GAP ESTIMATES OF THE SPECTRUM OF HILL'S EQUATION AND ACTION VARIABLES FOR KdV

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ABSTRACT. Consider the Schrödinger equation $-y'' + Vy = \lambda y$ for a potential V of period 1 in the weighted Sobolev space ($N \in \mathbb{Z}_{\geq 0}$, $\omega \in \mathbb{R}_{\geq 0}$)

$$H^{N,\omega}(S^1; \mathbb{C}) := \left\{ f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{i2\pi kx} \mid \|f\|_{N,\omega} < \infty \right\}$$

where $\hat{f}(k)$ ($k \in \mathbb{Z}$) denote the Fourier coefficients of f when considered as a function of period 1,

$$\|f\|_{N,\omega} := \left(\sum_k (1 + |k|)^{2N} e^{2\omega|k|} |\hat{f}(k)|^2 \right)^{1/2} < \infty,$$

and where S^1 is the circle of length 1. Denote by $\lambda_k \equiv \lambda_k(V)$ ($k \geq 0$) the periodic eigenvalues of $-\frac{d^2}{dx^2} + V$ when considered on the interval $[0, 2]$, with multiplicities and ordered so that $\operatorname{Re} \lambda_j \leq \operatorname{Re} \lambda_{j+1}$ ($j \geq 0$). We prove the following result.

Theorem. *For any bounded set $\mathcal{B} \subseteq H^{N,\omega}(S^1; \mathbb{C})$, there exist $n_0 \geq 1$ and $M \geq 1$ so that for $k \geq n_0$ and $V \in \mathcal{B}$, the eigenvalues $\lambda_{2k}, \lambda_{2k-1}$ are isolated pairs, satisfying (with $\{\lambda_{2k}, \lambda_{2k-1}\} = \{\lambda_k^+, \lambda_k^-\}$)*

- (i) $\sum_{k \geq n_0} (1 + k)^{2N} e^{2\omega k} |\lambda_k^+ - \lambda_k^-|^2 \leq M$,
- (ii) $\sum_{k \geq n_0} (1 + k)^{2N+1} e^{2\omega k} \left| (\lambda_k^+ - \lambda_k^-) - 2\sqrt{\hat{V}(k)\hat{V}(-k)} \right|^2 \leq M$.

1. INTRODUCTION AND SUMMARY OF THE RESULTS

The Korteweg-deVries equation (KdV) on the circle

$$(1.1) \quad \partial_t U(x, t) = -\partial_x^3 U(x, t) + 6U(x, t)\partial_x U(x, t)$$

is a completely integrable Hamiltonian system of infinite dimension. We choose as its phase space the Sobolev space $H^{N,\omega}(S^1)$, where S^1 is the circle of length 1, $\omega \in \mathbb{R}_{\geq 0}$ and $N \in \mathbb{Z}_{\geq 0}$. The Poisson structure is the one proposed by Gardner,

$$\{F_1, F_2\}_G := \int_{S^1} \frac{\partial F_1}{\partial V(x)} \frac{d}{dx} \frac{\partial F_2}{\partial V(x)} dx,$$

where F_1 and F_2 are C^1 functionals on $H^{N,\omega}(S^1)$ and $\frac{\partial F}{\partial V(x)}$ denotes the L^2 -gradient of F . The Gardner bracket is degenerate. Its symplectic leaves are given by

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$H_c^{N,\omega}(S^1) := c + H_0^{N,\omega}(S^1)$ with $c \in \mathbb{R}$, where

$$H_0^{N,\omega}(S^1) := \{f \in H^{N,\omega}(S^1) \mid \int_{S^1} f dx = 0\}.$$

Also we introduce the following weighted l^2 -spaces:

$$l_{N+\frac{1}{2},\omega}^2(\mathbb{N}; \mathbb{R}^2) := \{(x, y) = (x_j, y_j)_{j \geq 1} \mid \sum_{j \geq 1} j^{2N+1} e^{2\omega j} (x_j^2 + y_j^2) < \infty\}.$$

In section 3 we prove the following result.

Theorem 1. *Let $N \in \mathbb{Z}_{\geq 0}$ and $\omega \in \mathbb{R}_{\geq 0}$. Then there exists a map*

$$\Lambda^{(N,\omega)} : H_0^{N,\omega}(S^1) \rightarrow l_{N+\frac{1}{2},\omega}^2(N; \mathbb{R}^2)$$

with the following properties:

- (1) $\Lambda^{(N,\omega)}$ is a diffeomorphism;
- (2) $\Lambda^{(N,\omega)}$ and $(\Lambda^{(N,\omega)})^{-1}$ are real analytic;
- (3) the variables $(x_j^2 + y_j^2)/2$ associated to $\Lambda^{(N,\omega)}(V) = (x_j(V), y_j(V))_{j \geq 1}$ are action coordinates of KdV and its entire hierarchy.

We refer to $(x_j(V), y_j(V))_{j \geq 1}$ as Birkhoff coordinates of KdV (and its hierarchy). The map, associating to V Birkhoff coordinates, is referred to as Birkhoff map, and can be thought of as a nonlinear Fourier transform. Clearly, the Fourier transform $\mathcal{F}$ establishes a linear isomorphism between $H_0^{N,\omega}(S^1)$ and $l_{N,\omega}^2(\mathbb{N}; \mathbb{R}^2)$, $\mathcal{F}(V) = (Re \hat{V}(k), Im \hat{V}(k))_{k \geq 1}$, and Theorem 1 is an instance of (probably) many properties Fourier transform and Birkhoff map have in common. Theorem 1 has already been established in the case $\omega = 0$ [BKM1] (cf. also [Ka], [BBGK]). In order to prove that $\Lambda^{(N,\omega)}$ can be chosen as the restriction of Λ to $H_0^{N,\omega}$, one has to derive asymptotic estimates for the periodic eigenvalues of the Schrödinger operator $L := -\frac{d^2}{dx^2} + V$ for V in $H_0^{N,\omega}(S^1; \mathbb{C})$ considered on the interval $[0, 2]$. The periodic spectrum of L is discrete. Denote it by $(\lambda_k = \lambda_k(V))_{k \geq 0}$ (with multiplicities), where the λ_k 's are ordered in such a way that $Re \lambda_0 \leq Re \lambda_1 \leq \dots$ and, in case $Re(\lambda_k) = Re(\lambda_{k+1})$, $Im \lambda_k \leq Im \lambda_{k+1}$. For k sufficiently large, the eigenvalues come in isolated pairs $\{\lambda_{2k}, \lambda_{2k-1}\}$. The main result of this paper is the following one, proved in section 2:

Theorem 2. *Let $\mathcal{B} \subseteq H^{N,\omega}(S^1; \mathbb{C})$ be a bounded set of potentials ($N \in \mathbb{Z}_{\geq 0}$, $\omega \in \mathbb{R}_{\geq 0}$). Then there exists $n_0 \geq 1$ such that*

$$\sup_{V \in \mathcal{B}} \sum_{k \geq n_0} (1+k)^{2N+1} e^{2\omega k} \left| \lambda_k^+(V) - \lambda_k^-(V) - 2\sqrt{\hat{V}(k)\hat{V}(-k)} \right|^2 < \infty,$$

where $\{\lambda_k^+, \lambda_k^-\} = \{\lambda_{2k}, \lambda_{2k-1}\}$ (cf. Theorem 2.10 for the indexing λ_k^+, λ_k^- of the numbers $\lambda_{2k}, \lambda_{2k-1}$).

As a consequence of Theorem 2 one obtains

Corollary 3. *For a real valued potential $V \in L^2(S^1)$ to be an element of $H^{N,\omega}(S^1)$ it is necessary and sufficient that*

$$\sum_{k=1}^{\infty} (1+k)^{2N} e^{2\omega k} (\lambda_{2k}(V) - \lambda_{2k-1}(V))^2 < \infty.$$

In the case $\omega = 0$, Theorem 2 and Corollary 3 have been established by Marčenko [Ma] by different methods which might, however, not be adaptable to the case $\omega > 0$.

Theorem 1 can be used to prove by the 'inverse scattering method' that the Korteweg-deVries equation (1.1) is well-posed on the circle. To simplify the wording of the statement we restrict ourselves to the case where the initial data V is in $H_0^{N,\omega}(S^1)$ (cf. [BKM2] in case V has nonzero average).

Corollary 4. *Let $N \in \mathbb{Z}_{\geq 0}$ and $\omega \in \mathbb{R}_{\geq 0}$. There exists a solution operator $\mathcal{S} : H_0^{N,\omega}(S^1) \rightarrow C(\mathbb{R}; H_0^{N,\omega}(S^1))$ of (1.1) with the following properties:*

(i) *Given V_1, V_2 in $H_0^{N,\omega}(S^1)$, there exists $M > 0$ so that for any $t \in \mathbb{R}$*

$$\| \mathcal{S}(V_1)(t) - \mathcal{S}(V_2)(t) \|_{H_0^{N,\omega}(S^1)} \leq M(1 + |t|) \| V_1 - V_2 \|_{H_0^{N,\omega}(S^1)} .$$

(ii) *For any $0 < T < \infty$, $\mathcal{S} : H_0^{N,\omega}(S^1) \rightarrow C([-T, T]; H_0^{N,\omega}(S^1))$ is real analytic.*

Proof. The case $\omega = 0$, $N = 0$ can be treated as in [BKM2] (cf. also [Bo]). The same proof works for this more general situation. In fact, the case $\omega = 0$, $N \geq 1$ or $\omega > 0$, $N \geq 0$ is somewhat easier, as the frequencies of the KdV Hamiltonian are easily seen to be real analytic in these cases. $\square$

Remark. Results similar to the one presented for KdV hold for any of the equations in the KdV hierarchy.

2. PROOF OF THEOREM 2

In this section, we prove Theorem 2, stated in the introduction. First let us introduce some more notation.

Definition. $w := (w(k))_{k \in \mathbb{Z}}$ is said to be a weight if

- (i) $w(k) \geq 1$ ($k \in \mathbb{Z}$);
- (ii) there exists $M_w \geq 1$ such that $w(k) \leq M_w w(k-j)w(j)$ ($k, j \in \mathbb{Z}$).

Condition (ii) is referred to as the submultiplicative property of a weight.

Most frequently we will use the weight

$$(2.1) \quad w(k) := \left(1 + \left|\frac{k}{2}\right|\right)^N e^{\frac{\omega}{2}|k|},$$

where $N \in \mathbb{Z}_{\geq 0}$ and $\omega \in \mathbb{R}_{\geq 0}$. In that case, one can choose $M_w = 1$ in condition (ii) of the above definition. The reason for choosing $\frac{\omega}{2}$ rather than ω in (2.1) follows from the observation that

$$V = \sum_k \hat{V}(k) e^{i2\pi kx} = \sum_k \hat{V}(2k) e^{i\pi(2k)x}$$

for $V \in H^{N,\omega}(S^1; \mathbb{C})$, with $(\hat{V}(k))_{k \in \mathbb{Z}}$ denoting the Fourier coefficients of V considered as a function of period 2 and thus

$$\| V \|_{N,\omega}^2 = \sum_k \left(1 + \left|\frac{k}{2}\right|\right)^{2N} e^{2\omega|k|} |\hat{V}(k)|^2 = \sum_k \left(1 + \left|\frac{k}{2}\right|\right)^{2N} e^{\frac{\omega}{2}|k|} |\hat{V}(k)|^2 .$$

For $K \subseteq \mathbb{Z}$ and a weight w denote by $l_w(K)$ the complex Hilbert space $l_w^2(K) \equiv l_w^2(K; \mathbb{C})$,

$$l_w^2(K) := \{(a(k))_{k \in K} \mid \| a \|_w < \infty\}$$

where

$$\|a\|_w \equiv \|a\|_{l_w^2(K)} := \left(\sum_{k \in K} w(k)^2 |a(k)|^2 \right)^{1/2}.$$

Most frequently, we will use for K the set $\mathbb{Z}$ or $\mathbb{Z}(n) := \mathbb{Z} \setminus \{\pm n\}$. If necessary for clarity, we will sometimes write a_K for a sequence $(a(k))_{k \in K} \in l_w^2(K)$.

For a linear operator $A : l_{w_1}^2(K_1) \rightarrow l_{w_2}^2(K_2)$ we denote by $A(k, j)$ its matrix elements

$$(Aa)(k) := \sum_{j \in K_1} A(k, j)a(j) \quad (k \in K_2).$$

Definition. $\mathcal{S} : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$ is defined by $(\mathcal{S}a)(k) := a(k+1)$ ($k \in \mathbb{Z}$). $\mathcal{S}$ is called the shift operator. The restriction of $\mathcal{S}$ to $l_w^2(K)$ with values in $l_{\mathcal{S}w}^2(K)$ is denoted by $\mathcal{S}$ as well, and $\mathcal{S}^n = \mathcal{S} \circ \dots \circ \mathcal{S}$ denotes the n th iterate of $\mathcal{S}$. Notice that

$$\begin{aligned} \|\mathcal{S}^n a\|_{l_{\mathcal{S}^n w}^2(K)}^2 &= \sum_{k \in K} (\mathcal{S}^n w)(k)^2 \left| (\mathcal{S}^n a)(k) \right|^2 \\ (2.2) \quad &= \sum_{k \in K} w(k+n)^2 \left| a(k+n) \right|^2 \leq \|a\|_{l_w^2(\mathbb{Z})}^2. \end{aligned}$$

Definition. $\mathcal{J} : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$ is the involution given by

$$(\mathcal{J}a)(k) := a(-k).$$

To prove Theorem 2, it suffices to consider potentials $V \in H_0^{N, \omega}(S^1)$ (where S^1 is the circle of unit length), as adding a constant c to V simply shifts the periodic spectrum of $-\frac{d^2}{dx^2} + V$ by c .

