POISSON TRANSFORMS ON VECTOR BUNDLES

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ABSTRACT. Let G be a connected real semisimple Lie group with finite center, and K a maximal compact subgroup of G . Let (τ, V) be an irreducible unitary representation of K , and $G \times_K V$ the associated vector bundle. In the algebra of invariant differential operators on $G \times_K V$ the center of the universal enveloping algebra of $\text{Lie}(G)$ induces a certain commutative subalgebra Z_τ . We are able to determine the characters of Z_τ . Given such a character we define a Poisson transform from certain principal series representations to the corresponding space of joint eigensections. We prove that for most of the characters this map is a bijection, generalizing a famous conjecture by Helgason which corresponds to τ the trivial representation.

INTRODUCTION

Let G be a connected real semisimple Lie group with finite center, and K a maximal compact subgroup of G . Then G/K is a Riemannian symmetric space of noncompact type. We fix an Iwasawa decomposition $G = KAN$. Let M be the centralizer of A in K . Let $\mathfrak{g}$ and $\mathfrak{a}$ be the Lie algebras of G and A , respectively, and $\Sigma(\mathfrak{g}, \mathfrak{a})$ the root system for $\mathfrak{g}, \mathfrak{a}$. Let $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ be the positive roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the ordering given by N . Let $D(G/K)$ be the algebra of invariant differential operators on G/K . It is well-known that the characters of $D(G/K)$ are parametrized by $\lambda \in \mathfrak{a}_\mathbb{C}^*$, the complex dual space of $\mathfrak{a}$. Let $\mathcal{E}_\lambda(G/K)$ denote the space of joint eigenfunctions corresponding to λ . We write $g = k(g) \exp H(g) n(g)$ for each $g \in G$ according to $G = KAN$. For each $\phi \in C^\infty(K/M)$ we define $P_\lambda \phi \in C^\infty(G/K)$ by

$$P_\lambda \phi(g) = \int_K \phi(k) e^{-(\lambda + \rho)H(g^{-1}k)} dk.$$

Here ρ is the half sum of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (including multiplicities). It turns out that $P_\lambda \phi \in \mathcal{E}_\lambda(G/K)$. One can easily extend the definition of P_λ to the space $D'(K/M)$ (resp. $A'(K/M)$) of distributions (resp. analytic functionals) on K/M . In this paragraph we fix $\lambda \in \mathfrak{a}_\mathbb{C}^*$ such that $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$ is not in $-\mathbb{N} - \{0\}$, for each $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. It is proved by Helgason [Helg2] that P_λ defines a bijection from $C^\infty(K/M)_{K\text{-finite}}$ onto $\mathcal{E}_\lambda(G/K)_{K\text{-finite}}$. He also proves in the rank one case P_λ is a bijection from $A'(K/M)$ onto $\mathcal{E}_\lambda(G/K)$. He then conjectured this should be true for the higher rank case. The conjecture was eventually proved by six Japanese mathematicians in [KKMOOT]. It should be mentioned that a representation theoretic proof by Schmid, starting from the K -finite result, is indicated in [Sch]. Lewis, then a student of Helgason, made the following observation: Let $\mathcal{E}_\lambda^*(G/K)$ be the subspace of $\mathcal{E}_\lambda(G/K)$ where each element increases at most exponentially (see §2 for definition);

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then P_λ maps $D'(K/M)$ into $\mathcal{E}_\lambda^*(G/K)$. He was able to prove in the rank one case P_λ is a bijection from $D'(K/M)$ onto $\mathcal{E}_\lambda^*(G/K)$. See [Lew]. This result has been generalized to the higher rank case by Oshima and Sekiguchi [OS]. There is an alternative and independent proof by Wallach [Wall1]. By refining Wallach's idea van den Ban and Schlichtkrull have a third proof in [BS]. They define $\mathcal{E}_\lambda^\infty(G/K)$ as the subspace of $\mathcal{E}_\lambda(G/K)$ where each element and its derivatives increase at most exponentially (uniformly). Then they prove P_λ is a bijection from $C^\infty(K/M)$ onto $\mathcal{E}_\lambda^\infty(G/K)$. The bijectivity of P_λ from $D'(K/M)$ to $\mathcal{E}_\lambda^*(G/K)$ follows easily.

Let (τ, V) be an irreducible unitary representation of K . Let $G \times_K V$ be the associated vector bundle over G/K . The space of smooth sections of this vector bundle can be identified with

$$C^\infty \text{Ind}_K^G(\tau) = \{f \in C^\infty(G, V) | f(gk) = \tau(k^{-1})f(g), \forall g \in G, \forall k \in K\}.$$

Let D_τ denote the algebra of invariant differential operators on $C^\infty \text{Ind}_K^G(\tau)$. Notice when (τ, V) is the trivial representation we go back to the previous case. In the case where $\dim V = 1$, D_τ is commutative and its characters can be parameterized by $\lambda \in \mathfrak{a}_\mathbb{C}^*$. In [Shim] Shimeno is able to characterize the joint eigenspace of D_τ in terms of a Poisson transform for most of λ . Gaillard's results about the eigenforms of the Laplacian on hyperbolic spaces are illuminating. They show considerable variety even for a simple space. See [Ga] for details. van der Ven [Ven] considers vector-valued Poisson transforms in the rank one case, extending Gaillard's results. His emphasis, however, is on the singular eigenvalues. Minemura [Min] studies the properties of D_τ and obtains a result on the dimension of the spherical eigensections.

One of the difficulties people run into when trying to generalize the classical results is the complexity of D_τ , in particular its noncommutativity. The remedy used was either a condition on τ or a condition on (G/K) . We put a mild condition on $\mathfrak{g}$ (see beginning of §4) but no restriction on τ . We replace D_τ with a subalgebra Z_τ coming from $\mathcal{Z}(\mathfrak{g})$, the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Then we are able to determine the characters of Z_τ . It turns out they are given by $\lambda - \Lambda$, where $\lambda \in \mathfrak{a}_\mathbb{C}^*$, and Λ is given by the infinitesimal character of an irreducible representation of M contained in τ (see Proposition 1.11).

Let V be the representation space of τ , and

$$V = \bigoplus_{\sigma \in \widehat{M}} V(\sigma)$$

the isotypic decomposition of V into M -isotypic parts. We say $\sigma \in \tau$ if $V(\sigma) \neq 0$. Write

$$V(\Lambda) = \bigoplus_{\sigma \in \tau, \Lambda_\sigma = \Lambda} V(\sigma).$$

Here Λ_σ is given by the infinitesimal character of σ . Let $\tau(\Lambda)$ be the restriction of τ to M with representation space $V(\Lambda)$. We define a Poisson transform (see §1 for definition)

$$P_\lambda: C^\infty \text{Ind}_{MAN}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \rightarrow \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$$

by

$$P_\lambda \phi(g) = \int_K \tau(k) \phi(gk) dk.$$

Here $C^\infty \text{Ind}_{MAN}^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$ is the space defined by

$$\{\phi \in C^\infty(G, V(\Lambda)) | \phi(gman) = a^{\lambda-\rho} \tau(m^{-1}) \phi(g)\},$$

and $\mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$ the subspace of the total eigenspace where each element and its derivatives increase at most exponentially (uniformly). Let $C(\lambda)$ be the generalized Harish-Chandra C -function corresponding to τ (Proposition 2.3), $C_0(\lambda)$ the restriction of $C(\lambda)$ to $V(\Lambda)$, and $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ as defined after Remark 1.5.

Theorem. *Let $\lambda - \Lambda \in \mathfrak{h}_\mathbb{C}^*$, satisfying the following conditions:*

- (i) $2\langle \lambda - \Lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \notin \mathbb{Z}$, $\forall \alpha \in \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, with $\alpha|_{\mathfrak{a}} \neq 0$;
- (ii) $2\langle \lambda, \beta \rangle / \langle \beta, \beta \rangle \notin -\mathbb{N}$, $\forall \beta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$;
- (iii) $\det C_0(\lambda) \neq 0$.

Then P_λ is a bijection.

This generalizes the result of van den Ban and Schlichtkrull mentioned above which corresponds to τ the trivial representation.

We have similar results about distributions and K -finite sections, generalizing the above-mentioned results for τ trivial.

The main idea in the proof is to generalize the theory of asymptotic expansions developed in [Ban] and [BS]. By invoking Casselman's deep result [Ca] on globalization of Harish-Chandra modules, one might simplify our argument somehow. But we prefer a self-contained account. Besides, we think the theory of asymptotic expansions developed here is of interest on its own.

The paper is organized as follows: in Section 1 we study the invariant differential operators on $G \times_K V$. In Section 2 we introduce some function spaces on G . In Section 3 we state some results on the asymptotic expansion of an eigensection. In Section 4 we study the algebraic structure of a $(\mathfrak{g}, K)$ -module. In Sections 5 and 6 we prove the results stated in Section 3. In Section 7 we study the leading terms of the asymptotic expansion. In Section 8 we give an inversion formula to the Poisson transform. In Sections 9 and 10 we extend the Poisson transform to vector-valued distributions.

1. NOTATIONS AND PRELIMINARIES

Let G be a connected real semisimple Lie group with finite center and K a maximal compact subgroup of G . Then G/K is a Riemannian symmetric space. We fix an Iwasawa decomposition $G = KAN$, and let M be the centralizer of A in K , M' the normalizer of A in K , $W = M'/M$ the Weyl group. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$, and $\mathfrak{m}$ be the corresponding Lie algebras of G, K, A, N , and M , respectively, and $U(\mathfrak{g}), U(\mathfrak{k}), U(\mathfrak{a}), U(\mathfrak{n})$, and $U(\mathfrak{m})$ the corresponding universal enveloping algebras of the complexified Lie algebras. Let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system for $(\mathfrak{g}, \mathfrak{a})$, and $\Delta = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots for the ordering of $\Sigma(\mathfrak{g}, \mathfrak{a})$ given by N . Let $\mathcal{Z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. If $g \in G$ we write $g = k(g) \exp H(g) n(g)$ according to $G = KAN$.

Fix once and for all an irreducible unitary representation (τ, V) of K . Denote by $G \times_K V$ the associated vector bundle. Then the space of its smooth sections may be identified with the following space:

$$C^\infty \text{Ind}_K^G(\tau) = \{f \in C^\infty(G, V) | f(gk) = \tau(k)^{-1} f(g), \forall g \in G, \forall k \in K\}.$$

Let D_τ denote the algebra of differential operators on $C^\infty \text{Ind}_K^G(\tau)$ that commute with the left translations by elements of G . The remainder of this section will be

devoted to the study of this algebra. First for each $f \in C^\infty(G, V)$ and $X \in \mathfrak{g}$ we define $L_X f$ and $R_X f$ as follows:

$$\begin{aligned} L_X f(g) &= \left(\frac{d}{dt} f(\exp(-tX)g) \right) \Big|_{t=0}, \\ R_X f(g) &= \left(\frac{d}{dt} f(g \exp tX) \right) \Big|_{t=0}, \quad \forall g \in G. \end{aligned}$$

Then L and R define two representations of $\mathfrak{g}$ which we extend to representations of $U(\mathfrak{g})$. Let $\text{End}(V)$ denote the space of linear maps from V to itself. Then $U(\mathfrak{g}) \otimes \text{End}(V)$ is an associative algebra with the natural multiplication. Let $I(\tau)$ be the left ideal of $U(\mathfrak{g}) \otimes \text{End}(V)$ generated by $\{X \otimes 1 + 1 \otimes \tau(X) | X \in \mathfrak{k}\}$.

Proposition 1.1. *With the above notations, we have*

$$U(\mathfrak{g}) \otimes \text{End}(V) = (U(\mathfrak{a}) \otimes \text{End}(V)) \oplus (\mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau)).$$

Proof. It suffices to show the left-hand side is contained in the right-hand side. Suppose $u \otimes T \in U(\mathfrak{g}) \otimes \text{End}(V)$. By Poincaré-Birkhoff-Witt we can assume $u = u_1 u_2 u_3$, where $u_1 \in U(\mathfrak{n})$, $u_2 \in U(\mathfrak{a})$, and $u_3 \in U(\mathfrak{k})$. If $u_1 \in \mathfrak{n}U(\mathfrak{n})$ then $u \otimes T \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V)$. So we can assume $u = u_2 u_3$, where $u_2 \in U(\mathfrak{a})$, and $u_3 \in U(\mathfrak{k})$. Let $u_3 = X_1 \cdots X_j$, for $X_1, \dots, X_j \in \mathfrak{k}$. It is easy to show $u_2 u_3 \otimes T \in (U(\mathfrak{a}) \otimes \text{End}(V)) + I(\tau)$ by induction on j . $\square$

Define a K -action on $U(\mathfrak{g}) \otimes \text{End}(V)$ by

$$k(X \otimes T) = \text{Ad}(k)X \otimes \tau(k)T\tau(k)^{-1},$$

for each $k \in K$. Let $(U(\mathfrak{g}) \otimes \text{End}(V))^K$ be the fixed elements.

Proposition 1.2. *Let $\Gamma_1: U(\mathfrak{g}) \otimes \text{End}(V) \rightarrow U(\mathfrak{a}) \otimes \text{End}(V)$ be the projection map according to the decomposition in Proposition 1.1. Then Γ_1 is a homomorphism from $(U(\mathfrak{g}) \otimes \text{End}(V))^K$ into $U(\mathfrak{a}) \otimes \text{End}_M(V)$, where*

$$\text{End}_M(V) = \{T \in \text{End}(V) | \tau(m)T = T\tau(m), \forall m \in M\}.$$

Proof. Since M preserves $\mathfrak{n}$, it is easy to see Γ_1 maps $(U(\mathfrak{g}) \otimes \text{End}(V))^K$ into $U(\mathfrak{a}) \otimes \text{End}_M(V)$. We now check Γ_1 is a homomorphism.

Suppose $D_1, D_2 \in (U(\mathfrak{g}) \otimes \text{End}(V))^K$. Then

$$D_1 - \Gamma_1(D_1) \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau).$$

Hence

$$D_1 D_2 - \Gamma_1(D_1) D_2 \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau) D_2.$$

Assume $D_2 = \sum u_i \otimes T_i$, for $u_i \in U(\mathfrak{g})$, and $T_i \in \text{End}(V)$. Then for any $X \in \mathfrak{k}$,

$$\begin{aligned} (X \otimes 1 + 1 \otimes \tau(X)) D_2 &= \sum (X u_i \otimes T_i + u_i \otimes \tau(X) T_i) \\ &= \sum (\text{ad}(X) u_i \otimes T_i + u_i \otimes [\tau(X), T_i]) \\ &\quad + \sum (u_i X \otimes T_i + u_i \otimes T_i \tau(X)). \end{aligned}$$

Then first summation is zero since $D_2 \in (U(\mathfrak{g}) \otimes \text{End}(V))^K$. The second one is just $D_2(X \otimes 1 + 1 \otimes \tau(X))$. So we have proved $I(\tau) D_2 \subset I(\tau)$. Hence

$$D_1 D_2 - \Gamma_1(D_1) D_2 \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau).$$

However,

$$D_2 - \Gamma_1(D_2) \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau),$$

and

$$\Gamma_1(D_1)(\mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau)) \subset \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau).$$

Therefore

$$D_1 D_2 - \Gamma_1(D_1) \Gamma_1(D_2) \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V) + I(\tau).$$

This proves $\Gamma_1(D_1 D_2) = \Gamma_1(D_1) \Gamma_1(D_2)$. $\square$

For $D = \sum u_i \otimes T_i \in U(\mathfrak{g}) \otimes \text{End}(V)$, and $f \in C^\infty(G, V)$, we define

$$\mu_1(D)f = \sum T_i R_{u_i} f.$$

It is not difficult to show for $D \in (U(\mathfrak{g}) \otimes \text{End}(V))^K$ and $f \in C^\infty \text{Ind}_K^G(\tau)$, $\mu_1(D)f$ remains in $C^\infty \text{Ind}_K^G(\tau)$. So $\mu_1(D) \in D_\tau$. In fact μ_1 is a surjective homomorphism from $(U(\mathfrak{g}) \otimes \text{End}(V))^K$ onto D_τ .

