is the set of ranges of operators in $B(H)$. In particular, $OR(N)$ is a lattice.

Proof. Let A be an operator in $B(H)$. Then A satisfies condition b) of Theorem 5, and it follows from Proposition 16 that $r(A)$ is in $OR(N)$. ☐

Corollary 18. *Let N be a nest. Assume that $Q \neq 0$ and that for every P in $N, P > Q$, we have $dim(P \ominus P_{-}) < +\infty$. Let ξ be a vector in Q which is not QN-proper. There is no operator in $AlgN$ with range $Q^{\perp} + [\xi]$.*

Proof. Put $B = Q^{\perp} + \xi \otimes \xi$. It is clear that $r(B) = Q^{\perp} + [\xi]$. It follows from Proposition 16 that B must satisfy condition b) of Theorem 5. But an easy calculation shows that $\sum\limits_{P>Q} n_P(B) = 0$ and $n(B) = 1$. The conclusion follows. ☐

Proposition 19. *Let N be a nest such that $I = I_{-}$. Then there exist A, C in $AlgN$ such that $r(A) \cap r(C)$ is not the range of any operator in $AlgN$.*

Proof. A necessary and sufficient condition for a rank one operator $\eta \otimes \xi$ to lie in $AlgN$ is that there exists P in N such that ξ is in P and η is in $(P_{-})^{\perp}$ [22]. Therefore, if ξ is a vector in H which is not N-proper, there is no operator in $AlgN$ with range $[\xi]$. So it suffices to construct operators A and C in $AlgN$ such that $r(A) \cap r(C) = [\xi]$ and ξ is not N-proper.

Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of elements of N such that $P_0 = 0$, $P_{n+1} > P_n, P_n \neq I$, $dim(P_{n+1} \ominus P_n) \geq 2$ for each n, and P_n converges strongly to I. We consider unit vectors e_n, f_n in $P_{n+1} \ominus P_n$ with e_n orthogonal to f_n. We define $w_0 = f_0$, $w_1 = 2^{-1}f_1 - f_0$, $w_n = 2^{-n}f_n - (\sum_{i=0}^{n-1} 2^{-i}f_i)$ for each n. We consider the operators $A = \sum_{n=1}^{\infty} e_n \otimes e_{n-1}$ and $B = \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} e_n \otimes w_{n-1}$. It is clear that A and B are in $AlgN$. We set $C = A+B$, and we will prove that A and C are as required. Let x be in $r(A) \cap r(C)$. Then there exist y and z in H such that $x = Ay = Cz = Az + Bz$. We have $Bz = A(y-z)$, and since $r(A)$ and $r(B)$ are orthogonal we obtain that $Bz = 0$ and z is in $kerB$. It follows that $x = Ay = Az$, and so x is in $A(kerB)$. Conversely, assume that w is in $kerB$. We have $Aw = (A+B)w = Cw$, and so $A(kerB)$ is contained in $r(A) \cap r(C)$. Thus $r(A) \cap r(C) = A(kerB)$. We are going to show that the subspace $A(kerB)$ is spanned by a vector which is not N-proper. Let z be in $kerB$. Then $z = \sum_{n=1}^{\infty} z_n e_n + z_0 e_0 + r$, where z_n are complex numbers for $n = 0, 1, 2, ...$ and r is in $[\{e_n : n = 0, 1, 2, ...\}]^{\perp}$. Since $Bz = 0$ we obtain $\sum_{n=1}^{\infty} 2^{-\frac{n}{2}} z_n w_{n-1} = 0$. Now if $\sum_{n=0}^{\infty} \lambda_n w_n = 0$, then taking scalar products with f_0, f_1, ... successively, we find that $\lambda_0 = \sum_{k=1}^{\infty} \lambda_k$, $\lambda_1 = \sum_{k=2}^{\infty} \lambda_k$, ..., $\lambda_n = \sum_{k=n+1}^{\infty} \lambda_k$ for each n, and hence $\lambda_0 - \lambda_1 = \lambda_1$, $\lambda_1 - \lambda_2 = \lambda_2$, ...$\lambda_n - \lambda_{n+1} = \lambda_{n+1}$ for each n. Therefore $\lambda_n = 2^{-n}\lambda_0$. It follows that $z_{n+1} = 2^{-\frac{n}{2}} z_1$ for $n = 1, 2, ...$ We conclude that

$$z = \sum_{n=1}^{\infty} z_n e_n + z_0 e_0 + r = z_1 (\sum_{n=1}^{\infty} 2^{-\frac{n-1}{2}} e_n) + z_0 e_0 + r.$$