Express $-\frac{d^2}{dx^2} + V - \lambda$, acting on functions periodic of period 2, in Fourier space, as $A : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$, with

$$A(k, j) = \pi^2 k^2 \delta_{kj} + \hat{V}(k - j).$$

Recall that $\hat{V}(0) = \frac{1}{2} \int_0^2 V(x) dx = 0$, as $V \in H_0^{N, \omega}(S^1; \mathbb{C})$, and that $(\hat{V}(k))_{k \in \mathbb{Z}}$ denote the Fourier coefficients of V when considered as functions of period 2.

To analyze the eigenvalues $\lambda_{2n}, \lambda_{2n-1}$ near $n^2 \pi^2$ ($n \geq 1$), write $\lambda = n^2 \pi^2 + z$. Writing $l^2(\mathbb{Z})$ as a direct sum $l^2(\mathbb{Z}) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{Z}(n)$, $a = (a(-n), a(n), a_{\mathbb{Z}(n)})$ ($\mathbb{Z}(n) := \mathbb{Z} \setminus \{\pm n\}$), we see that $A - \lambda$ is of the form

$$\begin{aligned} (2.3) \quad & \begin{pmatrix} ((A - \lambda)(k, -n))_{k \in \mathbb{Z}} & ((A - \lambda)(k, n))_{k \in \mathbb{Z}} & ((A - \lambda)(k, j))_{\substack{k \in \mathbb{Z} \\ j \in \mathbb{Z}(n)}} \end{pmatrix} \\ &= \begin{pmatrix} -z & \hat{V}(-2n) & (\mathcal{S}^n \mathcal{J} \hat{V})_{\mathbb{Z}(n)}^T \\ \hat{V}(2n) & -z & (\mathcal{S}^{-n} \mathcal{J} \hat{V})_{\mathbb{Z}(n)}^T \\ (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} & (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} & B_n - z \end{pmatrix}, \end{aligned}$$

where the superscript T denotes the transpose and where $B_n : l^2(\mathbb{Z}(n)) \rightarrow l^2(\mathbb{Z}(n))$ is given by

$$\begin{aligned} (2.4) \quad & B_n := A_n - \pi^2 n^2 Id_n, \\ & A_n := (A_n(j, k))_{j, k \in \mathbb{Z}(n)}. \end{aligned}$$

The (possibly) complex number $\lambda = n^2\pi^2 + z$ is a periodic eigenvalue for $-\frac{d^2}{dx^2} + V$ if there exists $a = (a(k))_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$(A - \lambda)a = 0.$$

With $x := a(-n), y := a(n)$, this equation can be written as a system of three equations:

$$(2.5) \quad -zx + \hat{V}(-2n)y + \langle \mathcal{S}^n \mathcal{J} \hat{V}, a_{\mathbb{Z}(n)} \rangle = 0,$$

$$(2.6) \quad \hat{V}(2n)x - zy + \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, a_{\mathbb{Z}(n)} \rangle = 0,$$

$$(2.7) \quad (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)}x + (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)}y + (B_n - z)a_{\mathbb{Z}(n)} = 0.$$

Here

$$(2.8) \quad \langle a, b \rangle \equiv \langle a, b \rangle_{\mathbb{Z}(n)} = \sum_{k \in \mathbb{Z}(n)} a(k)b(k)$$

(no complex conjugation). To solve (2.7) for $a_{\mathbb{Z}(n)}$ we need to analyze the operator

$$z - B_n : l^2(\mathbb{Z}(n)) \rightarrow l^2(\mathbb{Z}(n)).$$

Denote by B_n^I the diagonal part of B_n ,

$$B_n^I(k, j) = \pi^2(k^2 - n^2)\delta_{kj} \quad (k, j \in \mathbb{Z}(n))$$

and define $B_n^{II} := B_n - B_n^I$. Notice that, with $M \geq 10$, for any $n \geq \frac{M}{2}$ and $|z| \leq M$, $z - B_n^I$ is invertible. Denote by $\|V\|$ the norm of V in $L^2(S^1)$, and introduce

$$(2.9) \quad T_n := B_n^{II}(z - B_n^I)^{-1} : l^2(\mathbb{Z}(n)) \rightarrow l^2(\mathbb{Z}(n)),$$

$$(2.10) \quad n_0 := \max\left(\frac{M+1}{2}, \|V\|\right), \quad M \geq 10.$$

Lemma 2.1. *For $n \geq n_0$ and $|z| \leq M$,*

$$(2.11) \quad \|T_n\|_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \frac{\|V\|}{5n} \leq \frac{1}{5}; \quad \|(z - B_n^I)^{-1}\|_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \frac{1}{5n};$$

$$(2.12) \quad z - B_n \text{ is invertible and } \|(z - B_n)^{-1}\|_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \frac{1}{4n}.$$

Remark. The conditions in Lemma 2.1 (and subsequent lemmas) are only assumed to insure that the quantities involved are well defined.

Proof. To obtain estimates (2.11) notice that $\|T_n\|_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \|T_n\|_{HS}$, where $\|T_n\|_{HS}$ denotes the Hilbert-Schmidt norm of T_n :

$$(2.13) \quad \begin{aligned} \|T_n\|_{HS}^2 &= \sum_{j, k \in \mathbb{Z}(n)} \frac{|\hat{V}(k-j)|^2}{|z - \pi^2(k^2 - n^2)|^2} \leq \|V\|^2 \sum_{k \neq \pm n} \frac{1}{|\pi^2|k^2 - n^2| - M|^2} \\ &\leq \frac{\|V\|^2}{75} \sum_{k \neq \pm n} \frac{1}{|k^2 - n^2|^2} \leq \frac{\pi^2}{300} \frac{\|V\|^2}{n^2} < \frac{\|V\|^2}{30n^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|(z - B_n^I)^{-1}\|_{\mathcal{L}(l^2(\mathbb{Z}(n)))}^2 &\leq \|(z - B_n^I)^{-1}\|_{HS}^2 \\ &\leq \sum_{k \neq \pm n} \frac{1}{|\pi^2|k^2 - n^2| - M|^2} \leq \frac{1}{30n^2}. \end{aligned}$$

To prove (2.12), write

$$z - B_n = z - B_n^I - B_n^{II} = (Id_n - B_n^{II}(z - B_n^I)^{-1})(z - B_n^I) = (Id_n - T_n)(z - B_n^I).$$

Then $(z - B_n)^{-1} = (z - B_n^I)^{-1}(Id_n - T_n)^{-1}$ and

$$\| (z - B_n)^{-1} \|_{\mathcal{L}(l^2(\mathbb{Z}(n)))} \leq \frac{1}{5n} \sum_{k \geq 0} \| T_n \|^k \leq \frac{1}{5n} \frac{1}{1 - \frac{1}{5}} = \frac{1}{4n}. \quad \square$$

In view of Lemma 2.1, (2.7) can be solved for $a_{\mathbb{Z}(n)}$, if $n \geq n_0$ and $|z| \leq M$:

$$a_{\mathbb{Z}(n)} = (z - B_n)^{-1}(\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} x + (z - B_n)^{-1}(\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} y.$$

If this is substituted into (2.5) and (2.6), we obtain (with $B_{-n} := B_n$)

$$(2.14) \quad \begin{pmatrix} -z + \alpha(-n, z) & \hat{V}(-2n) + \beta(-n, z) \\ \hat{V}(2n) + \beta(n, z) & -z + \alpha(n, z) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$(2.15) \quad \alpha(n, z) := \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n)^{-1}(\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} \rangle$$

and

$$(2.16) \quad \beta(n, z) := \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n)^{-1}(\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} \rangle.$$

To analyze (2.14) we begin by investigating $\alpha(n, z)$.

Lemma 2.2. *For $n \geq n_0$ and $|z| \leq M$,*

$$\alpha(n, z) = \alpha(-n, z).$$

Proof. By Lemma 2.3 (iii) below, $(z - B_n)^{-1}(k, j) = (z - B_n)^{-1}(-j, -k)$ and thus

$$\begin{aligned} \alpha(n, z) &= \sum_{k, j \neq \pm n} \hat{V}(n - k)(z - B_n)^{-1}(k, j) \hat{V}(j - n) \\ &= \sum_{k, j \neq \pm n} \hat{V}(-k - (-n))(z - B_n)^{-1}(-j, -k) \hat{V}((-n) - (-j)) \\ &= \sum_{k', j' \neq \pm n} \hat{V}((-n) - k')(z - B_n)^{-1}(k', j') \hat{V}(j' - (-n)) \\ &= \alpha(-n, z). \quad \square \end{aligned}$$

Lemma 2.3. *For $n \geq n_0$ and $j, k \in \mathbb{Z}(n)$,*

- (i) $A_n(k, j) = A_n(-j, -k)$,
- (ii) $(A_n - \lambda)^{-1}(k, j) = (A_n - \lambda)^{-1}(-j, -k)$,
- (iii) $(z - B_n)^{-1}(k, j) = (z - B_n)^{-1}(-j, -k)$.

Proof. (i) Recall that, for $k, j \in \mathbb{Z}(n)$, $A_n(k, j) = k^2 \pi^2 \delta_{kj} + \hat{V}(k - j)$. Therefore $A_n(k, j) = (-k)^2 \pi^2 \delta_{(-k)(-j)} + \hat{V}((-j) - (-k)) = A_n(-j, -k)$.

(ii) is a straightforward verification, and (iii) follows from (ii). $\square$

By Lemma 2.2, the system of equations (2.14) with $n \geq n_0$ has a nontrivial solution $\begin{pmatrix} x \\ y \end{pmatrix}$ for some $|z| \leq M$, if there exists $z \in \mathbb{C}$, $|z| \leq M$, such that

$$(2.17) \quad (z - \alpha(n, z))^2 - (\hat{V}(-2n) + \beta(-n, z))(\hat{V}(2n) + \beta(n, z)) = 0.$$

Equation (2.17) is solved in two steps:

$$(2.18) \quad z_n = \alpha(n, z_n) + \zeta \quad (\zeta \text{ in } \mathcal{D}_{\frac{M}{2}} := \{\zeta \in \mathbb{C} \mid |\zeta| < \frac{M}{2}\})$$

and, with $z(\zeta_n) := z_n(\zeta_n)$ given by (2.18),

$$(2.19) \quad \zeta_n^2 - \left(\hat{V}(-2n) + \beta(-n, z(\zeta_n)) \right) \left(\hat{V}(2n) + \beta(n, z(\zeta_n)) \right) = 0.$$

Let us first discuss equation (2.18). To solve it, we use the contractive mapping principle. For that purpose, we need

Lemma 2.4. *For $|z| \leq M$ and $n \geq n_0$,*

- (i) $|\alpha(n, z)| \leq \frac{1}{4n} \|V\|^2$,
- (ii) $|\frac{d}{dz}\alpha(n, z)| \leq \frac{1}{16n^2} \|V\|^2$.

Proof. (i) By (2.12) of Lemma 2.1, for $n \geq n_0$ and $|z| \leq M$,

$$|\alpha(n, z)| \leq \|V\| \|(z - B_n)^{-1}\| \|V\| \leq \frac{\|V\|^2}{4n}.$$

(ii) Notice that

$$\frac{d}{dz}\alpha(n, z) = \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, -(z - B_n)^{-2} \mathcal{S}^{-n} \hat{V} \rangle$$

and therefore

$$|\frac{d}{dz}\alpha(n, z)| \leq \|V\| \|(z - B_n)^{-1}\|^2 \|V\| \leq \frac{\|V\|^2}{16n^2}. \quad \square$$

Introduce

$$(2.20) \quad n_1 := \max(n_0, \|V\|^2)$$

Proposition 2.5. *For $\zeta \in \mathcal{D}_{M/2}$ and $n \geq n_1$, the equation*

$$z_n = \alpha(n, z_n) + \zeta$$

has a unique solution $z_n = z_n(\zeta)$ in $\mathcal{D}_M$, which depends analytically on $\zeta \in \mathcal{D}_{\frac{M}{2}}$.

