GOLUBEV SERIES FOR SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT. Let P be an elliptic system with real analytic coefficients on an open set $X \subset \mathbb{R}^n$, and let Φ be a fundamental solution of P . Given a locally connected closed set $\sigma \subset X$, we fix some massive measure m on σ . Here, a non-negative measure m is called massive, if the conditions $s \subset \sigma$ and $m(s) = 0$ imply that $\sigma \setminus s = \sigma$. We prove that, if f is a solution of the equation $Pf = 0$ in $X \setminus \sigma$, then for each relatively compact open subset U of X and every $1 < p < \infty$ there exist a solution f_e of the equation in U and a sequence f_α ($\alpha \in \mathbb{N}_0^n$) in $L^p(\sigma \cap U, m)$ satisfying $\|\alpha! f_\alpha\|_{L^p(\sigma \cap U, m)}^{1/|\alpha|} \rightarrow 0$ such that $f(x) = f_e(x) + \sum_\alpha \int_{\sigma \cap U} D_y^\alpha \Phi(x, y) f_\alpha(y) dm(y)$ for $x \in U \setminus \sigma$. This complements an earlier result of the second author on representation of solutions outside a compact subset of X .

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.1. Let P be a $(k \times k)$ -matrix of scalar partial differential operators with real analytic coefficients on an open set $X \subset \mathbb{R}^n$. Suppose further that P has a fundamental solution Φ which is real analytic outside the diagonal Δ of $X \times X$. By definition, $\Phi(x, y)$ is a $(k \times k)$ -matrix of distributions on $X \times X$ satisfying

$$\begin{cases} P(x, D_x) \Phi(x, y) &= \delta(x - y) I_k, \\ P'(y, D_y) \Phi(x, y) &= \delta(x - y) I_k \end{cases}$$

where P' is the transposed operator to P , and I_k is the identity $(k \times k)$ -matrix.

Recall that, according to a theorem of Malgrange, every elliptic differential operator with real analytic coefficients on X has a fundamental solution with the desired properties.

1.2. If U is an open subset of X , then denote by $S_P(U)$ the vector space of all weak solutions of the system $Pf = 0$ on U . Note that because of the analytic hypoellipticity of P , the solutions in $S_P(U)$ are actually real analytic functions in U . For a closed subset σ of X , solutions $f \in S_P(X \setminus \sigma)$ will be said to have singularities on σ .

In this article, we are interested in representations of solutions of the equation $Pf = 0$ in X having singularities on a closed subset σ of X . Before stating our principal result, we must first introduce one technical definition.

A (nonnegative) measure m on σ is said to be *massive*, if the two conditions $s \subset \sigma$ and $m(s) = 0$ imply that $\sigma \setminus s = \sigma$. In other words, every subset of σ of

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m -measure zero has empty interior. As the following example shows, a massive measure exists on every closed set σ .

Example 1.1. Let $\{y_j\}_{j \in \mathbb{N}}$ be a sequence of points of K , which is dense as a set in σ . Choose a sequence of positive numbers $\{\mu_j\}$ such that $\sum \mu_j < \infty$. For a set $s \subset \sigma$, we define $m(s) = \sum_{y_\nu \in s} \mu_\nu$. Then m is a massive measure on σ .

Let us fix some massive measure m on σ . Our main result is the following:

Theorem 1.1. Assume that K is a locally connected compact subset of σ , and $1 < p < \infty$. Then for each solution $f \in S_P(X \setminus \sigma)$ there exist both a solution $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$ and a sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subseteq [L^p(K, m)]^k$ such that

$$(1) \quad f(x) = f_e(x) + \sum_{\alpha \in \mathbb{N}_0^n} \int_K D_y^\alpha \Phi(x, y) c_\alpha(y) dm(y)$$

holds for all $x \in X \setminus \sigma$. Furthermore, $\|\alpha! c_\alpha\|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0$ when $|\alpha| \rightarrow \infty$.

We emphasize that $\overset{\circ}{K}$ is the interior of K on σ , i.e., in the induced topology of σ .

1.3. For holomorphic functions of one variable (i.e. for the operator $P = \partial/\partial \bar{z}$ in $\mathbb{C}^1$), for compact σ and $p = 2$, Theorem 1.1 is due to Havin [7]. Havin called the corresponding representation of the form (1) *Golubev-series*, since it was V.V. Golubev who posed the question whether such a formula held for every function analytic in $\hat{\mathbb{C}} \setminus K$ when K is a rectifiable simple arc and m the Lebesgue measure on K . For further details on the history of the problem cf. Havin [8]. More generally, we call representations of the form (1) *Golubev-series expansions* for solutions with singularities.

Baernstein [1] proved an analogous representation formula for functions holomorphic off the real axis. Using complex analysis and Hilbert space methods, the second author [15] showed Theorem 1.1 for the case of compact σ and $p = 2$ (see also [16]). Simonova [13] obtained an analogous representation theorem for functions harmonic off a hyperplane. Fischer and Tarkhanov [4] constructed a Golubev-series expansion for solutions of homogeneous elliptic systems with constant coefficients in $\mathbb{R}^n$, having singularities on a plane of a smaller dimension. They also derived Theorem 1.1 for the case of smooth σ and asked whether a result as formulated in Theorem 1.1 held for arbitrary locally connected sets σ .

The local connectedness of the compact set K we look at is a very delicate point in the literature. In fact it is related to the problem of extension of analytic functions on a neighborhood of K . (See Havin [8], Varfolomeev [17] and Rogers/Zame [12].)

In this paper, we prove the result by generalizing the ideas used in [15] in an appropriate way. Since the article [15] is in Russian and does not seem to be easily available, we have decided to present the paper in a self-contained way and do not use [15] as a reference.

1.4. The converse statement to Theorem 1.1 is quite easy to prove.

Lemma 1.1. Let K be a relatively compact subset of σ , and $1 \leq p < \infty$. For every sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subset [L^p(K, m)]^k$, satisfying $\|\alpha! c_\alpha\|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, the series $\sum_\alpha \int_K D_y^\alpha \Phi(x, y) c_\alpha(y) dm(y)$ converges for $x \in X \setminus K$ and defines an element in $S_P(X \setminus K)$.