Since $z_0 e_0 + r$ is in $kerA$, $Az = z_1 (\sum_{n=1}^{\infty} 2^{-\frac{n-1}{2}} e_{n-1})$. So the subspace $A(kerB)$ is spanned by the vector $\sum_{n=1}^{\infty} 2^{-\frac{n-1}{2}} e_{n-1}$, which is not N-proper. We conclude that $A(kerB)$ cannot be the range of any operator in $AlgN$. □

Theorem 20. *Let N be a nest. Then $OR(N)$ is a meet semi-lattice if and only if one of the following cases occurs:*

a) $Q = 0$.

b) There exist P in N, $P > Q$, such that $dim(P \ominus P_-) = +\infty$.

Proof. Suppose that $Q \neq 0$ and for every P in N, $P > Q$, we have $dim(P \ominus P_-) < +\infty$. We will show that $OR(N)$ is not a meet semi-lattice. It follows from Proposition 19 that there exist operators A and C in $AlgQN$ such that $r(A) \cap r(C) = [\xi]$ and ξ is not QN-proper. We define $A_1 = Q^{\perp} + A$, $C_1 = Q^{\perp} + C$. Then A_1 and C_1 are in $AlgN$ and $r(A_1) \cap r(C_1) = Q^{\perp} + [\xi]$. By Corollary 18, $Q^{\perp} + [\xi]$ is not in $OR(N)$.

Assume now that a) or b) holds. It follows from Corollary 17 that $OR(N)$ is the set of ranges of operators in $B(H)$, which is a lattice. □

Lemma 21. *Let N be a nest such that $I = I_-$. Then there exist partial isometries U_1 and U_2 in $AlgN$ with orthogonal domains and such that $r(U_1) = r(U_2) = H$.*

Proof. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of elements of N such that: $P_0 = 0$, $P_{n+1} > P_n, P_n \neq I$ for each n, and P_n converges strongly to I. It follows from Lemma 1 that there exists a sequence $\{M_n\}_{n=1}^{\infty}$ of closed mutually orthogonal infinite dimensional subspaces of H such that $M_n \subset (P_n)^\perp$ for every n. Let V_0 be a partial isometry with domain contained in M_1 and range P_1. For each $n \geq 1$, consider a partial isometry V_n with domain contained in M_{2n+1} and range $P_{2n+1} \ominus P_{2n-1}$. Put $U_1 = \sum_{n=0}^{\infty} V_n$. For each $n \geq 2$, consider a partial isometry W_n with domain contained in M_{2n} and range $P_{2n} \ominus P_{2n-2}$. Put $U_2 = \sum_{n=1}^{\infty} W_n$. □

Theorem 22. *Let N be a nest. Then $OR(N)$ is a join semi-lattice if and only if one of the following three cases occurs:*

a) $Q = 0$.

b) $Q = I$.

c) There exist P in N , $P > Q$, such that $dim(P \ominus P_-) = +\infty$.

Proof. Assume first that a) or c) holds. It follows from Corollary 17 that $OR(N)$ is a lattice. Assume that b) holds. Let W_1, W_2 be in $OR(N)$. Then $W_1 = r(A_1)$, $W_2 = r(A_2)$ for some operators A_1, A_2 in $AlgN$. It follows from Lemma 21 that there exist partial isometries U_1 and U_2 in $AlgN$ with orthogonal domains and such that $r(U_1) = r(U_2) = H$. We set $B = A_1U_1 + A_2U_2$. Then B is in $AlgN$ and $r(B) = W_1 + W_2$.