We define $\mu(D) = \mu_1(D \otimes 1)$, for each $D \in U(\mathfrak{g})^K$. By a theorem of Burnside which asserts that $\tau(U(\mathfrak{k})) = \text{End}(V)$, one can prove μ is a surjective homomorphism from $U(\mathfrak{g})^K$ onto D_τ , using the surjectivity of μ_1 .

For each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we introduce an important function Ψ_λ on G with values in $\text{End}(V)$ as follows:

$$\Psi_\lambda(nak) = a^{\lambda+\rho} \tau(k)^{-1},$$

for $n \in N$, $a \in A$, and $k \in K$. Here ρ is the half sum of the positive roots for $(\mathfrak{g}, \mathfrak{a})$. Notice that for each $v \in V$, the function: $g \rightarrow \Psi_\lambda(g) \cdot v$ belongs to $C^\infty \text{Ind}_K^G(\tau)$.

Proposition 1.3. *For each $D \in U(\mathfrak{g})^K$, and $v \in V$,*

$$\mu(D)(\Psi_\lambda \cdot v) = \Psi_\lambda \cdot (\Gamma_1(D \otimes 1)(\lambda + \rho)v).$$

Proof. Since both sides are left N -invariant and behave in the same way under the right K -action, it is sufficient to show they are equal when restricted to A . By definition

$$D \otimes 1 = D_1 + \Gamma_1(D \otimes 1) + D_2,$$

where $D_1 \in \mathfrak{n}U(\mathfrak{g}) \otimes \text{End}(V)$, and $D_2 \in I(\tau)$. It is easy to see that

$$\mu_1(D_1)(\Psi_\lambda \cdot v)|_A = 0,$$

and

$$\mu_1(D_2)(\Psi_\lambda \cdot v) = 0.$$

So

$$\mu(D)(\Psi_\lambda \cdot v)|_A = a^{\lambda+\rho} \Gamma_1(D \otimes 1)(\lambda + \rho)v. \quad \square$$

Corollary 1.4. *There exists a homomorphism $\Gamma': D_\tau \rightarrow U(\mathfrak{a}) \otimes \text{End}_M(V)$. Moreover, for each $D \in U(\mathfrak{g})^K$, $\Gamma'(\mu(D)) = \Gamma_1(D \otimes 1)$.*

Remark 1.5. It has been proved in Section 3 in [Min] that Γ' is injective, using results from [Lep].

In general D_τ is very complicated. For instance it is not abelian in most of the cases. For this reason we replace it by $\mu(\mathcal{Z}(\mathfrak{g}))$ which we denote by Z_τ .

Choose $\mathfrak{t}$ a maximal abelian subalgebra in $\mathfrak{m}$. Then $\mathfrak{h}_\mathbb{C} = (\mathfrak{t} + \mathfrak{a})_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the root system for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ be the set of positive roots for some ordering, and $\mathfrak{g}_\mathbb{C}^+$ (resp. $\mathfrak{g}_\mathbb{C}^-$) the sum of positive (resp. negative) root spaces. Choose an ordering such that $\mathfrak{n} \subset \mathfrak{g}_\mathbb{C}^+$. We consider each $\lambda \in \mathfrak{a}_\mathbb{C}^*$ (resp. $\mathfrak{t}_\mathbb{C}^*$) an element of $\mathfrak{h}_\mathbb{C}^*$ by the requirement that λ be zero in $\mathfrak{t}$ (resp. $\mathfrak{a}$). Let

$$P = \{\alpha \in \Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) | \alpha|_{\mathfrak{a}} \neq 0\}, \quad P_0 = \{\alpha \in \Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) | \alpha|_{\mathfrak{a}} = 0\}.$$

Write

$$\rho = \frac{1}{2} \sum_{\alpha \in P} \alpha, \quad \rho_0 = \frac{1}{2} \sum_{\alpha \in P_0} \alpha.$$

Let Θ be the Cartan involution of $\mathfrak{g}$ with fixed point set $\mathfrak{k}$ and extend it to an automorphism of $\mathfrak{g}_\mathbb{C}$. Then $\alpha \rightarrow -\Theta\alpha$ is a permutation of P , so $\rho|_{\mathfrak{t}} = 0$. Hence ρ can be viewed as the half sum of positive roots for $(\mathfrak{g}, \mathfrak{a})$.

Let $\gamma': \mathcal{Z}(\mathfrak{g}) \rightarrow U(\mathfrak{h}_\mathbb{C})$ be defined by

$$Z - \gamma'(Z) \in \mathfrak{g}_\mathbb{C}^- U(\mathfrak{g}),$$

for $Z \in \mathcal{Z}(\mathfrak{g})$. We define $\gamma(Z)(\mu) = \gamma'(Z)(\mu - \rho - \rho_0)$, for each $\mu \in \mathfrak{h}_\mathbb{C}^*$. This is the usual Harish-Chandra homomorphism.

Let $V = \bigoplus_{\sigma \in \widehat{M}} V(\sigma)$ be the decomposition into the M -isotypic parts. We say $\sigma \in \tau$ if $V(\sigma) \neq 0$.

For each irreducible representation (σ, V_σ) of M , we get a Lie algebra representation of $\mathfrak{m}$ by differentiation. We denote the representation by $d\sigma$. In general this is not irreducible. Fortunately it is a multiple of an irreducible representation of $\mathfrak{m}$. This fact can be seen in the following way. Let M_0 be the identity component of M . By structure theory (see 1.1.3.8 in [War]) one can find $Z(A)$, a finite subgroup of M where each element commutes with every element of M_0 . Choose an irreducible representation (σ, V_1) of M_0 in (σ, V_σ) . For each $z \in Z(A)$, $(\sigma, \sigma(z)V_1)$ gives an irreducible representation of M_0 in (σ, V_σ) , which is equivalent to (σ, V_1) . Since σ is irreducible, $V_\sigma = \sum_{z \in Z(A)} \sigma(z)V_1$. So by Schur's lemma the center $\mathcal{Z}(\mathfrak{m})$ of $U(\mathfrak{m})$ acts on V_σ by scalars. The action is determined by $\Lambda_\sigma \in \sqrt{-1}\mathfrak{t}^*$ as follows: For each Z in $\mathcal{Z}(\mathfrak{m})$, $d\sigma(Z) = \gamma(Z)(\Lambda_\sigma)I_{V_\sigma}$, where γ is the Harish-Chandra homomorphism for $(\mathfrak{m}, \mathfrak{t})$, and I_{V_σ} the identity map of V_σ . We choose Λ_σ the highest weight of σ plus ρ_0 .

Let $\Gamma: D_\tau \rightarrow U(\mathfrak{a}) \otimes \text{End}_M(V)$ be defined by

$$\Gamma(D)(\lambda) = \Gamma'(D)(\lambda + \rho).$$

Theorem 1.6. *For each $Z \in \mathcal{Z}(\mathfrak{g})$, and $\lambda \in \mathfrak{a}_\mathbb{C}^*$,*

$$\Gamma(\mu(Z))(\lambda)V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma)I_{V(\sigma)}.$$

We give a proof below using a well-known proposition about $\mathcal{Z}(\mathfrak{g})$. A more self-contained proof is in [Wall]. First for the proof and later use we recall the definition of Poisson transforms.

Let (δ, V_δ) be a finite dimensional representation of $B = MAN$, the minimal parabolic subgroup of G . Let $C^\infty \text{Ind}_B^G(\delta)$ be the space defined by

$$\{\phi \in C^\infty(G, V_\delta) | \phi(gman) = a^{-\rho}\delta^{-1}(man)\phi(g), \forall g \in G, \forall man \in B\}.$$

Let $C^\infty \text{Ind}_B^G(\delta)$ be endowed with the topology from $C^\infty(G, V_\delta)$. We will specify the topology on $C^\infty \text{Ind}_K^G(\tau)$ in the next section.

Definition 1.7. A Poisson transform is a continuous, linear, G -equivariant map from $C^\infty \text{Ind}_B^G(\delta)$ into $C^\infty \text{Ind}_K^G(\tau)$.

Given $T \in \text{Hom}_M(V_\delta, V)$, and $\phi \in C^\infty \text{Ind}_B^G(\delta)$, we write

$$P_T \phi(g) = \int_K \tau(k) T(\phi(gk)) dk.$$

One can easily check P_T is a Poisson transform.

Proposition 1.8. *The map $T \rightarrow P_T$ is a bijection from $\text{Hom}_M(V_\delta, V)$ onto the space of Poisson transforms.*

This result appears in [Ven]. We include a proof for completeness. Suppose P is a Poisson transform from $C^\infty \text{Ind}_B^G(\delta)$ into $C^\infty \text{Ind}_K^G(\tau)$. Define the Poisson kernel $p \in [C^\infty \text{Ind}_B^G(\delta)]' \otimes V$, the strong topological dual of $C^\infty \text{Ind}_B^G(\delta)$ tensored by V , by

$$\langle p, \phi \rangle = (P\phi)(e),$$

for each $\phi \in C^\infty \text{Ind}_B^G(\delta)$. By the G -equivariance of P the Poisson kernel completely determines P by

$$P\phi(x) = \langle p, L_{x^{-1}}\phi \rangle,$$

for any $\phi \in C^\infty \text{Ind}_B^G(\delta)$. Here $L_{x^{-1}}\phi(g) = \phi(xg)$.

By Section 9 there is a K -equivariant isomorphism between $(C^\infty \text{Ind}_B^G(\delta))'$ and $C^{-\infty} \text{Ind}_M^K(\check{\delta}|M)$, where $C^{-\infty} \text{Ind}_M^K(\check{\delta}|M)$ denotes the space of vector-valued distributions $f: C^\infty(K, \mathbb{C}) \rightarrow V_\delta^*$, such that

$$R_m f = \check{\delta}(m)^{-1} f,$$

for any $m \in M$. Here $\check{\delta}$ is the dual representation of $\delta|M$. And $R_m f(\phi) = f(R_{m^{-1}}\phi)$, with $(R_{m^{-1}}\phi)(k) = \phi(km^{-1})$. So

$$p \in C^{-\infty} \text{Ind}_M^K(\check{\delta}|M) \otimes V.$$

However, for $\phi \in C^\infty \text{Ind}_B^G(\delta)$,

$$\langle p, L_k \phi \rangle = P(L_k \phi)(e) = P\phi(k^{-1}) = \tau(k)(P\phi(e)) = \tau(k)(\langle p, \phi \rangle).$$

Hence $p \in (C^{-\infty} \text{Ind}_M^K(\check{\delta}|M) \otimes V)^K$. Let π be the representation of K in $V_\delta^* \otimes V$ defined by $\pi(k)(v \otimes w) = v \otimes \tau(k)w$, for $v \in V_\delta^*$, and $w \in V$. Then $p \in C^{-\infty}(K, V_\delta^* \otimes V)$, and $L_k p = \pi(k^{-1})p$. By Lemma 9.3 p must be smooth. Its transformation properties imply that p is determined by $p(e)$, which belongs to $(V_\delta^* \otimes V)_M \cong \text{Hom}_M(V_\delta, V)$.

Proof of Proposition 1.8. From the definition of P_T , it is immediate that the Poisson kernel of P_T evaluated at the identity is T . This shows the map $T \rightarrow P_T$ is injective. On the other hand, let P be a Poisson transform, and let p be its Poisson

kernel. Then

$$\begin{aligned} P\phi(x) &= \langle p, L_{x^{-1}}\phi \rangle \\ &= \int_K \langle p(k), \phi(xk) \rangle dk \\ &= \int_K \tau(k)p(e)\phi(xk)dk. \end{aligned}$$

This proves $P = P_{p(e)}$, whence the surjectivity. $\square$

Lemma 1.9.

$$\int_K F(k(g^{-1}k))dk = \int_K F(k)e^{-2\rho H(gk)}dk.$$

Let σ be a finite dimensional representation of M and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then $\sigma \otimes (-\lambda) \otimes 1$ defines a representation of B by $man \rightarrow a^{-\lambda}\sigma(m)$.

Corollary 1.10. For each $\phi \in C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$, $T \in \text{Hom}_M(V_\sigma, V)$,

$$P_T\phi(g) = \int_K \Psi_\lambda(k^{-1}g)T\phi(k)dk.$$

Proof.

$$\begin{aligned} P_T\phi(g) &= \int_K \tau(k)T\phi(gk)dk \\ &= \int_K \tau(k)T\phi(k(gk) \exp H(gk)n(gk))dk \\ &= \int_K e^{(\lambda-\rho)H(gk)}\tau(k)T\phi(k(gk))dk. \end{aligned}$$

By Lemma 1.9,

$$\begin{aligned} &\int_K e^{(\lambda-\rho)H(gk)}\tau(k)T\phi(k(gk))dk \\ &= \int_K e^{(\lambda+\rho)H(gk(g^{-1}k))}\tau(k(g^{-1}k))T\phi(k(gk(g^{-1}k)))dk \\ &= \int_K e^{(-\lambda+\rho)H(g^{-1}k)}\tau(k(g^{-1}k))T\phi(k)dk \\ &= \int_K \Psi_\lambda(k^{-1}g)T\phi(k)dk. \quad \square \end{aligned}$$

Proof of Theorem 1.6. Let δ be the restriction of τ to M with $V(\sigma)$ as the representation space. It is well-known that $L_Z\phi = \gamma(Z)(\Lambda_\sigma - \lambda)\phi$ for each $Z \in \mathcal{Z}(\mathfrak{g})$, and $\phi \in C^\infty \text{Ind}_B^G(\delta \otimes (-\lambda) \otimes 1)$. See [Vogan]. Let $*$ denote adjoint on $U(\mathfrak{g})$. By Corollary 5.31 on p. 324 in [Helg1],

$$\begin{aligned} R_Z P_T \phi &= L_{Z^*} P_T \phi = P_T L_Z \phi \\ &= P_T(\gamma(Z^*)(\Lambda_\sigma - \lambda)\phi) = P_{\gamma(Z^*)(\Lambda_\sigma - \lambda)} T \phi = P_{\gamma(Z)(-\Lambda_\sigma + \lambda)} T \phi. \end{aligned}$$

On the other hand, by Proposition 1.3 and Corollary 1.10,

$$R_Z P_T \phi = P_{\Gamma(\mu(Z))(\lambda)} T \phi.$$

So

$$P_{\gamma(Z)(-\Lambda_\sigma + \lambda)} T = P_{\Gamma(\mu(Z))(\lambda)} T.$$

By Proposition 1.8 we conclude

$$\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\lambda - \Lambda_\sigma)I_{V(\sigma)}. \quad \square$$

By definition a character of Z_τ is a homomorphism from Z_τ to $\mathbb{C}$.