Proof. First note that for $x \in X \setminus K$ we have

$$P(x, D) \int_K D_y^\alpha \Phi(x, y) c_\alpha(y) dm = \int_K D_y^\alpha \{P(x, D) \Phi(x, y)\} c_\alpha(y) dm = 0.$$

Thus the proof will be complete if we show that the series we look at converges uniformly on compact subsets of $X \setminus K$. It is well-known that a C^∞ function g on an open set $U \subset \mathbb{R}^n$ is real analytic if and only if for every compact set $K \subset U$ there are constants $a = a(g, K)$ and $c = c(g, K)$ such that

$$\sup_{y \in K} |D^\alpha g(y)| \leq c \cdot a^{|\alpha|} |\alpha|! \quad \text{for all } \alpha \in \mathbb{N}_0^n.$$

Now fix a compact set $\tilde{K} \subset X \setminus K$. Since the fundamental solution Φ is real analytic in a neighborhood of $\tilde{K} \times K$, there exist constants a and c , depending on Φ and $\tilde{K}$, such that

$$(2) \quad \sup_{(x, y) \in \tilde{K} \times K} \|D_y^\alpha \Phi(x, y)\| \leq c \cdot a^{|\alpha|} |\alpha|! \quad \text{for all } \alpha \in \mathbb{N}_0^n.$$

Using (2), for $\alpha \in \mathbb{N}_0^n$ we get

$$\begin{aligned} \sup_{x \in \tilde{K}} \left| \int_K D_y^\alpha \Phi(x, y) c_\alpha(y) dm(y) \right| &\leq c \cdot a^{|\alpha|} |\alpha|! \int_K |c_\alpha(y)| dm(y) \\ &\leq c \cdot a^{|\alpha|} |\alpha|! \|c_\alpha\|_{L^p(K, m)} m(K)^{\frac{1}{q}}, \end{aligned}$$

with $p^{-1} + q^{-1} = 1$. Therefore

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^n} \sup_{x \in \tilde{K}} \left| \int_K D_y^\alpha \Phi(x, y) c_\alpha(y) dm \right| &\leq c \cdot m(K)^{1/q} \sum_{\alpha \in \mathbb{N}_0^n} a^{|\alpha|} |\alpha|! \|c_\alpha\|_{L^p(K, m)} \\ &= c \cdot m(K)^{1/q} \sum_{j=0}^{\infty} a^j \sup_{|\alpha|=j} \|\alpha! c_\alpha\|_{L^p(K, m)} \left(\sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} \right) \\ &= c \cdot m(K)^{1/q} \sum_{j=0}^{\infty} (a \cdot n \sup_{|\alpha|=j} \|\alpha! c_\alpha\|_{L^p(K, m)}^{1/|\alpha|})^j, \end{aligned}$$

where we used that $\sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} = n^j$, $n = \dim \mathbb{R}^n$.

Now, since $\sup_{|\alpha|=j} \|\alpha! c_\alpha\|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0$ when $j \rightarrow \infty$, the last sum can be majorized by a geometric sum. Hence

$$\sum_{\alpha \in \mathbb{N}_0^n} \sup_{x \in \tilde{K}} \left| \int_K D_y^\alpha \Phi(x, y) c_\alpha(y) dm \right| \leq c(K, \tilde{K}) < \infty.$$

□

1.5. Let us distinguish the principal difficulty in the proof of Theorem 1.1.

Lemma 1.2. *Let K be a locally connected compact subset of X , m be a massive measure on K and $1 < p < \infty$. Then for every solution $f \in S_P(X \setminus K)$ there are a solution $f_e \in S_P(X)$ and a sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subset [L^p(K, m)]^k$ such that*

$$f(x) = f_e(x) + \sum_{\alpha \in \mathbb{N}_0^n} \int_K D_y^\alpha \Phi(x, y) c_\alpha(y) dm(y)$$

holds for all $x \in X \setminus K$. Furthermore, $\|\alpha! c_\alpha\|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0$ when $|\alpha| \rightarrow \infty$.

As Baernstein showed in [2], even for $P = \partial/\partial\bar{z}$ Lemma 1.2 is false for arbitrary compact K .

We now turn to the

Proof (of Theorem 1.1). Let $U \subset X$ be a relatively compact open set such that $U \cap \sigma = \overset{\circ}{K}$ and the set $K' = \partial U \cup \overset{\circ}{K}$ is locally connected. Fix some massive measure m' on K' whose restriction to K is m . The existence of such a measure follows from Example 1.1. Given a solution $f \in S_P(X \setminus \sigma)$, we consider the function f' which equals f in $U \setminus \sigma$ and is 0 in $X \setminus \bar{U}$. Then f' is a solution of the system $Pf' = 0$ with singularities on K' . Hence by Lemma 1.2 there exist a solution $f'_e \in S_P(X)$ and a sequence $\{c'_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subset [L^p(K', m')]^k$, satisfying $\|\alpha!c'_\alpha\|_{L^p(K', m')}^{1/|\alpha|} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, such that

$$f'(x) = f'_e(x) + \sum_{\alpha \in \mathbb{N}_0^n} \int_{K'} D_x^\alpha \Phi(x, y) c'_\alpha(y) dm'(y) \quad (x \in X \setminus K').$$

Set $c_\alpha := c'_\alpha|_K$, $\alpha \in \mathbb{N}_0^n$. Since $\|\alpha!c_\alpha\|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, the function f_e defined by

$$f_e(x) = f(x) - \sum_{\alpha \in \mathbb{N}_0^n} \int_K D_x^\alpha \Phi(x, y) c_\alpha(y) dm(y) \quad (x \in X \setminus \sigma)$$

belongs to $S_P(X \setminus \sigma)$ because of Lemma 1.1. Moreover, this function satisfies the equation $Pf_e = 0$ also in a neighborhood of each interior point of K , since we have

$$f_e(x) = f'_e(x) + \sum_{\alpha \in \mathbb{N}_0^n} \int_{K' \setminus K} D_x^\alpha \Phi(x, y) c'_\alpha(y) dm'(y) \quad \text{for } x \in U \setminus \sigma.$$

Thus $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$, as was to be proved. $\square$

The proof of Lemma 1.2 needs some preparation which we give in the following section by studying more thoroughly the topology on $S_{P'}(K)$. For the sake of simplicity, we restrict the following considerations to the case $k = 1$.

2. EQUIVALENT TOPOLOGIES IN $S_{P'}(K)$

2.1. Let K be any compact set in X . In this section, we study various topologies on $S_{P'}(K)$, where P' is the transposed operator to P . Define the space $S_{P'}(K)$ as follows. The function g belongs to $S_{P'}(K)$ if there exists an open set $U \supset K$ such that g is a solution of the equation $P'g = 0$ in U . If two such functions agree on some neighborhood of K , we identify them as elements in $S_{P'}(K)$.