Assume now that $Q \neq 0$, $Q \neq I$, and for every P in N, $P > Q$, we have $dim(P \ominus P_-) < +\infty$. We will show that $OR(N)$ is not a join semi-lattice. Put $A = Q^\perp$ and $B = e \otimes \xi$, where e is in $Q^\perp$ and ξ is a vector in Q which is not QN-proper. Then A and B are in $AlgN$ and $r(A) + r(B) = Q^\perp + [\xi]$. By Corollary 18 there is no operator in $AlgN$ with range $Q^\perp + [\xi]$. □

5. Nests with $I = I_-$

Let N be a nest on H. In this section we obtain information about the set $F_r(N)$

Let A and C be positive operators in $B(H)$. Consider the following condition:

(i) There exists a positive number λ such that $A \leq \lambda C$.

Put $A_1 = A^{\frac{1}{2}}$, $C_1 = C^{\frac{1}{2}}$. By [10, Th.1] condition (i) is equivalent to the following condition:

(ii) There exists an operator X in $B(H)$ such that $A_1 = C_1X$, $cokerX = cokerA_1$ and $r(X)$ is contained in $cokerC_1$.

We are interested in the following question:

Question: Assume (i) holds. Assume that A is in $F_r(N)$. Is it true that C is in $F_r(N)$?

The above question in the generality stated has a negative answer. However if we assume moreover that the nest N satisfies $I = I_-$ and that the range of the operator X is dense in $cokerC_1$, the answer is positive. This is shown in Theorem 23, below.

A similar question has been considered by Lowdenslager in [16] and by Douglas in [9] in the context of factorisation of operator functions. The results obtained there are used to prove Devinatz's Theorem [7].

The condition : "the range of the operator X is dense in $cokerC_1$" is implicit in the work of Douglas [9]. Theorem 23 and Corollaries 24 and 25 below are motivated by that paper. We remark that Theorem 23 improves Corollary 14 in the case of a nest with the property $I = I_-$.

Theorem 23. *Let A and C be operators in $B(H)$. We assume that there exists an operator X in $B(H)$ such that $A = CX$, $cokerX = cokerA$ and $r(X)$ is contained in $cokerC$. Then the following are equivalent:*

a) $r(X)$ is dense in $cokerC$.

b) i) $\overline{r(A)} = \overline{r(C)}$.

ii) Let N be a nest on H such that $I = I_-$. Assume that there exists an operator B in $AlgN$ such that $AA^* = BB^*$. Then there exists an operator D in $AlgN$ such that $CC^* = DD^*$.

Proof. a) implies b).

i) We have $A^* = X^*C^*$, and since $r(X)$ is dense in $cokerC$ we obtain that $KerA^* = KerC^*$. It follows that $\overline{r(A)} = \overline{r(C)}$.

ii) The subspace $A^{-1}(PrN)$ is dense in H by Theorem 5. It follows that $X(A^{-1}(PrN))$ is dense in $r(X)$ and hence in $cokerC$. But $C^{-1}(PrN)$ contains $X(A^{-1}(PrN)) + kerC$, which is dense in $cokerC + kerC = H$. Again by Theorem 5 we conclude that there exists an operator D in $AlgN$ such that $CC^* = DD^*$.

b) implies a).

Assume first that $cokerA$ is of finite dimension. Then $dimr(A) = dimr(X)$ and by condition i) $dimr(A) = dimr(C)$. So $dimr(X) = dimr(C)$, and since $r(X)$ is contained in $cokerC$ we have $r(X) = cokerC$. Assume now that $cokerA$ is of infinite dimension and that $r(X)$ is not dense in $cokerC$. We are going to construct a nest N on H with the property $I = I_-$ and such that AA^* is in $F_r(N)$ and CC^* is not in $F_r(N)$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $cokerA$. We set $P_n = [e_m : m \le n]$ for $n = 1, 2, ...$. We set $Q_n = r(A)^{\perp} + AP_n$. Let N be the nest $\{Q_n : n = 1, 2, ...\} \cup \{H, \{0\}\}$. It is clear that N satisfies $I = I_-$. Now $A^{-1}(Q_n) = KerA + P_n$ for $n = 1, 2, ...$, and so $A^{-1}(PrN)$ being equal to $KerA + \bigcup_{n=1}^{\infty} P_n$, is dense in H. It follows from Theorem 5 that AA^* is in $F_r(N)$. Put $R_n = X(P_n)$ for $n = 1, 2, ...$. We have $Q_n = r(A)^{\perp} + CR_n$, which by condition i) is equal to $r(C)^{\perp} + CR_n$, and so $C^{-1}(Q_n) = KerC + R_n$ for $n = 1, 2, ...$. It follows that $C^{-1}(PrN) = KerC + \bigcup_{n=1}^{\infty} R_n$. But then $C^{-1}(PrN)$ is contained in $KerC + r(X)$. Since $r(X)$ is not dense in $cokerC$, we see that $C^{-1}(PrN)$ is not dense in H. Hence, by Theorem 5, CC^* is not in $F_r(N)$. □

