SINGULAR LIMIT OF SOLUTIONS OF THE POROUS MEDIUM EQUATION WITH ABSORPTION

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ABSTRACT. We prove that as $m \rightarrow \infty$ the solutions $u^{(m)}$ of $u_t = \Delta u^m - u^p$, $(x, t) \in R^n \times (0, T)$, $T > 0$, $m > 1$, $p > 1$, $u \geq 0$, $u(x, 0) = f(x) \in L^1(R^n) \cap L^\infty(R^n)$, converges in $L^1_{loc}(R^n \times (0, T))$ to the solution of the ODE $v_t = -v^p$, $v(x, 0) = g(x)$, where $g \in L^1(R^n)$, $0 \leq g \leq 1$, satisfies $g - \Delta \tilde{g} = f$ in $\mathcal{D}'(R^n)$ for some function $\tilde{g} \in L^\infty_{loc}(R^n)$, $\tilde{g} \geq 0$, satisfying $\tilde{g}(x) = 0$ whenever $g(x) < 1$ for a.e. $x \in R^n$, $\int_E \tilde{g} dx \leq C|E|^{2/n}$ for $n \geq 3$ and $\int_E |\nabla \tilde{g}| dx \leq C|E|^{1/2}$ for $n = 2$, where $C > 0$ is a constant and E is any measurable subset of R^n .

In this paper we will show that as $m \rightarrow \infty$ the solutions $u = u^{(m)}$ of the equation

$$(0.1) \quad \begin{cases} u_t = \Delta u^m - \lambda u^p, u \geq 0, & \text{in } R^n \times (0, T), \\ u(x, 0) = f(x) \geq 0, x \in R^n, & f \in L^1(R^n) \cap L^\infty(R^n), \end{cases}$$

where $T > 0$, $p > 1$, $\lambda > 0$ converges to the solution of the following ODE:

$$(0.2) \quad \begin{cases} v_t = -\lambda v^p & \text{in } \mathcal{D}'(R^n \times (0, T)), \\ v(x, 0) = g(x), & x \in R^n, \end{cases}$$

where $g \in L^1(R^n)$, $0 \leq g \leq 1$, satisfies

$$(0.3) \quad g - \Delta \tilde{g} = f \quad \text{in } \mathcal{D}'(R^n)$$

for some function $\tilde{g} \in L^\infty_{loc}(R^n)$, $\tilde{g} \geq 0$, satisfying

$$(0.4) \quad \tilde{g}(x) = 0 \quad \text{whenever } g(x) < 1 \text{ for a.e. } x \in R^n$$

and

$$(0.5) \quad \begin{cases} \int_E \tilde{g}(x) dx \leq C|E|^{2/n} & \text{for } n \geq 3, \\ \int_E |\nabla \tilde{g}(x)| dx \leq C|E|^{1/2} & \text{for } n = 2 \end{cases}$$

for any measurable set $E \subset R^n$, where $C > 0$ is a constant independent of E and $m > p$.

When $\lambda > 0$, equation (0.1) models the flow of gases through a porous medium or thermal propagation with absorption where both the diffusion coefficient and the absorption coefficient are powers of the concentration u [K1][K2][VW]. When

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$\lambda = 0$, the above equation reduces to the well-known porous medium equation [A][P].

For $\lambda = 0$, Caffarelli and A.Friedman [CF] studied the asymptotic behaviour of solutions of (0.1) and showed that the solutions of (0.1) converge as $m \rightarrow \infty$ if f satisfies (0.1) and the following conditions:

$$\begin{aligned} f &\in C^1 \text{ in } \text{supp } f, \\ f(0) &> 1, f_r < 0 \text{ in } R^n \setminus \{0\} \cap \text{supp } f, \\ f_{r_{x_0}} &\leq 0 \text{ in } R^n \setminus B_1(0) \cap \text{supp } f \quad \forall x_0 \in B_{\varepsilon_0}(0) \end{aligned}$$

for some $\varepsilon_0 > 0$, where $r_{x_0} = |x - x_0|$, $B_r(0) = \{x : |x| < r\}$ and $f_{r_{x_0}}$ is the radial derivative of f with center at x_0 .

Their result was later generalized to the case of general $f \in L^1(R^n)$ by P.Bénilan, L.Boccardo and M.Herrero [BBH] and P.E.Sacks [S2]. Recently X.Xu [X] proved that one has similar result in the case of a hyperbolic equation. K.M.Hui [H1]–[H3] also obtained similar results in the case of the generalized p -Laplacian equation, the porous medium equation with convection term and in the case of equation (0.1) as $p \rightarrow \infty$.

Our result shows that the diffusion term is negligible compared with the other terms of the equation as $m \rightarrow \infty$. This is in sharp contrast to the case $p \rightarrow \infty$ [H1] in which the solution of (0.1) will converge to the solution of the porous medium equation with no absorption term.

We will organize our paper as follows. In section 1 we recall some existence and regularity results for equation (0.1) from [KPV] and [BC]. We will also recall a uniqueness result for (0.3) from [BBC] and prove some technical lemmas. In section 2 we will first prove the convergence result for the case $f \in C_0(R^n)$. We then prove our main convergence theorem by an approximation argument.

For simplicity we will assume that $T = 1$ and $\lambda = 1$ throughout the rest of the paper.

We start with some definitions. For any open set $\Omega_0 \subset R^n$, $h \in C(R)$, we say that u is a solution (respectively subsolution) of (cf. [DK])

$$(0.6) \quad u_t = \Delta u^m + h(u)$$

in $\overline{\Omega}_0 \times (0, 1)$ if (a) u is continuous and non-negative in $\overline{\Omega}_0 \times (0, 1)$, (b) $u \in L^\infty([0, 1]; L^1(\Omega_0)) \cap L^\infty(\Omega_0 \times (0, 1))$ and (c) u satisfies

$$(0.7) \quad \int_{\tau_1}^{\tau_2} \int_{\Omega} \left[u^m \Delta \eta + u \frac{\partial \eta}{\partial t} + h(u) \eta \right] dx dt = \int_{\tau_1}^{\tau_2} \int_{\partial \Omega} u^m \frac{\partial \eta}{\partial n} d\sigma ds + \int_{\Omega} u \eta dx \Big|_{\tau_1}^{\tau_2}$$

(respectively $\geq$) for all bounded open set $\Omega \subset \Omega_0$ with $\partial \Omega \in C^2$, $0 < \tau_1 \leq \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial \Omega \times [\tau_1, \tau_2]$ where $\partial/\partial n$ is the exterior normal derivative on $\partial \Omega$ and $d\sigma$ is the surface measure on $\partial \Omega$.

If u is a solution of (0.6) in $\overline{\Omega}_0 \times (0, 1)$, we say that u has initial trace or initial value $d\mu$ if

$$\lim_{t \rightarrow 0} \int u(x, t) \eta(x) dx = \int \eta d\mu \quad \forall \eta \in C_0^\infty(\overline{\Omega}_0)$$

We let $\rho \in C_0^\infty(R^n)$, $\rho \geq 0$, $\int \rho = 1$, and for any g we define

$$g * \rho_\varepsilon(x) = \int \rho_\varepsilon(x - y) g(y) dy, \quad \varepsilon > 0,$$

where $\rho_\varepsilon(y) = \rho(y/\varepsilon)/\varepsilon^n$. For any $r > 0$, $x_0 \in R^n$, let $B_r(x_0) = \{x \in R : |x - x_0| < r\}$. We will also assume $1 < p < m$, and let $u^{(m)}$ be the solution of (0.1) for the rest of the paper.

1.

In this section we will recall and extend some results of [BC], [BBC], [KPV] and [S2]. We will also prove some technical lemmas that will be used in the proof of the main theorem (Theorem 2.5) in section 2. We first recall a result of [KPV].

Theorem 1.1 (cf. Lemma 2.4 of [KPV]). *If $u_1^{(m)}, u_2^{(m)} \in L^\infty([0, 1]; L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$ are solutions of (0.1) in $R^n \times (0, 1)$ with initial values $f_1, f_2 \in L^1(R^n) \cap L^\infty(R^n)$ respectively, $f_1, f_2 \geq 0$, then*

$$(i) \quad \int_{R^n} (u_1^{(m)} - u_2^{(m)})_+(x, t) dx \leq \int_{R^n} (f_1 - f_2)_+(x) dx,$$

$$(ii) \quad \int_{R^n} |u_1^{(m)} - u_2^{(m)}|(x, t) dx \leq \int_{R^n} |f_1 - f_2|(x) dx$$

for all $0 < t < 1$. Hence $u_1^{(m)} \leq u_2^{(m)}$ if $f_1 \leq f_2$. In particular the solution of (0.1) in $R^n \times (0, 1)$ with initial value in $L^1(R^n) \cap L^\infty(R^n)$ is unique in the class $L^\infty([0, 1]; L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$.