For each U as above, let $S_{P'}(U)$ denote the space of solutions of the equation $P'g = 0$ in U with the topology of uniform convergence on compact subsets, i.e., the topology induced from $C(U)$. There is a natural map from $S_{P'}(U)$ into $S_{P'}(K)$, and we endow $S_{P'}(K)$ with the finest locally convex topology for which all these maps are continuous. We denote this topology by τ . Alternatively, the space $(S_{P'}(K), \tau)$ can be described as the inductive limit of the spaces $S_{P'}(U_\nu)$, where $\{U_\nu\}$ is any decreasing sequence of open sets containing K such that each neighborhood of K contains some U_ν , and such that each component of each U_ν meets $U_{\nu+1}$.

Remark 2.1. The space $(S_{P'}(K), \tau)$ is separated, a subset of this space is bounded iff it is contained and bounded in some $S_{P'}(U_\nu)$, and each closed bounded subset is compact. Proofs could be given by the same methods as in Koethe [9], p.379.

2.2. We will embed $S_{P'}(K)$ algebraically in a space $L^{(q)}$ whose topological dual consists of sequences of functions from $L^p(K, m)$. Lemma 1.2 follows from the Hahn-Banach Theorem once we show that the topology of $L^{(q)}$ restricted to $S_{P'}(K)$ is finer than the topology τ . To do this, we have first to study some Banach spaces.

Definition 2.1. Given positive numbers q and r , the space $l^q(r)$ is defined to consist of all sequences $\{\eta_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subseteq \mathbb{C}$ with $(\sum_{\alpha \in \mathbb{N}_0^n} |\eta_\alpha|^{q r^{|\alpha|}})^{1/q} < \infty$.

If K is an arbitrary compact subset of X and m is an arbitrary measure on K , then we denote by $l^q(r)^K$ the space of all functions $\eta(\cdot) = \{\eta_\alpha(\cdot)\}_{\alpha \in \mathbb{N}_0^n}$ on K with values in $l^q(r)$ such that $\eta_\alpha(\cdot) \in L^q(K, m)$ for every $\alpha \in \mathbb{N}_0^n$ and

$$(\sum_{\alpha \in \mathbb{N}_0^n} \|\eta_\alpha\|_{L^q(K, m)}^q r^{q|\alpha|})^{1/q} < \infty.$$

Lemma 2.1. For $q \in [1, \infty]$, the functional

$$(1) \quad \|\{\eta_\alpha\}\|_{l^q(r)^K} = (\sum_{\alpha \in \mathbb{N}_0^n} \|\eta_\alpha\|_{L^q(K, m)}^q r^{q|\alpha|})^{1/q}$$

defines a norm on $l^p(r)^K$.

Proof. The proof is an easy exercise from functional analysis. $\square$

Equipped with the norm (1), the space $l^q(r)^K$ is a Banach space, provided $q \in [1, \infty]$. Instead of proving this directly, we proceed by the following

Lemma 2.2. Let $r > 0, q \geq 1$ be arbitrary real numbers, and let $p \in [1, \infty]$ be the conjugate exponent to q . We have an isometrical isomorphism

$$(l^q(r)^K)' \cong l^p(\frac{1}{r})^K.$$

Proof. Assume that $q > 1$. Fix some $\theta = \{\theta_\alpha\}_{\alpha \in \mathbb{N}_0^n} \in l^p(\frac{1}{r})^K$. Then θ defines a linear functional on $l^q(r)^K$ via $\langle \theta, \eta \rangle = \sum_{\alpha \in \mathbb{N}_0^n} \int_K \langle \theta_\alpha(y), \eta_\alpha(y) \rangle dm(y)$, for $\eta = \{\eta_\alpha\} \in l^q(r)^K$. Since

$$(2) \quad \begin{aligned} |\langle \theta, \eta \rangle| &\leq \sum_{\alpha \in \mathbb{N}_0^n} (\|\theta_\alpha\|_{L^p(K, m)} r^{-|\alpha|}) (\|\eta_\alpha\|_{L^q(K, m)} r^{|\alpha|}) \\ &\leq \|\theta\|_{l^p(\frac{1}{r})^K} \cdot \|\eta\|_{l^q(r)^K}, \end{aligned}$$

this functional is continuous. Conversely, let $F \in (l^q(r)^K)'$. Given a multi-index $\alpha \in \mathbb{N}_0^n$, denote by e_α the element in $l^q(r)$ which is 1 in the α -th entry and 0 in all other entries. On $L^q(K, m)$, we may define a functional by juxtaposition $g \mapsto F(ge_\alpha)$ for $g \in L^q(K, m)$. Since F is continuous, this functional is continuous, too. By duality, there is a function $\theta_\alpha \in L^p(K, m)$ such that $F(ge_\alpha) = \int_K \langle \theta_\alpha(y), g(y) \rangle dm(y)$ for all $g \in L^q(K, m)$. Since for an element $\eta = \{\eta_\alpha\}$ in $l^q(r)^K$ we have $\eta = \sum_{\alpha \in \mathbb{N}_0^n} \eta_\alpha e_\alpha$ and the series converges in the norm of $l^q(r)^K$, it follows that

$$F(\eta) = \sum_{\alpha \in \mathbb{N}_0^n} F(\eta_\alpha e_\alpha) = \sum_{\alpha \in \mathbb{N}_0^n} \int_K \langle \theta_\alpha(y), \eta_\alpha(y) \rangle dm(y).$$

Put $\theta := \{\theta_\alpha\}_{\alpha \in \mathbb{N}_0^n}$. To complete the proof, it remains to show that θ is in $l^p(\frac{1}{r})^K$. To this end, we consider the sequence $\{\eta_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ of measurable functions on K given by

$$\eta_\alpha := \begin{cases} |\theta_\alpha|^{p-2} \bar{\theta}_\alpha r^{-p|\alpha|}, & \theta_\alpha \neq 0, \\ 0, & \theta_\alpha = 0. \end{cases}$$

Since $|\eta_\alpha|^q = |\theta_\alpha|^{p r^{-pq|\alpha|}}$ each function $\eta_\alpha(\cdot)$ is in $L^q(K, m)$. Hence it follows

$$\begin{aligned} \sum_{|\alpha| \leq N} \|\theta_\alpha\|_{L^p(K, m)}^p \left(\frac{1}{r}\right)^{p|\alpha|} &= |F(\sum_{|\alpha| \leq N} \eta_\alpha e_\alpha)| \leq \|F\|_{(l^q(r)^K)'} \sum_{|\alpha| \leq N} \eta_\alpha e_\alpha \|_{l^q(r)^K} \\ &= \|F\|_{(l^q(r)^K)'} \left(\sum_{|\alpha| \leq N} r^{-p|\alpha|} \|\theta_\alpha\|_{L^p(K, m)}^p\right)^{1/q}. \end{aligned}$$

Thus $(\sum_{|\alpha| \leq N} r^{-p|\alpha|} \|\theta_\alpha\|_{L^p(K, m)}^p)^{1/p} \leq \|F\|_{(l^q(r)^K)'}$ for every positive integer N . Together with (2) it follows that

$$\|\theta\|_{l^p(\frac{1}{r})^K} = \|F\|_{(l^q(r)^K)'},$$

as was to be proved.

