TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 351, Number 2, February 1999, Pages 533–545
S 0002-9947(99)02378-8

HOMOGENEOUS PROJECTIVE VARIETIES WITH DEGENERATE SECANTS

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Dedicated to Professor Satoshi Arima on the occasion of his 70th birthday

ABSTRACT. The *secant variety* of a projective variety X in $\mathbb{P}$, denoted by $\operatorname{Sec} X$, is defined to be the closure of the union of lines in $\mathbb{P}$ passing through at least two points of X, and the *secant deficiency* of X is defined by $\delta := 2\dim X + 1 - \dim \operatorname{Sec} X$. We list the homogeneous projective varieties X with $\delta > 0$ under the assumption that X arise from irreducible representations of complex simple algebraic groups. It turns out that there is no homogeneous, non-degenerate, projective variety X with $\operatorname{Sec} X \neq \mathbb{P}$ and $\delta > 8$, and the E_6-variety is the only homogeneous projective variety with largest secant deficiency $\delta = 8$. This gives a negative answer to a problem posed by R. Lazarsfeld and A. Van de Ven if we restrict ourselves to homogeneous projective varieties.

INTRODUCTION

The *secant variety* of a projective variety X in $\mathbb{P}$, denoted by $\operatorname{Sec} X$, is defined to be the closure of the union of lines in $\mathbb{P}$ passing through at least two points of X, and the *secant deficiency* of X is defined by

$$\delta := 2\dim X + 1 - \dim \operatorname{Sec} X.$$

In 1979, F. L. Zak proved a significant inequality,

$$3\dim X + 4 \leq 2\dim \mathbb{P}$$

for a smooth, non-degenerate X with $\operatorname{Sec} X \neq \mathbb{P}$, which had been conjectured by R. Hartshorne [Ht, Conjecture 4.2] (see also [FL], [LV], [Z]). From the viewpoint of Zak's inequality, projective varieties X which attain the equality, namely *Severi varieties*, were studied actively, and Zak finally found that there are exactly four Severi varieties (see [FR], [T], [LV], [Z]): It turns out that those varieties are all homogeneous and have $\delta = 1, 2, 4, 8$. For the extremal case of odd dimensional X, in which $3\dim X + 5 = 2\dim \mathbb{P}$, T. Fujita [F] gave a classification for 3-dimensional X and M. Ohno [O] recently gave classifications for 5-dimensional X and for 7-dimensional X under a certain condition, where those X of dimension 3,5,7 have $\delta = 1, 2, 3$, respectively. Thus several authors have studied projective varieties X with $\delta > 0$.

The purpose of this article is to list the homogeneous projective varieties X with $\delta > 0$ under the assumption that X arise from irreducible representations of complex simple algebraic groups. Zak already obtained a table of those X in case

Received by the editors April 9, 1996.
1991 *Mathematics Subject Classification.* Primary 14M17, 14N05, 17B10, 20G05.

of $2\dim X \geq \dim \mathbb{P}$. But we work without any dimensional condition. Although we as well as Zak need another step to investigate which X has $\operatorname{Sec} X \neq \mathbb{P}$, our strategy to pick up the candidates of X with $\delta > 0$ (not necessarily $\operatorname{Sec} X \neq \mathbb{P}$) is different and quite simple, as we will see below.

Let G be a complex simple algebraic group with Lie algebra $\mathfrak{g}$, let R be the root system of $\mathfrak{g}$, and fix a base Δ of R. Let λ be a dominant weight of $\mathfrak{g}$ with respect to Δ, $\rho : G \to GL(V)$ an irreducible, finite-dimensional representation of G with highest weight λ, and v_λ a maximal vector in V with weight λ. In this article we discuss projective varieties X in $\mathbb{P}_*(V)$ which is an orbit of the subspace spanned by v_λ under the action of G, where $\mathbb{P}_*(V)$ denotes the 1-dimensional subspaces of V. Denote by ω_i the i-th fundamental weight as in [B].

The result is

Theorem. *X in $\mathbb{P}_*(V)$ has $\delta > 0$ if and only if the type of $\mathfrak{g}$ and λ is one of the following:*
(A_1) $\omega_1; 2\omega_1$
(A_2) $\omega_1, \omega_2; 2\omega_1, 2\omega_2; \omega_1 + \omega_2$
(A_3) $\omega_1, \omega_3; \omega_2; 2\omega_1, 2\omega_3; \omega_1 + \omega_3$
$(A_{l\geq 4})$ $\omega_1, \omega_l; \omega_2, \omega_{l-1}; 2\omega_1, 2\omega_l; \omega_1 + \omega_l$
(B_2) $\omega_1; \omega_2; 2\omega_2$
$(B_{l=3,4})$ $\omega_1; \omega_2; \omega_l$
$(B_{l\geq 5})$ $\omega_1; \omega_2$
$(C_{l\geq 3})$ $\omega_1; \omega_2; 2\omega_2$
$(D_{l=4,5})$ $\omega_1; \omega_2; \omega_{l-1}, \omega_l$
$(D_{l\geq 6})$ $\omega_1; \omega_2$
(E_6) $\omega_1, \omega_6; \omega_2$
(E_7) ω_1
(E_8) ω_8
(F_4) $\omega_1; \omega_4$
(G_2) $\omega_1; \omega_2$

From this result one obtains the following table of homogeneous projective varieties with degenerate secants (see, for details, §3).