Proposition 1.11. *A character χ of Z_τ is given by $\lambda - \Lambda_\sigma$, where $\lambda \in \mathfrak{a}_\mathbb{C}^*$, and $\sigma \in \tau$. More specifically, $\chi(\mu(Z)) = \gamma(Z)(\lambda - \Lambda_\sigma)$, for each $Z \in \mathcal{Z}(\mathfrak{g})$.*

Lemma 1.12. *Let S be the common zeros of $p_1, \dots, p_m$ in $S(\mathfrak{h}_\mathbb{C})$. Assume in addition S is $\widetilde{W}$ invariant, $\widetilde{W}$ denoting the Weyl group for $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then one can find $q_1, \dots, q_n$ in $I(\mathfrak{h}_\mathbb{C})$, the $\widetilde{W}$ -invariants in $S(\mathfrak{h}_\mathbb{C})$, such that S is the common zeros of $q_1, \dots, q_n$.*

Proof. Write $R_i(X) = \prod_{s \in \widetilde{W}} (X - p_i^s)$, and $w = |\widetilde{W}|$. Then

$$R_i(X) = X^w + p_{i1}X^{w-1} + \dots + p_{iw}.$$

It is easy to see we can use p_{ij} as our $q_1, \dots, q_n$. $\square$

Proof of Proposition 1.11. Let $A = \mu \circ \gamma^{-1}: I(\mathfrak{h}_\mathbb{C}) \rightarrow Z_\tau$. By Theorem 1.6 $\ker(A) = \{\rho \in I(\mathfrak{h}_\mathbb{C}) | p|(-\Lambda_\sigma + \mathfrak{a}_\mathbb{C}^*) = 0, \text{ for all } \sigma \in \tau\}$. Here we use Remark 1.5 which asserts that Γ is injective. Suppose $\chi: Z_\tau \rightarrow \mathbb{C}$ is a character of Z_τ . Then there exists $\mu \in \mathfrak{h}_\mathbb{C}^*$, such that $\chi \circ A = \chi_\mu$, where χ_μ is the homomorphism defined by evaluation at μ . Obviously $p(\mu) = 0$, for all $p \in \ker(A)$. Let

$$S = \bigcup_{\sigma \in \tau, w \in \widetilde{W}} w(-\Lambda_\sigma + \mathfrak{a}_\mathbb{C}^*) \subset \mathfrak{h}_\mathbb{C}^*.$$

Obviously one can find $p_1, \dots, p_m$ in $S(\mathfrak{h}_\mathbb{C})$ such that S is the common zeros of $p_1, \dots, p_m$. Then by Lemma 1.12 we can find $q_1, \dots, q_n$ in $I(\mathfrak{h}_\mathbb{C})$ such that S is the common zeros of $q_1, \dots, q_n$. This shows $q_1, \dots, q_n$ are in $\ker(A)$. So $q_1(\mu) = \dots = q_n(\mu)$. Therefore $\mu \in S$, i.e. $\mu = w(\lambda - \Lambda_\sigma)$ for some $\lambda \in \mathfrak{a}_\mathbb{C}^*$, $\sigma \in \tau$, and $w \in \widetilde{W}$. $\square$

For $s \in M'$, define $s.(X \otimes T) = \text{Ad}(s)X \otimes \tau(s)T\tau(s^{-1})$, for $X \in U(\mathfrak{a})$, and $T \in \text{End}(V)$. The next proposition is about a property of the generalized Harish-Chandra homomorphism. It is a weak version of a conjecture by Lepowsky.

Proposition 1.13. *For each $s \in M'/M$, $s.\Gamma(D) = \Gamma(D)$, for each $D \in Z_\tau$.*

For the proof of this result we need more facts about Weyl groups. Let $\widetilde{W}_1 \subset \widetilde{W}$ be the subgroup where every element stabilizes $\mathfrak{a}$. It is well-known there is a surjective homomorphism $\widetilde{W}_1 \rightarrow M'/M$. The kernel $\widetilde{W}_0$ is the Weyl group for $(\mathfrak{m}, \mathfrak{t})$.

Lemma 1.14. *For each $s \in M'/M$, choose $w(s)$ in $\widetilde{W}_1$ in the preimage of s under the homomorphism above. Then $\Lambda_{\sigma^s} = w(s)\Lambda_\sigma$.*

Proof (by Vogan). Take a maximal torus T of M_0 . sTs^{-1} is another maximal torus. So there is $m \in M_0$, such that $msTs^{-1}m^{-1} = T$. To avoid cumbersome notations we assume $sTs^{-1} = T$. It is easy to see that $\text{Ad}(s)^*$, the transpose of $\text{Ad}(s)$, preserves $\Sigma(\mathfrak{m}, \mathfrak{t})$. We can also assume $\text{Ad}(s)^*$ preserves $\Sigma^+(\mathfrak{m}, \mathfrak{t})$. For $Z \in \mathcal{Z}(\mathfrak{m})$,

$$Z - \gamma'(Z) \in \mathfrak{m}^-U(\mathfrak{m}).$$

Hence

$$\mathrm{Ad}(s)Z - \mathrm{Ad}(s)\gamma'(Z) \in \mathfrak{m}^- U(\mathfrak{m}).$$

So

$$\begin{aligned} \sigma^s(Z) &= \sigma(\mathrm{Ad}(s)Z) = \mathrm{Ad}(s)\gamma'(Z)(\Lambda_\sigma - \rho_0) \\ &= \gamma'(Z)(\mathrm{Ad}(s)^*\Lambda_\sigma - \rho_0) = \gamma(Z)(\mathrm{Ad}(s)^*\Lambda_\sigma). \end{aligned}$$

Hence

$$\Lambda_{\sigma^s} = \mathrm{Ad}(s)^*\Lambda_\sigma = w(s)\Lambda_\sigma. \quad \square$$

Proof of Proposition 1.13. Take $Z \in \mathcal{Z}(\mathfrak{g})$ such that $D = \mu(Z)$. Then for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and $s \in M'$,

$$s.\Gamma(D)(\lambda)|V(\sigma) = s.\Gamma(\mu(Z))(\lambda)|V(\sigma) = \gamma(Z)(\mathrm{Ad}(s)^*\lambda - \Lambda_{\sigma^s})I_{V(\sigma)}.$$

By Lemma 1.14, $\Lambda_{\sigma^s} = w(s)\Lambda_\sigma$. So

$$\begin{aligned} s.\Gamma(D)(\lambda)|V(\sigma) &= \gamma(Z)(\mathrm{Ad}(s)^*\lambda - w(s)\Lambda_\sigma)I_{V(\sigma)} \\ &= \gamma(Z)(\lambda - \Lambda_\sigma)I_{V(\sigma)} \\ &= \Gamma(\mu(Z))(\lambda)|V(\sigma) \\ &= \Gamma(D)(\lambda)|V(\sigma). \quad \square \end{aligned}$$

Now let $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. Similarly as in Proposition 1.1 we get

$$U(\mathfrak{g}) \otimes \mathrm{End}(V) = U(\mathfrak{a}) \otimes \mathrm{End}(V) \oplus (\bar{\mathfrak{n}}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau)).$$

Then we define $\tilde{\Gamma}_1 : U(\mathfrak{g}) \otimes \mathrm{End}(V) \rightarrow U(\mathfrak{a}) \otimes \mathrm{End}(V)$ as the projection according to this decomposition.

Corollary 1.15. *For each $Z \in \mathcal{Z}(\mathfrak{g})$, and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,*

$$\tilde{\Gamma}_1(Z \otimes 1)(\lambda) = \Gamma(\mu(Z))(\lambda + \rho).$$

Proof. Take $s \in M'$ such that $\mathrm{Ad}(s)^*\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \Sigma^-(\mathfrak{g}, \mathfrak{a})$. By definition

$$Z \otimes 1 - \Gamma_1(Z \otimes 1) \in \mathfrak{n}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

Hence

$$s.(Z \otimes 1) - s.\Gamma_1(Z \otimes 1) \in \bar{\mathfrak{n}}U(\mathfrak{g}) \otimes \mathrm{End}(V) + I(\tau).$$

So

$$\tilde{\Gamma}_1(Z \otimes 1) = s.\Gamma_1(Z \otimes 1).$$

Hence

$$\begin{aligned} \tilde{\Gamma}_1(Z \otimes 1)(\lambda) &= \tau(s)\Gamma_1(Z \otimes 1)(\mathrm{Ad}(s)^*\lambda)\tau(s^{-1}) \\ &= \tau(s)\Gamma(\mu(Z))(\mathrm{Ad}(s)^*\lambda - \rho)\tau(s^{-1}) \\ &= \tau(s)\Gamma(\mu(Z))(\mathrm{Ad}(s)^*(\lambda + \rho))\tau(s^{-1}) \\ &= \Gamma(\mu(Z))(\lambda + \rho). \quad \square \end{aligned}$$

2. SOME FUNCTION SPACES ON G

In this section we introduce a certain growth condition on a function on G with values in V . It turns out the condition is satisfied by $P_T\phi$ for any $\phi \in C^\infty \text{Ind}_B^G(\delta)$, where δ is a certain finite dimensional representation of B .

For each $g \in G$, we denote by $\|g\|$ the operator norm of $\text{Ad}(g)$ on $\mathfrak{g}$, which is equipped with the inner product $\langle X, Y \rangle_\Theta = -K(X, \Theta Y)$. Here K is the Killing form on $\mathfrak{g}$.

Lemma 2.1. (i) $\|g\| = \|\Theta g\| = \|g^{-1}\| \geq 1$;
(ii) $\|g_1 g_2\| \leq \|g_1\| \|g_2\|$;
(iii) if $g = k_1 a k_2$ with $k_1, k_2 \in K$, $a \in A$, then

$$\|g\| = \exp \left(\max_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})} |\alpha(\log a)| \right);$$

(iv) there are constants $C_1, C_2 > 0$, such that if $x = \exp X$ with $X \in \mathfrak{p}$, then $e^{C_1|X|} \leq \|x\| \leq e^{C_2|X|}$. Here $\mathfrak{p}$ is the -1 eigenspace of Θ , and $|X| = \sqrt{\langle X, X \rangle_\Theta}$;
(v) $\|a\| \leq \|an\|$, for $a \in A$, and $n \in N$.

Proof. See [BS]. □

For any function $f: G \rightarrow V$ and $r \in \mathbb{R}$, we write

$$\|f\|_r = \sup_{g \in G} \|g\|^{-r} |f(g)|.$$

We say f increases at most exponentially if $\|f\|_r < \infty$, for some $r \in \mathbb{R}$. Let $C_r(G, V)$ denote the Banach space of continuous functions f on G with values in V with $\|f\|_r \leq \infty$.

Example 2.2. Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$, and σ a finite dimensional representation of M . Let $r(\lambda) = C_1^{-1} |\text{Re } \lambda - \rho|$, where C_1 is the constant in Lemma 2.1 (iv). Then for each $\phi \in C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$, and $T \in \text{Hom}_M(V_\sigma, V)$, $P_T\phi$ belongs to $C_{r(\lambda)}(G, V)$. This is in [BS] when τ is trivial and τ in general does not offer additional difficulties.

Write

$$C_r^\infty(G, V) = \{f \in C^\infty(G, V) | L_u f \in C_r(G, V), \forall u \in U(\mathfrak{g})\}.$$

We endow $C_r(G, V)$ with its standard topology: Let $X_1, \dots, X_p$ be a basis of $\mathfrak{g}$, and $X^I = X^{i_1} \dots X^{i_p} \in U(\mathfrak{g})$ for $I = (i_1, \dots, i_p) \in \mathbb{N}^p$. For $q \in \mathbb{N}$ and $f \in C^q(G, V)$, a q times continuously differentiable function from G to V , we define

$$\|f\|_{q,r} = \sum_{|I| \leq q} \|L_{X^I} f\|_r.$$

Endowed with this norm the space

$$C_r^q(G, V) = \{f \in C^q(G, V) | \|f\|_{q,r} < \infty\}$$

is a Banach space. Obviously $C_r^q \subset C_r^{q'}$ if $q' \leq q$, $C_r^\infty(G, V) = \bigcap_q C_r^q(G, V)$. The topology on $C_r^\infty(G, V)$ is given by the family of norms $\|\cdot\|_{q,r}$, $q \in \mathbb{N}$, on $C_r^\infty(G, V)$. We now consider for each $q \in \mathbb{N}$ the action of L and R on $C_r^q(G, V)$. Recall for $g, x \in G$, and $f \in C^q(G, V)$, $L_x f(g) = f(x^{-1}g)$, and $R_x f(g) = f(gx)$. Obviously L_x leaves $C_r^q(G, V)$ invariant. In fact $\|L_x f\|_{q,r} \leq C \|x\|^{r+s} \|f\|_{q,r}$, for each $f \in C_r^q(G, V)$, and $x \in G$. Here C and s are constants. On the other hand,

$\|R_x f\|_{q,r} \leq \|x\|^r \|f\|_{q,r}$. From Example 2.2, we see P_T maps $C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$ into $C_{r(\lambda)}^\infty(G, V)$ continuously.

Recall from Proposition 1.11 a character of Z_τ is given by $\lambda - \Lambda$, where $\lambda \in \mathfrak{a}_\mathbb{C}^*$, and Λ is the infinitesimal character of an irreducible representation of M in τ . Let $\mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau)$ denote the corresponding eigenspace of Z_τ . Let

$$\begin{aligned} \mathcal{E}_{\lambda-\Lambda, r} \text{Ind}_K^G(\tau) &= \mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) \cap C_r^\infty(G, V), \\ \mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) &= \bigcup_{r \in \mathbb{R}} \mathcal{E}_{\lambda-\Lambda, r} \text{Ind}_K^G(\tau). \end{aligned}$$

Our goal is to describe $\mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau)$ in terms of the Poisson transform, at least for “generic” $\lambda - \Lambda$. The following result due to Harish-Chandra is very important to us. See [Wall2].

Proposition 2.3. *Let $\overline{N} = \Theta N$. Then $C(\lambda)$ defined by*

$$C(\lambda) = \int_{\overline{N}} \tau(\mathbf{k}(\overline{n})) e^{-(\lambda+\rho)\mathbf{H}(\overline{n})} d\overline{n}$$

is holomorphic on $\{\lambda \in \mathfrak{a}_\mathbb{C}^ \mid \text{Re}\langle \lambda, \alpha \rangle > 0, \text{ for each } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}$. Moreover there exists a meromorphic continuation to $\mathfrak{a}_\mathbb{C}^*$.*

Proposition 2.4. *Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ such that $\text{Re}\langle \lambda, \alpha \rangle > 0$, for $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. Then*

$$\lim_{t \rightarrow \infty} e^{(-\lambda+\rho)(H)} P_T \phi(g \exp tH) = C(\lambda) T\phi,$$

for each $H \in \mathfrak{a}^+$, $T \in \text{Hom}_M(V_\sigma, V)$, and $\phi \in C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$. Here $\mathfrak{a}^+ = \{X \in \mathfrak{a} \mid \alpha(X) > 0, \forall \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}$.