Theorem 1.2. (0.1) has a unique solution

$$u^{(m)} \in L^\infty([0, 1]; L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$$

with

$$(i) \quad \int u^{(m)}(x, t) dx \leq \int f dx \quad \forall 0 < t < 1,$$

$$(ii) \quad \|u^{(m)}\|_{L^\infty(R^n \times (0, 1))} \leq \|f\|_{L^\infty(R^n)}.$$

Proof. The proof is similar to the proof of Theorem 1.3 of [H3]. We refer the reader to [H3] for the details. $\square$

Lemma 1.3. Let $0 \leq f \leq M$ with $\text{supp } f \subset B_R(0)$. Then the following hold.

(i) For any $\delta > 0$ and $0 < T_1 < 1$ there exists $m' > p$ depending only on M, R and T_1 such that

$$u^{(m)}(x, t) \leq 1 + \delta$$

and

$$(u^{(m)})^m(x, t) \leq \delta$$

for all $x \in R^n, T_1 \leq t < 1, m \geq m'$.

(ii) There exist $R_1 > R$ and $m' > p$ depending only on M, R such that

$$(1.1) \quad u^{(m)}(x, t) \equiv 0 \quad \forall |x| \geq R_1, 0 \leq t < 1, m \geq m'.$$

Proof. The proof is similar to the proof of Lemma 2.1 of [S2]. We let

$$w(x, t) = \frac{1}{(t + t_0)^{nk}} \left\{ a^2 - \frac{C_1 |x|^2}{(t + t_0)^{2k}} \right\}_+^{1/m-1}$$

be the Barenblatt solution of $u_t = \Delta u^m$ [HP], where

$$a^2 = C_1 t_0^{-2k} R^2 + M^{m-1} t_0^{n(m-1)k},$$

$k = 1/(n(m-1)+2)$, $C_1 = (m-1)/(2m(n(m-1)+2))$, $t_0 = C_1 R^2/M^{m-1}$. Since $u^{(m)}$ is a subsolution of $u_t = \Delta u^m$ and $u^{(m)}(x, 0) \leq M \leq w(x, 0)$, by the maximum principle,

$$u^{(m)}(x, t) \leq w(x, t) \quad \forall x \in R^n, 0 \leq t < 1.$$

Hence

$$\begin{aligned} (1.2) \quad u^{(m)}(x, t) &\leq \frac{a^{2/m-1}}{(T_1 + t_0)^{n/(n(m-1)+2)}} \\ &\leq \frac{1}{T_1^{n/(n(m-1)+2)}} \cdot \left(\frac{2C_1 R^2}{t_0^{2/(n(m-1)+2)}} \right)^{1/m-1} \\ &\leq \frac{2^{1/m-1} (C_1 R^2)^{n/(n(m-1)+2)} M^{2/(n(m-1)+2)}}{T_1^{n/(n(m-1)+2)}} \\ &\rightarrow 1 \quad \text{as } m \rightarrow \infty \quad \forall x \in R^n, T_1 \leq t < 1. \end{aligned}$$

By (1.2),

$$\begin{aligned} u^{(m)m}(x, t) &\leq \frac{2^{m/m-1} (C_1 R^2)^{nm/(n(m-1)+2)} M^{2m/(n(m-1)+2)}}{T_1^{nm/(n(m-1)+2)}} \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty \quad \forall x \in R^n, T_1 \leq t < 1. \end{aligned}$$

Hence (i) follows. We next observe that for each $0 \leq t < 1$,

$$\text{supp } u^{(m)}(\cdot, t) \subset \text{supp } w(\cdot, t) \subset B_{R_t}(0)$$

where

$$\begin{aligned} R_t &= \frac{a(t+t_0)^{1/(n(m-1)+2)}}{C_1^{1/2}} \leq \frac{2a}{C_1^{1/2}} \\ &\leq \frac{4R}{t_0^{1/(n(m-1)+2)}} \\ &\leq \frac{4R}{(C_1 R^2)^{1/(n(m-1)+2)}} \cdot M^{(m-1)/(n(m-1)+2)} \\ &\leq 4RM^{1/n} \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Hence (ii) follows. $\square$

Corollary 1.4. Suppose f is as in Lemma 1.3. Let $\Omega \subset R^n$ be a bounded open set with $\partial\Omega \in C^2$, and let $\eta \in C^\infty(R^n \times (0, 1))$. Then

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} u^{(m)m} \cdot \eta dx dt \rightarrow 0, \quad \int_{\tau_1}^{\tau_2} \int_{\partial\Omega} u^{(m)m} \frac{\partial \eta}{\partial n} d\sigma dt \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

for any $0 < \tau_1 \leq \tau_2 < 1$.

Proof. The corollary follows immediately from Lemma 1.3. $\square$

Lemma 1.5. Let $f \in C_0(R^n)$ and $h^{(m)}(x, t) = \int_0^t u^{(m)m}(x, \tau) d\tau$. Then $\{h^{(m)}\}_{m>p}$ is uniformly bounded on $R^n \times [0, 1)$. For any sequence $\{h^{(m_i)}\}_{i=1}^\infty$, $m_i \rightarrow \infty$ as $i \rightarrow \infty$, of $\{h^{(m)}\}_{m>p}$, there exist a subsequence $\{h^{(m'_i)}\}_{i=1}^\infty$ of $\{h^{(m_i)}\}_{i=1}^\infty$, a sequence of functions $\{h_k\}_{k=1}^\infty \subset L^\infty(R^n)$, $\tilde{g} \in L^\infty(R^n)$, $h_k, \tilde{g} \geq 0$, $\text{supp } h_k \subset B_{R_1}(0)$ where

R_1 is as in (ii) of Lemma 1.3, and a sequence $1 > \varepsilon_1 > \varepsilon_2 > \dots > 0$, $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, such that

$$(1.3) \quad \begin{cases} h^{(m'_i)}(\cdot, \varepsilon_k) \rightarrow h_k(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } i \rightarrow \infty, \quad \forall k = 1, 2, \dots, \\ h_k(\cdot) \rightarrow \tilde{g}(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } k \rightarrow \infty \end{cases}$$

for any compact subset $K \subset R^n$. Moreover, $\tilde{g}$ satisfies

$$(1.4) \quad \begin{cases} \tilde{g}(x) \leq Nf(x) \leq C(\|f\|_{L^1} + \|f\|_{L^\infty}) & \forall x \in R^n, \text{ for } n \geq 3, \\ |\nabla \tilde{g}(x)| \leq C(1 + \|f\|_{L^\infty}^{p-1})(\|f\|_{L^1} + \|f\|_{L^\infty}) & \forall x \in R^n, \text{ for } n = 1, 2 \end{cases}$$

and (0.5) for any measurable set $E \subset R^n$ where $C > 0$ is a constant independent of E and $m > p$, $N(x, y) = c_n|x - y|^{2-n}$, $c_n = 1/(n-2)|B_1(0)|$ for $n \geq 3$, and $Nf(x) = \int N(x, y)f(y)dy$.

Proof. We first observe that by Lemma 1.3 there exist $R_1 \geq 1$ and $m' > p$ such that (1.1) holds. For $n \in \mathbb{Z}^+$ and any $R > 0$, $\eta \in C(R^n)$, let G_R be the Green function for the ball $B_R(0)$ ($\Delta G_R = -\delta_0$),

$$G_R\eta(x) = \int_{B_R(0)} G_R(x, y)\eta(y)dy$$

and let

$$q_R(x, t) = \int_0^t u^{(m)m}(x, \tau)d\tau + G_R u^{(m)}(x, t) - G_R f(x) + \int_0^t G_R u^{(m)p}(x, \tau)d\tau.$$

Since $u^{(m)}$ is a solution of (0.1),

$$\int_{R^n} q(x, t)\Delta\eta(x)dx = 0 \quad \forall \eta \in C_0^\infty(B_R(0)), 0 \leq t < 1$$