The only-if-part is the main contribution of this work, while the if-part follows from well-known facts, results of Zak, and a recent result of M. Ohno, O. Yasukura and the author (see §3). Denote by $\widetilde{\alpha}$ the highest root of $\mathfrak{g}$, by μ the lowest weight of ρ, and by $(*, *)$ the inner product defined by the Killing form. The key to prove the only-if-part is a simple

Criterion.

$$(\lambda - \mu, \lambda - \widetilde{\alpha}) > 0 \Rightarrow \delta = 0.$$

It turns out, after proving the Theorem, that the converse is also true.

Using a result of Zak [Z, III, Corollary 1.7], we obtain from our table the following results for arbitrary homogeneous projective varieties X such that G is not necessarily simple. The first yields a partial answer to a problem posed by R. Lazarsfeld and A. Van de Ven [LV, §1f, Problem]:

Corollary 1. *There is no homogeneous, non-degenerate, projective variety X with $\operatorname{Sec} X \neq \mathbb{P}$ and $\delta > 8$. Furthermore, the E_6-variety is the only homogeneous projective variety with largest secant deficiency $\delta = 8$.*

Table of homogeneous projective varieties with degenerate secants

type	weight λ	representation	δ	X	$\dim \mathbb{P}+1$	$\operatorname{Sec} X \subseteq \mathbb{P}$	$-\varepsilon$
A_1	ω_1	standard	2	$\mathbb{P}^1$	2	$=$	1/4
	$2\omega_1$	2nd symm.	1	$v_2(\mathbb{P}^1)$	3	$=$	0
A_2	ω_1, ω_2	standard	3	$\mathbb{P}^2$	3	$=$	1/6
	$2\omega_1, 2\omega_2$	2nd symm.	1	$v_2(\mathbb{P}^2)$	6	$\neq$	0
	$\omega_1+\omega_2$	adjoint	1	$\mathbb{P}(T_{\mathbb{P}^2}) = \mathbb{P}^2\times\mathbb{P}^2\cap(1)$	8	$\neq$	0
A_3	ω_1, ω_3	standard	4	$\mathbb{P}^3$	4	$=$	1/8
	ω_2	2nd ext.	4	$\mathbb{G}(2,4)$	6	$=$	0
	$2\omega_1, 2\omega_3$	2nd symm.	1	$v_2(\mathbb{P}^3)$	10	$\neq$	0
	$\omega_1+\omega_3$	adjoint	1	$\mathbb{P}(T_{\mathbb{P}^3}) = \mathbb{P}^3\times\mathbb{P}^3\cap(1)$	15	$\neq$	0
$A_{l\geq 4}$	ω_1, ω_l	standard	$l+1$	$\mathbb{P}^l$	$l+1$	$=$	$1/2(l+1)$
	ω_2, ω_{l-1}	2nd ext.	4	$\mathbb{G}(2,l+1)$	$(l+1)l/2$	$\neq$ iff $l\geq 5$	0
	$2\omega_1, 2\omega_l$	2nd symm.	1	$v_2(\mathbb{P}^l)$	$(l+2)(l+1)/2$	$\neq$	0
	$\omega_1+\omega_l$	adjoint	1	$\mathbb{P}(T_{\mathbb{P}^l}) = \mathbb{P}^l\times\mathbb{P}^l\cap(1)$	$(l+1)^2-1$	$\neq$	0
B_2	ω_1	standard	3	Q^3	5	$=$	0
	ω_2	spin	4	$S_2=\mathbb{P}^3$	4	$=$	1/6
	$2\omega_2$	adjoint	1	$\mathbb{F}_1(Q^3)=v_2(\mathbb{P}^3)$	10	$\neq$	0
$B_{l=3,4}$	ω_1	standard	$2l-1$	Q^{2l-1}	$2l+1$	$=$	0
	ω_l	spin	6	S_l	2^l	$=$	1/20, 0
	ω_2	adjoint	1	$\mathbb{F}_1(Q^{2l-1})$	$2l^2+l$	$\neq$	0
$B_{l\geq 5}$	ω_1	standard	$2l-1$	Q^{2l-1}	$2l+1$	$=$	0
	ω_2	adjoint	1	$\mathbb{F}_1(Q^{2l-1})$	$2l^2+l$	$\neq$	0
$C_{l\geq 3}$	ω_1	standard	$2l$	$\mathbb{P}^{2l-1}$	$2l$	$=$	$1/2(l+1)$
	ω_2	2nd ext.	3	$\mathbb{G}(2,2l)\cap(1)$	$2l^2-l-1$	$\neq$	0
	$2\omega_1$	adjoint	1	$v_2(\mathbb{P}^{2l-1})$	$2l^2+l$	$\neq$	0
$D_{l=4,5}$	ω_1	standard	$2l-2$	Q^{2l-2}	$2l$	$=$	0
	ω_{l-1}, ω_l	half-spin	6	S_{l-1}	2^{l-1}	$=$	0
	ω_2	adjoint	1	$\mathbb{F}_1(Q^{2l-2})$	$2l^2-l$	$\neq$	0
$D_{l\geq 6}$	ω_1	standard	$2l-2$	Q^{2l-2}	$2l$	$=$	0
	ω_2	adjoint	1	$\mathbb{F}_1(Q^{2l-2})$	$2l^2-l$	$\neq$	0
E_6	ω_1, ω_6		8	X^{16}	27	$\neq$	0
	ω_2	adjoint	1	X^{20+1}	78	$\neq$	0
E_7	ω_1	adjoint	1	X^{32+1}	133	$\neq$	0
E_8	ω_8	adjoint	1	X^{56+1}	248	$\neq$	0
F_4	ω_4		7	$X^{16}\cap(1)$	26	$\neq$	0
	ω_1	adjoint	1	X^{14+1}	52	$\neq$	0
G_2	ω_1	"standard"	5	Q^5	7	$=$	1/12
	ω_2	adjoint	1	X^{4+1}	14	$\neq$	0