Proof. First we observe $k \rightarrow \tau(k) T\phi(g \exp tHk)$ is a function on K/M . By Theorem 5.20 in Chapter I in [Helg1],

$$\begin{aligned} P_T \phi(g \exp tH) &= \int_{\overline{N}} \tau(\mathbf{k}(\overline{n})) T\phi(g \exp tH \mathbf{k}(\overline{n})) e^{-2\rho \mathbf{H}(\overline{n})} d\overline{n} \\ &= \int_{\overline{N}} e^{-(\lambda+\rho)\mathbf{H}(\overline{n})} \tau(\mathbf{k}(\overline{n})) T\phi(g \exp tH \overline{n}) d\overline{n} \\ &= e^{(\lambda-\rho)tH} \int_{\overline{N}} e^{-(\lambda+\rho)\mathbf{H}(\overline{n})} \tau(\mathbf{k}(\overline{n})) T\phi(g a_t \overline{n} a_t^{-1}) d\overline{n}. \end{aligned}$$

Here $a_t = \exp tH$. So

$$e^{-(\lambda-\rho)tH} P_T \phi(g \exp tH) = \int_{\overline{N}} e^{-(\lambda+\rho)\mathbf{H}(\overline{n})} \tau(\mathbf{k}(\overline{n})) T\phi(g a_t \overline{n} a_t^{-1}) d\overline{n}$$

since $a_t \overline{n} a_t^{-1} \rightarrow e$, as $t \rightarrow \infty$. Formally we have

$$P_T \phi(g \exp tH) \rightarrow C(\lambda) T\phi(g),$$

as $t \rightarrow \infty$. To justify the exchange of two limits we use an argument due to Helgason. Let $\lambda = \xi + \sqrt{-1}\eta$, for $\xi, \eta \in \mathfrak{a}^*$. Our assumption on λ amounts to $A_\xi \in \mathfrak{a}^+$, where A_ξ is given by $\langle \mu, A_\xi \rangle = K(\xi, \mu)$, for each $\mu \in \mathfrak{a}^*$. It was proved by Harish-Chandra that

$$B(H, \mathbf{H}(\overline{n})) \geq 0, \quad B(H, \mathbf{H}(\overline{n}) - \mathbf{H}(a_t \overline{n} a_t^{-1})) \geq 0,$$

for each $H \in \mathfrak{a}^+$. Thus if we choose ε such that $0 < \varepsilon < 1$, $A_\rho - \varepsilon A_\xi \in \mathfrak{a}^+$, and put

$$C = \sup_{\overline{n}, t} |\tau(\mathbf{k}(\overline{n})) T\phi(g \mathbf{k}(a_t \overline{n} a_t^{-1}))| < \infty,$$

then

$$\begin{aligned}
& |e^{-(\lambda+\rho)H(\bar{n})}\tau(k(\bar{n}))T\phi(ga_t\bar{n}a_t^{-1})| \\
&= |e^{-(\lambda+\rho)H(\bar{n})}e^{(\lambda-\rho)H(a_t\bar{n}a_t^{-1})}\tau(k(\bar{n}))T\phi(gk(a_t\bar{n}a_t^{-1}))| \\
&\leq Ce^{-(\xi+\rho)H(\bar{n})}e^{(\xi-\rho)H(a_t\bar{n}a_t^{-1})} \\
&\leq Ce^{-(\xi+\rho)H(\bar{n})}e^{(\xi-\varepsilon\xi)H(a_t\bar{n}a_t^{-1})} \\
&\leq Ce^{-(\xi+\rho)H(\bar{n})}e^{(\xi-\varepsilon\xi)H(\bar{n})} \\
&\leq Ce^{(-\varepsilon\xi-\rho)H(\bar{n})}.
\end{aligned}$$

This being integrable over $\bar{N}$ justifies letting $t \rightarrow \infty$ under the integral sign and proves Proposition 2.4. $\square$

3. ASYMPTOTICS

By a formal expansion at a point $H_0 \in \mathfrak{a}^+$, we mean a formal sum

$$\sum_{\xi \in X} p_\xi(H, t) e^{t\xi(H)},$$

where X is a subset of $\mathfrak{a}_\mathbb{C}^*$ such that the subset $X(N)$ given by

$$X(N) = \{\xi \in X \mid \operatorname{Re} \xi(H_0) \geq N\}$$

is a finite set for each $N \in \mathbb{R}$, where p_ξ is a continuous function defined in a neighborhood of $\{H_0\} \times \mathbb{R}$ and polynomial in the last variable.

Let f be a function $\mathfrak{a}^+ \rightarrow V$. If $N \in \mathbb{R}$ we say the formal sum is asymptotic to f of order N at H_0 , if there exist a neighborhood of H_0 in $\mathfrak{a}^+$, say U , and constants $\varepsilon \geq 0$, $C \geq 0$, such that

$$\left| f(tH) - \sum_{\xi \in X(N)} p_\xi(H, t) e^{t\xi(H)} \right| \leq C e^{(N-\varepsilon)t},$$

for each $H \in U$, $t \geq 0$. Moreover, we say the formal expansion is an asymptotic expansion for f at H_0 if for every $N \in \mathbb{R}$ it is asymptotic to f of order N at H_0 . We write this as

$$f(tH) \sim \sum_{\xi \in X} p_\xi(H, t) e^{t\xi(H)} \quad (t \rightarrow \infty).$$

The following result shows that the p_ξ 's are essentially unique.

Proposition 3.1. *Let $X \subset \mathfrak{a}_\mathbb{C}^*$, $\sum_{\xi \in X} p_\xi(H, t) e^{t\xi(H)}$ and $\sum_{\xi \in X} q_\xi(H, t) e^{t\xi(H)}$ be formal expansions at H_0 , both assumed to be asymptotic to $f: \mathfrak{a}^+ \rightarrow V$. Then for each $\xi \in X$, there is a neighborhood U of H_0 , such that $p_\xi = q_\xi$ on $U \times \mathbb{R}$.*

Proof. See Proposition 3.1 in [BS]. $\square$

Let $\lambda - \Lambda$ be a character of Z_τ in the sense of Proposition 1.11, where $\lambda \in \mathfrak{a}_\mathbb{C}^*$, and Λ is given by the infinitesimal character of an irreducible representation of M . Let $X(\lambda, \Lambda)$ be the subset of $\mathfrak{a}_\mathbb{C}^*$ defined by

$$X(\lambda, \Lambda) = \{w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mathbb{N} \cdot \Delta \mid w \in \widetilde{W}, \sigma \in \tau, (w(\lambda - \Lambda) + \Lambda_\sigma) | \mathfrak{t} = 0\}.$$

Then we have the following results.

Theorem 3.2. (i) For each $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, $x \in G$, and $\xi \in X(\lambda, \Lambda)$, there exists a unique polynomial $p_{\lambda, \xi}(f, x, \cdot)$ on $\mathfrak{a}$ with values in V , such that

$$f(tH) \sim \sum_{\xi \in X(\lambda, \Lambda)} p_{\lambda, \xi}(f, x, tH) e^{t\xi(H)} \quad (t \rightarrow \infty),$$

at every $H_0 \in \mathfrak{a}^+$, and the polynomials have degree $\leq d$, where d is the number of elements in $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

(ii) Let $r \in \mathbb{R}$ and $\xi \in X(\lambda, \Lambda)$; there exists $r' \in \mathbb{R}$ such that $f \mapsto p_{\lambda, \xi}(f, \cdot, \cdot)$ is a continuous map of $\mathcal{E}_{\lambda-\Lambda, r}^\infty \text{Ind}_K^G(\tau)$ into $C_{r'}^\infty(G, V) \otimes P_d(\mathfrak{a})$, equivariant for the left action of G on $\mathcal{E}_{\lambda-\Lambda, r}^\infty \text{Ind}_K^G(\tau)$ to $C_{r'}^\infty(G, V) \otimes P_d(\mathfrak{a})$.

Theorem 3.3. Let Ω be an open set in $\mathfrak{a}_{\mathbb{C}}^*$. Let $\{f_\lambda\}_{\lambda \in \Omega}$ be a holomorphic family in $C_r^\infty \text{Ind}_K^G(\tau)$ such that $f_\lambda \in \mathcal{E}_{\lambda-\Lambda, r}^\infty \text{Ind}_K^G(\tau)$ for each $\lambda \in \Omega$. Fix $\lambda_0 \in \Omega$ and $\xi_0 \in X(\lambda_0, \Lambda)$. Let

$$\Xi(\lambda) = \{w(\lambda - \Lambda) + \Lambda_\sigma - \rho - \mu \in X(\lambda, \Lambda) | w(\lambda_0 - \Lambda) + \Lambda_\sigma - \rho - \mu = \xi_0\}.$$

There exist an open neighborhood $\Omega_0 \subset \Omega$ of λ_0 and a constant $r' \in \mathbb{R}$ such that the map $(\lambda, H) \mapsto \sum_{\xi \in \Xi(\lambda)} p_{\lambda, \xi}(f_\lambda, \cdot, H) e^{\xi(H)}$ is continuous from $\Omega \times \mathfrak{a}^+$ into $C_{r'}^\infty(G, V)$ and in addition holomorphic in λ .

We shall prove these results in Sections 5 and 6.

4. SOME ALGEBRAIC RESULTS

This section is a necessary preparation for the proof of the theorems stated in last section. It is strongly influenced by [Ban] and [BS].

Let E be the set of W -harmonic polynomials in $\mathfrak{a}^*$. It is well-known that $j: E \otimes I(\mathfrak{a}) \rightarrow S(\mathfrak{a})$ is bijective, where $j(e \otimes h) = eh$.

Now let $r: I(\mathfrak{h}_{\mathbb{C}}) \rightarrow I(\mathfrak{a})$ be the restriction map. **We assume r is surjective for the rest of the thesis.** According to [Helg3] if G/K is irreducible there are just four exceptions, and they only occur among symmetric spaces of exceptional groups. Pick a set of algebraically independent homogeneous generators of $I(\mathfrak{a})$, say, $p_1, \dots, p_m$. Choose homogeneous elements $q_1, \dots, q_m$ in $I(\mathfrak{h}_{\mathbb{C}})$, such that $r(q_i) = p_i$, for $i = 1, \dots, m$. Let $I_1(\mathfrak{h}_{\mathbb{C}})$ be the polynomial ring of $q_1, \dots, q_m$.

For any $\mu \in \mathfrak{h}_{\mathbb{C}}^*$, let

$$I_{1, \mu}^- = \{(T_\mu p)^- | p \in I_1(\mathfrak{h}_{\mathbb{C}})\}.$$

Here $(T_\mu p)(\nu) = p(\mu + \nu)$, for each $\nu \in \mathfrak{h}_{\mathbb{C}}^*$, and $(T_\mu p)^-(\lambda) = p(\mu + \lambda)$, for each $\lambda \in \mathfrak{a}^*$.

Proposition 4.1. Let $j_\mu: E \otimes I_{1, \mu}^- \rightarrow S(\mathfrak{a})$ be defined by

$$j_\mu(e \otimes h) = eh.$$

Then j_μ is bijective.

Proof. Observe $(T_\mu q_i)^- = p_i + r_i$, with $\deg r_i < \deg p_i$. Using the fact that j is bijective and by induction we are done. $\square$

Let $\mathcal{Z}_1(\mathfrak{g}) = \gamma^{-1}(I_1(\mathfrak{h}))$. Here γ is the Harish-Chandra homomorphism. For each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\Lambda = \Lambda_\sigma$ for some $\sigma \in \tau$, let

$$I(\lambda, \Lambda) = \{Z \in \mathcal{Z}_1(\mathfrak{g}) | \gamma(Z)(\lambda - \Lambda) = 0\}.$$

Recall $I(\tau)$ is the left ideal of $U(\mathfrak{g}) \otimes \text{End}(V)$ generated by $X \otimes 1 + 1 \otimes \tau(X)$, for all $X \in \mathfrak{k}$. Let $J(\lambda, \Lambda)$ be the left ideal generated by $I(\lambda, \Lambda)$ and $I(\tau)$. Let

$$\mathfrak{Y}_{\lambda, \Lambda} = U(\mathfrak{g}) \otimes \text{End}(V) / J(\lambda, \Lambda).$$

Our interest in $\mathfrak{Y}_{\lambda, \Lambda}$ comes from the fact that for $f \in \mathcal{E}_{\lambda - \Lambda} \text{Ind}_K^G(\tau)$, the map $u \otimes T \rightarrow TR_u f$ factors through $\mathfrak{Y}_{\lambda, \Lambda}$ since f is killed by $J(\lambda, \Lambda)$. We shall find below an underlying vector space for $\mathfrak{Y}_{\lambda, \Lambda}$ independent of λ .

Write $\mathfrak{Y} = U(\bar{\mathfrak{n}}) \otimes E \otimes \text{End}(V)$. We shall construct a linear bijection of $\mathfrak{Y}$ with $\mathfrak{Y}_{\lambda, \Lambda}$. First we identify $\mathfrak{Y}$ with a subspace of $U(\mathfrak{g}) \otimes \text{End}(V)$ as follows: $u \otimes e \otimes T \rightarrow (u \cdot e) \otimes T$, for $u \in U(\bar{\mathfrak{n}})$, $e \in E$, and $T \in \text{End}(V)$. Here $\cdot$ denotes the multiplication in $U(\mathfrak{a} + \bar{\mathfrak{n}})$. Let $\Psi: \mathfrak{Y} \otimes \mathcal{Z}_1(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes \text{End}(V) / I(\tau)$ be the map defined by

$$\Psi(y \otimes Z) = y \cdot (Z \otimes 1) + I(\tau),$$

for $y \in \mathfrak{Y}$, $Z \in \mathcal{Z}_1(\mathfrak{g})$. Here $\cdot$ means the multiplication in $U(\mathfrak{g}) \otimes \text{End}(V)$.

Proposition 4.2. *In the setting above, Ψ is bijective.*

Proof. By the Iwasawa decomposition we have

$$U(\mathfrak{g}) \otimes \text{End}(V) / I(\tau) \cong U(\bar{\mathfrak{n}}) \otimes U(\mathfrak{a}) \otimes \text{End}(V).$$

Via this isomorphism the degree on $U(\mathfrak{a})$ induces a degree (denoted by $\deg_{\mathfrak{a}}$) on $U(\mathfrak{g}) \otimes \text{End}(V) / I(\tau)$. Let $\mathfrak{Y} \otimes \mathcal{Z}_1(\mathfrak{g})$ be filtered by the total degree on $E \otimes \mathcal{Z}_1(\mathfrak{g})$. Notice

$$\deg_{\mathfrak{a}}(Z \otimes 1 - (T_{\rho - \Lambda_{\sigma}} \gamma(Z))^{-1} \otimes 1 + I(\tau)) < \deg(Z \otimes 1),$$

for $Z \in \mathcal{Z}_1(\mathfrak{g})$, and each $\sigma \in \tau$. So Ψ preserves the filtrations. It also follows that the graded map

$$gr \Psi: U(\bar{\mathfrak{n}}) \otimes gr(E \otimes \mathcal{Z}_1(\mathfrak{g})) \otimes \text{End}(V) \rightarrow U(\bar{\mathfrak{n}}) \otimes U(\mathfrak{a}) \otimes \text{End}(V)$$

associated to Ψ , is given by

$$u \otimes e \otimes Z \otimes T \rightarrow u \cdot e \cdot (T_{\rho - \Lambda_{\sigma}} \gamma(Z))^{-} \otimes T,$$

for $u \in U(\bar{\mathfrak{n}})$, $e \in E$, $Z \in \mathcal{Z}_1(\mathfrak{g})$, and $T \in \text{Hom}(V(\sigma), V)$ (here we use Proposition 1.15). This is bijective because of Proposition 4.1. $\square$

Corollary 4.3. (i) Ψ maps $\mathfrak{Y} \otimes I(\lambda, \Lambda)$ onto $J(\lambda, \Lambda)$ modulo $I(\tau)$; (ii) for each $u \in U(\mathfrak{g}) \otimes \text{End}(V)$ there exists a unique $y \in \mathfrak{Y}$, such that $u - y \in J(\lambda, \Lambda)$.

Proof. See Corollary 5.2 in [BS]. $\square$

From the corollary we obtain a linear bijection b_{λ} of $\mathfrak{Y}_{\lambda, \Lambda}$ onto $\mathfrak{Y}$, defined by $u - b_{\lambda}(u + J(\lambda, \Lambda)) \subset J(\lambda, \Lambda)$. Through this bijection $\mathfrak{Y}$ is equipped with a $(\mathfrak{g}, K)$ -module structure from $\mathfrak{Y}_{\lambda, \Lambda}$, by making b_{λ} a morphism of modules. Recall the $\mathfrak{g}$ action on $\mathfrak{Y}_{\lambda, \Lambda}$ is induced from left multiplication in $U(\mathfrak{g})$, and the K action is induced from the following K action on $U(\mathfrak{g}) \otimes \text{End}(V)$,

$$k.(u \otimes T) = \text{Ad}(k)u \otimes T\tau(k^{-1}),$$

for each $k \in K$, $u \in U(\mathfrak{g})$, and $T \in \text{End}(V)$. Notice the difference from the action we use to define $U(\mathfrak{g})^K$.