For $q = 1$, the proof follows the same lines with the obvious modifications. $\square$

Since the dual space to a normed space is a Banach space, Lemma 2.2 implies the following

Corollary 2.1. *Let $r > 0$ and $q > 1$. Then $l^q(r)^K$ is a reflexive Banach space.*

2.3. Note that if $r' > r'' > 0$, we have a continuous embedding $l^q(r')^K \hookrightarrow l^q(r'')^K$. Now let $\{r_\nu\}_{\nu \in \mathbb{N}}$ be some decreasing sequence of positive numbers tending to zero. The space $L^{(q)}$ is defined to be the inductive limit of the spaces $l^q(r_\nu)^K$. The space $L^{(q)}$ is separated. Each bounded set is contained and bounded in one of the $l^q(r_\nu)^K$. Moreover, $L^{(q)}$ is a (DF)-space, because it is the separated inductive limit of a sequence of normed, hence (DF)-, spaces (see Théorème 9 of Grothendieck [6]).

Our aim is to show that $S_{P'}(K)$ is topologically isomorphic to a subspace of $L^{(q)}$. Thus we proceed by constructing an embedding $S_{P'}(K) \hookrightarrow L^{(q)}$. More precisely, for each solution $g \in S_{P'}(K)$ we define

$$(3) \quad j(g) := \left\{ \frac{D^\alpha g}{\alpha!} |_K \right\}_{\alpha \in \mathbb{N}_0^n}.$$

Lemma 2.3. *For every $g \in S_{P'}(K)$, the sequence $j(g)$ is in $L^{(q)}$, and the mapping $j : S_{P'}(K) \rightarrow L^{(q)}$ is continuous and injective.*

Proof. Let $g \in S_{P'}(K)$. Then there is a neighborhood U of K in X such that $g \in S_{P'}(U)$. Now choose a function $\varphi \in \mathcal{D}(X)$ which is equal to 1 in a neighborhood of K . Since Φ is a left fundamental solution of P , we get $g = \Phi' P'(\varphi g)$ in a neighborhood of K .

The function $P'(\varphi g)$ is supported by the closure of the set of those points $x \in U$ such that $\text{grad } \varphi(x) \neq 0$. Let us denote this closure by σ . Then σ is a compact subset of $U \setminus K$, so there is a function $\psi \in \mathcal{D}(U \setminus K)$ which equals 1 in a neighborhood of σ .

Since $P'(\varphi g) = \psi P'(\varphi g)$, we have $g = \Phi'(\psi P'(\varphi g))$ in a neighborhood of K . Hence it follows for each multi-index α that

$$D^\alpha g(y) = \int P(x, D)(\psi(x) D_y^\alpha \Phi(x, y)) \cdot (\varphi(x) g(x)) dx \quad (y \in K).$$

Using estimate (2) with $\tilde{K} = \text{supp } \psi$, we get

$$\begin{aligned} \sup_{y \in K} |D^\alpha g(y)| &\leq c' a^{|\alpha| + \text{order } P} (|\alpha| + \text{order } P)! \sup_{x \in \text{supp } \varphi} |g(x)| \\ &\leq c'' (a')^{|\alpha|} |\alpha|! \sup_{x \in \text{supp } \varphi} |g(x)|, \end{aligned}$$

where a' is any number larger than a , and the constant c'' does not depend on $g \in S_{P'}(U)$ and α . It now follows that

$$\begin{aligned} (4) \quad \sum_{\alpha \in \mathbb{N}_0^n} \left\| \frac{D^\alpha g}{\alpha!} \right\|_{L^q(K, m)}^q r_\nu^{q|\alpha|} &\leq (c'')^q m(K) \left(\sum_{\alpha \in \mathbb{N}_0^n} ((a')^{|\alpha|} r_\nu^{|\alpha|} \frac{|\alpha|!}{\alpha!})^q \right) \sup_{x \in \text{supp } \varphi} |g(x)| \\ &= (c'')^q m(K) \left(\sum_{j=0}^{\infty} (na' r_\nu)^{qj} \right) \sup_{x \in \text{supp } \varphi} |g(x)|. \end{aligned}$$

Choose ν_0 large enough, such that $nar_{\nu_0} < 1$. Then (4) shows that $j(g) \in l^q(r_{\nu_0})^K$ as well as the continuity of the mapping $j : S_{P'}(U) \rightarrow l^q(r_{\nu_0})^K$.

Since a linear operator from $S_{P'}(K)$ into a locally convex space is continuous if and only if its restriction to each $S_{P'}(U)$ is continuous (for a proof cf. Bourbaki [3]), it follows that the mapping $j : S_{P'}(K) \rightarrow L^{(q)}$ is continuous.

To show that j is injective let $g \in S_{P'}(K)$ be such that $j(g) = 0$. This means that $D^\alpha g|_K \equiv 0$ in K for all $\alpha \in \mathbb{N}_0^n$, and hence, since g is real analytic, it follows $g \equiv 0$ in a neighborhood of K . $\square$

2.4. Now put

$$S_{P'}^{(q)} := j(S_{P'}(K)) \subseteq L^{(q)}.$$

We endow this space with the topology induced by $L^{(q)}$. We want to show

Lemma 2.4. *Let K be a locally connected compact subset of X , and $q > 1$. Then $S_{P'}^{(q)}$ is a closed subspace of $L^{(q)}$.*

For the proof of Lemma 2.4 we shall use the following result:

Lemma 2.5. *Assume that $\{L_\nu\}$ is a sequence of reflexive Banach spaces, such that L_ν is continuously embedded in $L_{\nu+1}$ for all ν , and L is the inductive limit of the sequence. Then a vector subspace Σ of L is closed if and only if for all ν the intersection $\Sigma \cap L_\nu$ is closed in L_ν .*

Proof. See Makarov [11]. $\square$

Proof (of Lemma 2.4). Using Lemma 2.5 it is sufficient to show that for each ν the subspace $S_{P'}^{(q)} \cap (l^q(r_\nu)^K)$ is closed in $l^q(r_\nu)^K$.