NOTATION: v_2 denotes the Veronese embedding, Q^n a quadric hypersurface of dimension n, $\mathbb{G}(k,m)$ the Grassmann variety of k-dimensional subspaces of an m-dimensional vector space, $\mathbb{F}_m(Q^n)$ the Fano variety of m-planes in Q^n, S_k the spinor variety, that is, an irreducible component of $\mathbb{F}_k(Q^{2k})$ embedded via a "square root" of the Plücker embedding, $\cap(1)$ cutting by a general hyperplane, and $\varepsilon := (\lambda-\mu, \lambda-\widetilde{\alpha})$.

The second is

Corollary 2 (Cf. [R]). *Let X be a homogeneous, non-degenerate, projective variety in $\mathbb{P}^N$, and let v_d be the d-uple embedding of $\mathbb{P}^N$. If $d\geq 2$ and $X\neq\mathbb{P}^N$, then $v_d(X)$ has non-degenerate secants.*

(see Acknowledgements)

1. A Proof of the Criterion

The criterion follows from two lemmas below.

Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G, denote by $\mathfrak{h}_{\mathbb{R}}^*$ the real vector space spanned by the roots R in the dual space $\mathfrak{h}^*$. By means of the Killing form on $\mathfrak{g}$, one can consider $\mathfrak{h}_{\mathbb{R}}^*$ as an Euclidean space with inner product $(*, *)$ such that the action of the Weyl group on $\mathfrak{h}_{\mathbb{R}}^*$ is orthogonal. Denote by R^+ the set of positive roots in $\mathfrak{h}_{\mathbb{R}}^*$. Let $\widetilde{\alpha}$ be the highest root of $\mathfrak{g}$, and let μ be the lowest weight of the representation ρ.

Let $\mathcal{W}$ be the Weyl chamber, that is, $\mathcal{W} := \{\omega \in \mathfrak{h}_{\mathbb{R}}^* | \alpha \in R^+ \Rightarrow (\omega, \alpha) \geq 0\}$, and denote by w_0 the involution on $\mathfrak{h}_{\mathbb{R}}^*$ such that $\mathcal{W}$ maps to $-\mathcal{W}$ (see [B, VI, §1, n° 6, Cor. 3]): We have $-\widetilde{\alpha} = w_0(\widetilde{\alpha})$.

For an element α and a subset S of $\mathfrak{h}_{\mathbb{R}}^*$, denote by $\alpha + S$ the set $\{\alpha + \beta \in \mathfrak{h}_{\mathbb{R}}^* | \beta \in S\}$, and by (α, S) the set $\{(\alpha, \beta) \in \mathbb{R} | \beta \in S\}$. For example, $\max(\alpha, S)$ means $\max\{(\alpha, \beta) \in \mathbb{R} | \beta \in S\}$.

Lemma 1.

$$(\lambda - \mu, \lambda - \widetilde{\alpha}) > 0 \Rightarrow (\lambda + R) \cap (\mu + R) = \emptyset.$$

Proof. We have that w_0 is orthogonal, $w_0(\lambda) = \mu$ and $w_0(R) = R$. So it follows that $(\lambda - \mu, R) = -(\lambda - \mu, R)$ and $(\lambda - \mu, \mu) = -(\lambda - \mu, \lambda)$, hence $(\lambda - \mu, \mu + R) = -(\lambda - \mu, \lambda + R)$. Thus,

$$\min(\lambda - \mu, \lambda + R) > 0 \Rightarrow (\mu + R) \cap (\lambda + R) = \emptyset.$$

On the other hand, since $-w_0(\mathcal{W}) = \mathcal{W}$ and $\lambda \in \mathcal{W}$, we have $-\mu = -w_0(\lambda) \in \mathcal{W}$, hence $\lambda - \mu \in \mathcal{W}$. Therefore it follows from the definition of $\mathcal{W}$ that if $\alpha \in R^+$, then $(\lambda - \mu, \alpha) \geq 0$. Hence, $\max(\lambda - \mu, R)$ is attained by the highest root $\widetilde{\alpha}$ (see, for example, [B, VI, §1, n° 8, Proposition 25]), and

$$\min(\lambda - \mu, \lambda + R) = (\lambda - \mu, \lambda - \widetilde{\alpha}).$$

□

Lemma 2.

$$\delta \leq \#\{(\lambda + R) \cap (\mu + R)\}.$$

In particular,

$$(\lambda + R) \cap (\mu + R) = \emptyset \Rightarrow \delta = 0.$$

Proof. According to [LV, p. 14], the deficiency δ in characteristic zero is equal to the dimension of the intersection $\mathfrak{g} \cdot v_\lambda \cap \mathfrak{g} \cdot v_\mu$ in V:

$$\delta = \dim(\mathfrak{g} \cdot v_\lambda \cap \mathfrak{g} \cdot v_\mu),$$

where v_λ, v_μ are weight vectors corresponding to λ and μ, respectively, and $\cdot$ means the action of $\mathfrak{g}$ on V via the differential $d\rho$.