Let τ_{λ} denote the resulting $\mathfrak{g}$ action on $\mathfrak{Y}$. Notice the action of $\bar{\mathfrak{n}}$ on $\mathfrak{Y}$ is just the left multiplication. The action of $\mathfrak{a}$ can be determined as follows: Let

$y \in \mathfrak{Y} \subset U(\mathfrak{g}) \otimes \text{End}(V)$, $H \in \mathfrak{a}$; then $H \cdot y$ can be written (modulo $I(\tau)$) as $\Psi(\sum y_i \otimes Z_i)$ according to Proposition 4.2. Then by the definition of τ_λ we have

$$(*) \quad \tau_\lambda(H)y = \sum \gamma(Z_i)(\lambda - \Lambda)y_i.$$

For each $k \in \mathbb{N}$, let $\bar{\mathfrak{n}}^k$ be the linear span of k times product of $\bar{\mathfrak{n}}$ in $U(\bar{\mathfrak{n}})$. Then τ_λ induces a representation τ_λ^k of $\mathfrak{a} + \mathfrak{m}$ on the finite dimensional space $\mathfrak{Y}/\bar{\mathfrak{n}}^k\mathfrak{Y}$. In particular τ_λ^1 is a representation of $\mathfrak{a} + \mathfrak{m}$ on $\mathfrak{Y}/\bar{\mathfrak{n}}\mathfrak{Y} \cong E \otimes \text{End}(V)$. By $(*)$ we know τ_λ and τ_λ^k are holomorphic in λ .

Let $\{\lambda_1, \dots, \lambda_l\}$ be the set of weights of τ_λ^1 restricted to $\mathfrak{a}$, and $\Lambda_k \subset -\mathbb{N} \cdot \Delta$ an enumeration of the weights of the $\mathfrak{a}$ -module $U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}})$.

Proposition 4.4. *For each $k \in \mathbb{N}$, $k \geq 1$, the set of weights of $(\tau_\lambda^k, \mathfrak{a})$ is*

$$\{\lambda_i + \mu | i = 1, \dots, l, \mu \in \Lambda_k\}.$$

Proof. By induction on k . It is trivial for $k = 1$. For $k > 1$, the induction step is a consequence of the following two exact sequences of $\mathfrak{a}$ -modules.

$$\begin{aligned} 0 \rightarrow \bar{\mathfrak{n}}^{k-1}U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}}) \otimes \mathfrak{Y}_{\lambda, \Lambda} \rightarrow \mathfrak{Y}_{\lambda, \Lambda}/\bar{\mathfrak{n}}^k \mathfrak{Y}_{\lambda, \Lambda} \rightarrow \mathfrak{Y}_{\lambda, \Lambda}/\bar{\mathfrak{n}}^{k-1} \mathfrak{Y}_{\lambda, \Lambda} \rightarrow 0, \\ 0 \rightarrow \bar{\mathfrak{n}}^{k-1}U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}}) \rightarrow U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}}) \rightarrow U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^{k-1} U(\bar{\mathfrak{n}}) \rightarrow 0. \end{aligned} \quad \square$$

Let $\bar{V}_k = \mathfrak{Y}/\bar{\mathfrak{n}}^k \mathfrak{Y}$, and $\tilde{V}_k$ be a finite dimensional subspace of $\mathfrak{Y}$ mapped bijectively onto $\bar{V}_k$ by the canonical projection. Let $\pi: \tilde{V}_k \rightarrow \bar{V}_k$ be the restriction of the canonical projection. Define $m: \mathfrak{Y} \rightarrow U(\mathfrak{g}) \otimes \text{End}(V)$ by

$$m(u \otimes e \otimes T) = (u \cdot e) \otimes T,$$

for $u \in U(\bar{\mathfrak{n}})$, $e \in E$, and $T \in \text{End}(V)$.

Let V_k be the image of $\tilde{V}_k$ under m , and $\eta: V_k \rightarrow \tilde{V}_k$ be the inverse of $m|_{\tilde{V}_k}$. Let $\mathcal{Z}(\mathfrak{a} + \mathfrak{m})$ be the center of $U(\mathfrak{a} + \mathfrak{m})$.

Proposition 4.5. *For $k \in \mathbb{N}$, $k \geq 1$, there exist*

- (i) *an algebra homomorphism $b_k(\lambda, \cdot): \mathcal{Z}(\mathfrak{a} + \mathfrak{m}) \rightarrow \text{End}(V_k)$,*
- (ii) *a linear map $y_\lambda: \mathcal{Z}(\mathfrak{a} + \mathfrak{m}) \otimes V_k \rightarrow \bar{\mathfrak{n}}^k U(\mathfrak{a} + \bar{\mathfrak{n}}) \otimes \text{End}(V)$, both depending polynomially on λ , such that for each $\lambda \in \mathfrak{a}_\mathbb{C}^*$, $D \in \mathcal{Z}(\mathfrak{a} + \mathfrak{m})$, and $v \in V_k$,*

$$Dv - b_k(\lambda, D)v - y_\lambda(D, v) \in J(\lambda, \Lambda).$$

Proof. Let $p_\lambda: U(\mathfrak{g}) \otimes \text{End}(V) \rightarrow \mathfrak{Y}$ be the map defined by

$$p_\lambda(u \otimes T) = \tau_\lambda(u)(1 \otimes 1 \otimes T),$$

for $u \in U(\mathfrak{g})$, and $T \in \text{End}(V)$. For $D \in \mathcal{Z}(\mathfrak{a} + \mathfrak{m})$, $\tilde{v} \in \tilde{V}_k$ we define the maps

$$\begin{aligned} \tilde{b}_k(\lambda, D) &= \pi^{-1} \circ \tau_\lambda^k \circ \pi \in \text{End } \tilde{V}_k, \\ \tilde{y}_\lambda(D, \tilde{v}) &= p_\lambda((D \otimes 1) \cdot m(\tilde{v})) - m(\tilde{b}_k(\lambda, D)\tilde{v}) \in \mathfrak{Y}. \end{aligned}$$

Then $b_k(\lambda, \cdot)$ and y_λ are defined by

$$b_k(\lambda, D) = m \circ \tilde{b}_k(\lambda, D) \circ \eta, \quad y_\lambda(D, v) = m(\tilde{y}_\lambda(D, \eta(v))),$$

for $D \in \mathcal{Z}(\mathfrak{a} + \mathfrak{m})$, $v \in V_k$. □

Corollary 4.6. *As a representation of $\mathfrak{a}$, $b_k(\lambda, \cdot)$ has the same weights as $(\tau_\lambda^k, \mathfrak{a})$, i.e. $\{\lambda_i + \mu | i = 1, \dots, l, \mu \in \Lambda_k\}$.*

Proof. By definition $b_k(\lambda, D) = m \circ \tilde{b}_k(\lambda, D) \circ \eta$, and $\eta = (m|_{\tilde{V}_k})^{-1}$. So $b_k(\lambda, \cdot)$ has the same weights as $\tilde{b}_k(\lambda, \cdot)$. Since $\tilde{b}_k(\lambda, \cdot) = \pi^{-1} \circ \tau_\lambda^k \circ \pi$, the proof is complete. $\square$

Let V_k^* be the dual space of V_k , and $b_k^*(\lambda, \cdot)$ be the transpose of $b_k(\lambda, \cdot)$. For each weight ξ of $b_k^*(\lambda, \cdot)$ we denote by $P_{\lambda, \xi}$ the projection map from V_k^* onto the generalized weight space of ξ , along the remaining generalized weight spaces. We now consider the holomorphic dependence of $P_{\lambda, \xi}$ on λ .

Proposition 4.7. *There exists for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and each weight ξ a unique polynomial $q_{\lambda, \xi}$ on $\mathfrak{a}$ with values in $\text{End}(V_k^*)$, such that*

$$P_{\lambda, \xi} q_{\lambda, \xi}(H) P_{\lambda, \xi} = q_{\lambda, \xi}(H),$$

$$\exp b_k^*(\lambda, H) = \sum_{\xi} e^{\xi(H)} q_{\lambda, \xi}(H),$$

for $H \in \mathfrak{a}$.

Proof. Let $V_k^*(\xi)$ be the generalized weight space of ξ . Then the restriction of $b_k^*(\lambda, \cdot)$ to $V_k^*(\xi)$ gives a representation of $\mathfrak{a}$. $\mathfrak{a}$ is abelian so in particular solvable. Hence by Lie's theorem one can find a basis such that $b_k^*(\lambda, H)|_{V_k^*(\xi)}$ corresponds to an upper triangular matrix, for each $H \in \mathfrak{a}$. The diagonal entries are $\xi(H)$. So there exists a unique polynomial $q_{\lambda, \xi}(H)$ on $\mathfrak{a}$ with values in $\text{End}(V_k^*)$, such that

$$\exp b_k^*(\lambda, H)|_{V_k^*(\xi)} = e^{\xi(H)} q_{\lambda, \xi}(H). \quad \square$$

Let F be an N -dimensional complex vector space, and τ_z a family of representations of $\mathfrak{a}$ in F , depending on a parameter $z \in \mathbb{C}^n$. For each weight ξ of τ_z let $P_{z, \xi}$ be the projection map from F onto the generalized weight space $V(\xi)$, along the remaining generalized weight spaces. Fix $z_0 \in \mathbb{C}^n$, and ξ_0 a weight of τ_{z_0} .

Lemma 4.8. *Given any neighborhood $N(\xi_0)$ of ξ_0 there exist a neighborhood $V(\xi_0)$ of ξ_0 in $N(\xi_0)$, and a neighborhood $\Omega(z_0)$ of z_0 , such that*

$$P(z) = \sum_{\xi \in V(\xi_0)} P_{z, \xi} \in \text{End}(F)$$

is holomorphic in z in $\Omega(z_0)$.

Proof. We use the argument in Chapter II in [Kato]. First let us consider the case where $\dim \mathfrak{a} = 1$. Pick a nonzero element $H_0 \in \mathfrak{a}$. Let

$$T(z) = \tau_z(H_0) \in \text{End}(F).$$

Then $\lambda_0 = \xi_0(H_0)$ is an eigenvalue of $T(z_0) = \tau_{z_0}(H_0)$. Define

$$R(z, \lambda) = (T(z) - \lambda)^{-1},$$

for $z \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$. By Theorem 1.5 in Section 3 of Chapter II in [Kato], $R(z, \lambda)$ is holomorphic in the two variables z and λ in each domain where λ is not an eigenvalue of $T(z)$. Moreover, for each (z_1, λ) in such a domain,

$$R(z, \lambda) = R(z_1, \lambda) + \sum_{I \in \mathbb{N}^n} R_I(\lambda) (z - z_1)^I,$$

where $R_I(\lambda)$ are determined by $R(z_1, \lambda)$, and they are holomorphic in λ . This is called the second Neumann series for the resolvent. It is uniformly convergent for sufficiently small $z - z_1$ and $\lambda \in \Gamma$ if Γ is a compact subset of the resolvent set of $T(z_1)$.

Let Γ be a closed positively oriented curve in the resolvent set of $T(z_0)$ enclosing λ_0 but no other eigenvalues of $T(z_0)$. Then

$$P(z) = -\frac{1}{2i\pi} \int_{\Gamma} R(z, \lambda) d\lambda$$

is holomorphic in z , for $z - z_0$ sufficiently small.

It is easy to see $P(z)$ is equal to the sum of the eigenprojections for all eigenvalues of $T(z)$ lying inside Γ . This basically takes care of the case $\dim \mathfrak{a} = 1$. In general we choose a basis $e_1, \dots, e_m$ for $\mathfrak{a}$. We can duplicate the above process to $T_i(z) = \tau_z(e_i)$, for $i = 1, \dots, m$. Thus we get $P_i(z)$, $i = 1, \dots, m$. Then the composition of P_i is our $P(z)$. $\square$

Fix $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$, and ξ_0 a weight of $b_k^*(\lambda_0, \cdot)$. For each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, let

$$\Xi(\lambda) = \{w(\lambda - \Lambda) + \Lambda_{\sigma} - \rho - \mu \in X(\lambda, \Lambda) | w(\lambda_0 - \Lambda) + \Lambda_{\sigma} - \rho - \mu = \xi_0\}.$$

Proposition 4.9. *There exist a neighborhood $\Omega_0(\lambda_0)$ of λ_0 and a neighborhood $V(\xi_0)$ of ξ_0 , such that*

$$P(\lambda) = \sum_{\xi \in V(\xi_0)} P_{\lambda, \xi} \in \text{End}(V_k^*)$$

is holomorphic in $\Omega_0(\lambda_0)$, and

$$\{\xi \in V(\xi_0) | \xi \text{ is a weight of } b_k^*(\lambda, \cdot)\} \cap X(\lambda, \Lambda) \subset \Xi(\lambda).$$

Proof. It follows at once from Lemma 4.8. $\square$

5. EXISTENCE OF ASYMPTOTIC EXPANSION

The methods we use in this section are similar to those used in [Ban], Section 12. Also see [BS], Section 6.

Fix $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $H_0 \in \mathfrak{a}^+$ and $r \in \mathbb{R}$. If A_1, A_2 are Banach spaces we denote by $B(A_1, A_2)$ the Banach space of bounded linear operators from A_1 to A_2 .

Proposition 5.1. *There exist, for each $N \in \mathbb{R}$,*

- (a) *open neighborhoods Ω of $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ and U of $H_0 \in \mathfrak{a}^+$,*
- (b) *constants $k, q \in \mathbb{N}, r' \geq r$, and $C, \varepsilon > 0$,*
- (c) *a continuous map*

$$\Phi: \Omega \times U \rightarrow B(C_r^q(G, V), V_k^* \otimes C_{r'}(G, V)),$$

holomorphic in the first variable, and

- (d) *a linear form $\eta \in (V_k^*)^*$, such that*
- (i) *$\Phi(\lambda, H)$ intertwines the left actions of G on $C_r^q(G, V)$ and $C_{r'}(G, V)$, for all $(\lambda, H) \in \Omega \times U$, and (ii)*

$$\|R_{\exp tH} f - (\eta \circ \exp b_k^*(\lambda, tH) \otimes 1) \Phi(\lambda, H) f\|_{r'} \leq C \|f\|_{q, r} e^{(N-\varepsilon)t},$$

for $f \in \mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) \cap C_r^q(G, V)$, $\lambda \in \Omega$, $H \in U$, $t \geq 0$.

Proof. In the same way as for Proposition 12.6 in [Ban]. $\square$

We now begin the proof of Theorem 3.2. Using Proposition 4.7 we can write

$$(\eta \circ \exp b_k^*(\lambda, tH) \otimes 1) \Phi(\lambda, H) = \sum_{\xi} p_{\lambda, \xi}(H, t) e^{t\xi(H)},$$

for $\lambda \in \Omega$, $H \in U$, $t \geq 0$, where the summation extends to the weights ξ of $b_k^*(\lambda, \cdot)$ which by Corollary 4.6 is the set

$$\{\lambda_i + \mu | i = 1, \dots, l, \mu \in \Lambda_k\},$$

and where $p_{\lambda, \xi}(H, t) = (\eta \circ q_{\lambda, \xi}(tH) \otimes 1)\Phi(\lambda, H) \in B(C_r^q, C_{r'}^q)$, which is continuous in H and polynomial in t . From (d) (ii) of Proposition 5.1 we have

$$\|R_{\exp tH} f - \sum_{\xi} e^{t\xi(H)} p_{\lambda, \xi}(H, t) f\|_{r'} \leq C \|f\|_{q, r} e^{t(N-\varepsilon)},$$

for $f \in \mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) \cap C_r^q(G, V)$. Since N is arbitrary we have for each $g \in G$,

$$f(g \exp tH) \sim \sum_{\xi \in \tilde{X}(\lambda, \Lambda)} (p_{\lambda, \xi}(H, t) f)(g) e^{t\xi(H)} \quad (t \rightarrow \infty).$$

Here $\tilde{X}(\lambda, \Lambda) = \{\lambda_i + \mu | i = 1, \dots, l, \mu \in -\mathbb{N} \cdot \Delta\}$.