Note that condition b)i) does not imply condition a). One can see that in the example constructed by Douglas in [9, p.120]. Also it is easy to see that condition b)ii) does not imply condition a). In fact, let A be an operator in $B(H)$ such that $r(A)$ is not dense in H. Put $X = A$, and $C = I$. Then we have $A = CX$, $cokerX = cokerA$ and $r(X)$ is contained in $cokerC$. Clearly b)ii) is satisfied. But since $r(A)$ is not dense in H, condition a) is not satisfied.

Theorem 23 is not valid for general nests. This is shown in the following example.

Example 1. Let H_1 be a Hilbert space and N_1 be a nest in H_1 such that $I = I_-$. Take a unit vector ξ in H which is not N-proper. Put $A = \xi \otimes \xi$. Then $A^{-1}(PrN_1) = [\xi]^{\perp}$, and hence $A^{-1}(PrN_1)$ has codimension one in H_1.

Let H_2 be a Hilbert space and $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis of H_2. Set $\eta = \sum_{n=1}^{\infty} n^{-1}e_n$ and define the operator B by $B = \sum_{n=1}^{\infty} n^{-1}e_n \otimes e_n$. Then $r(B)$ is dense in H_2. Take an orthonormal basis $\{f_n\}_{n=1}^{\infty}$ of $[\eta]^{\perp}$. Put $P_m = [f_1, f_2, ..., f_m]$ for

$m = 1, 2, ...$, and consider the nest

$$N_2 = \{[\eta] + (P_m)^{\perp} : m = 1, 2, ...\} \cup \{[\eta], \{0\}, H\}.$$

Note that $B^{-1}([\eta]) = \{0\}$.

Put $H = H_1 \oplus H_2$ and define a nest N on H by $N = N_1 \cup \{H_1 \oplus P\colon\ P \in N_2\}$. Note that $\overline{\bigcup_{P \in N, P = P_-} P} = H_1$.

Define operators X and Y in $B(H)$ by $Xx = Ax$ if x is in H_1, $Xx = Bx$ if x is in H_2, $Yx = Ax$ if x is in H_1, $Yx = x$ if x is in H_2. Then $X = YX$, and $r(X)$ is dense in $cokerY$. Now put $R = H_1 + [\eta]$. Then $n_R(X) = 1$ and $n(X) = 1$. It now follows from Theorem 5 that XX^* is in $F_r(N)$. On the other hand, for every S in N, $S > H_1$, we have $n_S(Y) = 0$ and $n(Y) = 1$. It follows from Theorem 5 that YY^* is not in $F_r(N)$. Hence Theorem 23 does not hold for general nests.

Theorem 23 has some useful corollaries.

Corollary 24. *Let N be a nest on H such that $I = I_-$. Let A and C be positive operators in $B(H)$. Put $A_1 = A^{\frac{1}{2}}$, $C_1 = C^{\frac{1}{2}}$. Assume that:*

i) There exists a positive number λ such that $A \leq \lambda C$.

ii) $KerA = KerC$.

iii) There exists a positive number μ such that $A \geq \mu C_1 A C_1$.

Then if A is in $F_r(N)$, C is in $F_r(N)$.