On the other hand, for a root α of $\mathfrak{g}$ we have $\dim \mathfrak{g}_\alpha \cdot v_\lambda \leq 1$ and $\mathfrak{g}_\alpha \cdot v_\lambda \subseteq V_{\lambda+\alpha}$. From the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ we obtain $\mathfrak{g} \cdot v_\lambda = \mathbb{C} \cdot v_\lambda \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \cdot v_\lambda$ in V since $\mathfrak{h} \cdot v_\lambda = \mathbb{C} \cdot v_\lambda$. Hence, we see that if $\dim(\mathfrak{g}_\alpha \cdot v_\lambda \cap \mathfrak{g}_\beta \cdot v_\mu) = 1$, then $\lambda + \alpha = \mu + \beta$. Therefore we have

$$\dim(\mathfrak{g} \cdot v_\lambda \cap \mathfrak{g} \cdot v_\mu) \leq \#\{(\lambda + R) \cap (\mu + R)\}.$$

□

2. Candidates

Proposition. *For a dominant weight λ of a complex simple Lie algebra $\mathfrak{g}$, $(\lambda - \mu, \lambda - \widetilde{\alpha}) \leq 0$ if and only if the type of $\mathfrak{g}$ and λ is one of the weights listed in Theorem.*

To show this proposition, realize $\mathfrak{h}_{\mathbb{R}}^*$ in a real vector space as in [B]. Then for a given $\lambda = \sum_{i=1}^{l} b_i\omega_i$ with non-negative integers b_i, one can compute the coordinates of λ and the corresponding lowest μ in $\mathfrak{h}_{\mathbb{R}}^*$ by virtue of $w_0(\lambda) = \mu$, hence those of $\lambda - \mu$ explicitly, where $\omega_1, \ldots, \omega_l$ are the fundamental weights. One can also compute those of the highest root $\widetilde{\alpha}$. Thus for a weight $\lambda = \sum_{i=1}^{l} b_i\omega_i$, setting

$$\varepsilon := (\lambda - \mu, \lambda - \widetilde{\alpha}),$$

one can write down the value ε in terms of the integers b_i. We compute below the set of non-trivial solutions (b_i) of non-negative integers b_i for an inequality,

$$\varepsilon \leq 0$$

in each type of $\mathfrak{g}$.

We denote by e_i the i-th canonical basis of $\mathbb{R}^m$ with $1 \leq i \leq m$, and consider $\sum_{i=a}^{b}$ void unless $a \leq b$.

Lemma A. *For any λ in case of type A_l with $l \geq 1$, $(\lambda - \mu, \lambda - \widetilde{\alpha}) \leq 0$ if and only if λ is one of the following: $\omega_1, 2\omega_1$ in case of $l = 1$; $\omega_1, \omega_2, 2\omega_1, 2\omega_2, \omega_1 + \omega_2$ in case of $l = 2$; $\omega_1, \omega_2, \omega_3, 2\omega_1, 2\omega_3, \omega_1 + \omega_3$ in case of $l = 3$; $\omega_1, \omega_2, \omega_{l-1}, \omega_l, 2\omega_1, 2\omega_l, \omega_1 + \omega_l$ in case of $l \geq 4$.*

Proof. In this case, $\mathfrak{h}_{\mathbb{R}}^* = \{(x_i) \in \mathbb{R}^{l+1} | \sum_{i=1}^{l+1} x_i = 0\} \subseteq \mathbb{R}^{l+1}$ and $\widetilde{\alpha} = \omega_1 + \omega_l = e_1 - e_{l+1}$. We have

$$\lambda = \sum_{i=1}^{l} b_i \left(\sum_{k=1}^{i} e_k - \frac{i}{l+1} \sum_{j=1}^{l+1} e_j \right).$$

Let W_0 be a linear transformation on $\mathbb{R}^{l+1}$ such that e_i maps to e_{l+2-i} with $1 \leq i \leq l+1$. We see from [B] that the restriction to $\mathfrak{h}_{\mathbb{R}}^*$ of W_0 gives the involution w_0, and we have

$$\lambda - \mu = \sum_{k=1}^{l+1} \left(\sum_{i=k}^{l} b_i - \sum_{j=l+2-k}^{l} b_j \right) e_k.$$

It follows that

$$2(l+1)(\lambda - \mu, \lambda) = \sum_{k=1}^{[\frac{l+1}{2}]} \left(\sum_{i=k}^{l-k+1} b_i \right)^2,$$

$$2(l+1)(\lambda - \mu, \widetilde{\alpha}) = 2 \sum_{k=1}^{l} b_k,$$

and

$$2(l+1)\varepsilon = \left(\sum_{i=1}^{l} b_i - 1 \right)^2 + \sum_{k=2}^{[\frac{l+1}{2}]} \left(\sum_{i=k}^{l-k+1} b_i \right)^2 - 1.$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \le 0$ is: $\{(1),(2)\}$ if $l = 1$; $\{(10)$, (01), (20), (02), $(11)\}$ if $l = 2$; $\{(100)$, (001), (010), (200), (002), $(101)\}$ if $l = 3$; $\{(10\cdots0)$, $(0\cdots01)$, $(010\cdots0)$, $(0\cdots010)$, $(20\cdots0)$, $(0\cdots02)$, $(10\cdots01)\}$ if $l \ge 4$. $\square$

Lemma B. *For any λ in case of type B_l with $l \ge 2$, $(\lambda-\mu, \lambda-\widetilde{\alpha}) \le 0$ if and only if λ is one of the following: $\omega_1, \omega_2, 2\omega_2$ in case of $l = 2$; $\omega_1, \omega_2, \omega_l$ in case of $l = 3, 4$; ω_1, ω_2 in case of $l \ge 5$.*