Hence for each $0 \leq t < 1$ $q_R(\cdot, t)$ is harmonic in $B_R(0)$ and satisfies $q_R(x, t) \equiv 0$ for all $|x| = R$, $R \geq R_1$, by (1.1). By the maximum principle,

$$q_R(x, t) \equiv 0 \quad \forall |x| \leq R, R \geq R_1, 0 \leq t < 1$$

Thus

$$(1.5) \quad h^{(m)}(x, t) = \int_0^t u^{(m)m}(x, \tau)d\tau = -G_R u^{(m)}(x, t) + G_R f(x) - \int_0^t G_R u^{(m)p}(x, \tau)d\tau$$

for all $|x| \leq R$, $R \geq R_1$, $0 \leq t < 1$, $m > m'$. Since $G_R \geq 0$ and $\int G_R(x, y)dy = (R^2 - |x|^2)/2n$, we have by (1.5),

$$\begin{aligned} 0 &\leq h^{(m)}(x, t) \leq G_{R_1} f(x) \\ &\leq \|f\|_{L^\infty} \cdot \int G_{R_1}(x, y)dy \\ &\leq \left(\frac{R_1^2 - |x|^2}{2n} \right) \cdot \|f\|_{L^\infty} \\ &\leq \frac{R_1^2}{2n} \|f\|_{L^\infty} \quad \forall |x| \leq R_1, 0 \leq t < 1, m > m'. \end{aligned}$$

Since $h^{(m)}(x, t) \equiv 0$ for all $|x| \geq R_1$, $0 \leq t \leq 1$, $m > p$, and

$$h^{(m)}(x, t) \leq (\|f\|_{L^\infty} + 1)^{m'} \quad \forall x \in R^n, 0 \leq t < 1, m' \geq m > p.$$

$\{h^{(m)}\}_{m>p}$ is uniformly bounded on $R^n \times [0, 1)$ for all $n \in \mathbb{Z}^+$. So any sequence $\{h^{(m_i)}\}_{i=1}^\infty$ of $\{h^{(q)}\}_{m>p}$ has a subsequence $\{h^{(m_{1,i})}\}_{i=1}^\infty$ such that $\{h^{(m_{1,i})}(\cdot, 1/2)\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K \subset R^n$. Similarly $\{h^{(m_{1,i})}\}_{i=1}^\infty$ has $\{h^{(m_{2,i})}\}_{i=1}^\infty$ such that $\{h^{(m_{2,i})}(\cdot, 1/3)\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K \subset R^n$. Repeating the argument $\{h^{(m_{k,i})}\}_{i=1}^\infty$ will have a subsequence $\{h^{(m_{k+1,i})}\}_{i=1}^\infty$ such that $\{h^{(m_{k+1,i})}(\cdot, 1/(k+2))\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K \subset R^n$ for all $k = 0, 1, \dots$, where $m_{0,i} = m_i$ for all $i \in \mathbb{Z}^+$.

The first part of the lemma then follows by a diagonalization argument as in the proof of Lemma 1.8 of [H3], and (1.3) holds. Since (1.3) holds, we may assume without loss of generality that for a.e. $x \in R^n$,

$$\begin{cases} h^{(m'_i)}(x, \varepsilon_k) \rightarrow h_k(x) & \text{as } i \rightarrow \infty \quad \forall k = 1, 2, \dots, \\ h_k(x) \rightarrow \tilde{g}(x) & \text{as } k \rightarrow \infty. \end{cases}$$

To prove (1.4) we observe that for $n \geq 3$, by letting $R \rightarrow \infty$ in (1.5), we have

$$\begin{aligned} h^{(m)}(x, t) &= -Nu^{(m)}(x, t) + Nf(x) - \int_0^t Nu^{(m)p}(x, \tau) d\tau \\ &\leq Nf(x) \leq C(\|f\|_{L^\infty} + \|f\|_{L^1}) \end{aligned}$$

for all $x \in R^n$, $0 \leq t < 1$, $m > m'$, where $C > 0$ is a constant independent of f and m . The first inequality of (1.4) then follows by putting $m = m'_i$, $t = \varepsilon_k$ and letting $i \rightarrow \infty$ and $k \rightarrow \infty$. For the second inequality of (1.4), observe that for $n = 2$ by (1.5) we have

$$\begin{aligned} \nabla h^{(m)}(x, t) &= -\nabla G_R u^{(m)}(x, t) + \nabla G_R f(x) - \int_0^t \nabla G_R u^{(m)p}(x, \tau) d\tau \\ &\rightarrow -\tilde{N}u^{(m)}(x, t) + \tilde{N}f(x) - \int_0^t \tilde{N}u^{(m)p}(x, \tau) d\tau \quad \text{as } R \rightarrow \infty \\ \Rightarrow |\nabla h^{(m)}(x, t)| &\leq \frac{(1 + \|f\|_{L^\infty}^{p-1})}{\pi} \cdot \left(\int_{R^2} \frac{1}{|x-y|} u^{(m)}(y, t) dy + \int_{R^2} \frac{1}{|x-y|} f(y) dy \right. \\ (1.6) \quad &\quad \left. + \int_0^t \int_{R^2} \frac{1}{|x-y|} u^{(m)}(y, \tau) dy d\tau \right) \end{aligned}$$

for all $x \in R^2$, $0 \leq t < 1$, $m > m'$ by Theorem 1.2, where

$$\tilde{N} = (\tilde{N}_1, \tilde{N}_2) = \frac{-1}{2\pi} \cdot \frac{x-y}{|x-y|^2}$$

and

$$\tilde{N}\eta = \frac{-1}{2\pi} \cdot \int_{R^2} \frac{x-y}{|x-y|^2} \eta(y) dy$$

for any function η of R^2 . Since

$$\begin{aligned} \int_{R^2} \frac{1}{|x-y|} u^{(m)}(y, t) dy &= \left(\int_{|x-y| \leq 1} + \int_{|x-y| > 1} \right) \frac{1}{|x-y|} u^{(m)}(y, t) dy \\ &\leq C(\|f\|_{L^\infty} + \|f\|_{L^1}) \quad \forall x \in R^2, 0 \leq t < 1, \end{aligned}$$

where $C > 0$ is a constant independent of f and $m > p$, we have

$$(1.7) \quad |\nabla h^{(m)}(x, t)| \leq C(1 + \|f\|_{L^\infty}^{p-1})(\|f\|_{L^\infty} + \|f\|_{L^1}) \quad \forall x \in R^2, 0 \leq t < 1,$$

where $C > 0$ is a constant independent of f and $m > p$. Hence for $n = 2$, $\{\nabla h^{(m)}(x, t)\}_{m > p}$ is uniformly bounded on $R^n \times [0, 1)$. By a diagonalization argument there exist a subsequence $\{\varepsilon_{k_j}\}$ of $\{\varepsilon_k\}$ and a subsequence $\{m''_i\}$ of $\{m'_i\}$ such that

$$\nabla h^{(m''_i)}(x, \varepsilon_{k_j}) \rightarrow \nabla h_{k_j}(x) \quad \text{weakly in } L^\infty(K)^* \text{ and a.e. } x \in R^2 \quad \text{as } i \rightarrow \infty$$

for all $j = 1, 2, \dots$ and

$$\nabla h_{k_j}(x) \rightarrow \nabla \tilde{g}(x) \quad \text{weakly in } L^\infty(K)^* \text{ and a.e. } x \in R^2 \quad \text{as } j \rightarrow \infty$$

for any compact subset $K \subset R^2$. Putting $m = m''_i$, $t = \varepsilon_{k_j}$ in (1.7) and letting $i \rightarrow \infty$, $j \rightarrow \infty$, we get the second inequality of (1.4) for $n = 2$.

For $n = 1$, since $u^{(m)}$ satisfies (0.1),

$$(1.8) \quad \begin{aligned} & \int_R h^{(m)}(x, t) \eta''(x) dx - \int_R \left(\int_0^t u^{(m)p}(x, \tau) d\tau \right) \eta(x) dx \\ &= \int_R u^{(m)}(x, t) \eta(x) dx - \int_R f \eta dx \quad \forall \eta \in C_0^\infty(R) \\ &\Rightarrow h^{(m)''}(\cdot, t) = h_1^{(m)}(\cdot, t) + u^{(m)}(\cdot, t) - f \quad \text{in } \mathcal{D}'(R) \quad \forall 0 \leq t < 1 \end{aligned}$$

where

$$h_1^{(m)}(x, t) = \int_0^t u^{(m)p}(x, \tau) d\tau.$$

Since $u^{(m)} \in C_0(R)$ by (1.1),

$$h_1^{(m)}(\cdot, t) + u^{(m)}(\cdot, t) - f \in C_0(R^n).$$

Thus $h^{(m)}(\cdot, t) \in C_0^2(R)$ for all $0 \leq t < 1$, $m > m'$. For any $x_0 \in R$, $x_0 \geq -R_1$, let

$$\eta(x) = \zeta(-R_1 - 2 - x) \zeta\left(\frac{x - x_0}{\varepsilon}\right)$$

where $\zeta \in C^\infty(R)$ satisfies $0 \leq \zeta \leq 1$, $\zeta(s) = 1$ if $s \leq -1$, $\zeta = 0$ if $s \geq 0$, $\zeta' \leq 0$, $\zeta'(-1) = \zeta'(0) = 0$. Then $0 \leq \eta \leq 1$, $\eta(x) \equiv 1$ for $-R_1 - 1 < x < x_0 - \varepsilon$, and $\eta(x) \equiv 0$ for $x > x_0$ or $x < -R_1 - 2$. Putting η into (1.8) and integrating by parts,

$$(1.9) \quad \begin{aligned} & \frac{1}{\varepsilon} \int_{x_0 - \varepsilon}^{x_0} h^{(m)'}(x, t) \zeta'((x - x_0)/\varepsilon) dx + \int_R \left(\int_0^t u^{(m)p}(x, \tau) d\tau \right) \eta(x) dx \\ &= - \int_R u^{(m)}(x, t) \eta(x) dx + \int_R f \eta dx. \end{aligned}$$