Proof. It follows from [10, Th.1] that there exists an operator X in $B(H)$ such that $A_1 = C_1 X$, $cokerX = cokerA_1$ and $r(X)$ contained in $cokerC_1$. It follows from Theorem 23 that in order to prove the assertion it suffices to show that $r(X)$ is dense in $cokerC_1$. Assume the contrary. Since $kerX^* = r(X)^{\perp}$, there exists a non-zero vector y in $kerX^* \cap cokerC_1$. Since $cokerC_1 = \overline{r(C_1)}$, there exists a sequence $\{y_n\colon\ n = 1, 2, ...\}$ such that $C_1 y_n$ converges to y. Then the sequence $\{A_1 y_n\colon\ n = 1, 2, ...\}$ converges to 0, since $A_1 y_n = X^* C_1 y_n$ and y is in $kerX^*$. We have $\langle Ay_n, y_n\rangle \geq \langle \mu C_1 A C_1 y_n, y_n\rangle$, and hence

$$\langle A_1 y_n, A_1 y_n\rangle \geq \mu \langle A_1 C_1 y_n, A_1 C_1 y_n\rangle$$

for $n = 1, 2, ...$. Taking limits we find that $0 \geq \mu\langle A_1 y, A_1 y\rangle$, which implies that $A_1 y = 0$. This is a contradiction, since y belongs to $cokerC_1 = cokerA_1$. Hence $r(X)$ is dense in $cokerC_1$. □

Corollary 25. *Let N be a nest on H such that $I = I_-$. Let A and C be positive operators in $B(H)$. Assume that:*

a) There exists a positive number λ such that $A \leq \lambda C$.

b) $KerA = KerC$.

c) A and C commute.

Then if A is in $F_r(N)$, C is in $F_r(N)$.

Proof. Put $\mu = \|C\|^{-1}$. The conclusion follows from Corollary 24. □

The following corollary answers a question posed by Shields in [23].

Corollary 26. *Let N be a nest on H such that $I = I_-$. Let A be a positive operator in $B(H)$ and $0 < \lambda \leq 1$. Then if A is in $F_r(N)$, A^{λ} is in $F_r(N)$.*

Proof. Without loss of generality we may assume that A is a contraction. The conclusion then follows from Corollary 25. □

Corollary 26 does not hold if we assume $\lambda \geq 1$. This is shown in the following example.

Example 2. We are going to show that there exist a nest N such that $I = I_-$ and a positive operator B in $B(H)$ with the following properties:

a) B is in $F_r(N)$.

b) B^2 is not in $F_r(N)$.

Let H be a Hilbert space and $\{e_n\}_{n=0}^{\infty}$ an orthonormal basis of H. We set $P_n = [e_m : m \leq n]$ for $n = 0, 1, 2, \ldots$. Let N be the nest $\{P_n : n = 0, 1, 2, \ldots\} \cup \{H, \{0\}\}$. Set $\xi = \sum_{n=1}^{\infty} n^{-1} e_n$ and $A = \sum_{n=1}^{\infty} n^{-1} e_n \otimes e_n + \xi \otimes e_0$. Put $B = AA^*$. Then $B = \sum_{n=1}^{\infty} n^{-2} e_n \otimes e_n + \psi \otimes e_0 + e_0 \otimes \psi + c e_0 \otimes e_0$, where $\psi = \sum_{n=1}^{\infty} n^{-2} e_n$ and c is a positive number. The set $A^{-1}(PrN\ )$ is dense in H, because it contains e_n for $n = 0, 1, 2, \ldots$. It follows from Theorem 5 that B is in $F_r(N)$. Let $x = \sum_{n=0}^{\infty} x_n e_n$ be in $B^{-1}(PrN\ )$. Then Bx is N-proper. The coefficient of e_n in Bx is $n^{-2}(x_n + x_0)$. Since Bx is N-proper, there exists n_0 such that $x_n + x_0 = 0$ for $n \geq n_0$. This implies that $x_0 = 0$, and so $B^{-1}(PrN)$ is orthogonal to e_0. It follows from Theorem 5 that B^2 is not in $F_r(N)$.