Proof. By Lemma 2.4, since $n \geq n_1$,

$$|\alpha(n, z)| \leq \frac{\|V\|^2}{4n} \leq \frac{1}{4} \quad (z \in \bar{\mathcal{D}}_M).$$

Therefore, $F(z) \equiv F_{n,\zeta}(z) := \zeta + \alpha(n, z)$ defines a map $F : \bar{\mathcal{D}}_M \rightarrow \bar{\mathcal{D}}_M$ (use $|F(z)| \leq |\zeta| + |\alpha(n, z)| \leq \frac{M}{2} + \frac{1}{4} < M$ for $M \geq 10$). F is a contraction, as for any pair $z_1, z_2 \in \bar{\mathcal{D}}_M$

$$\begin{aligned}
|F(z_1) - F(z_2)| &\leq \left(\sup_{|z| \leq M} \left| \frac{d}{dz} \alpha(n, z) \right| \right) |z_1 - z_2| \\
&\leq \frac{1}{16n} \left(\frac{1}{n} \|V\|^2 \right) \cdot |z_1 - z_2| \leq \frac{1}{n} \cdot \frac{1}{16} \cdot |z_1 - z_2| \leq \frac{1}{2} |z_1 - z_2|.
\end{aligned}$$

Thus, for any $\zeta \in \mathcal{D}_{\frac{M}{2}}$ and $n \geq n_1$, F admits a unique fixed point $z_n = z_n(\zeta)$, and $z_n(\zeta)$ depends analytically on ζ . $\square$

It remains to consider (2.19), which requires an estimate of $\beta(\pm n, z)$. First we need some auxiliary results. In (2.9) we introduced the operator

$$T_n := B_n^{II}(z - B_n^I)^{-1} \in \mathcal{L}(l^2(\mathbb{Z}(n))).$$

This operator can also be viewed as an element in $\mathcal{L}(l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)))$, where $w = (w(j))_{j \in \mathbb{Z}}$ is the weight $w(j) := (1 + |\frac{j}{2}|)^N e^{\frac{\omega}{2}|j|}$ and $(\mathcal{S}^n w)(j) := w(j + n)$. Denote by $W_n : l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)) \rightarrow l^2(\mathbb{Z}(n))$ the operator given by

$$(2.21) \quad W_n(k, j) := w(k + n) \delta_{kj}.$$

Notice that W_n is an isometry. Therefore the operator norm of

$$(2.22) \quad \tilde{T}_n := W_n T_n W_n^{-1}$$

is given by

$$(2.23) \quad \|\tilde{T}_n\|_{\mathcal{L}(l^2(\mathbb{Z}(n)))} = \|T_n\|_{\mathcal{L}(l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)))}.$$

Lemma 2.6. *For $n \geq n_0$ and $|z| \leq M$*

$$(2.24) \quad \|T_n\|_{\mathcal{L}(l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)))} \leq \frac{\|V\|_{N, \omega}}{5n}.$$

Proof. In view of (2.23), it suffices to estimate the Hilbert Schmidt norm of $\tilde{T}_n$ in $\mathcal{L}(l^2(\mathbb{Z}(n)))$. It follows from the submultiplicative property of the weight w (cf. the definition at the beginning of section 2) that

$$(2.25) \quad \frac{(\mathcal{S}^n w)(j)}{(\mathcal{S}^n w)(k)} \leq w(j - k).$$

Therefore

$$\begin{aligned}
\|\tilde{T}_n\|_{HS}^2 &= \sum_{j, k \neq \pm n} \frac{|\mathcal{S}^n w(j)|^2}{|\mathcal{S}^n w(k)|^2} |\hat{V}(j - k)|^2 \frac{1}{|z - \pi^2(k^2 - n^2)|^2} \\
&\leq \sum_{j, k \neq \pm n} |w(j - k)|^2 |\hat{V}(j - k)|^2 \frac{1}{75} \frac{1}{(k - n)^2 (k + n)^2} \\
&\leq \frac{1}{75} \sum_i (|w(i)| \hat{V}(i)|^2 \sum_{j-i \neq \pm n} \frac{1}{((j - i)^2 - n^2)^2}) \\
&\leq \|V\|_{N, \omega}^2 \frac{1}{30n^2},
\end{aligned}$$

where, for the last inequality, we argue in the same way as in the last steps of the inequality (2.13). $\square$

Let

$$(2.26) \quad n_2 := \max(n_1, \|V\|_{N, \omega}) = \max\left(\frac{M+1}{2}, \|V\|^2, \|V\|_{N, \omega}\right).$$

Proposition 2.7. For $n \geq n_2$,

$$(2.27) \quad \left(\sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} \sup_{|z| \leq M} |\beta(\pm n, z)|^2 \right)^{1/2} \leq \|V\|_{N,\omega}^2 \left(1 + \frac{1}{8} \|V\|_{N,\omega}\right);$$

$$(2.28) \quad \left(\sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N+4} e^{2\omega n} \sup_{|z| \leq M} \left| \frac{d}{dz} \beta(\pm n, z) \right|^2 \right)^{1/2} \leq \frac{1}{4} \|V\|_{N,\omega}^2 (1 + \|V\|_{N,\omega}).$$

Proof. The estimates for $\beta(n, z)$ and $\beta(-n, z)$ are obtained in the same way. Let us concentrate on $\beta(n, z)$.

(i) Proof of (2.27): By Lemma 2.6 and (2.26), $(Id_n - T_n) \in \mathcal{L}(l_{\mathcal{S}^w}^2(\mathbb{Z}(n)))$ is invertible for $n \geq n_2$. With

$$(Id_n - T_n)^{-1} = Id_n + T_n(Id_n - T_n)^{-1}$$

and $a_n \equiv a_n(z) \in l_w^2(\mathbb{Z}(n))$ defined by

$$(2.29) \quad \mathcal{S}^n a_n := (Id_n - T_n)^{-1} \mathcal{S}^n \hat{V} \in l_{\mathcal{S}^w}^2(\mathbb{Z}(n))$$

the expression $\beta(n, z)$ takes the form $\beta(n, z) = \beta_1(n, z) + \beta_2(n, z)$ with

$$(2.30A) \quad \beta_1(n, z) := \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} \mathcal{S}^n \hat{V} \rangle,$$

$$(2.30B) \quad \beta_2(n, z) := \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} T_n \mathcal{S}^n a_n \rangle.$$

The two terms $\beta_1(n, z)$ and $\beta_2(n, z)$ are estimated separately: For $n \geq n_2$

$$\begin{aligned} \sup_{|z| \leq M} |\beta_1(n, z)| &\leq \sum_{k \neq \pm n} |\hat{V}(n-k)| \frac{1}{8|k^2 - n^2|} |\hat{V}(n+k)| \\ &= \frac{1}{8} (b_{\hat{V}} * b_{\hat{V}})(2n), \end{aligned}$$

where

$$b_{\hat{V}}(j) := \frac{|\hat{V}(j)|}{j} \quad (j \neq 0) \quad \text{and} \quad b_{\hat{V}}(0) := 0.$$

Notice that $\|b_{\hat{V}}\|_{N+1, \frac{\omega}{2}} \leq 2^{N+1} \|V\|_{N,\omega}$.

Using the fact that $\|a * b\| \leq (\sum_k |a(k)|) \|b\|$ for $(a(k))_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$ and $(b(k))_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$, we conclude that

$$\begin{aligned} &\sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} \sup_{|z| \leq M} |\beta_1(n, z)| \\ &\leq \frac{1}{64} \sum_{n \neq 0} |(b_{\hat{V}} * b_{\hat{V}})(2n)|^2 \left(1 + \frac{|n|}{2}\right)^{2N+2} e^{\omega(2|n|)} \\ &\leq \frac{1}{64} \pi^2 (2 \|V\|_{N,\omega})^4 = \left(\frac{\pi}{2} \|V\|_{N,\omega}^2\right)^2. \end{aligned}$$

Direct computations furnish a slightly better estimate:

$$\begin{aligned}
 (2.31) \quad & \left(\sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} \sup_{|z| \leq M} |\beta_1(n, z)|^2 \right)^{1/2} \\
 & \leq \frac{1}{4} \left(\sum_{j \neq 0} \frac{e^{\frac{\omega}{2}|j|} |\hat{V}(j)|}{|j|} + \sum_{j \neq 0} \frac{e^{\frac{\omega}{2}|j|} |\hat{V}(j)|}{|j|} \right) \|V\|_{N, \omega} \\
 & \leq \frac{1}{2} \left(\sum_{j \neq 0} \frac{1}{j^2} \right)^{1/2} \|V\|_{N, \omega}^2 \leq \|V\|_{N, \omega}^2.
 \end{aligned}$$

Next let us estimate $\beta_2(n, z)$ in (2.30B):

$$\begin{aligned}
 & \left(1 + \frac{n}{2}\right)^{N+1} e^{\omega n} |\beta_2(n, z)| \\
 & = \left(1 + \frac{n}{2}\right)^{N+1} e^{\omega n} |\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} W_n^{-1} \tilde{T}_n W_n \mathcal{S}^n a_n \rangle| \\
 & \leq \frac{1}{4} \sum_{\substack{|k-n| \geq n \\ k \neq \pm n}} \left(1 + \frac{|k-n|}{2}\right)^N e^{\frac{\omega}{2}|n-k|} |\hat{V}(n-k)| \frac{1}{|n+k|} \\
 & \quad \times \frac{1}{\left(1 + \frac{|n+k|}{2}\right)^N} |(\tilde{T}_n W_n \mathcal{S}^n a_n)(k)| \\
 & + \frac{1}{4} \sum_{\substack{|k-n| < n \\ k \neq \pm n}} \frac{e^{\frac{\omega}{2}|n-k|} |\hat{V}(n-k)|}{|n-k|} |(\tilde{T}_n W_n \mathcal{S}^n a_n)(k)| \\
 & \leq \frac{1}{2} \|V\|_{N, \omega} \|\tilde{T}_n\| \|W_n \mathcal{S}^n a_n\|.
 \end{aligned}$$

By Lemma 2.6 and (2.23), $\|\tilde{T}_n\| \leq \frac{\|V\|_{N, \omega}}{5n}$. Further, $\|(Id_n - T_n)^{-1}\| \leq \frac{5}{4}$ for $n \geq n_2$, and therefore, by (2.29),

$$\|W_n \mathcal{S}^n a_n\|_{l^2(\mathbb{Z}(n))} = \|\mathcal{S}^n a_n\|_{l^2_{\mathcal{S}^n w}(\mathbb{Z}(n))} \leq \frac{5}{4} \|V\|_{N, \omega}.$$

Combining the estimates above, we obtain

$$\sup_{|z| \leq M} \left(1 + \frac{n}{2}\right)^{N+1} e^{\omega n} |\beta_2(n, z)| \leq \frac{\|V\|_{N, \omega}^3}{8n}.$$

Therefore

$$\begin{aligned}
 & \left(\sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2(N+1)} e^{2\omega n} \sup_{|z| \leq M} |\beta_2(n, z)|^2 \right)^{1/2} \\
 & \leq \frac{\|V\|_{N, \omega}^3}{8} \left(\sum_{n \geq n_2} \frac{1}{n^2} \right)^{1/2} \leq \frac{1}{8} \|V\|_{N, \omega}^3.
 \end{aligned}$$

Combined with (2.31), (2.30A) and (2.30B), the estimate (2.27).

(ii) Proof of (2.28): The derivative $\frac{d}{dz}\beta(n, z)$ is given by

$$(2.32) \quad \begin{aligned} \frac{d}{dz}\beta(n, z) &= \frac{d}{dz} \left(\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} \mathcal{S}^n \hat{V} \rangle + \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} T_n \mathcal{S}^n a_n \rangle \right) \\ &= -\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-2} \mathcal{S}^n \hat{V} \rangle - \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-2} T_n \mathcal{S}^n a_n \rangle \\ &\quad - \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} (Id_n - T_n)^{-1} T_n (z - B_n^I)^{-1} \mathcal{S}^n a_n \rangle \end{aligned}$$

where we used the fact that the derivative of

$$T_n \mathcal{S}^n a_n = T_n (Id_n - T_n)^{-1} \mathcal{S}^n \hat{V} = ((Id_n - T_n)^{-1} - Id_n) \mathcal{S}^n \hat{V}$$

is given by

$$\frac{d}{dz} T_n \mathcal{S}^n a_n = -(Id_n - T_n)^{-1} T_n (z - B_n^I)^{-1} \mathcal{S}^n a_n.$$

The three terms on the right hand side of (2.32) are estimated separately. The first one is estimated similarly as in (i): Using (2.31), one obtains

$$(2.32i) \quad \begin{aligned} &\left(\sum_{n \geq n_2} \sup_{|z| \leq M} \left(1 + \frac{n}{2}\right)^{2N+4} e^{2\omega n} |\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-2} \mathcal{S}^n \hat{V} \rangle|^2 \right)^{1/2} \\ &\leq \frac{1}{4} \|V\|_{N, \omega}^2. \end{aligned}$$

The second term on the right hand side of (2.32) is estimated similarly as in (i), and one obtains

$$(2.32ii) \quad \begin{aligned} &\left(\sum_{n \geq n_2} \sup_{|z| \leq M} \left(1 + \frac{n}{2}\right)^{2N+4} e^{2\omega n} |\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-2} T_n \mathcal{S}^n a_n \rangle|^2 \right)^{1/2} \\ &\leq \frac{1}{8} \|V\|_{N, \omega}^3. \end{aligned}$$

To estimate the last term on the right hand side of (2.32), we first notice that

$$(Id_n - T_n)^{-1} T_n = W_n^{-1} (Id_n - \tilde{T}_n)^{-1} \tilde{T}_n W_n.$$

Thus, with $\tilde{S}_n := (Id_n - \tilde{T}_n)^{-1} \tilde{T}_n$,

$$\begin{aligned} &(1 + \frac{n}{2})^{N+2} e^{\omega n} |\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} (Id_n - T_n)^{-1} T_n (z - B_n^I)^{-1} \mathcal{S}^n a_n \rangle| \\ &\leq \frac{1}{4} \sum_{\substack{|k-n| \geq n \\ k \neq \pm n}} \left(1 + \frac{|k-n|}{2}\right)^N e^{\frac{\omega}{2}|n-k|} |\hat{V}(n-k)| \\ &\quad \times \left| (\tilde{S}_n W_n (1+n) (z - B_n^I)^{-1} \mathcal{S}^n a_n)(k) \right| \\ &\quad + \frac{1}{4} \sum_{\substack{|k-n| < n \\ k \neq \pm n}} \frac{e^{\frac{\omega}{2}|n-k|} |\hat{V}(n-k)|}{|n-k|} \left| (\tilde{S}_n W_n (1+n) (z - B_n^I)^{-1} \mathcal{S}^n a_n)(k) \right| \\ &\leq \frac{1}{4} \|V\|_{N, \omega} \|\tilde{S}_n W_n (1+n) (z - B_n^I)^{-1} \mathcal{S}^n a_n\| \\ &\quad + \frac{1}{4} \|V\|_{0, \omega} \|\tilde{S}_n W_n (1+n) (z - B_n^I)^{-1} \mathcal{S}^n a_n\| \\ &\leq \frac{1}{2} \|V\|_{N, \omega} \|\tilde{T}_n\| \|(Id_n - \tilde{T}_n)^{-1}\| \|W_n (1+n) (z - B_n^I)^{-1} \mathcal{S}^n a_n\|. \end{aligned}$$