Since $h^{(m)}(\cdot, t) \in C_0^2(R)$,

$$\begin{aligned} & \left| \frac{1}{\varepsilon} \int_{x_0 - \varepsilon}^{x_0} h^{(m)'}(x, t) \zeta'((x - x_0)/\varepsilon) dx + h^{(m)'}(x_0, t) \right| \\ & \leq \frac{1}{\varepsilon} \int_{x_0 - \varepsilon}^{x_0} |h^{(m)'}(x, t) - h^{(m)'}(x_0, t)| |\zeta'((x - x_0)/\varepsilon)| dx \\ & \leq \sup_{x_0 - \varepsilon \leq x \leq x_0} |h^{(m)'}(x, t) - h^{(m)'}(x_0, t)| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ in (1.9), we get

$$h^{(m)'}(x_0, t) = \int_{-\infty}^{x_0} [h_1^{(m)}(x, t) + u^{(m)}(x, t) - f(x)] dx$$

$$\forall x_0 \in R, 0 \leq t < 1, m > m'$$

(1.10)

$$\Rightarrow |h^{(m)'}(x, t)| \leq C' = (2 + \|f\|_{L^\infty}^{p-1}) \|f\|_{L^1} < \infty \quad \forall x \in R, 0 \leq t < 1, m > m'.$$

Hence $\{h^{(m)'}(x, t)\}_{m>p}$ is uniformly bounded on $R \times [0, 1)$. The second inequality of (1.4) for $n = 1$ then follows by the same argument as in the case $n = 2$. We next observe that since

$$\begin{aligned} \int_E \frac{1}{|x-y|} dy &= \left(\int_{E \cap B_{R_0}(0)} + \int_{E \cap B_{R_0}(0)^c} \right) \frac{1}{|x-y|} dy \\ &\leq \int_{B_{R_0}(0)} \frac{1}{|x-y|} dy = 2|B_1(0)|^{1/2}|E|^{1/2} \end{aligned}$$

for any measurable set $E \subset R^2$ where $R_0 > 0$ is such that $|B_{R_0}(0)| = |E|$, we have

$$\begin{aligned} \int_E \left(\int_{R^2} \frac{1}{|x-y|} u^{(m)}(y, t) dy \right) dx &= \int_{R^2} \left(\int_E \frac{1}{|x-y|} dx \right) u^{(m)}(y, t) dy \\ &\leq 2|B_1(0)|^{1/2} \|f\|_{L^1} |E|^{1/2}. \end{aligned}$$

Hence by (1.6),

$$\int_E |\nabla h^{(m)}(x, t)| dx \leq \frac{6(1 + \|f\|_{L^\infty}^{p-1})}{\pi} |B_1(0)|^{1/2} \|f\|_{L^1} |E|^{1/2}$$

for any measurable set $E \subset R^2$. If we put $m = m_i''$, $t = \varepsilon_{k_j}$ in the above inequality and let $i \rightarrow \infty$, $j \rightarrow \infty$, the second inequality of (0.5) then follows by Fatou's Lemma. Similarly the first inequality of (0.5) follows by integrating the first inequality of (1.4). The lemma is proved. $\square$

We next recall a result of [BBC] and a result of [BC].

Theorem 1.6 (cf. [BBC]). *For any $0 \leq f \in L^1(R^n)$, there exist at most one function $g \in L^1$, $0 \leq g \leq 1$, $\|g\|_{L^1} \leq \|f\|_{L^1}$, and at most one function $\tilde{g} \in L^1_{loc}(R^n)$, $\tilde{g} \geq 0$, satisfying (0.3), (0.4), (0.5).*

Theorem 1.7. *There exists a constant $C > 0$ depending only on $\|f\|_{L^1}$, $\|f\|_{L^\infty}$, p , and independent of $m > p$, such that*

$$\int_{R^n} |u^{(m)}(x, t+h) - u^{(m)}(x, t)| dx \leq \frac{Ch}{t}$$

for all $0 < t \leq t+h < 1$, $0 \leq h \leq t/2$. Hence $u^{(m)} \in C((0, 1); L^1(R^n))$.

Proof. The theorem follows directly from the proof of Theorem 4 and Theorem 7 of [BC]. $\square$

2.

In this section we will use an adaptation of the argument in [H3] to prove the main theorem. We will first state a preliminary version of the main theorem.

Theorem 2.1. *Let $f \in C_0(R^n)$, $f \geq 0$. For any subsequence $\{u^{(m_i)}\}_{i=1}^\infty$, $m_i \rightarrow \infty$ as $i \rightarrow \infty$, of $\{u^{(m)}\}_{m \geq p}$, there exist a subsequence $\{u^{(m'_i)}\}_{i=1}^\infty$ of $\{u^{(m_i)}\}_{i=1}^\infty$ and a $u^{(\infty)} \in C((0, 1); L^1(R^n))$, $0 \leq u^{(\infty)} \leq 1$, such that*

$$u^{(m'_i)} \rightarrow u^{(\infty)} \text{ in } L^1_{loc}(R^n \times (0, 1)) \text{ as } i \rightarrow \infty.$$

Moreover $u^{(\infty)}$ satisfies $v_t = -v^p$ in $\mathcal{D}'(R^n \times (0, 1))$ with initial value $g \in L^1(R^n)$, $0 \leq g \leq 1$, $\|g\|_{L^1} = \|f\|_{L^1}$, satisfying $g - \Delta \tilde{g} = f$ for some function $\tilde{g} \in L^\infty(R^n)$, $\tilde{g} \geq 0$, satisfying (0.5) and (1.4).

Proof. The proof is similar to the proof of Theorem 2.1 of [H3] and Theorem 1 of [S2]. Since by Theorem 1.1,

$$\begin{aligned} & \int_{R^n} |u^{(m)}(x + x_0, t) - u^{(m)}(x, t)| dx \\ & \leq \int_{R^n} |f(x + x_0) - f(x)| dx \quad \forall x_0 \in R^n, 0 \leq t < 1 \end{aligned}$$

by Lemma 1.3(ii) and Theorem 1.7 $\{u^{(m)}\}_{m \geq p}$ is precompact in $L^1_{loc}(R^n \times (0, 1))$. Hence there exist a function $u^{(\infty)} \in L^1(R^n \times (0, 1))$ and a subsequence $\{u^{(m'_i)}\}_{i=1}^\infty$ of $\{u^{(m_i)}\}_{i=1}^\infty$ such that $\{u^{(m'_i)}\}_{i=1}^\infty$ converges to $u^{(\infty)}$ in $L^1_{loc}(R^n \times (0, 1))$ as $i \rightarrow \infty$. Without loss of generality we may assume that $u^{(m_i)} \rightarrow u^{(\infty)}$ in $L^1_{loc}(R^n \times (0, 1))$ and for a.e. $(x, t) \in R^n \times (0, 1)$ as $i \rightarrow \infty$. Then $0 \leq u^{(\infty)} \leq 1$ by Lemma 1.3.