Proposition 27. *Let N be a nest on H such that $I = I_-$. Let A and C be operators in $F_r(N)$. Then $A + C$ is in $F_r(N)$.*

Proof. Put $A_1 = A^{\frac{1}{2}}$, $C_1 = C^{\frac{1}{2}}$. It follows from Theorem 2.2. of [11] that $r(A_1) + r(C_1) = r((A + C)^{\frac{1}{2}})$. By Theorem 13 $r(A_1)$ and $r(C_1)$ are in $OR(N)$, and so $r(A_1) + r(C_1)$ is in $OR(N)$ by Theorem 22. Using Theorem 13 again, we see that $A + C$ is in $F_r(N)$. $\square$

The following proposition characterises the positive operators with closed range which belong to $F_r(N)$. We say that a subspace V of H is N-proper if $V \cap PrN$ is dense in V.

Proposition 28. *Let N be a nest such that $I = I_-$. Let A be a positive operator in $B(H)$ with closed range. Then A is in $F_r(N)$ if and only if $r(A)$ is N-proper.*

Proof. Put $A_1 = A^{\frac{1}{2}}$. It is easy to see that since $r(A)$ is closed, $r(A) = r(A_1)$ and hence $r(A_1)$ is closed. By [2, Prop. 6], $(A_1)^{-1}(PrN)$ is dense in H if and only if $r(A_1)$ is N-proper. The proposition now follows from Theorem 5. $\square$

The following theorem provides a decomposition of a positive operator A into a "factorable" and a "completely non-factorable" part with respect to a nest N satisfying $I = I_-$. An analogous decomposition has been obtained in [21] and [1] for special cases.

Theorem 29. *Let N be a nest such that $I = I_-$. Let A be a positive operator in $B(H)$. There exist operators B and C in $B(H)$ with the following properties:*

a) $B \geq 0$, $C \geq 0$, $A = B + C$.

b) B is in $F_r(N)$.

c) If E is in $AlgN$ and satisfies $EE^* \leq C$, then $E = 0$.

d) If F is in $AlgN$ and satisfies $FF^* \leq A$, then $FF^* \leq B$.

Moreover, the operators B and C are unique with these properties.

Proof. Put $A_1 = A^{\frac{1}{2}}$. Put $H_1 = \overline{(A_1)^{-1}(PrN)}$ and let R be the orthogonal projection onto H_1. Define $B_1 = A_1 R$, $C_1 = A_1 R^{\perp}$. Then $A_1 = B_1 + C_1$ and $B_1(C_1)^* = 0$. We show that $r(C_1) \cap PrN = \{0\}$. Let x be a vector in H such that $C_1 x$ is in PrN. Then $A_1 R^{\perp} x$ is in PrN, and so $R^{\perp} x$ is in H_1 and hence it is 0. Since $C_1 x = C_1 R^{\perp} x$, we conclude that $C_1 x = 0$.

Put $B = B_1(B_1)^*$, $C = C_1(C_1)^*$. We have $B \geq 0$, $C \geq 0$ and $A = A_1(A_1)^* = B_1(B_1)^* + C_1(C_1)^* = B + C$. So a) is satisfied. We show that b) is satisfied. The space $(A_1)^{-1}(PrN) + R^{\perp}H$ is dense in H and is contained in $(B_1)^{-1}(PrN)$. Therefore $(B_1)^{-1}(PrN)$ is dense in H, and it follows from Theorem 5 that B is in $F_r(N)$.

Assume E is in $AlgN$ and satisfies $EE^* \leq C$. Then $r(E) \subseteq r(C_1)$ by [10]. But $r(C_1) \cap PrN = \{0\}$. Since $E^{-1}(PrN)$ is dense in H, $E = 0$.

Now we prove that d) is satisfied. By [10] there exists a contraction X in $B(H)$ such that $F = A_1 X$. The operator X sends $F^{-1}(PrN\ )$ into $(A_1)^{-1}(PrN)$, and since $F^{-1}(PrN\ )$ is dense in H we conclude that the range of X is contained in H_1. Therefore $F = A_1 X = (A_1 R + A_1 R^{\perp})X = A_1 R X$. So $FF^* \leq B$.

Assume now that the operators B_0 and C_0 have the properties a) through d). Since $B = DD^*$ for some D in $AlgN$, it follows from d) that $B \leq B_0$. Similarly $B_0 \leq B$, and so $B = B_0$ and $C = C_0$. $\square$

Using the results of this section, one may obtain analogous results for the set $F_l(N)$ for a nest N with the property $0 = 0_-$.

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