Assume that for a solution $g \in S_{P'}(K)$ the image $j(g)$ is in $l^q(r_\nu)^K$. Then for all points $y \in K$, except perhaps for a set of zero measure m , we have

$$\left(\sum_{\alpha \in \mathbb{N}_0^n} \left| \frac{D^\alpha g(y)}{\alpha!} \right|^{q r_\nu^{q|\alpha|}} \right)^{1/q} < \infty.$$

Since the measure m is supposed to be massive, this inequality holds for a set σ_g of points $y \in K$ which is dense in K . So

$$\limsup_{|\alpha| \rightarrow \infty} \left| \frac{D^\alpha g(y)}{\alpha!} \right|^{1/|\alpha|} \leq \frac{1}{r_\nu} \quad \text{for all } y \in \sigma_g.$$

We shall construct a complex neighborhood U_ν of K into which all the elements of $j^{-1}(l^q(r_\nu)^K)$ have (single valued) holomorphic extensions. This is the only place where we use the local connectedness of K .

For each $y \in K$ choose a neighborhood O_y in $\mathbb{C}^n$ such that $O_y \subset \Delta(y, r_\nu)$ and such that $K \cap O_y$ is connected. This is possible, since K is assumed to be locally connected. Here $\Delta(y, r) = \{z \in \mathbb{C}^n : |z_i - y_i| < r \ (i = 1, \dots, n)\}$ is the polydisk in $\mathbb{C}^n$ with center y and radius r . Choose r_y such that $\Delta(y, 2r_y) \subset O_y$. Define $U_\nu = \bigcup_{y \in K} \Delta(y, r_y)$. Then U_ν is a neighborhood of K in $\mathbb{C}^n$.

Let $g \in j^{-1}(l^q(r_\nu)^K)$ and $z \in U_\nu$. Define $\tilde{g}(z) = \sum_\alpha \frac{D^\alpha g(y)}{\alpha!} (z - y)^\alpha$ where y is any point of σ_g such that $z \in \Delta(y, r_y)$. The series converges, since $|z_i - y_i| < \frac{1}{2r_\nu}$ for all $i = 1, \dots, n$. We have to show that $\tilde{g}(z)$ does not depend on y .

Suppose that $z \in \Delta(y', r_{y'}) \cap \Delta(y'', r_{y''})$, where $y', y'' \in \sigma_g$. Let $r_{y''} \leq r_{y'}$. Then $|y''_i - y'_i| < r_{y'} + r_{y''} \leq 2r_{y'}$ for all $i = 1, \dots, n$; hence $y'' \in \Delta(y', 2r_{y'}) \subset O_{y'}$. We conclude that both y' and y'' belong to the connected set $K \cap O_{y'}$. Let U be an open set in $\mathbb{C}^n$ containing K , into which g has a (single valued) holomorphic extension. Then $K \cap O_{y'} \subset U \cap \Delta(y', r_\nu)$, and we denote by O the component of the set on the right which contains y' . Obviously, y'' is in O , too. The equation $g(z) = \sum_\alpha \frac{D^\alpha g(y')}{\alpha!} (z - y')^\alpha$ is valid for all $z \in O$. Hence the series

$$\tilde{g}(z) = \sum_\alpha \frac{D^\alpha g(y'')}{\alpha!} (z - y'')^\alpha \quad \text{about } y''$$

is a rearrangement of the series

$$g(z) = \sum_\alpha \frac{D^\alpha g(y')}{\alpha!} (z - y')^\alpha \quad \text{about } y',$$

and uniqueness of $\tilde{g}(z)$ follows.

It is obvious that $\tilde{g}$ is holomorphic in U_ν . Moreover, it is easily verified that $\tilde{g}$ and g agree on $U_\nu \cap U$. We may assume that the coefficients of the differential operator P have holomorphic extensions to U_ν . Then $P'\tilde{g} \equiv 0$ in U_ν , since the function $P'\tilde{g}$ is holomorphic in U_ν and vanishes on an open subset of each component of U_ν .

Thus every solution $g \in j^{-1}(l^q(r_\nu)^K)$ has a (single valued) extension to the complex neighborhood U_ν of K . Now, let $\{\eta^{(j)}\}$ be a sequence in $S_{P'}^{(q)} \cap l^q(r_\nu)^K$ which converges to an element $\eta = \{\eta_\alpha\}$ in $l^q(r_\nu)^K$. We would like to prove that η is in $S_{P'}^{(q)} \cap l^q(r_\nu)^K$, too. By definition of $S_{P'}^{(q)}$, for every $j = 1, 2, \dots$ there is a $g_j \in S_{P'}(K)$ such that $\eta_\alpha^{(j)} = \frac{D^\alpha g_j}{\alpha!} |_K \ (\alpha \in \mathbb{N}_0^n)$. Moreover, as was already proved, each element g_j is represented by a holomorphic function $g_j(z)$ in the complex neighborhood U_ν of K satisfying $P'g_j = 0$ there.

The convergence $\eta^{(j)} \rightarrow \eta$ in $l^p(r_\nu)^K$ means that

$$\lim_{j \rightarrow \infty} \left(\int_K \sum_{\alpha \in \mathbb{N}_0^n} r_\nu^{q|\alpha|} \left| \frac{D^\alpha g_j(y)}{\alpha!} - \eta_\alpha(y) \right|^q dm(y) \right)^{1/q} = 0.$$

Hence it follows that there exists a subsequence $\{g_{j_s}\}$ such that for all points $y \in K$, except for a set of zero measure m , we have

$$(5) \quad \lim_{j_s \rightarrow \infty} \left(\sum_{\alpha \in \mathbb{N}_0^n} r_\nu^{q|\alpha|} \left| \frac{D^\alpha g_{j_s}(y)}{\alpha!} - \eta_\alpha(y) \right|^q \right)^{1/q} = 0.$$

Since the measure m is massive, equality holds for a set σ of points $y \in K$ which is dense in K . We now use compactness of K to conclude the following. There are a finite number of points $y^{(1)}, \dots, y^{(n)}$ in σ and a positive $r < r_\nu$ such that K is contained in the union $U = \Delta(y^{(1)}, r) \cup \dots \cup \Delta(y^{(n)}, r)$, and $\overline{U} \subset U_\nu$. Our purpose is to show that the sequence $\{g_{j_s}\}$ converges to some function g in $S_{P'}(U)$. Since the space $S_{P'}(U)$ is complete, it suffices to prove that this sequence is a Cauchy sequence in $S_{P'}(U)$, i.e., in each of the spaces $C(k)$, where k is a compact subset of U . Obviously, we may restrict ourselves to compact sets k lying in one of the polydisks $\Delta(y^{(1)}, r), \dots, \Delta(y^{(n)}, r)$.

Let k be a compact subset of $\Delta(y, r)$ where $\Delta(y, r)$ is one of the polydisks previously mentioned. Denote by d the distance from k to the n -skeleton of $\Delta(y, r)$, i.e., $\partial_n \Delta(y, r) = \{\zeta \in \mathbb{C}^n : |\zeta_i - y_i| = r \ (i = 1, \dots, n)\}$. The distance is taken in the polydisk-norm.

