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HAAR MEASURE AND THE ARTIN CONDUCTOR

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Abstract. Let G be a connected reductive group, defined over a local, non-archimedean field k. The group $G(k)$ is locally compact and unimodular. In *On the motive of a reductive group*, Invent. Math. **130** (1997), by B. H. Gross, a Haar measure $|\omega_G|$ was defined on $G(k)$, using the theory of Bruhat and Tits. In this note, we give another construction of the measure $|\omega_G|$, using the Artin conductor of the motive M of G over k. The equivalence of the two constructions is deduced from a result of G. Prasad.

1. The Root Datum and Motive of G

In this section, k is an arbitrary field and G is a connected reductive group over k. We let $\overline{k}$ be an algebraic closure of k, k_s the separable closure of k in $\overline{k}$, and $\Gamma = Gal(k_s/k)$.

Let $T \subset B \subset G$ be a maximal torus, contained in a Borel subgroup, defined over k_s. Let $\Psi = \Psi(G, B, T)$ be the based root datum defined by this choice. We recall (cf. [Sp], pg. 3-12) that:

$$\Psi = \left(X^\bullet(T), \Delta^\bullet(T,B), X_\bullet(T), \Delta_\bullet(T,B)\right), \tag{1.1}$$

with $X^\bullet(T)$ and $X_\bullet(T)$ the character and cocharacter groups of T respectively, and $\Delta^\bullet$ and $\Delta_\bullet$ the simple roots and coroots determined by B respectively. Let $W = N_G(T)/T$ be the Weyl group of Ψ. The finite group W acts as automorphisms of $X^\bullet(T)$, and is generated by the reflections:

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\alpha \tag{1.2}$$

for $\alpha \in \Delta^\bullet$.

The Galois group Γ acts as automorphisms of Ψ, i.e. as automorphisms of the group $X^\bullet(T)$ preserving the finite set $\Delta^\bullet$, as follows. If $\sigma \in \Gamma$, then we can find $g \in G(k_s)$ such that

$$Int(g)(\sigma T) = g\sigma(T)g^{-1} = T,$$
$$Int(g)(\sigma B) = g\sigma(B)g^{-1} = B,$$

with g well-defined up to left multiplication by $T(k_s)$. Hence it induces a well-defined automorphism

$$\psi(\sigma) : X^\bullet(T) \longrightarrow X^\bullet(T)$$

preserving $\Delta^\bullet$. Hence we get a group homomorphism $\psi : \Gamma \longrightarrow Aut(\Psi)$. Via ψ, Γ acts on $Aut(\Psi)$ by inner automorphisms.

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Similarly, if $f : G \longrightarrow G$ is any automorphism of G over k_s, it induces an automorphism $\psi(f)$ of Ψ, which depends only on the image of f in the quotient group $Out_{k_s}(G)$ of outer automorphisms. The resulting map $Out_{k_s}(G) \longrightarrow Aut(\Psi)$ is an isomorphism which respects the respective Galois actions on the two groups (cf. [Sp], pg. 10).

The Galois group Γ also acts on W, via the formula

$$\sigma(s_\alpha) = s_{\sigma(\alpha)} \tag{1.3}$$

and the semi-direct product $W \rtimes \Gamma$ acts on the rational vector space

$$E = X^\bullet(T) \otimes \mathbb{Q}. \tag{1.4}$$

Let $R = Sym^\bullet(E)^W$, which is a graded $\mathbb{Q}[\Gamma]$-module. Let R_+ be the ideal of elements of positive degree in R, and define

$$V = R_+/R_+^2 = \bigoplus_{d \geq 1} V_d. \tag{1.5}$$

This is a graded $\mathbb{Q}[\Gamma]$-module, and Chevalley proved that $dim(V) = dim(E)$ (cf. [Ch]). Steinberg extended the proof to show that E and V are isomorphic Γ-modules (cf. [St], pg. 22). We sketch a proof of this result that does not involve the classification of irreducible root systems.

Proposition 1.6. *The $\mathbb{Q}[\Gamma]$-modules E and V are isomorphic.*

Proof. By the criterion in [Se, pg. 104], it suffices to show that for all $\sigma \in \Gamma$, the fixed spaces E^σ and V^σ have the same dimension.