Lemma 5.2. *Let $X \subset \mathfrak{a}_{\mathbb{C}}^*$ and $f: \mathfrak{a}^+ \rightarrow V$. Assume that for each $H_0 \in \mathfrak{a}^+$ there is a given formal expansion*

$$\sum_{\xi \in X} p_{\xi, H_0}(H, t) e^{t\xi(H)}$$

which is an asymptotic expansion for f at H_0 . Then for each $\xi \in X$ there exists a unique continuous function $p_{\xi}: \mathfrak{a}^+ \rightarrow V$ such that for each $H_0 \in \mathfrak{a}^+$ there is a neighborhood U with

$$p_{\xi, H_0}(H, t) = p_{\xi}(tH),$$

for $H \in U$, and $t > 0$.

Proof. See Corollary 3.4 in [BS]. $\square$

As can be seen in the proof of Proposition 12.6 in [Ban], for $t > 0$, $H \in U$ with $tH \in U$, $\Phi(\lambda, tH) = \Phi(\lambda, H)$. Thus for $t > 0$, $H \in U$ with $tH \in U$, $(p_{\lambda, \xi}(H, t) f)(g) = (p_{\lambda, \xi}(tH, 1) f)(g)$. By Lemma 5.2, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $r \in \mathbb{R}$, and $\xi \in \tilde{X}(\lambda, \Lambda)$, there exist constants $r' \in \mathbb{R}$, $q \in \mathbb{N}$, and a unique continuous map $p_{\lambda, \xi}(\cdot, \cdot, \cdot): \mathfrak{a}^+ \rightarrow B(\mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) \cap C_r^q(G, V), C_{r'}^q(G, V))$, such that

$$f(g \exp tH) \sim \sum_{\xi \in \tilde{X}(\lambda, \Lambda)} p_{\lambda, \xi}(f, g, tH) e^{t\xi(H)} \quad (t \rightarrow \infty),$$

at every $H_0 \in \mathfrak{a}$, for $f \in \mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) \cap C_r^q(G, V)$.

To complete the proof of Theorem 3.2 it remains to show (1) we can replace $\tilde{X}(\lambda, \Lambda)$ by $X(\lambda, \Lambda)$, (2) $p_{\lambda, \xi}(f, g, H)$ is a polynomial in H with order $\leq d$. We shall finish the proof in the next section. We now consider the holomorphic dependence in λ in order to prove Theorem 3.3.

Let $r \in \mathbb{R}$ and Ω be an open set in $\mathfrak{a}_{\mathbb{C}}^*$. Let $\{f_{\lambda}\}_{\lambda \in \Omega}$ be a holomorphic family in $C_r^{\infty}(G, V)$, and $f_{\lambda} \in \mathcal{E}_{\lambda-\Lambda}^{\infty} \text{Ind}_K^G(\tau)$, for each $\lambda \in \Omega$. We now study the asymptotic expansion of f_{λ} . Fix $\lambda_0 \in \Omega$, and $\xi_0 \in \tilde{X}(\lambda_0, \Lambda)$.

Proposition 5.3. *There exist a neighborhood $\Omega(\lambda_0)$ of λ_0 in Ω and a neighborhood $V(\xi_0)$ of ξ_0 in $\mathfrak{a}_{\mathbb{C}}^*$, such that*

$$(\lambda, H) \rightarrow \sum_{\xi \in V(\xi_0)} p_{\lambda, \xi}(f_{\lambda}, \cdot, H) e^{\xi(H)}$$

is continuous from $\Omega(\lambda_0) \times U$ to $C_{r'}^{q'}(G, V)$ for some $q' \in \mathbb{N}$, $r' \in \mathbb{R}$, and in addition holomorphic in λ . Moreover, we can choose $V(\xi_0)$ such that $V(\xi_0) \cap X(\lambda, \Lambda) \subset \Xi(\lambda)$.

Proof. It follows from Proposition 4.9. $\square$

6. DIFFERENTIAL EQUATIONS FOR THE COEFFICIENTS

In this section we derive certain differential equations for the vector-valued functions $p_{\lambda, \xi}(f, g, \cdot)$ on $\mathfrak{a}^+$, where $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, and $g \in G$.

Fix $Z \in \mathcal{Z}(\mathfrak{g})$, and $D = \mu(Z) \in Z_\tau$. We can choose finitely many x_i in $\overline{\mathfrak{n}}U(\overline{\mathfrak{n}})$, and $v_i \in U(\mathfrak{a}) \otimes \text{End}(V)$, such that

$$Z - \tilde{\Gamma}_1(Z \otimes 1) - \sum x_i v_i \in I(\tau),$$

and $\text{ad}(\mathfrak{a})$ acts on x_i by a weight $-\eta_i \neq 0$, where $\eta_i \in \mathbb{N} \cdot \Delta$, and $v_i, \tilde{\Gamma}_1(Z \otimes 1) \in U(\mathfrak{a}) \otimes \text{End}(V)$ which can be interpreted as differential operators with constant coefficients on $C^\infty(\mathfrak{a}, V)$.

Proposition 6.1. *Let $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$. Then the functions $p_{\lambda, \xi}(f, \cdot, \cdot)e^\xi$ from $G \times \mathfrak{a}^+$ to V satisfy the following recursive equations:*

$$\begin{aligned} & 1 \otimes \partial(\tilde{\Gamma}_1(Z \otimes 1) - \gamma(Z)(\lambda - \Lambda))(p_{\lambda, \xi}(f, \cdot, \cdot)e^\xi) \\ &= - \sum_{i, \xi + \eta_i \in \tilde{X}(\lambda, \Lambda)} R_{x_i} \otimes e^{-\eta_i} \partial(v^i)(p_{\lambda, \xi + \eta_i}(f, \cdot, \cdot)e^{\xi + \eta_i}), \end{aligned}$$

for all $\xi \in \tilde{X}(\lambda, \Lambda)$.

The proof is the same as for Proposition 7.1 in [BS].

Proof of Theorem 3.2. Let

$$V = \bigoplus_{\Lambda_1 \in \mathfrak{t}^*} V(\Lambda_1),$$

where $V(\Lambda_1) = \bigoplus_{\sigma \in \tau, \Lambda_\sigma = \Lambda_1} V(\sigma)$. Let $P(\Lambda_1)$ be the projection from V to $V(\Lambda_1)$. By Corollary 1.15 $\Gamma_1(Z \otimes 1)V(\Lambda_1) = (T_{\rho-\Lambda_1}\gamma(Z))^- \otimes I_{V(\Lambda_1)}$. For $\xi_1, \xi_2 \in \mathfrak{a}_\mathbb{C}^*$, we say $\xi_1 \prec \xi_2$ if there exists $\eta \in \mathbb{N} \cdot \Delta$ such that $\xi_2 = \xi_1 + \eta$. This defines a partial order on $\mathfrak{a}_\mathbb{C}^*$. For each $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, define $E(\lambda, \Lambda, f)$ by

$$E(\lambda, \Lambda, f) = \{\xi \in \tilde{X}(\lambda, \Lambda) | p_{\lambda, \xi}(f, \cdot, \cdot) \neq 0\}.$$

Let $E_L(\lambda, \Lambda, f)$ denote the set of maximal elements in $E(\lambda, \Lambda, f)$. Suppose $\xi \in E_L(\lambda, \Lambda, f)$. Then $p_{\lambda, \xi}(f, \cdot, \cdot) \neq 0$. So one can find $g \in G$, $\Lambda_1 \in \mathfrak{t}^*$, such that $P(\Lambda_1)p_{\lambda, \xi}(f, g, \cdot) \neq 0$.

Since the right-hand side of the equation in Proposition 6.1 is zero because ξ is maximal in $E(\lambda, \Lambda, f)$,

$$\partial(\tilde{\Gamma}_1(Z \otimes 1) - \gamma(Z)(\lambda - \Lambda))(p_{\lambda, \xi}(f, g, \cdot)e^\xi) = 0.$$

So

$$\partial((T_{-\Lambda_1+\rho}\gamma(Z))^- - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda, \xi}(f, g, \cdot)e^\xi) = 0.$$

We extend $p_{\lambda, \xi}(f, g, \cdot)e^\xi$ to a function on $\mathfrak{a}^+ + \sqrt{-1}\mathfrak{t} \subset \mathfrak{h} = \mathfrak{a} + \sqrt{-1}\mathfrak{t}$, by abuse of notation still denoted by $p_{\lambda, \xi}(f, g, \cdot)e^\xi$, by the requirement that it be constant in the $\mathfrak{t}$ direction. Hence

$$\partial((T_{-\Lambda_1+\rho}\gamma(Z)) - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda, \xi}(f, g, \cdot)e^\xi) = 0.$$

So

$$\partial((\gamma(Z)) - \gamma(Z)(\lambda - \Lambda))(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi - \Lambda_1 + \rho}) = 0.$$

By Theorem 3.13, Chapter III in [Helg1], $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)e^{\xi - \Lambda_1 + \rho} = \sum q_i e^{\mu_i}$, where q_i are polynomials on $\mathfrak{h}$, $\mu_i \in \mathfrak{h}_{\mathbb{C}}^*$. Recall that $p_{\lambda,\xi}(f, g, tH)$ is a polynomial in t . We conclude $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a polynomial on $\mathfrak{h}$, and

$$\xi - \Lambda_1 + \rho = w(\lambda - \Lambda),$$

for some $w \in \widetilde{W}$. Also $P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)$ is a $\widetilde{W}(w(\lambda - \Lambda))$ -harmonic, where $\widetilde{W}(\mu) = \{w \in \widetilde{W} | w\mu = \mu\}$, for each $\mu \in \mathfrak{h}_{\mathbb{C}}^*$. So

$$\deg(P(\Lambda_1)p_{\lambda,\xi}(f, g, \cdot)) \leq d.$$

Here d is the number of elements in $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. It follows that we can replace $\widetilde{X}(\lambda, \Lambda)$ by $X(\lambda, \Lambda)$ since $E_L(\lambda, \Lambda, f) \subset X(\lambda, \Lambda)$.

By induction on ξ using Proposition 6.1 one can easily show $p_{\lambda,\xi}(f, g, \cdot)$ is a polynomial with degree $\leq d$. Note we only need to show it for $g = e$. So this completes the proof of Theorem 3.2.

The proof of Theorem 3.3 follows from Proposition 5.3. $\square$

7. LEADING EXPONENTS

We further consider the properties of a leading term in the asymptotic expansion of $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$.

Proposition 7.1. *For each $\xi \in E_L(\lambda, \Lambda, f)$, $man \in B$, $H \in \mathfrak{a}$, and $g \in G$,*

$$p_{\lambda,\xi}(f, gman, H) = e^{\xi(\log a)} \tau(m)^{-1} p_{\lambda,\xi}(f, g, H + \log a).$$

Proof. The same as for Theorem 8.4 in [BS]. $\square$

Let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. We introduce conditions on $\lambda - \Lambda$ and λ as follows:

$$\mathfrak{A}_1 = \{\lambda - \Lambda | \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \Lambda \in \mathfrak{t}_{\mathbb{C}}^*, \langle \lambda - \Lambda, \alpha^\vee \rangle \notin \mathbb{Z}, \forall \alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}), \alpha|_{\mathfrak{a}} \neq 0\},$$

$$\mathfrak{A}_2 = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* | \langle \lambda, \beta^\vee \rangle \notin -\mathbb{N}, \forall \beta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}.$$

Let

$$\widetilde{W}_0 = \{w \in \widetilde{W} | w|_{\mathfrak{a}} = \text{id}\}, \quad \widetilde{W}_1 = \{w \in \widetilde{W} | w\mathfrak{a} = \mathfrak{a}\}.$$

Proposition 7.2. *Suppose $\lambda - \Lambda \in \mathfrak{A}_1$. We have*

(i) *if $w(\lambda - \Lambda) = \lambda - \Lambda$ for some $w \in \widetilde{W}$, then $w \in \widetilde{W}_0$;*

(ii) *if there exist $w \in \widetilde{W}$, $\sigma \in \tau$ such that*

$$(w(\lambda - \Lambda) + \Lambda_\sigma)|_{\mathfrak{t}} = 0,$$

then $w \in \widetilde{W}_1$, and $\Lambda_\sigma = w\Lambda$.

Proof. (i) Since $w(\lambda - \Lambda) = \lambda - \Lambda$, $w = w_{\alpha_1} \cdots w_{\alpha_s}$, where $\alpha_j \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and $\langle \lambda - \Lambda, \alpha_j \rangle = 0$. Then we conclude $\alpha_j|_{\mathfrak{a}} = 0$ from $\mathfrak{A}_1$. So $w \in \widetilde{W}_0$. (ii) For any $\beta \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ with $\beta|_{\mathfrak{a}} = 0$, we have $\langle w(\lambda - \Lambda) + \Lambda_\sigma, \beta \rangle = 0$ since $(w(\lambda - \Lambda) + \Lambda_\sigma)|_{\mathfrak{t}} = 0$. Hence

$$\begin{aligned} \frac{2\langle \lambda - \Lambda, w^{-1}\beta \rangle}{\langle \beta, \beta \rangle} &= -\frac{2\langle \Lambda_\sigma, \beta \rangle}{\langle \beta, \beta \rangle}, \\ \frac{2\langle \lambda - \Lambda, w^{-1}\beta \rangle}{\langle w^{-1}\beta, w^{-1}\beta \rangle} &= -\frac{2\langle \Lambda_\sigma, \beta \rangle}{\langle \beta, \beta \rangle}. \end{aligned}$$

The right-hand side being integral forces $w^{-1}\beta|\mathfrak{a} = 0$. This shows w preserves $\mathfrak{t}$. Therefore w preserves $\mathfrak{a}$. Trivially $\Lambda_\sigma = w\Lambda$. $\square$

Proposition 7.3. *Let $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$. Suppose $\lambda-\Lambda \in \mathfrak{A}_1$, and ξ in $E_L(\lambda, \Lambda, f)$. Then $\xi \in W\lambda - \rho$, and $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$ for each $g \in G$.*

Proof. In the last section we showed if $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot) \not\equiv 0$, then there exists $w \in \widetilde{W}$, such that $\xi - \Lambda_\sigma + \rho = w(\lambda - \Lambda)$. So

$$(w(\lambda - \Lambda) + \Lambda_\sigma)|\mathfrak{t} = 0.$$

By Proposition 7.2 (ii) $w \in \widetilde{W}_1$. So $\xi + \rho = w\lambda$. Hence $\xi \in W\lambda - \rho$. $\square$

We also showed that $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is $\widetilde{W}(w(\lambda - \Lambda))$ -harmonic. Since $w \in \widetilde{W}_1$, $w(\lambda - \Lambda) \in \mathfrak{A}_1$. By Proposition 7.2 (i) $\widetilde{W}(w(\lambda - \Lambda)) \subset \widetilde{W}_0$. We conclude $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$. This shows $p_{\lambda,\xi}(f, g, \cdot)$ is constant in $\mathfrak{a}$ since $\sigma \in \tau$ is arbitrary. In this case we denote it by $p_{\lambda,\xi}(f, g)$.