We claim that $u^{(\infty)} \in C((0, 1); R^n)$. To prove the claim we observe first that by Theorem 1.7, $\forall 0 < \tau_1 < \tau_2 < 1$, there exists $C > 0$ such that

$$\begin{aligned} (2.1) \quad & \int_{R^n} |u^{(m_i)}(x, s') - u^{(m_i)}(x, s)| dx \leq C(s' - s) \\ & \forall \tau_1 < s < s' < \tau_2, 0 < s' - s < s/2, i \in \mathbb{Z}^+. \end{aligned}$$

For any t, t' satisfying $\tau_1 < t < t' < \tau_2$, $0 < t' - t < t/4$, we let

$$h_0 = \min\left(t - \tau_1, \tau_2 - t', \frac{t' - t}{2}, \frac{3t - 2t'}{5}\right).$$

Note that $3t - 2t' > 3t - 2(5t/4) = t/2 > 0$. Then for any $0 < h_1, h_2 < h_0$, $s \in (t - h_1, t + h_1)$, $s' \in (t' - h_2, t' + h_2)$, we have $\tau_1 < s < s' < \tau_2$,

$$s' - s > t' - h_2 - (t + h_1) \geq t' - t - 2h_0 \geq 0$$

and

$$\begin{aligned} & s' - s < t' + h_2 - (t - h_1) \leq t' - t + 2h_0 \leq t' - t + 2\left(\frac{3t - 2t'}{5}\right) = \frac{t + t'}{5} \\ \Rightarrow \quad & \frac{s}{2} > \frac{t - h_1}{2} \geq \frac{t - \frac{3t - 2t'}{5}}{2} = \frac{t + t'}{5} \geq s' - s. \end{aligned}$$

Hence (2.1) holds for all $|s - t| < h_1$, $|s' - t'| < h_2$. By integrating (2.1) over $|s - t| < h_1$, $|s' - t'| < h_2$, we have

$$\begin{aligned} & \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(m_i)}(x, s') - u^{(m_i)}(x, s)| dx ds ds' \\ & \leq C \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} (s' - s) ds ds'. \end{aligned}$$

Hence for all $\tau_1 < t < t' < \tau_2$, $0 < t' - t < t/4$, $0 < h_1$, $h_2 < h_0$,

$$\begin{aligned} & \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(\infty)}(x, s') - u^{(\infty)}(x, s)| dx ds ds' \\ & \leq \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(\infty)}(x, s') - u^{(m_i)}(x, s')| dx ds ds' \\ & \quad + \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(m_i)}(x, s') - u^{(m_i)}(x, s)| dx ds ds' \\ & \quad + \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(m_i)}(x, s) - u^{(\infty)}(x, s)| dx ds ds' \\ & \leq 2h_1 \int_{t'-h_2}^{t'+h_2} \int_{R^n} |u^{(\infty)}(x, s') - u^{(m_i)}(x, s')| dx ds' \\ & \quad + C \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} (s' - s) ds ds' \\ & \quad + 2h_2 \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(m_i)}(x, s) - u^{(\infty)}(x, s)| dx ds. \end{aligned}$$

Since by Lemma 1.3, there exists $R_1 > 0$ and $m' > p$ such that (1.1) holds, it follows that $u^{(m_i)} \rightarrow u^{(\infty)}$ in $L^1(R^n \times (\tau_1, \tau_2))$ as $i \rightarrow \infty$. Hence by letting $i \rightarrow \infty$, we get

$$\int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(\infty)}(x, s') - u^{(\infty)}(x, s)| dx ds ds' \leq C \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} (s' - s) ds ds'$$

for all $\tau_1 < t < t' < \tau_2$, $0 < t' - t < t/4$, $0 < h_1$, $h_2 < h_0$. Dividing both side by $4h_1h_2$ and letting first $h_1 \rightarrow 0$ and then $h_2 \rightarrow 0$, we get, by the Lebesgue differentiation theorem [St],

$$(2.2) \quad \int_{R^n} |u^{(\infty)}(x, t') - u^{(\infty)}(x, t)| dx \leq C(t' - t)$$

for a.e. t' , t , $\tau_1 < t < t' < \tau_2$, $0 < t' - t < t/4$. By redefining $u^{(\infty)}$ on a set of measure zero if necessary, we can assume without loss of generality that (2.1) holds for all t' , t satisfying $\tau_1 < t < t' < \tau_2$, $0 < t' - t < t/4$. Hence $u^{(\infty)} \in C((0, 1); L^1(R^n))$. Putting $h(u) = -u^p$, $u = u^{(m_i)}$ in (0.7) and letting $i \rightarrow \infty$, we find that $u^{(\infty)}$ satisfies

$$(2.3) \quad \int_0^1 \int_{\Omega} (v\eta_t - v^p\eta) dx d\tau = 0$$

for any bounded open set $\Omega \subset R^n$ with $\partial\Omega \in C^2$, $\eta \in C^\infty(\Omega \times [0, 1])$, $\eta \equiv 0$ on $\partial\Omega \times [0, 1]$ and $\eta(\cdot, t) \equiv 0$ for t close to 0 and 1. Since $u^{(\infty)} \in C((0, 1); L^1(R^n))$, by putting

$$\eta(x, t) = \psi(x, t) \zeta\left(\frac{t - \tau_2}{\varepsilon}\right) \zeta\left(\frac{\tau_1 - t}{\varepsilon}\right)$$

into (2.3), where ζ is as in the proof of Lemma 1.5, $0 < \tau_1 < \tau_2 < 1$, $\psi \in C^\infty(\Omega \times [0, 1])$, $\psi \equiv 0$ on $\partial\Omega \times [0, 1]$, and letting $\varepsilon \rightarrow 0$, by the same argument as the proof of Theorem 3.2 of [DKe] we get that $u^{(\infty)}$ satisfies

$$(2.4) \quad \int_{\Omega} v\psi dx \Big|_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \int_{\Omega} (v\psi_t - v^p\psi) dx d\tau$$

for any bounded open set $\Omega \subset R^n$ with $\partial\Omega \in C^2$, $\psi \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\psi \equiv 0$ on $\partial\Omega \times [\tau_1, \tau_2]$.

To show that g satisfies (0.3) we let $h^{(m)}$ be as in Lemma 1.5 and Ω be a bounded open subset of R^n with $\partial\Omega \in C^2$. Then by Lemma 1.5 there exists a constant $C_1 > 0$ such that $\|h^{(m)}\|_{L^\infty(R^n \times [0, 1])} \leq C_1$ for all $m > p$, and there exist a subsequence $\{h^{(m'_i)}\}_{i=1}^\infty$ of $\{h^{(m_i)}\}_{i=1}^\infty$, a sequence of functions $\{h_k\}_{k=1}^\infty \subset L^\infty(R^n)$, $\tilde{g} \in L^\infty(R^n)$, $h_k, \tilde{g} \geq 0$, and a sequence $1 > \varepsilon_1 > \varepsilon_2 > \dots > 0$, $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, such that (1.3), (1.4), and (0.5) hold.

Since by Theorem 1.2,

$$\int_{R^n} u^{(m_i)}(x, \varepsilon_k) dx \leq \int_{R^n} f(x) dx, \quad \forall i, k = 1, 2, \dots,$$

letting $i \rightarrow \infty$, we get by Fatou's Lemma,

$$\int_{R^n} u^{(\infty)}(x, \varepsilon_k) dx \leq \int_{R^n} f(x) dx, \quad \forall k = 1, 2, \dots$$

Hence there exists a subsequence $\{u^{(\infty)}(x, \varepsilon'_k)\}_{k=1}^\infty$ of $\{u^{(\infty)}(x, \varepsilon_k)\}_{k=1}^\infty$ such that $u^{(\infty)}(x, \varepsilon'_k) \rightarrow d\mu$ weakly in $L^1(R^n)$ for some measure $d\mu \in L^1(R^n)$. Since $0 \leq u^{(\infty)} \leq 1$, we have $d\mu = gdx$ for some function $g \in L^1(R^n)$ with $0 \leq g \leq 1$, $\|g\|_{L^1} \leq \|f\|_{L^1}$. We may assume without loss of generality that $u^{(\infty)}(x, \varepsilon_k) \rightarrow g$ weakly in $L^1(R^n)$ as $k \rightarrow \infty$. Putting $\tau_1 = \varepsilon_k$ in (2.4) and letting first $k \rightarrow \infty$ and then $\tau_2 \rightarrow 0$, we get

$$\lim_{t \rightarrow 0} \int u^{(\infty)} \eta dx = \int g \eta dx \quad \forall \eta \in C_0^\infty(R^n).$$