By Lemma 2.6 and (2.23), $|||\tilde{T}_n||| \leq \frac{\|V\|_{N,\omega}}{5n}$ and $|||(Id_n - \tilde{T}_n)^{-1}||| \leq \frac{5}{4}$.
Further, for $n \geq n_2$

$$|||(1+n)(z - B_n^I)^{-1}|||_{\mathcal{L}(l_{S^n_w}^2(\mathbb{Z}(n)))} \leq \sup_{\substack{|z| \leq M \\ k \neq \pm n}} \left| \frac{1+n}{z - \pi^2(k-n)(k+n)} \right| \leq \frac{1}{4}.$$

Thus, with (2.29),

$$\|W_n(1+n)(z - B_n^I)^{-1}\mathcal{S}^n a_n\|_{l^2(\mathbb{Z}(n))} \leq \frac{1}{4} \cdot \frac{5}{4} \|V\|_{N,\omega}.$$

Combining these estimates leads to

$$\begin{aligned} (2.32\text{iii}) \quad & \left(\sum_{n \geq n_2} \sup_{|z| \leq M} \left(1 + \frac{n}{2}\right)^{2N+4} \right. \\ & \times e^{2\omega n} |\langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, (z - B_n^I)^{-1} (Id_n - T_n)^{-1} T_n (z - B_n^I)^{-1} \mathcal{S}^n a_n \rangle|^2 \Big)^{1/2} \\ & \leq \frac{1}{32} \|V\|_{N,\omega}^3 \left(\sum_{n \geq n_2} \frac{1}{n^2} \right)^{1/2} \leq \frac{1}{32} \|V\|_{N,\omega}^3. \end{aligned}$$

From (2.32) and (2.32i)-(2.32iii) the estimate (2.28) follows. $\square$

We are now ready to investigate (2.19).

Let

$$(2.33) \quad r_n := \max(|\hat{V}(\pm 2n)|) + \max_{|z| \leq M} |\beta(\pm n, z)|.$$

Notice that, by Proposition 2.7, for $n \geq n_2$,

$$(2.34) \quad r_n \leq \|V\| + \|V\|_{N,\omega}^2 \left(1 + \frac{1}{8} \|V\|_{N,\omega}\right).$$

Proposition 2.8. *Assume that $M \geq 10$ satisfies*

$$\|V\| + \|V\|_{N,\omega}^2 \left(1 + \frac{1}{8} \|V\|_{N,\omega}\right) \leq \frac{M}{4}.$$

Then, for $n \geq n_2$, equation (2.19) has exactly two (counted with multiplicity) solutions ζ_n^+, ζ_n^- in $\overline{\mathcal{D}_{r_n}}$.

Proof. The result follows from Rouché's theorem. Clearly $\zeta^2 = 0$ has two roots in $\mathcal{D}_{r_n}$. For $|\zeta| = Kr_n$ with $1 < K < 2$,

$$\sup_{|z| \leq M} \left| (\hat{V}(2n) + \beta(n, z))(\hat{V}(-2n) + \beta(-n, z)) \right| \leq r_n^2 < |\zeta^2|.$$

As $\beta(\pm n, z_n(\zeta))$ depend analytically on ζ for $|\zeta| < \frac{M}{2}$ and $Kr_n < 2r_n \leq \frac{M}{2}$, we deduce from Rouché's theorem that equation (2.19) has precisely two roots in $\mathcal{D}_{Kr_n}$. As the two roots are independent of K and $1 < K < 2$ is arbitrary close to 1, we conclude that $\zeta_n^\pm \in \overline{\mathcal{D}_{r_n}}$. $\square$

Let $z_n^\pm = z(\zeta_n^\pm) = \zeta_n^\pm + \alpha(n, z_n^\pm)$, where $\zeta_n^\pm$ are given by Proposition 2.8. Then

$$(2.35) \quad |z_n^+ - z_n^-| \leq |\zeta_n^+ - \zeta_n^-| + \left(\sup_{|z| \leq M} \left| \frac{d}{dz} \alpha(n, z) \right| \right) |z_n^+ - z_n^-|.$$

By Lemma 2.4 (ii), (2.20) and as $n_2 \geq n_1$,

$$(2.36) \quad \sup_{|z| \leq M} \left| \frac{d}{dz} \alpha(n, z) \right| \leq \frac{1}{2}.$$

Together with $|\zeta_n^+ - \zeta_n^-| \leq |\zeta_n^+| + |\zeta_n^-| \leq 2r_n$, estimate (2.35) leads to

$$|z_n^+ - z_n^-| \leq 4r_n.$$

In view of the definition (2.33) of r_n , Proposition 2.7 and $|z_n^+ - z_n^-| = |\lambda_n^+ - \lambda_n^-|$, we conclude that

$$(2.37) \quad \begin{aligned} & \left(\sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N} e^{2\omega n} |\lambda_n^+ - \lambda_n^-|^2 \right)^{1/2} \\ & \leq 8(\|V\|_{N,\omega} + \|V\|_{N,\omega}^2 (1 + \frac{1}{8} \|V\|_{N,\omega})) < \infty. \end{aligned}$$

Next we want to obtain asymptotics for $\lambda_n^+ - \lambda_n^-$. Rewrite (2.19) in two ways:

$$(2.38) \quad (\zeta_n^\pm)^2 - \delta(n)\delta(-n) - \eta_n^\pm = 0,$$

with $\eta_n^\pm = \eta(z_n^\pm)$, where $\delta(n), \eta(z)$ are defined either by (alternative 1)

$$\begin{aligned} \delta(n) &\equiv \delta^I(n) := \hat{V}(2n) + \beta(n, 0), \\ \eta(z) &\equiv \eta^I(z) := (\beta(-n, z) - \beta(-n, 0))\hat{V}(2n) \\ &\quad + (\beta(n, z) - \beta(n, 0))\hat{V}(-2n) \\ &\quad + \beta(-n, z)\beta(n, z) - \beta(-n, 0)\beta(n, 0) \end{aligned}$$

or by (alternative 2)

$$\begin{aligned} \delta(n) &\equiv \delta^{II}(n) := \hat{V}(2n), \\ \eta(z) &\equiv \eta^{II}(z) := \beta(-n, z)\hat{V}(2n) + \beta(n, z)\hat{V}(-2n) + \beta(-n, z)\beta(n, z). \end{aligned}$$

Introduce s_n , given by (alternative 1)

$$\begin{aligned} s_n \equiv s_n^I &:= \sup_{|z| \leq M} \left(|\beta(-n, z) - \beta(-n, 0)| |\hat{V}(2n)| + |\beta(n, z) - \beta(n, 0)| |\hat{V}(-2n)| \right. \\ &\quad \left. + |\beta(-n, z)| |\beta(n, z)| + |\beta(-n, 0)| |\beta(n, 0)| \right) \end{aligned}$$

or by (alternative 2)

$$s_n \equiv s_n^{II} := \sup_{|z| \leq M} \left(|\beta(-n, z)| |\hat{V}(2n)| + |\beta(n, z)| |\hat{V}(-2n)| + |\beta(-n, z)| |\beta(n, z)| \right).$$

Use $|\beta(n, z) - \beta(n, 0)| \leq M \sup_{|z| \leq M} \left| \frac{d}{dz} \beta(n, z) \right|$ and Proposition 2.7 to obtain

$$(2.39^I) \quad \begin{aligned} & \sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} s_n^I \\ & \leq \frac{M}{2} \|V\|_{N,\omega}^3 (1 + \|V\|_{N,\omega}) + 2 \|V\|_{N,\omega}^4 (1 + \frac{1}{8} \|V\|_{N,\omega})^2 \end{aligned}$$

and

$$(2.39^{II}) \quad \begin{aligned} & \sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N+1} e^{2\omega n} s_n^{II} \\ & \leq 2 \|V\|_{N,\omega}^3 (1 + \frac{1}{8} \|V\|_{N,\omega}) + \|V\|_{N,\omega}^4 (1 + \frac{1}{8} \|V\|_{N,\omega})^2. \end{aligned}$$

Proposition 2.9. *Assume that $M \geq 10$ satisfies*

$$\|V\| + \|V\|_{N,\omega}^2 \left(1 + \frac{1}{8} \|V\|_{N,\omega}\right) \leq \frac{M}{4}.$$

Then for $n \geq n_2$ and $\delta(n), s_n$ given by either $\delta^I(n), s_n^I$ or $\delta^{II}(n), s_n^{II}$, the roots ζ_n^+, ζ_n^- can be labeled in such a way that

- (i) $|\zeta_n^+ - (\delta(n)\delta(-n))^{1/2}| \leq 5s_n^{1/2},$
- (ii) $|\zeta_n^- + (\delta(n)\delta(-n))^{1/2}| \leq 5s_n^{1/2}.$

Proof. We consider two different cases:

Case 1. $|\delta(n)\delta(-n)| \leq 4s_n$. This is an easy case for which (i) and (ii) are proved in the same way. Let us concentrate on (i). Then

$$\begin{aligned} |(\zeta_n^+ - (\delta(n)\delta(-n))^{1/2})^2| &\leq 2|(\zeta_n^+)^2| + 2|\delta(n)\delta(-n)| \\ &\leq 2|\delta(n)\delta(-n) + \eta_n^+| + 2|\delta(n)\delta(-n)| \\ &\leq 4|\delta(n)\delta(-n)| + 2|\eta_n^+| \leq 18s_n \leq (5s_n^{1/2})^2, \end{aligned}$$

where for the second inequality, we used (2.38).

Thus $|\zeta_n^+ - (\delta(n)\delta(-n))^{1/2}| \leq 5s_n^{1/2}.$

Case 2. $|\delta(n)\delta(-n)| \geq 4s_n$. Without any loss of generality we may assume that $s_n > 0$. In particular, $|\delta(n)\delta(-n)| > 0$. The equation (2.38) can then be rewritten as

$$(2.40) \quad (\zeta_n)^2 = \delta(-n)\delta(n) \left(1 + \frac{\eta(z(\zeta_n))}{\delta(-n)\delta(n)}\right)$$

where $z(\zeta_n)$ is given by (2.18). With

$$\xi := \frac{\zeta_n}{(\delta(-n)\delta(n))^{1/2}}$$

formula (2.40) leads to

$$(2.41) \quad (\xi)^2 = \left(1 + \frac{\eta(z(\zeta_n))}{\delta(-n)\delta(n)}\right).$$

As $|\delta(n)\delta(-n)| \geq 4s_n$, one concludes that $\left|\frac{\eta(z(\zeta_n))}{\delta(-n)\delta(n)}\right| \leq \frac{1}{4}.$

Denoting by $(1+w)^{1/2}$ the branch of the square root determined by $(1)^{1/2} = +1$, we obtain the equations

$$(2.42^\pm) \quad \xi = \pm F(\xi) = \pm \left(1 + \frac{\eta(z)}{\delta(n)\delta(-n)}\right)^{1/2}$$

with $z = z(\zeta_n) = z((\delta(n)\delta(-n))^{1/2} \cdot \xi)$. Let us first consider (2.42⁺): Introduce $\mathcal{D}_{1/4}(1) := \{\xi \in \mathbb{C} \mid |\xi - 1| < \frac{1}{4}\}$ and notice that for $\xi \in \mathcal{D}_{1/4}(1), \zeta_n := (\hat{V}(2n)\hat{V}(-2n))^{1/2}\xi$ satisfies $|\zeta_n| \leq \frac{M}{4} \cdot \frac{5}{4} < \frac{M}{2}.$

As $|(1+x)^{1/2} - 1| \leq |x|$ for $x \in \mathcal{D}_{1/4}(0)$ and case 2 holds, we conclude that F maps $\overline{\mathcal{D}_{1/4}(1)}$ into itself. Moreover F is continuous and therefore, according to Brower's fixed point theorem, admits at least one fixed point, denoted by ξ^I , i.e.

$$\xi^I = +F(\xi^I) = + \left(1 + \frac{\eta(z^I)}{\delta(n)\delta(-n)}\right)^{1/2}$$

where $z^I = z(\hat{V}(-2n)\hat{V}(2n))^{1/2} \xi^I$. Then, as we are in case 2,

$$\begin{aligned} |\xi^I - 1| &\leq \left| \left(1 + \frac{\eta(z^I)}{\delta(n)\delta(-n)} \right)^{1/2} - 1 \right| \leq \frac{|\eta(z^I)|}{|\delta(n)\delta(-n)|} \\ &\leq \frac{1}{2} \frac{s_n^{1/2}}{|\delta(n)\delta(-n)|^{1/2}} \end{aligned}$$

and, with $\zeta^I := (\delta(n)\delta(-n))^{1/2} \xi^I$,

$$(2.43) \quad |\zeta^I - (\delta(-n)\delta(n))^{1/2}| \leq \frac{1}{2} s_n^{1/2}.$$

The same arguments can be used to show that there exists a solution $\zeta^{II} \in \mathcal{D}_{1/4}(-1)$ of (2.42⁻) so that, with $\zeta^{II} := (\delta(n)\delta(-n))^{1/2} \xi^{II}$,

$$(2.44) \quad |\zeta^{II} + (\delta(n)\delta(-n))^{1/2}| \leq \frac{1}{2} s_n^{1/2}.$$

It remains to show that $\{\zeta^I, \zeta^{II}\} = \{\zeta_n^+, \zeta_n^-\}$. First notice that $\zeta^I \neq \zeta^{II}$. Otherwise, we obtain a contradiction, by combining (2.43), (2.44), $s_n > 0$ and the inequality case 2 as follows: Assume $\zeta^I = \zeta^{II} = \zeta^*$. Then

$$\begin{aligned} 0 < 2(2 s_n^{1/2}) &\leq |2(\delta(-n)\delta(n))^{1/2}| \leq |(\delta(n)\delta(-n))^{1/2} - \zeta^* + (\delta(n)\delta(-n))^{1/2} + \zeta^*| \\ &\leq \frac{1}{2} s_n^{1/2} + \frac{1}{2} s_n^{1/2} = s_n^{1/2}, \end{aligned}$$

and $0 < 4s_n^{1/2} \leq s_n^{1/2}$ gives the claimed contradiction.