Proof. In this case, $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^l$, $\widetilde{\alpha} = e_1 + e_2$ and $w_0 = -1$. We have

$$\lambda - \mu = 2\lambda = \sum_{k=1}^{l} \left(2\sum_{i=k}^{l} b_i - b_l \right) e_k.$$

It follows that

$$2(2l-1)(\lambda-\mu, \lambda) = \frac{1}{2}\sum_{k=1}^{l} \left(2\sum_{i=k}^{l} b_i - b_l \right)^2,$$

and

$$2(2l-1)(\lambda-\mu, \widetilde{\alpha}) = \sum_{k=1}^{2} \left(2\sum_{i=1}^{l} b_i - b_l \right).$$

For the case $l = 2$, we have

$$12\varepsilon = (2b_1 + b_2 - 1)^2 + (b_2 - 1)^2 - 2.$$

For any $l \ge 3$, we have

$$\begin{aligned} 4(2l-1)\varepsilon = &\left(2\sum_{i=1}^{l-1} b_i + b_l - 1 \right)^2 + \left(2\sum_{i=2}^{l-1} b_i + b_l - 1 \right)^2 \\ &+ \sum_{k=3}^{l-1} \left(2\sum_{i=k}^{l-1} b_i + b_l \right) + b_l^2 - 2. \end{aligned}$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \le 0$ is: $\{(10)$, (01), $(02)\}$ if $l = 2$; $\{(10\cdots0)$, $(010\cdots0)$, $(0\cdots01)\}$ if $l = 3, 4$; $\{(10\cdots0)$, $(010\cdots0)\}$ if $l \ge 5$. $\square$

Lemma C. *For any λ in case of type C_l with $l \ge 3$, $(\lambda-\mu, \lambda-\widetilde{\alpha}) \le 0$ if and only if $\lambda = \omega_1, \omega_2$ or $2\omega_1$.*

Proof. In this case, $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^l$, $\widetilde{\alpha} = 2e_1$ and $w_0 = -1$. We have

$$\lambda - \mu = 2\lambda = 2\sum_{k=1}^{l} \left(\sum_{i=k}^{l} b_i \right) e_k.$$

It follows that

$$4(l+1)(\lambda-\mu, \lambda) = 2\sum_{k=1}^{l} \left(\sum_{i=k}^{l} b_i \right)^2,$$

$$4(l+1)(\lambda-\mu, \widetilde{\alpha}) = 4\sum_{i=1}^{l} b_i,$$

and

$$2(l+1)\varepsilon = \left(\sum_{i=1}^{l} b_i - 1\right)^2 + \sum_{k=2}^{l}\left(\sum_{i=k}^{l} b_i\right)^2 - 1.$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \leq 0$ is $\{(10\cdots0),\ (010\cdots0),\ (20\cdots0)\}$. $\square$

Lemma D. *For any λ in case of type D_l with $l \geq 4$, $(\lambda-\mu, \lambda-\widetilde{\alpha}) \leq 0$ if and only if λ is one of the following: $\omega_1, \omega_2, \omega_{l-1}, \omega_l$ in case of $l = 4, 5$; ω_1, ω_2 in case of $l \geq 6$.*

Proof. In this case, $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^l$ and $\widetilde{\alpha} = e_1 + e_2$. We have

$$\lambda = \sum_{k=1}^{l}\left(\sum_{i=k}^{l} b_i - \frac{b_{l-1}+b_l}{2}\right) e_k.$$

In case of even l with $l \geq 4$, we have $w_0 = -1$ and

$$\lambda - \mu = 2\lambda = \sum_{k=1}^{l}\left(2\sum_{i=k}^{l} b_i - b_{l-1} - b_l\right) e_k.$$

In case of odd l with $l \geq 5$, we see from [B] that w_0 is equal to a linear transformation of $\mathbb{R}^l$ such that e_i maps to $-e_i$ with $1 \leq i \leq l-1$ and e_l maps to e_l, and we have

$$\lambda - \mu = \sum_{k=1}^{l-1}\left(2\sum_{i=k}^{l} b_i - b_{l-1} - b_l\right) e_k.$$

For any $l \geq 4$ we have

$$4(l-1)(\lambda-\mu, \lambda) = \frac{1}{2}\sum_{k=1}^{2[\frac{l}{2}]}\left(2\sum_{i=k}^{l} b_i - b_{l-1} - b_l\right)^2,$$

and

$$4(l-1)(\lambda-\mu, \widetilde{\alpha}) = \sum_{k=1}^{2}\left(2\sum_{i=k}^{l} b_i - b_{l-1} - b_l\right).$$

Therefore, for even l we have

$$\begin{aligned}8(l-1)\varepsilon = {} & \left(2\sum_{i=1}^{l-2} b_i + b_{l-1} + b_l - 1\right)^2 + \left(2\sum_{i=2}^{l-2} b_i + b_{l-1} + b_l - 1\right)^2 \\ & + \sum_{k=3}^{l-2}\left(2\sum_{i=k}^{l-2} b_i + b_{l-1} + b_l\right)^2 + 2b_{l-1}^2 + 2b_l^2 - 2,\end{aligned}$$

and for odd l we have

$$\begin{aligned}8(l-1)\varepsilon = {} & \left(2\sum_{i=1}^{l-2} b_i + b_{l-1} + b_l - 1\right)^2 + \left(2\sum_{i=2}^{l-2} b_i + b_{l-1} + b_l - 1\right)^2 \\ & + \sum_{k=3}^{l-2}\left(2\sum_{i=k}^{l-2} b_i + b_{l-1} + b_l\right)^2 + (b_{l-1} + b_l)^2 - 2.\end{aligned}$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \leq 0$ is: $\{(10\cdots0),\ (010\cdots0),\ (0\cdots010),\ (0\cdots01)\}$ if $l = 4, 5$; $\{(10\cdots0),\ (010\cdots0)\}$ if $l \geq 6$. $\square$