Putting $h(u) = -u^p$, $u = u^{(m)}$, $\tau_2 = \varepsilon_k$, $m = m'_i$ in (0.7) and letting first $\tau_1 \rightarrow 0$ and then $i \rightarrow \infty$, we get by Corollary 1.4 and Lemma 1.5,

$$(2.5) \quad \begin{aligned} & \int_{R^n} h_k(x) \Delta \eta(x) dx - \int_0^{\varepsilon_k} \int_{R^n} u^{(\infty)p}(x, \tau) \eta(x) dx d\tau \\ &= \int_{R^n} u^{(\infty)}(x, \varepsilon_k) \eta(x) dx - \int_{R^n} f \eta dx \end{aligned}$$

for all $\eta \in C_0^\infty(R^n)$, $k = 1, 2, \dots$. Letting $k \rightarrow \infty$, we get by Lemma 1.5,

$$g - \Delta \tilde{g} = f \text{ in } \mathcal{D}'(R^n)$$

for some function $\tilde{g} \in L^\infty(R^n)$, $\tilde{g} \geq 0$, satisfying (1.4). Let $R_1 > 0$ be as in (ii) of Lemma 1.3 and let $\psi \in C_0^\infty(R^n)$, $0 \leq \psi \leq 1$, be such that $\psi(x) = 1$ for $|x| \leq R_1 + 1$, $\psi(x) = 0$ for $|x| \geq R_1 + 2$. Since $\text{supp } h_k \subset B_{R_1}(0)$ by Lemma 1.5 and

$$\text{supp } u^{(m)} \subset B_{R_1}(0) \quad \Rightarrow \quad \text{supp } u^{(\infty)} \subset \overline{B_{R_1}(0)}$$

putting $\eta = \psi$ in (2.5) we have

$$\begin{aligned} \int_0^{\varepsilon_k} \int_{R^n} u^{(\infty)p}(x, \tau) dx d\tau &= \int_{R^n} u^{(\infty)}(x, \varepsilon_k) dx - \int_{R^n} f dx \\ \Rightarrow \int_{R^n} g dx &= \int_{R^n} f dx \quad \text{as } k \rightarrow \infty. \end{aligned}$$

This completes the proof of Theorem 2.1. $\square$

Lemma 2.2. *If $u^{(\infty)}$ and g is as in Theorem 2.1, then for a.e. $x \in R^n$,*

$$(2.6) \quad u^{(\infty)}(x, t) - g(x) = - \int_0^t u^{(\infty)p}(x, s) ds \quad \forall 0 < t < 1$$

and

$$(2.7) \quad \lim_{t \rightarrow 0} u^{(\infty)}(x, t) = g(x).$$

Proof. Putting $\psi(x, t) = \rho_\varepsilon(y - x)$ in (2.4) and letting first $\tau_1 \rightarrow 0$ and then $\varepsilon \rightarrow 0$, we get (2.6). Since $0 \leq u^{(\infty)} \leq 1$, (2.7) is a direct consequence of (2.6). $\square$

Lemma 2.3. *Let $f, g, \tilde{g}$ be as in Theorem 2.1. Then (0.4) holds.*

Proof. Let $u^{(m'_i)}, u^{(\infty)}$ be as in Theorem 2.1, and let $v^{(m)}$ be the solution of

$$\begin{cases} v_t = \Delta v^m, v \geq 0, & \text{in } R^n \times (0, 1), \\ v(x, 0) = f(x) & \forall x \in R^n. \end{cases}$$

By the result of [BBH], there exists $g_1 \in L^1(R^n)$, $0 \leq g_1 \leq 1$, $\tilde{g}_1 \geq 0$, satisfying

$$\begin{cases} g_1 - \Delta \tilde{g}_1 = f & \text{in } \mathcal{D}'(R^n), \\ \tilde{g}_1(x) = 0 & \text{whenever } g_1(x) < 1 \text{ a.e. } x \in R^n, \\ \int_{R^n} g_1 = \int_{R^n} f \end{cases}$$

and $v^{(m)}(x, t) \rightarrow g_1(x)$ in $L^1(R^n)$ uniformly in t on compact subsets of $(0, 1)$ as $m \rightarrow \infty$. Since $u^{(m)}$ satisfies $u_t \leq \Delta u^m$, by the maximum principle,

$$\begin{aligned} u^{(m)}(x, t) &\leq v^{(m)}(x, t) \quad \forall (x, t) \in R^n \times (0, 1) \\ \Rightarrow u^{(\infty)}(x, t) &\leq g_1(x) \quad \text{a.e. } x \in R^n, 0 < t < 1 \text{ as } m = m'_i \rightarrow \infty \\ \Rightarrow g(x) &\leq g_1(x) \quad \text{a.e. } x \in R^n \text{ as } t \rightarrow 0. \end{aligned}$$

Since $\int_{R^n} g dx = \int_{R^n} f dx = \int_{R^n} g_1 dx$, $g(x) = g_1(x)$ a.e. $x \in R^n$. Thus $\tilde{g} = \tilde{g}_1$ and the lemma follows. $\square$

As a consequence of Theorem 1.6, Theorem 2.1, Lemmas 2.2, 2.3, and the uniqueness of a solution of the ODE (2.6), (2.7), we have

Theorem 2.4. *Suppose $f \in C_0(R^n)$, $f \geq 0$. Then there exists a unique function $u^{(\infty)} \in C((0, 1); R^n)$, $0 \leq u^{(\infty)} \leq 1$, such that $u^{(m)} \rightarrow u^{(\infty)}$ in $L^1_{loc}(R^n \times (0, 1))$ as $m \rightarrow \infty$. Moreover $u^{(\infty)}$ satisfies (0.2) with initial value $g \in L^1(R^n)$, $0 \leq g \leq 1$, $\|g\|_{L^1(R^n)} = \|f\|_{L^1(R^n)}$, satisfying $g - \Delta \tilde{g} = f$ for some function $\tilde{g} \in L^\infty(R^n)$, $\tilde{g} \geq 0$, satisfying (0.4), (0.5), and (1.4).*

We are now ready to state and prove the main theorem.

Theorem 2.5. *For any $f \in L^1(R^n) \cap L^\infty(R^n)$, $f \geq 0$, there exists a unique function $u^{(\infty)} \in C((0, 1); R^n)$, $0 \leq u^{(\infty)} \leq 1$, such that $u^{(m)}$ converges to $u^{(\infty)}$ in $L^1_{loc}(R^n \times (0, 1))$ as $m \rightarrow \infty$. Moreover $u^{(\infty)}$ satisfies (0.2) with initial value $g \in L^1(R^n)$, $0 \leq g \leq 1$, $\|g\|_{L^1(R^n)} \leq \|f\|_{L^1(R^n)}$, satisfying $g - \Delta \tilde{g} = f$ for some function $\tilde{g} \in L^\infty_{loc}(R^n)$, $\tilde{g} \geq 0$, satisfying (0.4) and (0.5).*

Proof. The proof is a modification of the proof of Theorem 2.10 of [H3]. We first choose a sequence $\{f_j\}_{j=1}^\infty \subset C_0(R^n)$ such that $\|f_j\|_{L^\infty(R^n)} \leq \|f\|_{L^\infty(R^n)} + 1$, $\|f_j\|_{L^1(R^n)} \leq \|f\|_{L^1(R^n)} + 1$ for all $j = 1, 2, \dots$, and $\|f_j - f\|_{L^1(R^n)} \rightarrow 0$ as $j \rightarrow \infty$.

For all $j = 1, 2, \dots$, let $u_j^{(m)}$ be the solution of (0.1) in $R^n \times (0, 1)$ with initial value $u_j^{(m)}(x, 0) = f_j(x)$ and let $u^{(\infty)}$ be the unique function given by Theorem 2.4 such that $u_j^{(m)} \rightarrow u_j^{(\infty)}$ in $L^1_{loc}(R^n \times (0, 1))$ as $m \rightarrow \infty$. Then $u_j^{(\infty)}$ satisfies (0.2) with initial value $g_j \in L^1(R^n)$, $0 \leq g_j \leq 1$, satisfying

$$(2.8) \quad \begin{cases} \int_{R^n} g_j \leq \int_{R^n} f_j \leq \int_{R^n} f dx + 1, \\ g_j - \Delta \tilde{g}_j = f \quad \text{in } \mathcal{D}'(R^n) \text{ for some } \tilde{g}_j \in L^\infty(R^n), \tilde{g}_j \geq 0, \text{ satisfying (1.4), (0.5),} \\ \tilde{g}_j(x) = 0 \quad \text{whenever } g_j(x) < 1 \text{ a.e. } x \in R^n \end{cases}$$

for all $j = 1, 2, \dots$. As in the proof of Theorem 2.1, by Theorems 1.1 and 1.7 $\{u^{(m)}\}_{m \geq p}$ is precompact in $L^1_{loc}(R^n \times (0, 1))$. Thus any subsequence $\{u^{(m_i)}\}_{i=1}^\infty$, $m_i \rightarrow \infty$ as $i \rightarrow \infty$, of $\{u^{(m)}\}_{m \geq p}$ has a subsequence $\{u^{(m'_i)}\}_{i=1}^\infty$ such that $u^{(m'_i)} \rightarrow u^{(\infty)}$ in $L^1_{loc}(R^n \times (0, 1))$ and $u^{(m'_i)}(x, t) \rightarrow u^{(\infty)}(x, t)$ a.e. $(x, t) \in R^n \times (0, 1)$ as $i \rightarrow \infty$ for some function $u^{(\infty)} \in L^1(R^n \times (0, 1))$. By Theorem 1.1,

$$(2.8) \quad \begin{aligned} \int_{R^n} |u_j^{(m'_i)} - u^{(m'_i)}|(x, t) dx &\leq \int_{R^n} |f_j - f|(x) dx \quad \forall i, j = 1, 2, \dots \\ \Rightarrow \int_{\tau_1}^{\tau_2} \int_{R^n} |u_j^{(m'_i)} - u^{(m'_i)}|(x, t) dx dt \\ &\leq (\tau_2 - \tau_1) \int_{R^n} |f_j - f|(x) dx \quad \forall 0 < \tau_1 \leq \tau_2 < 1. \end{aligned}$$

Letting $i \rightarrow \infty$, we get by Fatou's Lemma,

$$(2.10) \quad \int_{\tau_1}^{\tau_2} \int_{R^n} |u_j^{(\infty)} - u^{(\infty)}|(x, t) dx dt \leq (\tau_2 - \tau_1) \int_{R^n} |f_j - f|(x) dx \rightarrow 0 \text{ as } j \rightarrow \infty$$

for all $0 < \tau_1 \leq \tau_2 < 1$.