Further notice that $\zeta^I, \zeta^{II}, \zeta_n^+, \zeta_n^-$ are all solutions of (2.40). But according to Proposition 2.8, equation (2.40) has precisely two solutions. Therefore $\{\zeta^I, \zeta^{II}\} = \{\zeta_n^+, \zeta_n^-\}$. This proves Proposition 2.9 in case 2. $\square$

We are now ready to prove Theorem 2. It is contained in the following

Theorem 2.10. *Let $S^1 = \mathbb{R}/\mathbb{Z}$, $N \in \mathbb{Z}_{\geq 0}$, $\omega \in \mathbb{R}_{\geq 0}$ and $M \geq 10$.*

Then, for any $V \in H^{N,\omega}(S^1; \mathbb{C})$ with $\|V\| + \|V\|_{N,\omega}^2 (1 + \frac{1}{8} \|V\|_{N,\omega}) \leq \frac{M}{4}$,

$$(i) \quad \left(\sum_{n \geq M^2} \left(1 + \frac{n}{2} \right)^{2N+1} e^{2\omega n} \left| (\lambda_n^+ - \lambda_n^-) - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2} \right|^2 \right)^{1/2} \leq 3M^2,$$

$$\begin{aligned} (ii) \quad &\left(\sum_{n \geq M^2} \left(1 + \frac{n}{2} \right)^{2N+2} e^{2\omega n} \right. \\ &\quad \left. \times \left| (\lambda_n^+ - \lambda_n^-) - 2((\hat{V}(-2n) + \beta(-n, 0))(\hat{V}(2n) + \beta(n, 0)))^{1/2} \right|^2 \right)^{1/2} \leq 5M^2, \end{aligned}$$

where, for $n \geq M^2$, $\lambda_n^\pm = n^2\pi^2 + z_n^\pm$ (and thus $\{\lambda_n^+, \lambda_n^-\} = \{\lambda_{2n}, \lambda_{2n-1}\}$) with $(\lambda_n)_{n \geq 0}$ denoting the periodic spectrum (ordered as explained in the introduction) of $-\frac{d^2}{dx^2} + V$ considered on the interval $[0, 2]$ and $(\hat{V}(k))_{k \in \mathbb{Z}}$ are the Fourier coefficients of V when considered as functions with period 2.

Proof. Without any loss of generality we assume that $V \in H_0^{N,\omega}(S^1; \mathbb{C})$. Statements (i) and (ii) are proved in the same way, so we concentrate on (i). Notice that $M^2 \geq n_2 := \max(\frac{M+1}{2}, \|V\|^2, \|V\|_{N,\omega})$.

For $n \geq M^2$, $\lambda_n^+ - \lambda_n^- = z_n^+ - z_n^-$. Furthermore, $z_n^\pm = \alpha(n, z_n^\pm) + \zeta_n^\pm$, and, by Proposition 2.9,

$$|(\zeta_n^+ - \zeta_n^-) - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2}| \leq 10s_n^{1/2}.$$

Therefore

$$\begin{aligned} & |\lambda_n^+ - \lambda_n^- - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2}| \\ (2.45) \quad & \leq |\zeta_n^+ - \zeta_n^- - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2}| + \left(\sup_{|z| \leq M} \left| \frac{d}{dz} \alpha(n, z) \right| \right) |z_n^+ - z_n^-| \\ & \leq 10s_n^{1/2} + \frac{\|V\|^2}{n^2} |z_n^+ - z_n^-|, \end{aligned}$$

where for the last inequality, we used Lemma 2.4 (ii) and (2.26). By (2.37)

$$\begin{aligned} (2.46) \quad & \left(\sum_{n \geq M^2} \left(1 + \frac{n}{2}\right)^{2N+4} e^{2\omega n} \left(\frac{\|V\|^2}{n^2} |z_n^+ - z_n^-|^2\right)^{1/2} \right) \\ & \leq 2 \cdot \|V\|^2 8(\|V\|_{N,\omega} + \|V\|_{N,\omega}^2 (1 + \frac{1}{8} \|V\|_{N,\omega})) \leq 2M^2. \end{aligned}$$

By (2.39^{II}),

$$(2.47) \quad 10 \cdot \left(\sum_{n \geq M^2} \left(1 + \frac{n}{2}\right)^{2N+1} e^{2\omega n} s_n \right)^{1/2} \leq 5M.$$

As $M \geq 10$, $5M \leq M^2$, and therefore from (2.45), (2.46) and (2.47) we obtain

$$(2.48) \quad \left(\sum_{n \geq M^2} \left(1 + \frac{n}{2}\right)^{2N+1} e^{2\omega n} \left| \lambda_n^+ - \lambda_n^- - 2(\hat{V}(-2n)\hat{V}(2n))^{1/2} \right|^2 \right)^{1/2} \leq 3M^2. \quad \square$$

Remark 1. Theorem 2.10 can be improved for real valued potentials. It leads to a result obtained by Marčenko [Ma]:

Theorem 2.10 A. *Let $N \in \mathbb{Z}_{\geq 0}$ and $M \geq 10$. Then, for any $V \in H^{N,\omega}(S^1; \mathbb{R})$ with $\|V\| + \|V\|_{N,\omega}^2 (1 + \frac{1}{8} \|V\|_{N,\omega}) \leq \frac{M}{4}$,*

$$(2.49) \quad \sum_{n \geq M^2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} \left| (\lambda_n^+ - \lambda_n^-) - 2|\hat{V}(-2n)\hat{V}(2n)|^{1/2} \right|^2 \leq 3M^2.$$

Remark for Theorem 2.10 A. Recall that

$$(2.19) \quad (\zeta_n)^2 - \hat{V}(2n)\hat{V}(-2n) - \eta_n = 0.$$

If V is real valued, then $|\hat{V}(2n)| = |\hat{V}(-2n)|$. This is a quantitative version of the following statement:

$$(2.50) \quad \hat{V}(2n) \cdot \hat{V}(-2n) = 0 \iff \hat{V}(2n) = 0 \text{ and } \hat{V}(-2n) = 0.$$

To improve on Theorem 2.10 as in Theorem 2.10 A, it seems that one needs to restrict to potentials satisfying a vanishing condition which is a quantitative version of (2.50).

Proof of Theorem 2.10 A. Without any loss of generality, $V \in H_0^{N,\omega}(S^1; \mathbb{R})$, i.e. $\hat{V}(0) = 0$. If V is real valued, then $\hat{V}(-k) = \overline{\hat{V}(k)}$, and

$$(2.51) \quad \beta(-n, z) = \overline{\beta(n, \bar{z})} \quad (|z| \leq M, n \geq n_2),$$

$$(2.52) \quad \overline{\alpha(n, z)} = \alpha(n, \bar{z}) \quad (|z| \leq M, n \geq n_2).$$

Further, $-\frac{d^2}{dx^2} + V$ is selfadjoint, and therefore, the periodic spectrum of $-\frac{d^2}{dx^2} + V$ is contained in $\mathbb{R}$. Following the proof of Theorem 2.10, we know that for $n \geq 2M^2$ we have $\lambda_n^\pm = n^2\pi^2 + z_n^\pm$, and thus $z_n^\pm \in \mathbb{R}$. Moreover, $\zeta_n^\pm = z_n^\pm - \alpha(n, z_n^\pm) \in \mathbb{R}$, as $\alpha(n, z_n^\pm) \in \mathbb{R}$ by (2.52).

Therefore, equation (2.19) can be written as

$$(\zeta_n^\pm)^2 = |\hat{V}(2n) + \beta(n, z(\zeta_n^\pm))|^2,$$

which leads to

$$\zeta_n^\pm = \pm |\hat{V}(2n) + \beta(n, z(\zeta_n^\pm))|.$$

Thus

$$\begin{aligned} |\zeta_n^\pm - (\pm |\hat{V}(2n)|)| &\leq | |\hat{V}(2n) + \beta(n, z(\zeta_n^\pm))| - |\hat{V}(2n)| | \\ &\leq |\beta(n, z(\zeta_n^\pm))| \end{aligned}$$

and, by Proposition 2.7,

$$(2.53) \quad \begin{aligned} &\left(\sum_{n \geq n_2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} \left| (\zeta_n^+ - \zeta_n^-) - 2|\hat{V}(2n)| \right| \right)^{1/2} \\ &\leq \|V\|_{N,\omega}^2 \left(1 + \frac{1}{8} \|V\|_{N,\omega}\right) \leq \frac{M}{4}. \end{aligned}$$

Combining with (2.45) and (2.46), one obtains

$$\begin{aligned} &\left(\sum_{n \geq M^2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} \left| (\lambda_n^+ - \lambda_n^-) - 2|\hat{V}(2n)| \right|^2 \right)^{1/2} \\ &\leq 2M^2 + \frac{M}{4} \leq 3M^2. \quad \square \end{aligned}$$

Remark 2. If V is an even, possibly complex valued potential, Theorem 2.10 can be reformulated in a way which leads to an improvement. In the case when $V \in H^{N,\omega}(S^1; \mathbb{C})$ is even, i.e. $V(x) = V(-x)$, it is a well known fact that for n sufficiently large, $\{\lambda_n^+, \lambda_n^-\} = \{\mu_n, \nu_n\}$, where $(\mu_n)_{n \geq 1}$ denote the Dirichlet eigenvalues and $(\nu_n)_{n \geq 0}$ denote the Neumann eigenvalues of $-\frac{d^2}{dx^2} + V$ considered on the unit interval. Further, the eigenfunctions of the Dirichlet eigenvalues are odd, whereas the eigenfunctions of the Neumann eigenvalues are even.

Theorem 2.10 B. Let $N \in \mathbb{Z}_{\geq 0}$, $\omega \in \mathbb{R}_{\geq 0}$ and $M \geq 10$.

Then, for any **even** potential $V \in H^{N,\omega}(S^1, \mathbb{C})$ satisfying

$$\|V\| + \|V\|_{N,\omega}^2 \left(1 + \frac{1}{8} \|V\|_{N,\omega}\right) \leq \frac{M}{4},$$

we have

$$\left(\sum_{n \geq M^2} \left(1 + \frac{n}{2}\right)^{2n+2} e^{2\omega n} |(\mu_n - \nu_n) + 2\hat{V}(2n)|^2 \right)^{1/2} \leq 3M^2.$$

Proof. In the case where V is even one verifies that

$$\hat{V}(2k) = \hat{V}(-2k); \quad \beta(-k, z) = \beta(k, z) .$$

Therefore, equation (2.19) leads to $\zeta_n^2 = (\hat{V}(2n) + \beta(n, z(\zeta_n)))^2$ and, in turn,

$$(2.53^\pm) \quad \zeta_n^\pm = \pm (\hat{V}(2n) + \beta(n, z(\zeta_n^\pm))).$$

Then, with $\varepsilon_n = \pm$,

$$\mu_n = n^2 \pi^2 + \alpha(n, z(\zeta_n^{\varepsilon_n})) + \varepsilon_n (\hat{V}(2n) + \beta(n, z(\zeta_n^{\varepsilon_n})))$$

and

$$\nu_n = n^2 \pi^2 + \alpha(n, z(\zeta_n^{-\varepsilon_n})) - \varepsilon_n (\hat{V}(2n) + \beta(n, z(\zeta_n^{-\varepsilon_n}))).$$

To determine the sign ε_n , recall that the eigenfunction $y_2(x, \mu_n)$ is odd. Its Fourier coefficients $(a(k; n))_{k \in \mathbb{Z}}$ therefore satisfy $a(-k; n) = -a(k; n)$.