Lemma $\mathbf{E}_6$. *For any λ in case of type E_6, $(\lambda - \mu, \lambda - \widetilde{\alpha}) \leq 0$ if and only if $\lambda = \omega_1, \omega_2$ or ω_6.*

Proof. In this case,

$$\mathfrak{h}_{\mathbb{R}}^* = \{(x_i) \in \mathbb{R}^8 | x_6 = x_7 = -x_8\} \subseteq \mathbb{R}^8$$

and

$$\widetilde{\alpha} = \frac{1}{2}\left(\sum_{i=1}^{5} e_i - e_6 - e_7 + e_8\right).$$

We have

$$\begin{aligned}
\lambda = {} & \frac{1}{2}(b_2 - b_3)e_1 + \frac{1}{2}(b_2 + b_3)e_2 + \left(\frac{1}{2}(b_2 + b_3) + b_4\right) e_3 \\
& + \left(\frac{1}{2}(b_2 + b_3) + b_4 + b_5\right) e_4 \\
& + \left(\frac{1}{2}(b_2 + b_3) + b_4 + b_5 + b_6\right) e_5 \\
& + \left(\frac{2}{3}b_1 + \frac{1}{2}b_2 + \frac{5}{6}b_3 + b_4 + \frac{2}{3}b_5 + \frac{1}{3}b_6\right)(-e_6 - e_7 + e_8).
\end{aligned}$$

Let W_0 be a linear transformation on $\mathbb{R}^8$ defined by a matrix

$$-\frac{1}{2}\begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix},$$

where we set

$$W := \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

We see from [B] that the restriction to $\mathfrak{h}_{\mathbb{R}}^*$ of W_0 gives the involution w_0 (To obtain this form of matrix W_0 representing w_0, impose an extra condition that the linear transformation leaves $e_5 + e_7$ and $e_6 + e_8$ invariant). Using W_0, we have

$$\begin{aligned}
\lambda - \mu = {} & \left(b_2 - \frac{1}{2}(b_3 + b_5)\right) e_1 + \left(b_2 + \frac{1}{2}(b_3 + b_5)\right) e_2 \\
& + \left(b_2 + 2b_4 + \frac{1}{2}(b_3 + b_5)\right) e_3 + \left(b_2 + 2b_4 + \frac{3}{2}(b_3 + b_5)\right) e_4 \\
& + \left(b_2 + (b_1 + b_6) + 2b_4 + \frac{3}{2}(b_3 + b_5)\right)(e_5 - e_6 - e_7 + e_8).
\end{aligned}$$

It follows that

$$\begin{aligned}
24(\lambda-\mu,\lambda) &= \left(b_2-\frac{1}{2}(b_3+b_5)\right)\left(\frac{1}{2}(b_2-b_3)\right) \\
&\quad+\left(b_2+\frac{1}{2}(b_3+b_5)\right)\left(\frac{1}{2}(b_2+b_3)\right) \\
&\quad+\left(b_2+2b_4+\frac{1}{2}(b_3+b_5)\right)\left(\frac{1}{2}(b_2+b_3)+b_4\right) \\
&\quad+\left(b_2+2b_4+\frac{3}{2}(b_3+b_5)\right)\left(\frac{1}{2}(b_2+b_3)+b_4+b_5\right) \\
&\quad+\left(b_2+(b_1+b_6)+2b_4+\frac{3}{2}(b_3+b_5)\right) \\
&\qquad\times\left(\frac{1}{2}(b_2+b_3)+b_4+b_5+b_6\right) \\
&\quad+3\left(b_2+(b_1+b_6)+2b_4+\frac{3}{2}(b_3+b_5)\right) \\
&\qquad\times\left(\frac{2}{3}b_1+\frac{1}{2}b_2+\frac{5}{6}b_3+b_4+\frac{2}{3}b_5+\frac{1}{3}b_6\right) \\
&= b_2^2+\frac{1}{2}(b_3+b_5)^2+(b_2+b_3+2b_4+b_5)^2 \\
&\quad+\frac{1}{2}(2b_1+2b_2+3b_3+4b_4+3b_5+2b_6)^2,
\end{aligned}$$

$$\begin{aligned}
24(\lambda-\mu,\widetilde{\alpha}) &= \frac{1}{2}\left\{\left(b_2-\frac{1}{2}(b_3+b_5)\right)+\left(b_2+\frac{1}{2}(b_3+b_5)\right)\right. \\
&\quad+\left(b_2+2b_4+\frac{1}{2}(b_3+b_5)\right)+\left(b_2+2b_4+\frac{3}{2}(b_3+b_5)\right) \\
&\quad\left.+4\left(b_2+(b_1+b_6)+2b_4+\frac{3}{2}(b_3+b_5)\right)\right\} \\
&= b_2+(b_2+b_3+2b_4+b_5)+(2b_1+2b_2+3b_3+4b_4+3b_5+2b_6),
\end{aligned}$$

and

$$\begin{aligned}
24\varepsilon &= \left(b_2-\frac{1}{2}\right)^2+\frac{1}{2}(b_3+b_5)^2+\left(b_2+b_3+2b_4+b_5-\frac{1}{2}\right)^2 \\
&\quad+\frac{1}{2}(2b_1+2b_2+3b_3+4b_4+3b_5+2b_6-1)^2-1.
\end{aligned}$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \leq 0$ is $\{(100000), (010000), (000001)\}$. $\square$