Corollary 7.4. *If $\lambda - \rho \in E_L(\lambda, \Lambda, f)$, and in addition λ is regular, i.e., $W(\lambda) = \{w \in W | w\lambda = \lambda\} = e$, then*

$$p_{\lambda,\lambda-\rho}(f, g) = P(\Lambda)p_{\lambda,\lambda-\rho}(f, g).$$

Proof. If for some $\sigma \in \tau$, such that $P(\Lambda_\sigma)p_{\lambda,\xi}(f, g) \not\equiv 0$, then there exists $w \in \widetilde{W}_1$, with

$$w\lambda = (\lambda - \rho) + \rho, w\Lambda_\sigma = \Lambda.$$

λ being regular implies $w \in \widetilde{W}_0$. But then $P(\Lambda) = P(\Lambda_\sigma)$ by definition. $\square$

By Appendix II in [KKMOOT] if $\lambda \in \mathfrak{A}_2$, then $\lambda - \rho$ is always maximal in $W\lambda - \rho$. So we have the following definition.

Definition 7.5. Let $\lambda - \Lambda \in \mathfrak{A}_1$, and $\lambda \in \mathfrak{A}_2$. For $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, $\beta_\lambda(f)$ is defined by

$$\beta_\lambda(f) = p_{\lambda,\lambda-\rho}(f, \cdot).$$

We call β_λ the boundary value map.

Theorem 7.6. *Let $\lambda - \Lambda \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$. Then*

(i) β_λ maps $\mathcal{E}_{\lambda-\Lambda,r}^\infty \text{Ind}_K^G(\tau)$ linearly, continuously, and G -equivariantly into $C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$ for each $r \in \mathbb{R}$, where $\tau(\Lambda)$ is the restriction of τ to M with representation space $V(\Lambda)$.

(ii) Let $\Omega \subset \mathfrak{a}_\mathbb{C}^*$ be open, $\{f_\lambda\}_{\lambda \in \Omega}$ a holomorphic family in $\mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$; then $\lambda \rightarrow \beta_\lambda(f_\lambda)$ is holomorphic in $\Omega \cap \mathfrak{A}_2$.

Proof. (i) comes from Theorem 3.2; (ii) is a result of Theorem 3.3. $\square$

Finally we notice for certain λ we can obtain the boundary value map by a simple limit procedure.

Lemma 7.7. *Let $\lambda - \Lambda \in \mathfrak{A}_1$. If $\text{Re}\langle \lambda, \alpha \rangle > 0$, for each $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, then*

$$\beta_\lambda f(g) = \lim_{t \rightarrow \infty} e^{(-\lambda + \rho)(tH)} f(g \exp tH),$$

for $f \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, and $H \in \mathfrak{a}^+$.

Proof. The condition on λ implies that $\operatorname{Re} \xi(H) < \operatorname{Re}(\lambda - \rho)(H)$ for all $\xi \in X(\lambda, \Lambda)$ with $\xi \neq \lambda - \rho$. Then the result follows from Theorem 3.2 and the very definition of asymptotic expansion. $\square$

For each $\phi \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$, we define $P_\lambda \phi$ by

$$P_\lambda \phi(g) = \int_K \tau(k) \phi(gk) dk.$$

From the proof of Theorem 1.6 we conclude $P_\lambda \phi \in \mathcal{E}_{\lambda-\Lambda, r} \operatorname{Ind}_K^G(\tau)$. By Example 2.2 $P_\lambda \phi \in \mathcal{E}_{\lambda-\Lambda, r}^\infty \operatorname{Ind}_K^G(\tau)$.

Corollary 7.8. *Under the same conditions as in Lemma 7.7,*

$$\beta_\lambda P_\lambda \phi = C(\lambda) \phi,$$

for each $\phi \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.

Proof. By Proposition 2.4 and Lemma 7.7. $\square$

8. THE INVERSION OF THE POISSON TRANSFORM

Let $C(\lambda)$ be the generalized Harish-Chandra C -function given by

$$C(\lambda) = \int_{\overline{N}} e^{-(\lambda+\rho)H(\overline{n})} \tau(k(\overline{n})) d\overline{n}.$$

Recall $P_\lambda: C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1) \rightarrow \mathcal{E}_{\lambda-\Lambda}^\infty \operatorname{Ind}_K^G(\tau)$ is defined by

$$P_\lambda \phi(g) = \int_K \tau(k) \phi(gk) dk.$$

Theorem 8.1. *Let $\lambda - \Lambda \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$, and $C_0(\lambda)$ the restriction of $C(\lambda)$ to $V(\Lambda)$. Then*

$$\beta_\lambda P_\lambda \phi = C_0(\lambda) \phi,$$

for each $\phi \in C^\infty \operatorname{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.

Proof. If $\operatorname{Re} \langle \lambda, \alpha \rangle > 0$, for all $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$, then by Corollary 7.8,

$$\beta_\lambda P_\lambda \phi = C_0(\lambda) \phi.$$

Since $P_\lambda \phi$ is a holomorphic family in $\mathcal{E}_{\lambda-\Lambda}^\infty \operatorname{Ind}_K^G(\tau)$, by Theorem 7.6 the left-hand side is holomorphic. The right-hand side is meromorphic on $\mathfrak{a}_\mathbb{C}^*$. Hence two sides must coincide. $\square$

Corollary 8.2. *If in addition we assume $\det C_0(\lambda) \neq 0$, then β_λ is surjective. Hence P_λ is injective.*

Theorem 8.3. *Let $\lambda - \Lambda \in \mathfrak{A}_1$, and $\lambda \in \mathfrak{A}_2$, and $\det C_0(\lambda) \neq 0$. Then P_λ is bijective, and the inverse of P_λ is given by $C_0(\lambda)^{-1} \beta_\lambda$.*

For the proof we recall a definition which can be found in [Wall], Section 11.6. Let $\mathfrak{V}$ be a finitely generated $(\mathfrak{g}, K)$ -module.

Definition 8.4. $\mathfrak{V}_{mod}^*$ denotes the set of all $\mu \in \mathfrak{V}^*$, such that there exists $d_\mu \in \mathbb{R}$ and for each $\nu \in \mathfrak{V}$ there exist an analytic function $f_{\mu, \nu}$ and a constant $C_{\mu, \nu} > 0$ with the following properties:

- (i) $L_u f_{\mu, \nu}(k) = \mu(k^{-1} \cdot (u \cdot \nu))$, for $u \in U(\mathfrak{g})$, $k \in K$,
- (ii) $|f_{\mu, \nu}(g)| \leq C_{\mu, \nu} \|g\|^{d_\mu}$, for each $g \in G$.

Recall that $(C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))'$ is the strong topological dual of $C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$. The following result can also be found in [Wall], Section 11.7.

Proposition 8.5. *Let $(C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))_{K\text{-finite}}$ denote the space of K -finite elements in $C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$. Then*

$$(C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))_{K\text{-finite}}^*_{\text{mod}} = (C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))'.$$

Before we go ahead with the proof of Theorem 8.3, we mention the following result about the irreducibility of the principal series representations. Let $\sigma \in \widehat{M}$.

Lemma 8.6. *As a $(\mathfrak{g}, K)$ -module $C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$ is irreducible if $\lambda - \Lambda \in \mathfrak{A}_1$.*

Proof. This is a direct consequence of Theorem 1.1 in [SV]. $\square$

Proof of Theorem 8.3. It suffices to show β_λ is injective. Assume the opposite. Then there exists $f_0 \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$, such that $\beta_\lambda f_0 = 0$, and $f_0 \neq 0$. We can assume $f_0(e) \neq 0$ since β is G -equivariant. Define f_K by

$$f_K(g) = \int_K \text{tr } \tau(k) f_0(kg) dk.$$

Then f_K is K -finite, and $f_K(e) = \frac{1}{\dim(\tau)} f_0(e) \neq 0$. Let

$$\mathfrak{W} = L_{U(\mathfrak{g})} L_K f_K.$$

Then $\mathfrak{W}$ is a finitely generated $(\mathfrak{g}, K)$ -module. Let $\mathfrak{W}_1$ be an irreducible submodule of $\mathfrak{W}$. By the subrepresentation theorem and Lemma 8.4 there exists $\sigma \in \widehat{M}$, such that $\mathfrak{W}_1 \cong C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$. So there is a $(\mathfrak{g}, K)$ map

$$P_\sigma: C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}} \rightarrow \mathfrak{W}.$$

It is easy to see $\Lambda = \Lambda_\sigma$. Define $\mu \in \mathfrak{W}^* \otimes V$ by

$$\mu(\nu) = \nu(e),$$

for each $\nu \in \mathfrak{W}$.

Taking $f_{\mu, \nu} = \nu \in \mathcal{E}_{\lambda-\Lambda}^\infty \text{Ind}_K^G(\tau)$ in Definition 8.4, we can verify that (i) and (ii) are satisfied. So $\mu \in \mathfrak{W}_{\text{mod}}^* \otimes V$. Hence

$$\mu^\sharp = \mu \circ P_\sigma \in (C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}})^*_{\text{mod}} \otimes V.$$

Then by Proposition 8.5,

$$\mu^\sharp \in (C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1))' \otimes V.$$

Now define $P_\sigma^\sharp: C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1) \rightarrow C^\infty \text{Ind}_K^G(\tau)$ by

$$P_\sigma^\sharp \phi(g) = \mu^\sharp(L_{g^{-1}} \phi).$$

Since P_σ is a $\mathfrak{g}$ map and eigensections are analytic we can show that for ϕ in $C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$,

$$P_\sigma \phi = P_\sigma^\sharp \phi,$$

by showing they are identical at e along with their derivatives.

We observe that $P_\sigma^\sharp$ is a linear, continuous, and G -equivariant map from $C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$ to $C^\infty \text{Ind}_K^G(\tau)$. By Proposition 1.8 we conclude $\sigma \in \tau$, and there exists $T \in \text{Hom}_M(V_\sigma, V)$ such that $P_\sigma^\sharp = P_T$. Hence

$$P_\sigma = P_T: C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}} \rightarrow \mathfrak{W}.$$

Taking $\sigma \in C^\infty \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$ such that $0 \neq f = P_T \phi$, then $f = P_\lambda(T\phi)$. Notice $T\phi \in C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)_{K\text{-finite}}$. So

$$B_\lambda f = \beta_\lambda P_\lambda(T\phi) = C(\lambda)T\pi \neq 0.$$

This contradicts the assumption $f \in \mathfrak{W} \subset \ker(\beta_\lambda)$. $\square$

9. VECTOR-VALUED DISTRIBUTIONS

Suppose K is a Lie group and V a finite dimensional space over $\mathbb{C}$. Let $C^{-\infty}(K, V)$ denote all continuous $\mathbb{C}$ -linear maps from $C_c^\infty(K, \mathbb{C})$ to V . Let M be a compact subgroup of K , and (π, V) a finite dimensional representation of M . Let $C^{-\infty} \text{Ind}_M^K(\pi)$ be the space defined by

$$\{f \in C^{-\infty}(K, V) \mid R_m f(\phi) = \pi(m^{-1})f(\phi), \forall \phi \in C_c^\infty(K, \mathbb{C}), \forall m \in M\}.$$

Here $R_m f(\phi) = f(R_{m^{-1}}\phi)$, with $R_{m^{-1}}\phi(k) = \phi(km^{-1})$.

Let $(\check{\pi}, V^*)$ be the dual representation of (π, V) , and $\langle, \rangle$ the nondegenerate bilinear form on $V \times V^*$. Let $(C_c^\infty \text{Ind}_M^K(\pi))'$ be the strong dual of $C_c^\infty \text{Ind}_M^K(\pi)$. For each $T \in (C_c^\infty \text{Ind}_M^K(\pi))'$, $\phi \in C_c^\infty(K, \mathbb{C})$, and $v \in V$, we define $\xi_1(T)(\phi) \in V^*$ by

$$\langle v, \xi_1(T)(\phi) \rangle = T(\xi_1(\phi, v)),$$

where $\xi_1(\phi, v)(k) = \int_M \phi(km)\pi(m)v dm$. It is easy to show that

$$\xi_1(T) \in C^{-\infty} \text{Ind}_M^K(\check{\pi}).$$

Proposition 9.1. *The map $\xi_1: (C_c^\infty \text{Ind}_M^K(\pi))' \rightarrow C^{-\infty} \text{Ind}_M^K(\check{\pi})$ is bijective.*

Proof. Define

$$\eta_1: C^{-\infty} \text{Ind}_M^K(\check{\pi}) \rightarrow (C_c^\infty \text{Ind}_M^K(\pi))'$$

as follows: for each $f \in C^{-\infty} \text{Ind}_M^K(\check{\pi})$, and $\phi \in C_c^\infty \text{Ind}_M^K(\pi)$, the map

$$f_\phi: u \rightarrow f(\langle \phi, u \rangle)$$

is a linear map from V^* to V^* . Then we define

$$\eta_1(f) = \text{tr}(f_\phi).$$

It is a long but rather straightforward calculation to show ξ_1 and η_1 are inverses to each other. $\square$

Now let $G = KAN$, and (δ, V_δ) be a finite dimensional representation of $B = MAN$. Let

$$C^\infty \text{Ind}_B^G(\delta) = \{f \in C^\infty(G, V_\delta) \mid R_{man} f = a^{-\rho} \delta^{-1}(man)f, \forall man \in B\},$$

$$C^{-\infty} \text{Ind}_B^G(\delta) = \{f \in C^{-\infty}(G, V_\delta) \mid R_{man} f = a^{-\rho} \delta^{-1}(man)f, \forall man \in B\}.$$

For $T \in (C^\infty \text{Ind}_B^G(\delta))'$, $\xi(T)$ is defined by

$$\langle v, \xi(T)(\phi) \rangle = T(\xi(\phi, v)),$$

for each $v \in V_\delta$, and $\phi \in C_c^\infty(G, \mathbb{C})$. Here $\xi(\phi, v)$ is defined as follows: for each $g \in G$,

$$\xi(\phi, v)(g) = \int_{MAN} \phi(gman) a^\rho \delta(man) v dm dadn.$$

Now we show $\xi(T) \in C^{-\infty} \text{Ind}_B^G(\check{\delta})$. By definition,

$$\langle v, \xi(T)(R_{(man)^{-1}}\phi) \rangle = T(\xi(R_{(man)^{-1}}\phi, v)).$$

However, it is a simple calculation to see

$$\xi(R_{(man)^{-1}}\phi, v) = \xi(\phi, a^{-\rho}\delta(man)v).$$

Hence

$$\begin{aligned} \langle v, R_{man}\xi(T)(\phi) \rangle &= \langle v, \xi(T)(R_{(man)^{-1}}\phi) \rangle \\ &= T(\xi(\phi, a^{-\rho}\delta(man)v)) \\ &= \langle a^{-\rho}\delta(man)v, \xi(T)(\phi) \rangle \\ &= \langle v, a^{-\rho}\check{\delta}((man)^{-1})T(\phi) \rangle. \end{aligned}$$

This proves $\xi(T) \in C^{-\infty} \text{Ind}_B^G(\check{\delta})$.

Theorem 9.2. *Let ξ be defined as above. Then ξ is a G -equivariant bijection from $(C^\infty \text{Ind}_B^G(\delta))'$ to $C^{-\infty} \text{Ind}_B^G(\check{\delta})$.*

Lemma 9.3. *Let L be a Lie group and (π, V) a finite dimensional representation of L on V . Suppose $f \in C^{-\infty}(L, V)$, satisfying*

$$R_l f = \pi(l^{-1})f,$$

for each $l \in L$. Let dl be the right invariant Haar measure on L . Then there exists a unique vector $v \in V$, such that

$$f(\phi) = \int_L \phi(l) \pi(l^{-1}) v dl,$$

for each $\phi \in C_c^\infty(L, \mathbb{C})$.