Hence $u^{(\infty)}$ is the limit of the functions $\{u_j^{(\infty)}\}_{j=1}^\infty$ in $L^1_{loc}(R^n \times (0, 1))$ as $j \rightarrow \infty$. Thus $u^{(\infty)}$ is unique and $0 \leq u^{(\infty)} \leq 1$, since $0 \leq u_j^{(\infty)} \leq 1$ for all $j = 1, 2, \dots$.

Putting $v = u_j^{(\infty)}$ in (2.3) and letting $j \rightarrow \infty$, we see that $u^{(\infty)}$ satisfies (2.3). We next claim that $u^{(\infty)} \in C((0, 1); R^n)$. To prove the claim we first observe that by the proof of Theorem 2.1 there exists a constant $C > 0$ independent of $j = 1, 2, \dots$ such that (2.2) holds for $u_j^{(\infty)}$ for all $j = 1, 2, \dots$. For any t, t' satisfying $0 < \tau_1 < t < t' < \tau_2 < 1$, $0 < t' - t < t/8$, we let

$$h_0 = \min \left(t - \tau_1, \tau_2 - t', \frac{t' - t}{2}, \frac{5t - 4t'}{9} \right).$$

As in the proof of Lemma 1.5, for any $0 < h_1, h_2 < h_0$, $s \in (t - h_1, t + h_1)$, $s' \in (t' - h_2, t' + h_2)$, we have $\tau_1 < s < s' < \tau_2$, $0 < s' - s < s/4$. Hence for any t, t' satisfying $0 < \tau_1 < t < t' < \tau_2 < 1$, $0 < t' - t < t/8$, we have

$$\begin{aligned}
& \int_{R^n} |u_j^{(\infty)}(x, s') - u_j^{(\infty)}(x, s)| dx \leq C(s' - s) \\
& \quad \forall |s - t| < h_1, |s' - t'| < h_2, j = 1, 2, \dots \\
& \Rightarrow \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u_j^{(\infty)}(x, s') - u_j^{(\infty)}(x, s)| dx ds ds' \\
& \leq C \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} (s' - s) ds ds' \quad \forall j = 1, 2, \dots \\
& \Rightarrow \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(\infty)}(x, s') - u^{(\infty)}(x, s)| dx ds ds' \\
& \leq \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u^{(\infty)}(x, s') - u_j^{(\infty)}(x, s')| dx ds ds' \\
& \quad + \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u_j^{(\infty)}(x, s') - u_j^{(\infty)}(x, s)| dx ds ds' \\
& \quad + \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} \int_{R^n} |u_j^{(\infty)}(x, s) - u^{(\infty)}(x, s)| dx ds ds' \\
& \leq C \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} (s' - s) ds ds' + 8h_1 h_2 \|f_j - f\|_{L^1} \\
& \rightarrow C \int_{t'-h_2}^{t'+h_2} \int_{t-h_1}^{t+h_1} (s' - s) ds ds' \quad \text{as } j \rightarrow \infty
\end{aligned}$$

by (2.10) for some constant depending on $\|f\|_{L^1}$, $\|f\|_{L^\infty}$ and τ_1, τ_2 . Dividing both sides by $4h_1 h_2$ and letting first $h_1 \rightarrow 0$ and then $h_2 \rightarrow 0$, we get by Lebesgue's differentiation theorem that (2.2) holds for $u^{(\infty)}$, $\tau_1 < t < t' < \tau_2$, $0 < t' - t < t/8$. Hence $u^{(\infty)} \in C((0, 1); L^1(R^n))$. By the same argument as the proof of Theorem 2.1, $u^{(\infty)}$ satisfies (2.4) and has an initial trace g , $0 \leq g \leq 1$; and $\|g\|_{L^1(R^n)} \leq \|f\|_{L^1(R^n)}$ by Theorem 1.1.

We will next show that $g_j \rightarrow g$ in $L^1(R^n)$ as $j \rightarrow \infty$. Observe that, since both $u_j^{(\infty)}$ and $u^{(\infty)} \in C((0, 1); L^1(R^n))$, dividing both side of (2.10) by $(\tau_2 - \tau_1)$ and letting $\tau_2 - \tau_1 \rightarrow 0$, we get

$$(2.11) \quad \int_{R^n} |u_j^{(\infty)} - u^{(\infty)}|(x, t) dx \leq \int_{R^n} |f_j - f|(x) dx \quad \forall 0 < t < 1, j = 1, 2, \dots$$

Since for a.e. $x \in R^n$, $\lim_{t \rightarrow 0} u^{(\infty)}(x, t) = g(x)$ and $\lim_{t \rightarrow 0} u_j^{(\infty)}(x, t) = g_j(x)$ for all $j = 1, 2, \dots$ by the argument of Lemma 2.2, letting $t \rightarrow 0$ in (2.11) we get by Fatou's Lemma,

$$\int_{R^n} |g - g_j|(x) dx \leq \int_{R^n} |f - f_j|(x) dx \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Hence $g_j \rightarrow g$ in $L^1(R^n)$ as $j \rightarrow \infty$. By passing to a subsequence if necessary we may assume without loss of generality that $g_j(x) \rightarrow g(x)$ a.e. $x \in R^n$. We next observe that since $\tilde{g}_j$ satisfies (1.4),

$$\begin{aligned}\tilde{g}_j(x) &\leq C(\|f_j\|_\infty + \|f_j\|_{L^1}) \\ &\leq C(\|f\|_\infty + \|f\|_{L^1} + 2) \quad \forall x \in R^n, n \geq 3, j = 1, 2, \dots,\end{aligned}$$

for some constants $C > 0$. Hence $\{\tilde{g}_j\}_{j=1}^\infty$ is uniformly bounded in R^n for $n \geq 3$ and there exist a function $\tilde{g} \in L^\infty(R^n)$ and a subsequence $\{\tilde{g}_{j_k}\}_{k=1}^\infty$ of $\{\tilde{g}_j\}_{j=1}^\infty$ such that $\tilde{g}_{j_k} \rightarrow \tilde{g}$ weakly in $(L^\infty(K))^*$ for any compact subset K of R^n , $n \geq 3$. Without loss of generality we may assume that $\tilde{g}_j \rightarrow \tilde{g}$ weakly in $(L^\infty(K))^*$ for any compact subset K of R^n and a.e. $x \in R^n$ as $j \rightarrow \infty$.