Lemma $\mathbf{E}_7$. *For any λ in case of type E_7, $(\lambda-\mu,\lambda-\widetilde{\alpha}) \leq 0$ if and only if $\lambda=\omega_1$.*

Proof. In this case, $\mathfrak{h}_{\mathbb{R}}^* = \{(x_i) \in \mathbb{R}^8 | x_7 + x_8 = 0\} \subseteq \mathbb{R}^8$, $\widetilde{\alpha} = -e_7 + e_8$ and $w_0 = -1$. We have

$$\begin{aligned}\lambda - \mu = 2\lambda =& (b_2 - b_3)e_1 + (b_2 + b_3)e_2 \\ &+ (b_2 + b_3 + 2b_4)e_3 + (b_2 + b_3 + 2b_4 + 2b_5)e_4 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6)e_5 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)e_6 \\ &+ (2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 + b_7)(-e_7 + e_8).\end{aligned}$$

It follows that

$$\begin{aligned}36(\lambda - \mu, \lambda) = &\frac{1}{2}\{(b_2 - b_3)^2 + (b_2 + b_3)^2 \\ &+ (b_2 + b_3 + 2b_4)^2 + (b_2 + b_3 + 2b_4 + 2b_5)^2 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6)^2 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)^2 \\ &+ 2(2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 + b_7)^2\}, \\ 36(\lambda - \mu, \widetilde{\alpha}) = &2(2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 + b_7),\end{aligned}$$

and

$$\begin{aligned}72\varepsilon = &(b_2 - b_3)^2 + (b_2 + b_3)^2 \\ &+ (b_2 + b_3 + 2b_4)^2 + (b_2 + b_3 + 2b_4 + 2b_5)^2 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6)^2 + (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)^2 \\ &+ 2(2b_1 + 2b_2 + 3b_3 + 4b_4 + 3b_5 + 2b_6 + b_7 - 1)^2 - 2.\end{aligned}$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \leq 0$ is $\{(1000000)\}$. $\square$

Lemma E$_8$. *For any λ in case of type E_8, $(\lambda - \mu, \lambda - \widetilde{\alpha}) \leq 0$ if and only if $\lambda = \omega_8$.*

Proof. In this case, $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}^8$, $\widetilde{\alpha} = e_7 + e_8$ and $w_0 = -1$. We have

$$\begin{aligned}\lambda - \mu = 2\lambda =& (b_2 - b_3)e_1 + (b_2 + b_3)e_2 \\ &+ (b_2 + b_3 + 2b_4)e_3 + (b_2 + b_3 + 2b_4 + 2b_5)e_4 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6)e_5 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7)e_6 \\ &+ (b_2 + b_3 + 2b_4 + 2b_5 + 2b_6 + 2b_7 + 2b_8)e_7 \\ &+ (4b_1 + 5b_2 + 7b_3 + 10b_4 + 8b_5 + 6b_6 + 4b_7 + 2b_8)e_8.\end{aligned}$$

It follows that

$$\begin{aligned}60(\lambda-\mu,\lambda) = \frac{1}{2}\{&(b_2-b_3)^2+(b_2+b_3)^2\\&+(b_2+b_3+2b_4)^2+(b_2+b_3+2b_4+2b_5)^2\\&+(b_2+b_3+2b_4+2b_5+2b_6)^2\\&+(b_2+b_3+2b_4+2b_5+2b_6+2b_7)^2\\&+(b_2+b_3+2b_4+2b_5+2b_6+2b_7+2b_8)^2\\&+(4b_1+5b_2+7b_3+10b_4+8b_5+6b_6+4b_7+2b_8)^2\},\\60(\lambda-\mu,\widetilde{\alpha}) = (&b_2+b_3+2b_4+2b_5+2b_6+2b_7+2b_8)\\&+(4b_1+5b_2+7b_3+10b_4+8b_5+6b_6+4b_7+2b_8),\end{aligned}$$

and

$$\begin{aligned}120\varepsilon = {}&(b_2-b_3)^2+(b_2+b_3)^2\\&+(b_2+b_3+2b_4)^2+(b_2+b_3+2b_4+2b_5)^2\\&+(b_2+b_3+2b_4+2b_5+2b_6)^2+(b_2+b_3+2b_4+2b_5+2b_6+2b_7)^2\\&+(b_2+b_3+2b_4+2b_5+2b_6+2b_7+2b_8-1)^2\\&+(4b_1+5b_2+7b_3+10b_4+8b_5+6b_6+4b_7+2b_8-1)^2-2.\end{aligned}$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \le 0$ is $\{(00000001)\}$. □

Lemma $\mathbf{F}_4$. *For any λ in case of type F_4, $(\lambda-\mu,\lambda-\widetilde{\alpha}) \le 0$ if and only if $\lambda=\omega_1$ or ω_4.*

Proof. In this case, $\mathfrak{h}^*_{\mathbb{R}}=\mathbb{R}^4$, $\widetilde{\alpha}=e_1+e_2$ and $w_0=-1$. We have

$$\lambda-\mu=2\lambda=(2b_1+4b_2+3b_3+2b_4)e_1+(2b_1+2b_2+b_3)e_2+(2b_2+b_3)e_3+b_3e_4.$$