Proof. We use an argument due to Helgason. For ϕ and ψ in $C_c^\infty(L, \mathbb{C})$, we define $\phi * \psi$ in $C_c^\infty(L, \mathbb{C})$ by

$$\phi * \psi(x) = \int_L \phi(l) \psi(xl^{-1}) dl.$$

Then

$$f(\phi * \psi) = \int_L \phi(l) f(R_{l^{-1}}\psi) dl = \int_L \phi(l) \pi(l^{-1}) f(\psi) dl.$$

Choose a sequence ψ_n such that $\check{\psi}_n \rightarrow \delta$, the delta function, as $n \rightarrow +\infty$. Here $\check{\psi}_n(l) = \psi_n(l^{-1})$. Let $v_n = f(\psi_n)$. Then

$$(*) \quad f(\phi * \psi_n) = \int_L \phi(l) \pi(l^{-1}) v_n dl.$$

We can choose an appropriate ϕ (e.g. close to δ), such that $\int_L \phi(l) \pi(l^{-1}) dl$ is invertible. Since $\phi * \psi_n \rightarrow \phi$, by letting $n \rightarrow +\infty$ in $(*)$, we conclude there exists $v \in V$, such that $v_n \rightarrow v$, and

$$f(\phi) = \int_L \phi(l) \pi(l^{-1}) v dl.$$

The uniqueness follows from the fact that there is ϕ such that $\int_L \phi(l)\pi(l^{-1})dl$ is invertible. $\square$

Proof of Theorem 9.2. First we construct the inverse η of ξ as follows: Take $f \in C^{-\infty} \text{Ind}_B^G(\delta)$, and $\psi \in C^\infty(K, \mathbb{C})$. Then $\phi \rightarrow f(\psi \otimes \phi)$ defines a continuous linear map from $C_c^\infty(A \times N, \mathbb{C})$ to V_δ^* , where

$$(\psi \otimes \phi)(kan) = \psi(k)\phi(an).$$

It is easy to check this map satisfies all the conditions as in Lemma 9.3 if we take $L = AN$, $\pi(an) = a^\rho \check{\delta}(an)$. So there exists a unique element in V_δ^* , which we denote by $f^-(\psi)$, such that

$$f(\psi \otimes \phi) = \int_{A \times N} \phi(an) a^\rho \check{\delta}^{-1}(an) f^-(\psi) dadn.$$

Notice $a^{2\rho} dadn$ gives a right invariant Haar measure on AN . It is fairly easy to see $f^- \in C^{-\infty} \text{Ind}_M^K(\check{\delta}|M)$. Then by Proposition 9.1 $\eta_1(f^-)$ gives an element in $(C^\infty \text{Ind}_M^K(\delta|M))'$. Since $C^\infty \text{Ind}_M^K(\delta|M) \cong C^\infty \text{Ind}_B^G(\delta)$, one can view $\eta_1(f^-)$ as an element in $(C^\infty \text{Ind}_B^G(\delta))'$. Finally we define $\eta(f)$ by

$$\eta(f) = \eta_1(f^-).$$

The final step of the proof is to show $\eta \circ \xi = \text{id}$, and $\eta \circ \xi = \text{id}$. For each $T \in (C^\infty \text{Ind}_B^G(\delta))'$, $\psi \in C^\infty(K, \mathbb{C})$, and $\phi \in C_c^\infty(A \times N, \mathbb{C})$,

$$\xi(T)(\psi \otimes \phi) = \int_{A \times N} \phi(an) a^\rho \check{\delta}^{-1}(an) (\xi(T))^- dadn.$$

So for each $v \in V$,

$$(**) \quad \langle v, \xi(T)(\psi \otimes \phi) \rangle = \langle v, \int_{A \times N} \phi(an) a^\rho \check{\delta}^{-1}(an) (\xi(T))^- (\psi) dadn \rangle.$$

By definition

$$\begin{aligned} \xi(\psi \otimes \phi, v)(k) &= \int_{MAN} (\psi \otimes \phi)(kman) a^\rho \delta(man) v dm dadn \\ &= \int_{MAN} \psi(km) \delta(m) \phi(an) a^\rho \delta(an) v dm dadn \\ &= \xi_1(\psi, v_1), \end{aligned}$$

where $v_1 = \int_{A \times N} a^\rho \phi(an) \delta(an) v dadn$. So by (**)

$$\begin{aligned} \langle v, \xi(T)(\psi \otimes \phi) \rangle &= T(\xi_1(\psi, v_1)) \\ &= \langle v_1, \xi_1(T)(\psi) \rangle \\ &= \langle v, \int_{A \times N} \phi(an) a^\rho \check{\delta}^{-1}(an) \xi_1(T)(\psi) dadn \rangle. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{A \times N} \phi(an) a^\rho \check{\delta}^{-1}(an) (\xi(T))^- (\psi) dadn \\ &= \int_{A \times N} \phi(an) a^\rho \check{\delta}^{-1}(an) \xi_1(T)(\psi) dadn. \end{aligned}$$

By comparing both sides we have $\xi_1(T) = (\xi(T))^-$. Hence

$$T = \xi_1^{-1}((\xi(T))^-) = \eta_1((\xi(T))^-) = \eta(\xi(T)).$$

Similarly we can verify $\xi \circ \eta = \text{id}$. Note it is enough to check on functions of the form $\psi \otimes \phi$. $\square$

Now suppose V_δ is a Hilbert space. Let δ^* be the representation defined as follows: for each $g \in G, w, v \in V_\delta$, we have $\langle \delta(g)v, w \rangle = \langle v, \delta(g)^t w \rangle$; then $\delta^*(g) = \delta(g^{-1})^t$. Let $C^{-\infty} \text{Ind}_B^G(\delta^*)$ be the space of conjugate linear maps f from $C_c^\infty(G, \mathbb{C})$ to V_δ , such that

$$R_{man}f = a^{-\rho}\delta^*((man)^{-1})f.$$

For each $T \in (C^\infty \text{Ind}_B^G(\delta))'$, and $\phi \in C_c^\infty(G, \mathbb{R})$, $\xi(T)(\phi)$ is defined by

$$\langle v, \xi(T)(\phi) \rangle = T(\xi(\phi, v)),$$

for each $v \in V_\delta$. Here

$$\xi(\phi, v)(g) = \int_{MAN} \phi(gman)a^\rho \delta(man)v dm dadn.$$

Corollary 9.4. ξ is a bijection from $(C^\infty \text{Ind}_B^G(\delta))'$ onto $C^{-\infty} \text{Ind}_B^G(\delta^*)$.

Let σ be a unitary representation of M and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. $\sigma \otimes \bar{\lambda} \otimes 1$ is the representation of B defined by $man \rightarrow a\bar{\lambda}\sigma(m)$. Then

$$(\sigma \otimes \bar{\lambda} \otimes 1)^* = \sigma \otimes (-\lambda) \otimes 1.$$

Corollary 9.5. The map

$$\xi: (C^\infty \text{Ind}_B^G(\sigma \otimes \bar{\lambda} \otimes 1))' \rightarrow C^{-\infty} \text{Ind}_B^G(\sigma \otimes (-\lambda) \otimes 1)$$

is a bijection.

10. DISTRIBUTION BOUNDARY VALUES

We now introduce a weak growth condition in the eigenspace $\mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau)$. Recall in Section 2 we have

$$C_r^q(G, V) = \{f \in C^q(G, V) \mid \|f\|_{q,r} < \infty\},$$

$q \in \mathbb{N}$ and $r \in \mathbb{R}$. $C_r^\infty(G, V) = \bigcap_q C_r^q(G, V)$. We define $\mathfrak{F}$ to be the space

$$\mathfrak{F} = \bigcap_r C_r^\infty(G, V) = \bigcap_{q,r} C_r^q(G, V),$$

endowed with the projective limit topology for the intersection over q and r (i.e., the topology given by the family of forms $\|\cdot\|_{q,r}$). Using the same argument as on p. 142 in [BS] we conclude $\mathfrak{F}$ is a Fréchet space. It follows from Section 2 that L and R act smoothly on $\mathfrak{F}$.

Let $\mathfrak{F}'$ be the space dual to $\mathfrak{F}$, equipped with the strong dual topology. For each $T \in \mathfrak{F}', q \in \mathbb{N}$, and $r \in \mathbb{R}$, we define

$$\|T\|'_{q,r} = \sup\{T(\varphi) \mid \varphi \in \mathfrak{F}, \|\varphi\|_{q,r} \leq 1\}.$$

The space $C_r^q(G, V)' = \{T \in \mathfrak{F}' \mid \|T\|'_{q,r} < \infty\}$ with this norm is the dual space of $C_r^q(G, V)$. Moreover, we have $\mathfrak{F}' = \bigcup_{q,r} C_r^q(G, V)'$. By duality $\mathfrak{F}'$ is the inductive limit of these spaces. Using Lemma 2.1 we can prove that for some $b \in \mathbb{R}$, $\int_G \|g\|^b dg < \infty$. It follows that there is a continuous injection of $C_r^0(G, V)$ into $C_{b-r}^0(G, V)'$ defined by integration over G . Hence there is a continuous injection of $C_r^0(G, V)$ into $\mathfrak{F}'$.

Let $q' \geq q$, and $r \in \mathbb{R}$. For each $T \in C_r^q(G, V)'$, and $\varphi \in C_r^{q'}(G, \mathbb{R})$, we define an element $L^\vee(\varphi)T$ in $C_r^{q'-q}(G, V)$ by

$$\langle v, L^\vee(\varphi)T(x) \rangle = T(R_{x^{-1}}\varphi \cdot v).$$

Note if $f \in C_r^0(G, V)$, and $\varphi \in C_{b-r}^0(G, \mathbb{C})$, then

$$L^\vee(\varphi)f(x) = \int_G \varphi(g)f(gx)dg.$$

Lemma 10.1. *Let $q, q' \in \mathbb{N}$ with $q \leq q'$. There exist $s \geq 0$ and $C \geq 0$ such that*

$$\|L^\vee(\varphi)T\|_{q'-q, r} \leq C\|T\|'_{q', r}\|\varphi\|_{q', r-s},$$

for all $r \in \mathbb{R}$, $T \in C_r^q(G, V)'$, and $\varphi \in C_{r-s}^{q'}(G, \mathbb{R})$.

Proof. See Lemma 11.1 in [BS]. $\square$

Let $\mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$ denote the closed subspace $\mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) \cap \mathfrak{F}'$. We call the elements of $\mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$ eigensections of weak moderate growth. Notice if $f \in \mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$, and $\varphi \in C_c^\infty(G, \mathbb{R})$, then $L^\vee(\varphi)f \in \mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$ by Lemma 10.1. For $\lambda - \mathbf{L} \in \mathfrak{A}_1$, $\lambda \in \mathfrak{A}_2$, and $f \in \mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$, we define a vector-valued distribution $\overline{\beta}_\lambda f$ on G by

$$\overline{\beta}_\lambda f(\varphi) = \beta_\lambda(L^\vee(\varphi)f)(e),$$

for each $\varphi \in C_c^\infty(G, \mathbb{R})$.

Proposition 10.2. *$\overline{\beta}_\lambda$ is a linear, continuous, and G -equivariant map from $\mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$ to $C^{-\infty} \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$.*

Proof. It suffices to show $\overline{\beta}_\lambda f \in C^{-\infty} \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$. By definition,

$$\begin{aligned} L^\vee(R_{(man)^{-1}}\varphi)f(x) &= f(R_{x^{-1}}R_{(man)^{-1}}\varphi) \\ &= f(R_{(manx)^{-1}}\varphi) \\ &= L^\vee(\varphi)f(manx). \end{aligned}$$

However, β_λ is G -equivariant. Hence

$$\begin{aligned} B_\lambda(L^\vee(R_{(man)^{-1}}\varphi)f)(e) &= \beta_\lambda(L^\vee(\varphi)f)(man) \\ &= \tau(\Lambda)(m^{-1})a^{\lambda-\rho}\beta_\lambda(L^\vee(\varphi)f)(e). \end{aligned}$$

This proves $\overline{\beta}_\lambda f \in C^{-\infty} \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$. $\square$

For each $T \in (C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$, we define $\overline{P}_\lambda T$ as follows:

$$\langle v, \overline{P}_\lambda T(g) \rangle = T(P(\Lambda)L_g\Phi_\lambda \cdot v),$$

for each $v \in V$. Here $\Phi_\lambda(x)$ is the transpose of $\Psi_\lambda(x^{-1})$, and $P(\Lambda)$ the projection from V to $V(\Lambda)$. The motivation of this definition is from Corollary 1.10.

Proposition 10.3. *$\overline{P}_\lambda T \in \mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$, for $T \in (C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$. And $\overline{P}_\lambda$ is linear, continuous, and G -equivariant.*

Proof. Similar to the proof for Corollary 11.3 in [BS]. $\square$

Lemma 10.4. *Let $T \in (C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$, and $\varphi \in C_c^\infty(G, \mathbb{R})$. Then $L^\vee(\varphi)\overline{P}_\lambda T = P_\lambda(L^\vee(\varphi)\xi(T))$. Here ξ is the isomorphism in Corollary 9.5, and $L^\vee(\varphi)\xi(T)(x) = \xi(T)(R_{x^{-1}}\varphi)$.*

Proof. $L^\vee(\varphi)$, $\overline{P}_\lambda$, and P_λ are continuous. So it is enough to check for $T \in C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (-\lambda) \otimes 1)$. The proof follows from the G -equivariance of P_λ . $\square$

By a similar argument we get

Lemma 10.5. *Let $f \in \mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$, and $\varphi \in C_c^\infty(G, \mathbb{R})$. Then*

$$L^\vee(\varphi)\overline{\beta}_\lambda f = \beta_\lambda(L^\vee(\varphi)f).$$

Theorem 10.6. *Under the same conditions as in Theorem 8.3, $\overline{P}_\lambda$ defines a G -equivariant topological isomorphism from $(C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$ onto $\mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$. And $\eta \circ C_0(\lambda)^{-1} \circ \overline{\beta}_\lambda$ gives the inverse of $\overline{P}_\lambda$.*

Proof. By Theorem 8.1 and Lemma 10.4, 10.5,

$$L^\vee(\varphi)\overline{\beta}_\lambda \overline{P}_\lambda T = \beta_\lambda P_\lambda L^\vee(\varphi)\xi(T) = C_0(\lambda)L^\vee(\varphi)\xi(T),$$

for $T \in (C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes \overline{\lambda} \otimes 1))'$. Similarly, for each $f \in \mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$,

$$L^\vee(\varphi)\overline{P}_\lambda \eta(C_0(\lambda)^{-1}\overline{\beta}_\lambda f) = P_\lambda C_0(\lambda)^{-1}\beta_\lambda L^\vee(\varphi)f = L^\vee(\varphi)f.$$

So we have

$$\overline{\beta}_\lambda \circ \overline{P}_\lambda = C_0(\lambda) \circ \xi, \quad \overline{P}_\lambda \circ \eta \circ C_0(\lambda)^{-1}\overline{\beta}_\lambda = \text{id}. \quad \square$$

Remark 10.7. Let $\mathcal{E}_{\lambda-\Lambda, r} \text{Ind}_K^G(\tau) = \mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau) \cap C_r(G, V)$ be equipped with the Banach space topology inherited from $C_r(G, V)$. Then $\mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau)$ is identical with the inductive limit topology for the union $\mathcal{E}_{\lambda-\Lambda}^* \text{Ind}_K^G(\tau) = \bigcup_r \mathcal{E}_{\lambda-\Lambda, r} \text{Ind}_K^G(\tau)$. See p. 146 in [BS].

A classical result asserts that the left K -finite elements in $\mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau)$ increase at most exponentially. So by the remark above we easily get

Corollary 10.8. *Under the same conditions as in Theorem 8.3, P_λ is a bijection from $C^\infty \text{Ind}_B^G(\tau(\Lambda) \otimes (\lambda) \otimes 1)_{K\text{-finite}}$ to $\mathcal{E}_{\lambda-\Lambda} \text{Ind}_K^G(\tau)_{K\text{-finite}}$.*

Remark 10.9. I think by Schmid's method indicated in [Sch] one should be able to get a bijection on the level of hyperfunctions from Corollary 10.8.

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