Thus, in equation (2.14), $x = -y$. Together with

$$-z(\zeta_n^{\varepsilon_n}) + \alpha(n, z(\zeta_n^{\varepsilon_n})) = \zeta_n^{\varepsilon_n} = \varepsilon_n (\hat{V}(2n) + \beta(n, z(\zeta_n^{\varepsilon_n}))),$$

equation (2.14) implies

$$(\varepsilon_n (\hat{V}(2n) + \beta(n, z(\zeta_n^{\varepsilon_n}))) + \hat{V}(2n) + \beta(n, z(\zeta_n^{\varepsilon_n})))x = 0 .$$

Since, in view of (2.5)-(2.7), $(x, y) \neq (0, 0)$ (for $n \geq n_0$) (otherwise, $a(k, n) = 0$ for all k) we then conclude that $\varepsilon_n = -1$ as claimed. As a consequence,

$$\mu_n - \nu_n = -2\hat{V}(2n) - \beta(n, z_n^+) - \beta(n, z_n^-) + \alpha(n, z_n^-) - \alpha(n, z_n^+) .$$

As in the proof of Theorem 2.10 A, one then obtains, by Proposition 2.7, combined with (2.45) and (2.46),

$$\left(\sum_{n \geq 2M^2} \left(1 + \frac{n}{2}\right)^{2N+2} e^{2\omega n} |\mu_n - \nu_n + 2\hat{V}(2n)|^2 \right)^{1/2} \leq 3M^2 . \quad \square$$

As an application of Theorem 2.10 we obtain asymptotic estimates of the eigenvalues $\lambda_n^\pm$,

$$(2.54^\pm) \quad \lambda_n^\pm = n^2 \pi^2 + \alpha\left(n, \frac{z_n^+ + z_n^-}{2}\right) \pm ((\hat{V}(-2n) + \beta(-n, 0))(\hat{V}(2n) + \beta(n, 0)))^{1/2} + l_{N+1, \omega}^2(n),$$

and of $\tau_n := \frac{\lambda_n^+ + \lambda_n^-}{2}$,

$$(2.55) \quad \tau_n = n^2 \pi^2 + \alpha\left(n, \frac{z_n^+ + z_n^-}{2}\right) + l_{N+1, \omega}^2(n),$$

where, by abuse of notation, we mean by $(l_{N+1, \omega}^2(n))_{n \geq 1}$ an element in $l_{N+1, \omega}^2(\mathbb{N})$.

We finish this section with a brief discussion of the linear space E_n spanned by eigenfunctions f_n^+ , f_n^- corresponding to simple eigenvalues $\lambda_n^+ \neq \lambda_n^-$, and of the root space E_n corresponding to double eigenvalues $\lambda_n^+ = \lambda_n^-$.

For a function $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik\pi x}$ in E_n , the part $\sum_{k \neq \pm n} \hat{f}(k) e^{ik\pi x}$ is small when compared with $\hat{f}(n) e^{in\pi x} + \hat{f}(-n) e^{-in\pi x}$. This follows from the following result, which we will use in section 3.

In view of (2.7), we introduce $a_{x, y, z} \in l^2(\mathbb{Z}(n))$:

$$(2.56) \quad a_{x, y, z} := x(z - B_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)} + y(z - B_n)^{-1} (\mathcal{S}^{-n} \hat{V})_{\mathbb{Z}(n)} .$$

Proposition 2.11. Assume $V \in H_0^{N,\omega}(S^1; \mathbb{C})$. Then for $|z| \leq M$, $n \geq n_2$ and $x, y \in \mathbb{C}$

- (i) $\| (z - B_n)^{-1} (\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l_{\mathcal{S}^n w}^2(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|_{N,\omega}$;
- (ii) $\| (z - B_n)^{-1} (\mathcal{S}^{-n} V)_{\mathbb{Z}(n)} \|_{l_{\mathcal{S}^{-n} w}^2(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|_{N,\omega}$;
- (iii) $\left(\sum_{k \neq \pm n} e^{|k|-n|\omega|} \left(1 + \left| \frac{|k|-n}{2} \right| \right)^{2N} |a_{x,y,z}(k)|^2 \right)^{1/2} \leq \frac{|x|+|y|}{n} \| V \|_{N,\omega}$;
- (iv) $\| (z - B_n)^{-1} (\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|$;
- (v) $\| (z - B_n)^{-1} (\mathcal{S}^{-n} V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} \leq \frac{1}{n} \| V \|$;
- (vi) $\sum_{k \neq \pm n} |a_{x,y,z}(k)| \leq \frac{|x|+|y|}{n} \| V \|$.

Remark. Notice that, with $w_1 := \mathcal{S}^n w$, $w_2 := \mathcal{S}^{-n} w$, the function

$$w_1 \wedge w_2(k) := \min(w_1(k), w_2(k))$$

is given by

$$\begin{aligned} w_1 \wedge w_2(k) &= \left(1 + \left| \frac{|k|-n}{2} \right| \right)^N e^{\frac{\omega}{2} |k|-n|} \\ &= \begin{cases} w_1(k) & \text{for } k \leq 0, \\ w_2(k) & \text{for } k \geq 0. \end{cases} \end{aligned}$$

Furthermore,

$$\sup_k \frac{w_1(k)}{w_2(k)} \leq (1+n)^N e^{n\omega} \quad \text{and} \quad \sup_k \frac{w_2(k)}{w_1(k)} \leq (1+n)^N e^{n\omega}.$$

This implies that $w_1 \wedge w_2$ is a weight with $M_{w_1 \wedge w_2} = ((1+n)^N e^{n\omega})^2$. Thus $M_{w_1 \wedge w_2}$ is increasing in n .

Proof. (i) By Lemma 2.6,

$$\| |T_n| \|_{\mathcal{L}(l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)))} \leq \frac{\| V \|_{N,\omega}}{2n}$$

and thus, for $|z| \leq M$ and $n \geq n_2$,

$$\begin{aligned} \| (z - B_n)^{-1} \|_{\mathcal{L}(l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)))} &= \| (z - B_n^I)^{-1} \cdot (Id_n - T_n)^{-1} \|_{\mathcal{L}(l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)))} \\ &\leq 2 \| (z - B_n^I)^{-1} \|_{\mathcal{L}(l_{\mathcal{S}^n w}^2(\mathbb{Z}(n)))} \leq 2 \cdot \frac{1}{2n}, \end{aligned}$$

where for the last inequality we used Lemma 2.1. This implies that

$$\| (z - B_n)^{-1} (\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l_{\mathcal{S}^n w}^2(\mathbb{Z}(n))} \leq \frac{1}{n} \| \mathcal{S}^n V \|_{\mathcal{S}^n w} \leq \frac{1}{n} \| V \|_{N,\omega}.$$

(ii) Using the same arguments as in the proof of Lemma 2.6, one shows that

$$\| |T_n| \|_{\mathcal{L}(l_{\mathcal{S}^{-n} w}^2(\mathbb{Z}(n)))} \leq \frac{\| V \|_{N,\omega}}{2n}.$$

One then argues as in the proof of (i) to conclude (ii).

(iii) Notice that for $k \in \mathbb{Z}$, $n \geq 1$

$$| |k| - n | = \min(|k - n|, |k + n|).$$

Thus

$$\left(1 + \left| \frac{|k|-n}{2} \right| \right)^N e^{\frac{\omega}{2} |k|-n|} \leq \min((\mathcal{S}^n w)(k), (\mathcal{S}^{-n} w)(k))$$

and (iii) follows from (i), (ii) and (2.56).

(iv) We have

$$\begin{aligned}
& \| (z - B_n)^{-1} (\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} = \| (z - B_n^I)^{-1} (Id_n - T_n)^{-1} (\mathcal{S}^n V)_{\mathbb{Z}(n)} \|_{l^1(\mathbb{Z}(n))} \\
&= \sum_{k \neq \pm n} \frac{1}{|z - \pi^2(k^2 - n^2)|} |(Id_n - T_n)^{-1} (\mathcal{S}^n V)(k)| \\
&\leq \left(\sum_{k \neq \pm n} \frac{1}{|z - \pi^2(k^2 - n^2)|^2} \right)^{1/2} \| (Id_n - T_n)^{-1} (\mathcal{S}^n V)_{\mathbb{Z}(n)} \| \\
&\leq \frac{1}{2n} \cdot 2 \cdot \| V \|,
\end{aligned}$$

where for the last inequality we used Lemma 2.1 (and its proof).

(v) is proved in the same way as (iv).

(vi) follows from (iv) and (v). $\square$

3. PROOF OF THEOREM 1

To prove Theorem 1, we follow the same scheme used in [BKM1, section 2]. Recall that the map Λ (cf. [BBGK]) is constructed in two steps. First let us consider the map $\Phi : L_0^2(S^1) \rightarrow l^2(\mathbb{R}^2)$, introduced and analyzed in [Ka]. For $V \in L_0^2(S^1)$, denote by E_n the image of the Riesz projector ($n \geq 1$)

$$(3.1) \quad P_n := \frac{1}{2\pi i} \int_{\Gamma_n} (z - (-\frac{d^2}{dx^2} + V))^{-1} dz,$$

where Γ_n is a counterclockwise oriented circle with center $\tau_n(V) := (\lambda_n^+ + \lambda_n^-)/2$ of radius bigger than $\frac{\gamma_n}{2} = \frac{\lambda_n^+ - \lambda_n^-}{2}$, but sufficiently small so that all eigenvalues different from λ_n^+ , λ_n^- are outside of Γ_n . Here, for convenience, we set $\lambda_n^+ \equiv \lambda_{2n}$ and $\lambda_n^- \equiv \lambda_{2n-1}$.

We choose in E_n a basis

$$\begin{aligned}
G_{2n-1}(x) &\equiv G_n^-(x) = \sum_{k \in \mathbb{Z}} \hat{G}_n^-(k) e^{ik\pi x}, \\
G_{2n}(x) &\equiv G_n^+(x) = \sum_{k \in \mathbb{Z}} \hat{G}_n^+(k) e^{ik\pi x}
\end{aligned}$$

normalized as follows (the normalization conditions are written in such a way that they remain unchanged if we consider small complex valued perturbations of V):

$$(3.2) \quad \sum_{k \in \mathbb{Z}} \hat{G}_n^\pm(k) \hat{G}_n^\pm(-k) = 1,$$

$$(3.3) \quad 0 = G_n^-(0) = \sum_{k \in \mathbb{Z}} \hat{G}_n^-(k),$$

$$(3.4) \quad 0 = \langle G_n^-, G_n^+ \rangle = \sum_{k \in \mathbb{Z}} \hat{G}_n^-(k) \hat{G}_n^+(-k).$$

The signs of G_n^- and G_n^+ are determined in such a way that (with $' = \frac{d}{dx}$)

$$(3.5) \quad \operatorname{Re}((G_n^-)'(0)) > 0$$

$$(3.6) \quad \operatorname{Re} \left(\det \begin{pmatrix} G_n^+(0) & G_n^-(0) \\ (G_n^+)'(0) & (G_n^-)'(0) \end{pmatrix} \right) > 0.$$

The map $\Phi(V) := (\Phi_n(V))_{n \geq 1}$ is then defined by

$$(3.7) \quad \Phi_n(V) := \begin{pmatrix} \int_{S^1} G_n^+(x) \left(-\frac{d^2}{dx^2} + V - \tau_n\right) G_n^+(x) dx \\ \int_{S^1} G_n^-(x) \left(-\frac{d^2}{dx^2} + V - \tau_n\right) G_n^-(x) dx \end{pmatrix}.$$

According to [Ka] (cf. also [BBGK, section 4]), $\Phi : L_0^2(S^1) \rightarrow l^2(\mathbb{R}^2)$ is real analytic. Denote by $\Phi^{(N,\omega)}$ the restriction $\Phi^{(N,\omega)} := \Phi|_{H_0^{N,\omega}(S^1)}$. Note that Theorem 2 implies that $\Phi^{(N,\omega)}(H_0^{N,\omega}(S^1)) \subseteq l_{N,\omega}^2(\mathbb{R}^2)$.

Since $\Phi : L_0^2(S^1) \rightarrow l^2(\mathbb{R}^2)$ is real analytic, for any $V_0 \in L_0^2(S^1)$, there exists a neighborhood $\mathcal{U}$ of V_0 in $L_0^2(S^1; \mathbb{C})$ so that Φ can be extended to an analytic map, $\Phi : \mathcal{U} \rightarrow l^2(\mathbb{N}; \mathbb{C}^2)$. Thus $\Phi^{(N,\omega)}$ extends to a map on $\mathcal{U} \cap H_0^{N,\omega}(S^1; \mathbb{C})$.

Proposition 3.1. *Assume $V_0 \in H_0^{N,\omega}(S^1; \mathbb{R})$ ($N \in \mathbb{Z}_{\geq 0}, \omega \in \mathbb{R}_{\geq 0}$).*

(i) *Then there exist a neighborhood $\mathcal{U}$ of V_0 in $H_0^{N,\omega}(S^1; \mathbb{C})$ and $1 \leq C < \infty$ such that Φ_n is analytic on $\mathcal{U}$ ($n \geq 1$) and, for any $V \in \mathcal{U}$,*

$$\sum_{n \geq 1} (1+n)^{2N+2} e^{2n\omega} \left\| \Phi_n(V) - \begin{pmatrix} \frac{\hat{V}(2n) + \hat{V}(-2n)}{2} \\ \frac{\hat{V}(2n) - \hat{V}(-2n)}{2i} \end{pmatrix} \right\|^2 \leq C.$$

(ii) $\Phi^{(N,\omega)} : H_0^{N,\omega}(S^1) \rightarrow l_{N,\omega}^2(\mathbb{R}^2)$ *is real analytic.*

Proof. (ii) From (i) we conclude that $\Phi^{(N,\omega)}(\mathcal{U})$ is a bounded subset of $l_{N,\omega}^2(\mathbb{N}; \mathbb{C}^2)$. Moreover, Φ_n is analytic on $\mathcal{U}$ for any $n \geq 1$. As $V_0 \in H_0^{N,\omega}(S^1)$ is arbitrary, this implies that $\Phi^{(N,\omega)} : H_0^{N,\omega}(S^1) \rightarrow l_{N,\omega}^2(\mathbb{R}^2)$ is real analytic (cf. [PT, Appendix A]).