We may regard some branch of $(g_{j_s}(z) - g_{j_t}(z))^q$ in $\Delta(y, r)$ to yield a holomorphic function there. By Cauchy's Theorem we have for all $z \in \Delta(y, r)$:

(6)

$$(g_{j_s}(z) - g_{j_t}(z))^q = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\partial_n \Delta(y, r)} \frac{(g_{j_s}(\zeta) - g_{j_t}(\zeta))^q}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \wedge \dots \wedge d\zeta_n.$$

The Taylor-series expansion for $(g_{j_s}(\zeta) - g_{j_t}(\zeta))$, centered at y , converges uniformly in the closure of $\Delta(y, r)$. So (6) implies for $z \in k$:

$$\begin{aligned} |g_{j_s}(z) - g_{j_t}(z)| &\leq \left(\frac{1}{(2\pi d)^n \int_{\partial_n \Delta(y, r)} |g_{j_s}(\zeta) - g_{j_t}(\zeta)|^q |d\zeta_1| \wedge \dots \wedge |d\zeta_n|} \right)^{1/q} \\ &= \left(\frac{1}{(2\pi d)^n} \int_{\partial_n \Delta(y, r)} \left| \sum_{\alpha \in \mathbb{N}_0^n} \frac{D^\alpha (g_{j_s}(y) - g_{j_t}(y))}{\alpha!} (\zeta - y)^\alpha \right|^q |d\zeta_1| \wedge \dots \wedge |d\zeta_n| \right)^{1/q}. \end{aligned}$$

Using Hölder's inequality and taking into account that $r < r_\nu$, we get

$$\begin{aligned} \sup_{z \in k} |g_{j_s}(z) - g_{j_t}(z)| &\leq \left(\frac{1}{(2\pi d)^n} \int_{\partial_n \Delta(y, r)} \left(\sum_{\alpha \in \mathbb{N}_0^n} \frac{|\zeta - y|^\alpha}{|r_\nu|^\alpha} \right)^{(q/p)} |d\zeta_1| \wedge \dots \wedge |d\zeta_n| \right)^{1/q} \\ &\cdot \left(\sum_{\alpha \in \mathbb{N}_0^n} \left| \frac{D^\alpha (g_{j_s}(y) - g_{j_t}(y))}{\alpha!} r_\nu^{|\alpha|} \right|^q \right)^{1/q} \\ &= \left(\frac{r}{d} \right)^{n/q} \left(\sum_{\alpha} \left(\frac{r}{r_\nu} \right)^{p|\alpha|} \right)^{1/p} \left(\sum_{\alpha} \left| \frac{D^\alpha g_{j_s}(y) - D^\alpha g_{j_t}(y)}{\alpha!} \right|_{q, r_\nu^{q|\alpha|}} \right)^{1/q} \\ &\leq \left(\frac{r}{d} \right)^{n/q} \left(\frac{r_\nu^p}{r_\nu^p - r^p} \right)^{n/p} \left\{ \left(\sum_{\alpha} r_\nu^{|\alpha|q} \left| \frac{D^\alpha g_{j_s}(y)}{\alpha!} - \eta_\alpha(y) \right|^q \right)^{\frac{1}{q}} + \right. \\ &\quad \left. + \left(\sum_{\alpha} r_\nu^{|\alpha|q} \left| \frac{D^\alpha g_{j_t}(y)}{\alpha!} - \eta_\alpha(y) \right|^q \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

By (5) it follows that $\sup_{z \in k} |g_{j_s}(z) - g_{j_t}(z)| \rightarrow 0$ when both j_s and j_t tend to infinity. This is just what we wanted to prove. Thus, there is a solution $g \in S_{P'}(U)$ such that $g_{j_s} \rightarrow g$ in $S_{P'}(U)$. Because of Lemma 2.3, we obtain $\eta = j(g)$. Hence $\eta \in S_{P'}^{(q)}$, as was to be proved. $\square$

The main result of this section consists of the following.

Theorem 2.1. *Assume that K is a locally connected compact subset of X , and $q > 1$. Then the mapping $j^{-1} : S_{P'}^{(q)} \rightarrow S_{P'}(K)$ is continuous.*

Proof. The assertion follows from Lemma 2.4 and a version of the Open Mapping Theorem, but we prefer the direct proof. As was already mentioned, the mapping $j^{-1} : S_{P'}^{(q)} \rightarrow S_{P'}(K)$ is continuous, iff each restriction $j^{-1} : S_{P'}^{(q)} \cap l^q(r_\nu)^K$ is continuous (see Bourbaki [3]). Let $\{g_j\}$ be a sequence of $S_{P'}(K)$ such that the sequence $\{\frac{D^\alpha g_j}{\alpha!}\}_{\alpha \in \mathbb{N}_0^n}$ converges to zero in $l^q(r_\nu)^K$. By the same way as we proceeded in the proof of Lemma 2.4, we find a complex neighborhood U_ν of K such that every element g_j is represented by a holomorphic function $g_j(z)$ in U_ν satisfying $P'g_j = 0$ there.

Choose a positive $r < r_\nu$ such that the set $U = \bigcup_{y \in K} \Delta(y, r)$ is contained in U_ν together with its closure. Then we claim that $\{g_j\}$ tends to zero uniformly on compact subsets of U . In fact, otherwise there would exist a compact set $k \subset U$, an $\varepsilon > 0$ and a subsequence $\{g_{j_s}\}$ such that $\sup_{z \in k} |g_{j_s}(z)| \geq \varepsilon$ for all j_s . But then it follows just in the same way as in the proof of Lemma 2.4 that some subsequence of $\{g_{j_s}\}$ should tend to zero uniformly on compact subsets of U . This contradiction implies our statement. Hence $g_j \rightarrow 0$ in $S_{P'}(K)$, as was to be proved. $\square$

Combining Theorem 2.1 and Lemma 2.3, we obtain the

Corollary 2.2. *Under the conditions of Theorem 2.1, the mapping $j : S_{P'}(K) \rightarrow S_{P'}^{(q)}$ is a topological isomorphism of the space $(S_{P'}(K), \tau)$ onto the space $S_{P'}^{(q)}$ equipped with the topology induced by $L^{(q)}$.*

3. PROOF OF THE MAIN LEMMA AND REMARKS

3.1. In order to prove Lemma 1.2, we shall use the fact that each solution $f \in S_P(X \setminus K)$ may be written as the sum of a solution in $S_P(X)$ and a solution in $S_P(X \setminus K)$ which is *regular at infinity*. The latter notion can be introduced as follows:

Denote by $\hat{X}$ the one point compactification of X , i.e., the union of X and the symbolic point ∞ . The topology in $\hat{X}$ is defined by the following system of neighborhoods: If $x \in X$, then we take the usual neighborhood basis, and if $x = \infty$, then we take the family of complements of all compact subsets in X . Let U be a neighbourhood of ∞ . A function $f \in S_P(U)$ which has the representation (in a neighborhood of ∞ , possibly smaller than U) $f = \Phi(F)$, for some distribution F with compact support, in K , is called *regular at infinity*. Here $\Phi(F)$ is the value of the pseudo-differential operator Φ on F . For smooth functions F with compact support $\Phi(F)$ is defined by $\Phi(F) = \int_{\mathbb{R}^n} \Phi(\cdot, y) F(y) dy$. For distributions F with compact support, $\Phi(F)$ is defined by duality.