For $n = 1, 2$, since

$$\begin{aligned}|\nabla \tilde{g}_j| &\leq C(1 + \|f_j\|_{L^\infty}^{p-1})(\|f_j\|_{L^\infty} + \|f_j\|_{L^1}) \quad \forall j = 1, 2, \dots, \\ (2.12) \quad &\leq C(2 + \|f\|_{L^\infty}^{p-1})(\|f\|_{L^\infty} + \|f\|_{L^1} + 2) \quad \forall j = 1, 2, \dots\end{aligned}$$

by (1.4), $\{\tilde{g}_j\}_{j=1}^\infty$ is uniformly Lipschitz continuous. Hence by the Ascoli Theorem either $\{\tilde{g}_j\}_{j=1}^\infty$ has a subsequence (which we may assume to be the sequence itself) that converges uniformly on compact subsets of R^n to a continuous function, or $\tilde{g}_j(x) \rightarrow \infty$ for all $x \in R^n$ as $j \rightarrow \infty$ for $n = 1, 2$. We claim that the latter case is not possible. To prove the claim we let $\tilde{g} = \lim_{j \rightarrow \infty} \tilde{g}_j$ and let

$$\begin{aligned}E &= \{x \in R^n : g_j(x) \rightarrow g(x) \text{ and } \tilde{g}_j \rightarrow \tilde{g}(x) \text{ as } j \rightarrow \infty\}, \\ E_1 &= \bigcup_{j=1}^\infty \{x \in R^n : g_j(x) < 1 \text{ and } \tilde{g}_j \neq 0\}, \\ E_0 &= E \cap \{g < 1\} \setminus E_1.\end{aligned}$$

For any $x_0 \in E_0$, since $g_j(x_0) \rightarrow g(x_0) < 1$ as $j \rightarrow \infty$, there exists $j_0 \in \mathbb{Z}^+$ such that $g_j(x_0) < 1 \quad \forall j \geq j_0$. Since $x_0 \notin E_1$, $\tilde{g}_j(x_0) = 0$ for all $j \geq j_0$. Letting $j \rightarrow \infty$, we have $\tilde{g}(x_0) = 0$. Since $|\{g < 1\} \setminus E_0| = 0$ and $|E_1| = 0$ by Lemma 2.3, $\tilde{g}(x_0) = 0$ a.e. $x_0 \in \{g < 1\}$. Since $\int g < \infty$, $|\{g < 1\}| > 0$. Hence there exists x_0 such that $\tilde{g}(x_0) = 0$. Thus the claim follows. Hence $\tilde{g}_j$ converges uniformly to $\tilde{g}$ on every compact subset of R^n as $j \rightarrow \infty$ for $n = 1, 2$. Thus $\tilde{g} \in L_{loc}^\infty(R^n)$, $\tilde{g} \geq 0$, and (0.4) holds. We are now ready to show that $g, \tilde{g}$ satisfy (0.3) and (0.5).

Since $\{|\nabla \tilde{g}_j|\}_{j=1}^\infty$ is uniformly bounded in R^n for $n = 1, 2$ by (2.12), we may assume without loss of generality that for $n = 1, 2$,

$$\nabla \tilde{g}_j \rightarrow \nabla \tilde{g} \quad \text{weakly in } (L^\infty(K))^* \quad \text{and a.e. } x \in R^n \quad \text{as } j \rightarrow \infty.$$

Putting $\tilde{g} = \tilde{g}_j$ in (0.5), and letting $j \rightarrow \infty$, we get by Fatou's Lemma that (0.5) holds for $\tilde{g}$.

We next observe that by Theorem 2.1, $u_j^{(\infty)}$ satisfies, for all $\eta \in C_0^\infty(R^n)$, $0 < t < 1$,

$$\begin{aligned} & \int_{R^n} \tilde{g}_j \Delta \eta dx - \int_0^t \int_{R^n} u_j^{(\infty)p} \eta dx d\tau = \int_{R^n} u_j^{(\infty)}(x, t) \eta(x) dx - \int_{R^n} f_j \eta dx \\ \Rightarrow & \int_{R^n} \tilde{g} \Delta \eta dx - \int_0^t \int_{R^n} u^{(\infty)p} \eta dx d\tau \\ & = \int_{R^n} u^{(\infty)}(x, t) \eta(x) dx - \int_{R^n} f \eta dx \text{ as } j \rightarrow \infty \\ \Rightarrow & \int_{R^n} \tilde{g} \Delta \eta dx = \int_{R^n} g(x) \eta(x) dx - \int_{R^n} f \eta dx \quad \text{as } t \rightarrow 0 \\ \Rightarrow & g - \Delta \tilde{g} = f \quad \text{in } \mathcal{D}'(R^n). \end{aligned}$$

Hence (0.3) holds. By the same argument as the proof of Lemma 2.2, $u^{(\infty)}$ and g satisfy (2.6) and (2.7). The theorem then follows from Lemma 1.5, Theorem 1.6 and the uniqueness of a solution of the ODE (2.6), (2.7). $\square$

REFERENCES

- [A] D.G.Aronson, *The porous medium equation*, Some Problems in Nonlinear Diffusion, Lecture Notes in Mathematics 1224, Springer-Verlag, New York, 1986, pp. 1–46. MR **88a**:35130
- [BBC] P.Bénilan, H.Brezis and M.G.Crandall, *A semilinear elliptic equation in L^1* , Ann. Scuola Norm. Sup. Pisa, (4) 2 (1975), 523–555. MR **52**:11299
- [BBH] P.Bénilan, L.Boccardo and M.Herrero, *On the limit of solutions of $u_t = \Delta u^m$ as $m \rightarrow \infty$* , Some Topics in Nonlinear PDE's, Proceedings Int. Conf. Torino 1989, M.Bertsch et. al., eds., Rend. Sem. Mat. Univ. Politec. Torino **1989**, Special Issue (1991), 1–13. MR **93c**:35069
- [BC] P.Bénilan and M.G.Crandall, *Regularizing effects of homogeneous evolution equations*, Contributions to analysis and geometry, D.N.Clark, G.Pecelli and R.Sacksteder ed., The Johns Hopkins Univ. Press, (1981), pp. 23–39. MR **83g**:47063
- [CF] L.A.Caffarelli and A.Friedman, *Asymptotic behaviour of solutions of $u_t = \Delta u^m$ as $m \rightarrow \infty$* , Indiana Univ. Math. J. (1987), 711–728. MR **88m**:35075
- [DK] B. E. J. Dahlberg and C. Kenig, *Nonnegative solutions of the generalized porous medium equations*, Revista Matemática Iberoamericana **2** (1986), 267–305. MR **88k**:35223
- [DKe] J.I.Diaz and R.Kersner, *On a nonlinear degenerate parabolic equation in infiltration or evaporation through a porous medium*, J. Diff. Equations **69** (1987), 368–403. MR **88i**:35088
- [HP] M.A.Herrero and M.Pierre, *The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$* , Trans. of A.M.S. **291** (1985), 145–158. MR **86i**:35065
- [H1] K.M.Hui, *Asymptotic behaviour of solutions of $u_t = \Delta u^m - u^p$ as $p \rightarrow \infty$* , Nonlinear Anal., TMA **21**(3) (1993), 191–195. MR **94h**:35103
- [H2] K.M.Hui, *Singular limit of solutions of the generalized p -laplacian equation*, Nonlinear Analysis, TMA **24**(9) (1995), 1327–1345. MR **96f**:35094
- [H3] K.M.Hui, *Singular limit of solutions of $u_t = \Delta u^m - A \cdot \nabla(u^q/q)$ as $q \rightarrow \infty$* , Trans. A.M.S. **347**(5) (1995), 1687–1712. MR **95h**:35105
- [K1] A.S.Kalashnikov, *The effect of absorption on heat propagation in a medium in which thermal conductivity depends on the temperature (in Russian)*, Zh. Vychisl. Mat. i Mat. Fiz. **16** (1976), 689–696; English transl. in USSR Comput. Math. and Math. Phys. **16** (1976). MR **54**:14575
- [K2] A.S.Kalashnikov, *Some questions of the qualitative theory of nonlinear degenerate parabolic second-order equations*, Uspeki Mat. Nauk. **42** (1987), no. 2, 135–176; English transl. in Russian Math. Surveys **42** (1987), no. 2. MR **88h**:35054
- [KPV] S.Kamin, L.A.Peletier and J.L.Vazquez, *A nonlinear diffusion-absorption equation with unbounded initial data*, Nonlinear Diffusion Equations and Their Equilibrium States, 3 (Gregynog, 1989), Progr. Nonlinear Differential Equations Appl., vol. 7, Birkhäuser, Boston, 1992, pp. 243–263. MR **93f**:35103

- [P] L.A.Peletier, *The porous medium equation*, Applications of Nonlinear Analysis in the Physical Sciences, H.Amann, N.Bazley, K.Kirchgassner, editors, Pitman, Boston, 1981, pp. 229–241. MR **83k**:76076
- [S1] P.E.Sacks, *Continuity of solutions of a singular parabolic equation*, Nonlinear Analysis, TMA **7** (1983), 387–409. MR **84d**:35081
- [S2] P.E.Sacks, *A singular limit problem for the porous medium equation*, J. Math. Anal. Appl. **140** (1989), 456–466. MR **90f**:35099
- [St] E.M.Stein, *Singular integral and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1971. MR **44**:7280
- [VW] J.L.Vazquez and M.Walias, *Existence and uniqueness of diffusion-absorption equations with general data*, Differential and Integral Eq. **7**(1) (1994), 15–36. MR **94k**:35170
- [X] X.Xu, *Asymptotics behaviour of solutions of hyperbolic conservation laws $u_t + (u^m)_x = 0$ as $m \rightarrow \infty$ with inconsistent values*, Proceedings of the Royal Soc. Edinburgh **113A** (1989), 61–71. MR **90k**:35043

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