(i) As $\Phi^{(N,\omega)}$ is the restriction of Φ and Φ is locally bounded, it suffices to find $\mathcal{U}, C, n_3 \geq 1$ such that for $V \in \mathcal{U}$

$$(3.8) \quad \sum_{n \geq n_3} (1+n)^{2N+2} e^{2n\omega} \left\| \Phi_n(V) - \begin{pmatrix} (\hat{V}(2n) + \hat{V}(-2n))/2 \\ (\hat{V}(2n) - \hat{V}(-2n))/2i \end{pmatrix} \right\|^2 \leq C.$$

To prove (3.8) we consider the cases where $\lambda_n^+ = \lambda_n^-$ and $\lambda_n^+ \neq \lambda_n^-$ separately. The statement follows from Lemma 3.2 and Lemma 3.3 below.

Let us first treat the case where $\lambda_n^+ = \lambda_n^-$ is a double eigenvalue of $-\frac{d^2}{dx^2} + V$ for $V \in H_0^{N,\omega}(S^1, \mathbb{C})$. Let $\mathcal{M} = \{n \geq 2n_2 \mid \lambda_n^+ = \lambda_n^-\}$, where n_2 is given by (2.26). For $n \in \mathcal{M}$ we have $z_n^+ = z_n^-$, and therefore

$$(3.9) \quad \zeta_n^+ = \zeta_n^-.$$

Together with $(\zeta_n^\pm)^2 = A$, where $A = (\hat{V}(2n) + \beta(n, z_n^+))(\hat{V}(-2n) + \beta(-n, z_n^+))$, it follows that

$$(3.10) \quad \zeta_n^+ = \zeta_n^- = 0.$$

For $n \in \mathcal{M}$, G_n^+ and G_n^- are of the form

$$(3.11) \quad G_n^\pm = x_n^\pm e^{-in\pi x} + y_n^\pm e^{in\pi x} + \sum_{k \neq \pm n} a_{x_n^\pm, y_n^\pm, z_n^\pm}(k) e^{ik\pi x},$$

$$(3.12) \quad G_n^- = x_n^- e^{-in\pi x} + y_n^- e^{in\pi x} + \sum_{k \neq \pm n} a_{x_n^-, y_n^-, z_n^+}(k) e^{ik\pi x},$$

where $a_{x_n^-, y_n^-, z_n^+}$ is given by (2.56).

The normalization condition (3.3) leads to

$$(3.13) \quad \begin{aligned} & x_n^- (1 + \sum_{k \neq \pm n} (z_n^+ - B_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)}(k)) \\ & + y_n^- \left(1 + \sum_{k \neq \pm n} (z_n^+ - B_n)^{-1} (\mathcal{S}^n \hat{V})_{\mathbb{Z}(n)}(k) \right) = 0, \end{aligned}$$

which, in view of Proposition 2.11, (iv), (v), yields, as $n \geq 2n_2$ for $n \in \mathcal{M}$,

$$(3.14) \quad \begin{aligned} |x_n^-| & \leq |y_n^-| (1 + \frac{1}{n} \|V\|) \frac{1}{1 - \frac{1}{n} \|V\|} \\ & \leq |y_n^-| (1 + \frac{4\|V\|}{n}) \leq 3|y_n^-|, \end{aligned}$$

and, similary,

$$(3.15) \quad |y_n^-| \leq |x_n^-| (1 + \frac{4\|V\|}{n}) \leq 3|x_n^-|.$$

Using (3.14) and (3.15), one obtains from the normalization condition (3.3) the estimate

$$\begin{aligned} \frac{2}{3}|x_n^-|^2 & \leq 2|x_n^- y_n^-| = |1 - \sum_{k \neq \pm n} a_{x_n^-, y_n^-, z_n^+}(k) a_{x_n^-, y_n^-, z_n^+}(-k)| \\ & \leq 1 + \|a_{x_n^-, y_n^-, z_n^+}\|^2 \leq 1 + \frac{|x_n^-| + |y_n^-|}{2} \frac{\|V\|}{n} \leq 1 + 2|x_n^-| \frac{\|V\|}{n}. \end{aligned}$$

It follows that for $n \geq 3n_2$,

$$(3.16) \quad |x_n^-| \leq \sqrt{\frac{16}{10}} + \frac{2}{3} \frac{\|V\|}{n} \leq 2.$$

Similarly, one can show that for $n \geq 3n_2$

$$(3.17) \quad |y_n^-| \leq \sqrt{\frac{16}{10}} + \frac{2}{3} \frac{\|V\|}{n} \leq 2.$$

Therefore, (3.13) leads to, using Proposition 2.11 (vi),

$$(3.18) \quad |x_n^- + y_n^-| \leq \frac{4}{n} \|V\|,$$

and (3.2) implies, for $n \geq 3n_2$, using Proposition 2.11 (iii),

$$(3.19) \quad |2x_n^- y_n^- - 1| \leq \left(\frac{4\|V\|}{n} \right)^2 \leq \frac{4\|V\|}{n}.$$

Estimates (3.18) and (3.19) imply, with (3.5), $n \in \mathcal{M}$, $n \geq 3n_2$,

$$(3.20) \quad |x_n^- - \frac{i}{\sqrt{2}}| \leq \frac{10\|V\|}{n}; \quad |y_n^- + \frac{i}{\sqrt{2}}| \leq \frac{10\|V\|}{n}.$$

Using (3.2) and (3.4) and the above estimates, we conclude, by similar computations, that, for $n \in \mathcal{M}$ with $n \geq 20n_2$,

$$(3.21) \quad \begin{aligned} |x_n^+| &\leq 3, & |x_n^+ - \frac{1}{\sqrt{2}}| &\leq \frac{200 \|V\|}{n}; \\ |y_n^+| &\leq 3, & |y_n^+ - \frac{1}{\sqrt{2}}| &\leq \frac{200 \|V\|}{n}. \end{aligned}$$

For $n \in \mathcal{M}$, $n \geq 20n_2$, one obtains

$$\begin{aligned} \left(-\frac{d^2}{dx^2} + V - \tau_n\right) G_n^+ &= (\lambda_n^+ - \tau_n) G_n^+ + (\hat{V}(-2n) + \beta(-n, z_n^+)) y_n^+ e^{-in\pi x} \\ &\quad + (\hat{V}(2n) + \beta(n, z_n^+)) x_n^+ e^{in\pi x}, \end{aligned}$$

which leads to

$$\begin{aligned} \langle G_n^+, \left(-\frac{d^2}{dx^2} + V - \tau_n\right) G_n^+ \rangle &= (\hat{V}(2n) + \hat{V}(-2n)) x_n^+ y_n^+ + x_n^+ y_n^+ (\beta(n, z_n^+) + \beta(-n, z_n^+)) \\ &= \frac{1}{2} (\hat{V}(2n) + \hat{V}(-2n)) + l_{N+1, \omega}^2(n) \end{aligned}$$

and

$$\begin{aligned} \langle G_n^-, \left(-\frac{d^2}{dx^2} + V - \tau_n\right) G_n^+ \rangle &= x_n^- y_n^+ \hat{V}(-2n) + y_n^- x_n^+ \hat{V}(2n) + x_n^- y_n^+ \beta(-n, z_n^+) + y_n^- x_n^+ \beta(n, z_n^+) \\ &= \frac{i}{2} (\hat{V}(-2n) - \hat{V}(2n)) + l_{N+1, \omega}^2(n), \end{aligned}$$

where, by abuse of notation, we mean by $(l_{N+1, \omega}^2(n))_{n \geq 1}$ an element in $l_{N+1, \omega}^2$ uniformly bounded for $V \in \mathcal{U}$, where $\mathcal{U}$ is a sufficiently small neighborhood of $V_0 \in H_0^{N, \omega}(S^1, \mathbb{R})$ in $H_0^{N, \omega}(S^1; \mathbb{C})$. Using the above estimates, one obtains

Lemma 3.2. *Assume that $V_0 \in H_0^{N, \omega}(S^1, \mathbb{R})$. Then there exist a neighborhood $\mathcal{U}$ of V_0 in $H^{N, \omega}(S^1; \mathbb{C})$, $n_3 \geq n_2$ and $1 \leq C < \infty$ so that for $V \in \mathcal{U}$,*

$$\sum_{\substack{n \geq n_3 \\ \lambda_n^+ = \lambda_n^-}} (1+n)^{2N+2} e^{2n\omega} \|\Phi_n(V) - \frac{(\hat{V}(2n) + \hat{V}(-2n))/2}{(\hat{V}(2n) - \hat{V}(-2n))/2i}\|^2 \leq C.$$

Let us now consider those $n \geq n_2$ with $\lambda_n^+ \neq \lambda_n^-$. Denote by $f_n^\pm$ eigenfunctions of $\lambda_n^\pm$ normalized so that

$$\sum_{k \in \mathbb{Z}} \hat{f}_n^\pm(k)^2 = 1.$$

Then G_n^+ and G_n^- are linear combinations of f_n^+ and f_n^- ,

$$(3.22) \quad \begin{aligned} G_n^+ &= \alpha_n^+ f_n^+ + \alpha_n^- f_n^- \\ &= x_n^+ e^{-in\pi x} + y_n^+ e^{in\pi x} + \sum_{k \neq \pm n} b_n^+(k) e^{ik\pi x} \end{aligned}$$

with

$$\begin{aligned} b_n^+ &:= \alpha_n^+ \hat{f}_n^+(-n)(z_n^+ - B_n)^{-1} \mathcal{S}^n \hat{V} + \alpha_n^- \hat{f}_n^-(-n)(z_n^- - B_n)^{-1} \mathcal{S}^n \hat{V} \\ &\quad + \alpha_n^+ \hat{f}_n^+(n)(z_n^+ - B_n)^{-1} \mathcal{S}^{-n} \hat{V} + \alpha_n^- \hat{f}_n^-(n)(z_n^- - B_n)^{-1} \mathcal{S}^{-n} \hat{V} \end{aligned}$$

and

$$\begin{aligned} (3.23) \quad G_n^- &= \beta_n^+ f_n^+ + \beta_n^- f_n^- \\ &= x_n^- e^{-in\pi x} + y_n^- e^{in\pi x} + \sum_{k \neq \pm n} b_n^-(k) e^{ik\pi x} \end{aligned}$$

with b_n^- given by an expression similar to b_n^+ .

For $V_0 \in L_0^2(S^1; \mathbb{R})$ there exist a neighborhood $\mathcal{U}$ of V_0 in $L_0^2(S^1; \mathbb{C})$ and $n_3 \geq n_2$ so that for $n \geq n_3$ (cf. [BBGK, Lemma 4.10], [Ka, Proposition 8])

$$(3.24^+) \quad x_n^+ = \frac{1}{\sqrt{2}} + O\left(\frac{1}{n}\right), \quad y_n^+ = \frac{1}{\sqrt{2}} + O\left(\frac{1}{n}\right),$$

$$(3.24^-) \quad x_n^- = \frac{i}{\sqrt{2}} + O\left(\frac{1}{n}\right), \quad y_n^- = -\frac{i}{\sqrt{2}} + O\left(\frac{1}{n}\right),$$

$$(3.25^+) \quad f_n^+(x) = \frac{e^{-i\theta_n}}{\sqrt{2}} e^{-in\pi x} + \frac{e^{i\theta_n}}{\sqrt{2}} e^{in\pi x} + O\left(\frac{1}{n}\right),$$

$$(3.25^-) \quad f_n^-(x) = i \frac{e^{-i\theta_n}}{\sqrt{2}} e^{-in\pi x} - i \frac{e^{i\theta_n}}{\sqrt{2}} e^{in\pi x} + O\left(\frac{1}{n}\right).$$

As $\lambda_n^+ \neq \lambda_n^-$, the normalization conditions for $G_n^\pm$ imply

$$(\alpha_n^+)^2 + (\alpha_n^-)^2 = 1, \quad (\beta_n^+)^2 + (\beta_n^-)^2 = 1, \quad \alpha_n^+ \beta_n^+ + \alpha_n^- \beta_n^- = 0,$$

which leads to

$$(3.26) \quad \alpha_n^+ = \frac{e^{i\theta_n} + e^{-i\theta_n}}{2} + O\left(\frac{1}{n}\right), \quad \alpha_n^- = \frac{e^{i\theta_n} - e^{-i\theta_n}}{2i} + O\left(\frac{1}{n}\right),$$

$$(3.27) \quad \beta_n^+ = -\frac{e^{i\theta_n} - e^{-i\theta_n}}{2i} + O\left(\frac{1}{n}\right), \quad \beta_n^- = \frac{e^{i\theta_n} + e^{-i\theta_n}}{2} + O\left(\frac{1}{n}\right).$$

When written in Fourier space, $(-\frac{d^2}{dx^2} + V - \tau_n)G_n^+$ takes the form (cf. (2.3))

$$(3.28) \quad \begin{pmatrix} -x_n^+(z_n^+ + z_n^-)/2 + y_n^+ \hat{V}(-2n) + \langle \mathcal{S}^n \mathcal{J} \hat{V}, b_n^+ \rangle \\ x_n^+ \hat{V}(2n) - y_n^+(z_n^+ + z_n^-)/2 + \langle \mathcal{S}^{-n} \mathcal{J} \hat{V}, b_n^+ \rangle \\ x_n^+ \mathcal{S}^n \hat{V} + y_n^+ \mathcal{S}^{-n} \hat{V} + (B_n - (z_n^+ + z_n^-)/2) b_n^+ \end{pmatrix}.$$