Of course, this notion depends on our particular choice of the fundamental solution Φ , while the space of solutions regular at infinity does not depend on Φ on the whole.

Let us denote by $S_P^{(r)}(X \setminus K)$ the subspace of $S_P(X \setminus K)$ consisting of the solutions regular at infinity.

Lemma 3.1. *For each compact set $K \subset X$, it follows that*

$$S_P(X \setminus K) = S_P(X) \oplus S_P^{(r)}(X \setminus K).$$

The sum on the right is topological.

Proof. Let G_P be a Green operator for P , i.e., a bidifferential operator of order $\text{ord}(P) - 1$ on X with the property that $dG_P(g, f) = (\langle g, Pf \rangle_x - \langle P'g, f \rangle_x) dx$ for all g and f , which are smooth enough in X . Here $dx = dx_1 \wedge \dots \wedge dx_n$. Given a solution $f \in S_P(X \setminus K)$, we define the functions f_e and f_r in the following way. Let $x \in X$. Choose an open set $U \subset\subset X$ with piecewise smooth boundary such that $K \subset U$ and $x \in U$. Set $f_e(x) = -\int_{\partial U} G_P(\Phi(x, \cdot), f)$. It follows from the Green formula that $f_e(x)$ does not depend on the particular choice of U . Obviously, $f_e \in S_P(X)$. Now let $x \in X \setminus K$. Let $U \subset\subset X$ be an open set with piecewise smooth boundary such that $K \subset U$ and $x \notin \bar{U}$. Set $f_r(x) = \int_{\partial U} G_P(\Phi(x, \cdot), f)$. Again, f_r does not depend on the choice of U . It is clear that $f_r \in S_P^{(r)}(X \setminus K)$. By the Green formula we get $f = f_e + f_r$. The rest of the proof is obvious. $\square$

Thus, every solution $f \in S_P(X \setminus K)$ may be written in the form $f = f_e + f_r$, with $f_e \in S_P(X)$ and $f_r \in S_P^{(r)}(X \setminus K)$, and this representation is unique.

3.2. Given a solution $f \in S_P(X \setminus K)$, we define a linear functional F_f on $S_{P'}(K)$ as follows. Let $g \in S_{P'}(K)$. This means that there is a neighborhood U of K such that $g \in S_{P'}(U)$. Choose a new neighborhood U_g of K such that $U_g \subset\subset U$ and the boundary of U_g is piecewise smooth. Put

$$(1) \quad \langle F_f, g \rangle = \int_{\partial U_g} G_P(g, f) \quad (g \in S_{P'}(K)).$$

It follows from the Green formula, that the value $\langle F_f, g \rangle$ does not depend on the particular choice of U_g . Moreover, F_f is a continuous linear functional on $S_{P'}(K)$.

Lemma 3.2. *If $f \in S_P(X \setminus K)$, then*

$$(2) \quad \langle F_f, \Phi(x, \cdot) \rangle = f_r(x) \quad \text{for } x \in X \setminus K.$$

Proof. In fact, if $x \in X \setminus K$, then $\Phi(x, \cdot)$ satisfies $P'\Phi(x, \cdot) = 0$ in the neighborhood $X \setminus \{x\}$ of K . So the left-hand side of (2) is well-defined. To finish the proof, it only remains to look at the proof of Lemma 3.1. $\square$

3.3. We proceed now by applying Theorem 2.1. Therefore, we are interested in a representation of functionals $F \in (L^{(q)})'$, where $1 < q < \infty$.

Lemma 3.3. *Let $1 < q < \infty$ and p be the conjugate exponent to q . To each continuous linear functional F on $(L^{(q)})$ there is a sequence $f = \{f_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ in $L^p(K, m)$ such that $\|f_\alpha\|_{L^p(K, m)}^{1/|\alpha|} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, such that*

$$\langle F, \eta \rangle = \sum_{\alpha \in \mathbb{N}_0^n} \int_K \langle f_\alpha(y), \eta_\alpha(y) \rangle dm(y) \quad \text{for all } \eta = \{\eta_\alpha\} \in L^{(q)}.$$

Proof. Let $\eta \in l^q(r_\nu)^K$. Then $\eta = \sum_{\alpha \in \mathbb{N}_0^n} \eta_\alpha e_\alpha$, and the series converges with respect to the norm of $l^q(r_\nu)^K$. Since F is a continuous functional on $L^{(q)}$, its restriction to each of the $l^q(r_\nu)^K$ is continuous, too. Therefore, we have $\langle F, \eta \rangle = \sum_\alpha \langle F, \eta_\alpha e_\alpha \rangle$ for all $\eta = \{\eta_\alpha\} \in L^{(q)}$. For a fixed multi-index α , we consider the linear functional on $L^q(K, m)$ defined by $g \mapsto \langle F, g e_\alpha \rangle$ ($g \in L^q(K, m)$). This functional is obviously continuous, so by duality there is a function $f_\alpha \in L^p(K, m)$ such that $\langle F, g e_\alpha \rangle = \int_K \langle f_\alpha(y), g(y) \rangle dm(y)$ for all $g \in L^q(K, m)$. Hence $\langle F, \eta \rangle = \sum_\alpha \int_K \langle f_\alpha(y), \eta_\alpha(y) \rangle dm(y)$ for all $\eta \in L^{(q)}$. The expression on the right hand side of this equality is a continuous linear functional on $L^{(q)}$, and thus on each of the

spaces $l^q(r_\nu)^K$. Hence it follows by Lemma 2.2 that $\{f_\alpha\} \in l^p(\frac{1}{r_\nu})^K$ for every ν . Then $\sum_\alpha (\|f_\alpha\|_{L^p(K,m)}^{\frac{1}{|\alpha|}} \frac{1}{r_\nu})^{p|\alpha|} < \infty$, showing that $\limsup_{|\alpha| \rightarrow \infty} \|f_\alpha\|_{L^p(K,m)}^{\frac{1}{|\alpha|}} \leq r_\nu$ for all ν . Since $r_\nu \rightarrow \infty$, the assertion follows. $\square$