It follows that

$$\begin{aligned}18(\lambda-\mu,\lambda)&=\frac{1}{2}\{(2b_1+4b_2+3b_3+2b_4)^2+(2b_1+2b_2+b_3)^2+(2b_2+b_3)^2+b_3^2\},\\18(\lambda-\mu,\widetilde{\alpha})&=(2b_1+4b_2+3b_3+2b_4)+(2b_1+2b_2+b_3),\end{aligned}$$

and

$$36\varepsilon=(2b_1+4b_2+3b_3+2b_4-1)^2+(2b_1+2b_2+b_3-1)^2+(2b_2+b_3)^2+b_3^2-2.$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \le 0$ is $\{(1000), (0001)\}$. □

Lemma $\mathbf{G}_2$. *For any λ in case of type G_2, $(\lambda-\mu,\lambda-\widetilde{\alpha}) \le 0$ if and only if $\lambda=\omega_1$ or ω_2.*

Proof. In this case, $\mathfrak{h}^*_{\mathbb{R}}=\{(x_i)\in\mathbb{R}^3 \mid \sum_{i=1}^3 x_i=0\}\subseteq\mathbb{R}^3$, $\widetilde{\alpha}=-e_1-e_2+2e_3$ and $w_0=-1$. We have

$$\lambda-\mu=2\lambda=2\{-b_2e_1-(b_1+b_2)e_2+(b_1+2b_2)e_3\}.$$

It follows that

$$\begin{aligned}24(\lambda-\mu,\lambda)&=2\{(-b_2)^2+(-(b_1+b_2))^2+(b_1+2b_2)^2\},\\24(\lambda-\mu,\widetilde{\alpha})&=2\{b_2+(b_1+b_2)+2(b_1+2b_2)\},\end{aligned}$$

and

$$12\varepsilon = \left(b_2 - \frac{1}{2}\right)^2 + \left(b_1 + b_2 - \frac{1}{2}\right)^2 + (b_1 + 2b_2 - 1)^2 - \frac{3}{2}.$$

Thus the set of non-trivial solutions (b_i) for $\varepsilon \leq 0$ is $\{(10), (01)\}$. □

3. Proofs of main results

Proof of Theorem. If X corresponding to λ has $\delta > 0$, then it follows from the Criterion and Proposition that λ is one of the dominant weights listed in the statement of the Theorem.

We show the converse. For the adjoint representation, the required results follow from one of the main theorems in [KOY], which with the same notations as in the Introduction asserts that *if G is simple and of rank ≥ 2, and if ρ is the adjoint representation, then the corresponding variety X has $\delta = 1$.* For the other cases, using well-known facts [FH] and results of Zak [LV, Appendix], [Z], one can show that each X has $\delta > 0$: for ω_1, ω_6 in case of E_6, the corresponding variety X is well-known as the Severi variety of the largest dimension, and X for ω_4 in case of F_4 is its hyperplane section; for the remaining dominant weights, the corresponding variety X is either a projective space, its Veronese embedding, a quadric hypersurface, a Grassmann variety of lines in a projective space, its hyperplane section, or a spinor variety. □

Moreover, using results in [FH], [KOY], [LV], [Z], one can verify that a projective variety X obtained from each weight listed in the Theorem enjoys the properties stated in our table.

Finally we prove two results for arbitrary homogeneous projective varieties mentioned in the Introduction. Without loss of generality, we may assume for any homogeneous projective variety X that X is obtained from an irreducible representation of a semi-simple algebraic group G (see [Z, III, §1], [FH, Prop. 9.17]).

Proof of Corollary 1. Suppose that $\operatorname{Sec} X \neq \mathbb{P}$ and $\delta > 8$ for some X. According to a result of Zak [Z, III, Corollary 1.7], if G were not simple, and if $\delta > 0$, then $\delta = 2$; this is a contradiction. Hence G must be simple. But we see from our table that there is no such X. □

Proof of Corollary 2. Suppose that $v_d(X)$ has $\delta > 0$ for some X and $d \geq 2$. According to a result of Zak [Z, III, Corollary 1.7], if G were not simple, and if $\delta > 0$, then X would be isomorphic to some Segre product $\mathbb{P}^a \times \mathbb{P}^b$, hence $\mathcal{O}_X(d) = \mathcal{O}_{v_d(X)}(1) = \mathcal{O}_{\mathbb{P}^a}(1) \boxtimes \mathcal{O}_{\mathbb{P}^b}(1)$: This is a contradiction since the last line bundle is indivisible by $d \geq 2$. Hence G is simple.

If X is corresponding to λ, then its d-uple embedding $v_d(X)$ is corresponding to $d\lambda$. In our table, dominant weights of the form $d\lambda$ for some $d \geq 2$ and for some dominant weight λ, are $2\omega_1, 2\omega_l$ in case of A_l, $2\omega_2$ in case of B_l, and $2\omega_l$ in case of C_l, and all X in those cases are projective spaces. □

Acknowledgements

The author would like to thank Professors O. Yasukura, M. Ohno, S. H. Chin and colleagues of KIN-YOH seminar at Waseda University for many useful conversations during the preparation of this work.

After the first draft of this paper was written, Professors J. Roberts and J. M. Landsberg independently showed me that the conclusion of Corollary 2 holds without the assumption that X is homogeneous: Those three proofs including the proof here are completely different from each other.

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