The terms appearing in this expression are discussed separately. Since $\alpha(n, z) = \alpha(-n, z)$, one obtains

$$\begin{aligned}
 \langle S^n \mathcal{J}\hat{V}, b_n^+ \rangle &= \alpha_n^+ \hat{f}_n^+(-n) \alpha(n, z_n^+) + \alpha_n^- \hat{f}_n^-(-n) \alpha(n, z_n^-) \\
 &\quad + \alpha_n^+ \hat{f}_n^+(n) \beta(-n, z_n^+) + \alpha_n^- \hat{f}_n^-(n) \beta(-n, z_n^-) \\
 (3.29) \quad &= x_n^+ \frac{\alpha(n, z_n^+) + \alpha(n, z_n^-)}{2} + x_n^+ \frac{\alpha(n, z_n^+) - \alpha(n, z_n^-)}{2} \\
 &\quad + \alpha_n^- \hat{f}_n^-(-n) (\alpha(n, z_n^+) - \alpha(n, z_n^-)) \\
 &\quad + \alpha_n^+ \hat{f}_n^+(n) \beta(-n, z_n^+) + \alpha_n^- \hat{f}_n^-(n) \beta(-n, z_n^-).
 \end{aligned}$$

Taking into account that $-\zeta_n^\pm = -z_n^\pm + \alpha(n, z_n^\pm)$ and thus

$$-\frac{z_n^+ + z_n^-}{2} + \frac{\alpha(n, z_n^+) + \alpha(n, z_n^-)}{2} = -\frac{\zeta_n^+ + \zeta_n^-}{2},$$

the first component in (3.28) can be written as

$$(3.30) \quad -x_n^+ \left(\frac{z_n^+ + z_n^-}{2} \right) + y_n^+ \hat{V}(-2n) + \langle S^n \mathcal{J}\hat{V}, b_n^+ \rangle = \hat{V}(-2n) y_n^+ + g_n^+(-n),$$

where

$$\begin{aligned}
 (3.31) \quad g_n^+(-n) &= (x_n^+ + \alpha_n^- \hat{f}_n^-(-n)) (\alpha(n, z_n^+) - \alpha(n, z_n^-)) - x_n^+ \frac{\zeta_n^+ + \zeta_n^-}{2} \\
 &\quad + \alpha_n^+ \hat{f}_n^+(n) \beta(-n, z_n^+) + \alpha_n^- \hat{f}_n^-(n) \beta(-n, z_n^-).
 \end{aligned}$$

In view of (3.24)–(3.27), Lemma 2.4(ii), Proposition 2.7 and Theorem 2.10 we conclude that $g_n^+(-n) = l_{N+1, \omega}^2(n)$, uniformly bounded for V in a sufficiently small neighborhood $\mathcal{U}$ of V_0 in $H_0^{N, \omega}(S^1, \mathbb{C})$.

The second component in (3.28) is analyzed similarly, to yield

$$(3.32) \quad x_n^+ \hat{V}(2n) - y_n^+ (z_n^+ + z_n^-)/2 + \langle S^{-n} \mathcal{J}\hat{V}, b_n^+ \rangle = \hat{V}(2n) x_n^+ + g_n^+(n),$$

where

$$\begin{aligned}
 (3.33) \quad g_n^+(n) &= (-y_n^+ + \alpha_n^+ \hat{f}_n^+(n)) \frac{\alpha(n, z_n^+) - \alpha(n, z_n^-)}{2} - y_n^+ \frac{\zeta_n^+ + \zeta_n^-}{2} \\
 &\quad + \alpha_n^+ \hat{f}_n^+(-n) \beta(n, z_n^+) + \alpha_n^- \hat{f}_n^-(-n) \beta(n, z_n^-),
 \end{aligned}$$

and we again conclude that $g_n^+(-n) = l_{N+1, \omega}^2(n)$.

Finally we analyze the third component in (3.28):

We compute (with $\gamma_n := \lambda_n^+ - \lambda_n^-$)

$$\begin{aligned}
(B_n - \frac{z_n^+ + z_n^-}{2})b_n^+ &= -(\alpha_n^+ \hat{f}_n^+(-n) + \alpha_n^- \hat{f}_n^-(-n))\mathcal{S}^n \hat{V} \\
&\quad - (\alpha_n^+ \hat{f}_n^+(n) + \alpha_n^- \hat{f}_n^-(-n))\mathcal{S}^{-n} \hat{V} \\
&\quad + \alpha_n^+ \hat{f}_n^+(-n) \frac{\gamma_n}{2} (z_n^+ - B_n)^{-1} \mathcal{S}^n \hat{V} \\
&\quad + \alpha_n^- \hat{f}_n^-(-n) \frac{\gamma_n}{2} (z_n^- - B_n)^{-1} \mathcal{S}^n \hat{V} \\
&\quad + \alpha_n^+ \hat{f}_n^+(n) \frac{\gamma_n}{2} (z_n^+ - B_n)^{-1} \mathcal{S}^{-n} \hat{V} \\
&\quad + \alpha_n^- \hat{f}_n^-(n) \frac{\gamma_n}{2} (z_n^- - B_n)^{-1} \mathcal{S}^{-n} \hat{V}.
\end{aligned}$$

Thus

$$\begin{aligned}
(3.34) \quad x_n^+ \mathcal{S}^n \hat{V} + y_n^+ \mathcal{S}^{-n} \hat{V} + (B_n - \frac{z_n^+ + z_n^-}{2})b_n^+ \\
= \alpha_n^+ \hat{f}_n^+(-n) \frac{\gamma_n}{2} (z_n^+ - B_n)^{-1} \mathcal{S}^n \hat{V} \\
+ \alpha_n^- \hat{f}_n^-(-n) \frac{\gamma_n}{2} (z_n^- - B_n)^{-1} \mathcal{S}^n \hat{V} \\
+ \alpha_n^+ \hat{f}_n^+(n) \frac{\gamma_n}{2} (z_n^+ - B_n)^{-1} \mathcal{S}^{-n} \hat{V} \\
+ \alpha_n^- \hat{f}_n^-(n) \frac{\gamma_n}{2} (z_n^- - B_n)^{-1} \mathcal{S}^{-n} \hat{V} \\
= l_{N+1,\omega}^2(n).
\end{aligned}$$

Combining (3.30)-(3.34), we obtain, for $n \geq n_3$ with $\lambda_n^+ \neq \lambda_n^-$,

$$\langle G_n^+, (-\frac{d^2}{dx^2} + V - \tau_n)G_n^+ \rangle = \hat{V}(-2n)(y_n^+)^2 + \hat{V}(2n)(x_n^+)^2 + l_{N+1,\omega}^2(n),$$

$$\langle G_n^-, (-\frac{d^2}{dx^2} + V - \tau_n)G_n^+ \rangle = \hat{V}(-2n)x_n^- y_n^+ + \hat{V}(2n)y_n^- x_n^+ + l_{N+1,\omega}^2(n).$$

In view of (3.24 $^\pm$) one then concludes the following:

Lemma 3.3. *Assume $V_0 \in H^{N,\omega}(S^1; \mathbb{R})$. Then there exist a neighborhood $\mathcal{U}$ of V_0 in $H^{N,\omega}(S^1; \mathbb{C})$, $n_3 \geq n_2$ and $1 \leq C < \infty$ such that for any $V \in \mathcal{U}$ and $n \geq n_3$*

$$\sum_{\substack{n \geq n_3 \\ \lambda_n^+ \neq \lambda_n^-}} (1+n)^{2N+2} e^{2n\omega} \left\| \Phi_n(V) - \begin{pmatrix} \frac{\hat{V}(2n) + \hat{V}(-2n)}{2} \\ \frac{\hat{V}(2n) - \hat{V}(-2n)}{2i} \end{pmatrix} \right\|^2 \leq C.$$

Proposition 3.4. *The map $\Phi^{(N,\omega)} : H_0^{N,\omega}(S^1) \rightarrow l_{N,\omega}^2(\mathbb{R}^2)$ has the following properties:*

- (i) $\Phi^{(N,\omega)}$ is bijective and real analytic;

(ii) $(\Phi^{(N,\omega)})^{-1}$ is real analytic;

Proof. As $\Phi|_{H_0^{N,\omega}(S^1)} = \Phi|_{H_0^{N,\omega}(S^1)}$ and $\Phi : L_0^2(S^1) \rightarrow l^2(\mathbb{R}^2)$ is bijective and bianalytic, we conclude that $\Phi^{(N,\omega)}$ is one-to-one and, for $V \in H_0^N(S^1)$, $d_V \Phi^{(N,\omega)}$ is also one-to-one. From Corollary 3, stated in the introduction, and the fact that Φ is onto we conclude that $\Phi^{(N,\omega)}$ is onto. By Proposition 3.1, $\Phi^{(N,\omega)}$ is real analytic. To prove statement (ii), it suffices to show that $d_V \Phi^{(N,\omega)}$ is onto for an arbitrary element V in $H_0^{N,\omega}(S^1)$. By (i) and the Fredholm alternative it suffices to prove that $d_V \Phi^{(N,\omega)} = A + K$, where $A : H_0^{N,\omega}(S^1) \rightarrow l_{N,\omega}^2(\mathbb{R}^2)$ is a linear isomorphism and K is a compact operator. This follows from Proposition 3.1. $\square$

Next we consider the map $\Lambda : L_0^2(S^1) \rightarrow l_{1/2}^2(\mathbb{R}^2)$, defined in [BBGK] by $\Lambda(V) = (\Lambda_n(V))_{n \geq 1}$ with $\Lambda_n(V) = \xi_n(V) \Phi_n(V)$. Here

$$\xi_n(V)^2 = \frac{2\mathcal{I}_n(V)}{\gamma_n(V)/2}$$

and $\mathcal{I}_n(V)$ ($n \geq 1$) denote the action variables of KdV with respect to the Gardner bracket ([FM]; cf. [BBGK])

$$\mathcal{I}_n(V) = \frac{2}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \mu \frac{\dot{\Delta}(\mu)}{(\Delta(\mu) - 4)^{1/2}} d\mu.$$

Define $\Lambda^{(N,\omega)} := \Lambda|_{H_0^{N,\omega}(S^1)}$. From the asymptotics, valid uniformly on sets of potentials bounded in $L_0^2(S^1; \mathbb{C})$.

$$(3.35) \quad \xi_n = \frac{1}{\sqrt{n\pi}} \left(1 + O\left(\frac{\log n}{n}\right)\right)$$

(cf. [BBGK]) we conclude that $\Lambda^{(N,\omega)}(H_0^{N,\omega}(S^1)) \subset l_{N+\frac{1}{2},\omega}^2(\mathbb{R}^2)$.

Proposition 3.5. *The map $\Lambda^{(N,\omega)} : H_0^{N,\omega}(S^1) \rightarrow l_{N+\frac{1}{2},\omega}^2(\mathbb{R}^2)$ has the following properties:*

- (i) $\Lambda^{(N,\omega)}$ is bijective and real analytic;
- (ii) $(\Lambda^{(N,\omega)})^{-1}$ is real analytic;
- (iii) for any $0 \leq \varepsilon < 1/2$,

$$\Lambda_n^{(N,\omega)}(V) = \frac{1}{\sqrt{n\pi}} \left(\frac{\hat{V}(2n) + \hat{V}(-2n)}{2}, \frac{\hat{V}(2n) - \hat{V}(-2n)}{2i} \right) + l_{N+1+\varepsilon,\omega}^2(n)$$

uniformly on sets of potentials bounded in $H_0^{N,\omega}(S^1; \mathbb{C})$.

Proof. We argue as in the proof of Proposition 3.4 to conclude from [BBGK, section 2] that $\Lambda^{(N,\omega)}$ is bijective and real analytic and that $d_V \Lambda^{(N,\omega)}$ is one-to-one for any $V \in H_0^{N,\omega}(S^1)$. Statement (iii) follows from (3.35) and Proposition 3.1. To prove statement (ii) it suffices to prove that $d_V \Lambda^{(N,\omega)}$ is onto for any V in $H_0^{N,\omega}(S^1)$.

Write

$$(3.36) \quad d_V \Lambda_n(W) = \xi_n(V) d_V \Phi_n(W) + d_V \xi_n(W) \Phi_n(V)$$

and introduce, for V fixed,

$$A := (A_n)_{n \geq 1} : H_0^{N,\omega}(S^1) \rightarrow l_{N+\frac{1}{2},\omega}^2(\mathbb{R}^2)$$

with $A_n(W) := \frac{1}{\sqrt{n\pi}} d_V \Phi_n(W)$. By Proposition 3.4, A is a linear isomorphism. Moreover, by [BBGK, Lemma 4.18], for any $0 \leq \varepsilon < 1/2$,

$$(3.37) \quad d_V \xi_n(W) = O\left(\frac{1}{n^{1+\varepsilon}}\right) \|W\|_{L^2}.$$

Substituting (3.35) and (3.37) into (3.36), we conclude that, for any $0 \leq \varepsilon < 1/2$, there exists $C_\varepsilon > 0$ such that

$$\| (d_V \Lambda_n(W) - A_n(W))_{n \geq 1} \|_{l_{N+1+\varepsilon, \omega}^2} \leq C_\varepsilon \|W\|_{L^2}.$$

Estimate (3.21) implies that $d_V \Lambda^{(N, \omega)} = A + K$, where K is compact. By the Fredholm alternative and the fact that $d_V \Lambda^{(N, \omega)}$ is one-to-one, we conclude that $d_V \Lambda^{(N, \omega)}$ is onto. This implies statement (ii). $\square$

Proof of Theorem 1. Theorem 1 follows from Proposition 3.5. $\square$

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