3.4. We now turn to the

Proof (of Lemma 1.2). Assume that $f \in S_P(X \setminus K)$. We consider the continuous linear functional F_f on $S_{P'}(K)$ given by formula (1). The composition $F = F_f \circ j^{-1}$ defines a linear functional on the space $S_{P'}^{(q)}$, as follows from Lemma 2.3. Because of Theorem 2.1, the functional F is continuous. By the Hahn-Banach Theorem, F can be continuously extended to the whole space $L^{(q)}$. According to Lemma 3.3, there exists a sequence $\{f_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ in $L^p(K, m)$, satisfying $\|f_\alpha\|_{L^p(K,m)}^{\frac{1}{|\alpha|}} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, such that

$$\langle F, j(g) \rangle = \sum_\alpha \int_K \langle f_\alpha(y), \frac{D^\alpha g(y)}{\alpha!} \rangle dm(y) \text{ for all } g \in S_{P'}(K).$$

Now putting $g = \Phi(x, \cdot)$, where x is a fixed point of $X \setminus K$, and using Lemma 3.2 we derive the assertion of Lemma 1.2 with $c_\alpha = f_\alpha/\alpha!$ ($\alpha \in \mathbb{N}_0^n$), since

$$\langle F, j(\Phi(x, \cdot)) \rangle = \langle F_f, \Phi(x, \cdot) \rangle = f_r(x) = f(x) - f_e(x).$$

$\square$

3.5. When K is a single point, the representation asserted by Lemma 1.2 is just the Laurent expansion of f .

Corollary 3.1. *Let y_0 be a fixed point of X . Then for every solution $f \in S_P(X \setminus \{y_0\})$ there exist a solution $f_e \in S_P(X)$ and a sequence $\{c_\alpha\}_{\alpha \in \mathbb{N}_0^n} \subset \mathbb{C}^k$, satisfying $|\alpha! c_\alpha|^{1/|\alpha|} \rightarrow 0$ when $|\alpha| \rightarrow \infty$, such that*

$$(3) \quad f(x) = f_e(x) + \sum_\alpha D_y^\alpha \Phi(x, y_0) c_\alpha \quad (x \in X \setminus \{y_0\}).$$

Proof. The assertion follows by using $m(y_0) = 1$ as a massive measure on $K = \{y_0\}$. $\square$

The coefficients $\{c_\alpha\}$ will not be uniquely determined by f , since

$$P'(y_0, D_y) \Phi(x, y_0) = \delta(x - y_0) I_k$$

becomes zero off y_0 .

The Laurent-series expansions for solutions of general elliptic equations were first studied by Lopatinskii [10].

3.6. If $O \subset\subset X$ is an open set whose boundary is locally connected, then each solution f of $Pf = 0$ in O has a representation (1) for $x \in O$ with $K = \partial O$. The only thing we have to do is to construct a massive measure m on ∂O , and to extend f to a function satisfying the equation in the complement of ∂O . The assertion follows by Lemma 1.2.

3.7. Theorem 1.1 implies that arbitrary singularities of solutions of elliptic equations may be locally separated into atomic (i.e. one-point) singularities.

Corollary 3.2. *Assume that K is a locally connected compact subset of σ , and $\{y_\nu\}$ is a dense sequence of points of K . Then every solution $f \in S_P(X \setminus \sigma)$ can be written in the form $f = f_e + \sum_\nu f_\nu$, where $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$ and $f_\nu \in S_P(X \setminus \{y_\nu\})$, and the series converges in the topology of $S_P(X \setminus K)$.*

Proof. We use the massive measure m on K constructed in Example 1.1. By Theorem 1.1

$$f(x) = f_e(x) + \sum_\alpha \left(\sum_\nu D_y^\alpha \Phi(x, y_\nu) c_\alpha(y_\nu) \mu_\nu \right) \text{ for } x \in X \setminus \sigma,$$

where $f_e \in S_P((X \setminus \sigma) \cup \overset{\circ}{K})$ and $\lim_{|\alpha| \rightarrow \infty} (\sum_\nu |\alpha! c_\alpha(y_\nu)|^p \mu_\nu)^{1/(p|\alpha|)} = 0$. The last condition allows to rearrange the summations and to derive $f = f_e + \sum_\nu f_\nu$ with

$$f_\nu = \sum_\alpha D_y^\alpha \Phi(x, y_\nu) c_\alpha(y_\nu) \mu_\nu,$$

as was to be proved. $\square$

3.8. For the Laplace operator we obtain the following result (which seems to be new).

Corollary 3.3. *Let $K \subset \sigma$ be a locally connected compact set, and $1 < p < \infty$. Then every harmonic function f in $X \setminus \sigma$ has the form*

$$f(x) = f_e(x) + \sum_{j=0}^{\infty} \int_K \frac{h_j(y, x-y)}{|x-y|^{n+2(j-1)}} dm(y) \quad (x \in X \setminus \sigma)$$

where f_e is a harmonic function in $(X \setminus \sigma) \cup \overset{\circ}{K}$, and $h_j(y, z)$ are homogeneous harmonic polynomials of degree j in z with coefficients in $L^p(K, m)$ with respect to y , such that $\lim_{j \rightarrow \infty} (\frac{1}{j!} \int_K |h_j(y, Dz) h_j(y, z)|^{p/2} dm(y))^{1/p_j} = 0$.

Proof. It suffices to transform formula (1) by means of the Hecke identity (cf. Stein [14]). $\square$

3.9. We finish this section by mentioning one more aspect of Theorem 1.1. It is a natural question to ask whether a given solution $f \in S_P(X \setminus \{y_0\})$ admits a representation (3) with a finite number of summands. This is obviously the case iff f has a finite order of growth near y_0 , i.e., $|f(x)| \leq c|x - y_0|^{-\gamma}$ in some deleted neighborhood of y_0 . In other words, y_0 has to be a pole of f . Therefore, the solutions $f \in S_P(X \setminus K)$ for which the expansions (1) have only a finite number of terms are analogues of solutions with poles in general. Such solutions can be characterized as follows.

Theorem 3.1. *Let K be an arbitrary compact set in X , m be a massive measure on K , and $1 < p < \infty$. A solution $f \in S_P(X \setminus K)$ has a representation (1) with a finite number of terms iff the functional F_f given by (1) is continuous on $S_{P'}(K)$ with respect to the topology defined by the family of seminorms $\|D^\alpha g\|_{L^q(K, m)}$ ($\alpha \in \mathbb{N}_0^n$).*

Proof. See Tarkhanov [15]. $\square$

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