For any graded Γ-module $A = \bigoplus A_m$, we define the Poincare series of σ by

$$P(A, \sigma)(t) = \sum tr(\sigma | A_m) t^m.$$

Then $P(A \otimes B) = P(A)P(B)$. Steinberg showed that there is an isomorphism of graded Γ-modules:

$$S^\bullet(E) \cong S^\bullet(\oplus V_d) \otimes A.$$

Here A is finite dimensional, with basis $\{b_w\}_{w \in W}$, and Γ-action given by:

$$\sigma(b_w) = b_{\sigma(w)}.$$

The degree of b_w is the length $l(w)$ of w, with respect to the generators s_α furnished by $\Delta^\bullet$. This isomorphism yields the following identity of Poincare series:

$$det(1 - \sigma t | E)^{-1} = \prod_{d \geq 1} det(1 - \sigma t^d | V_d)^{-1} \cdot \sum_{w \in W^\sigma} t^{l(w)}.$$

In particular, the quotient

$$\frac{\prod_{d \geq 1} det(1 - \sigma t^d | V_d)}{det(1 - \sigma t | E)}$$

is a polynomial $P(t)$, with $P(1) \neq 0$. Hence $dim(V^\sigma) = \sum_{d \geq 1} dim(V_d^\sigma) = dim(E^\sigma)$, as required. □

As in [Gr], we define the **motive** M of G as the Artin-Tate motive

$$M = \bigoplus_{d \geq 1} V_d(1 - d) \tag{1.7}$$

over k. This depends only on the isogeny class of the quasi-split inner form G_{qs} of G over k. Indeed, if T_{qs} is a maximal torus contained in a Borel subgroup $B_{qs} \subset G_{qs}$ over k, then,

$$E \cong X^{\bullet}(T_{qs}) \otimes \mathbb{Q} \tag{1.8}$$

as a $W \rtimes \Gamma$-module (cf. [Sp], pg. 12).

We also define the invariant

$$d(G) \in Hom(\Gamma, \mathbb{Z}^{\times}) = H^1(\Gamma, \mathbb{Z}^{\times}) \tag{1.9}$$

as the character of Γ on $\wedge^{top} X^{\bullet}(T)$, or equivalently as the representation $det(E)$ of Γ. This is analogous to, but simpler than Kottwitz's invariant $e(G) \in H^2(\Gamma, \mu_2)$ (cf. [K]).

The canonical ring homomorphism $ch : \mathbb{Z} \longrightarrow k$ induces a map $\mathbb{Z}^{\times} \longrightarrow \mu_2$. We let

$$\delta(G) \in H^1(\Gamma, \mu_2) = k^{\times}/k^{\times 2} \tag{1.10}$$

be the image of the invariant $d(G)$. This is trivial when $char(k) = 2$, and can be computed in general as follows. Let K be the étale k-algebra of dimension 2 corresponding to $d(G)$. Write $K = k + k\alpha$, and suppose α satisfies the non-zero quadratic polynomial $a\alpha^2 + b\alpha + c = 0$ over k. Then $\delta(G) \equiv b^2 - 4ac \ (mod\ k^{\times 2})$.

2. Automorphisms of G

Let f be an automorphism of G over k_s. Let $\psi(f)$ be the corresponding automorphism of the based root datum Ψ, and let $Lie(f)$ be the corresponding automorphism of the Lie algebra $\mathfrak{g}$ over k_s. The former depends only on the image of f in $Out_{k_s}(G)$; similarly we have the following:

Lemma 2.1. *The automorphism $\wedge^{top} Lie(f)$ of $\wedge^{top}\mathfrak{g}$ depends only on the image of f in $Out_{k_s}(G)$.*

Proof. The action of inner automorphisms on $\wedge^{top}\mathfrak{g}$ gives a homomorphism $G^{ad} \longrightarrow \mathbb{G}_m$ of algebraic groups over k. This is trivial as G^{ad} is connected with trivial center. □

Proposition 2.2.

$$ch(det(\psi(f))) = det(Lie(f)) \in ch(\mathbb{Z}^{\times}) = \mu_2(k).$$

Proof. Let $\{T, B, X_\alpha : \alpha \in \Delta^{\bullet}\}$ be a pinning of G over k_s, where X_α is a basis of the one-dimensional root space $\mathfrak{g}_\alpha$. By the previous lemma, we may assume that the automorphism f preserves the pinning (cf. [Sp], pg. 10). Then $Lie(f)$ preserves a Chevalley basis of $\mathfrak{g}$ over k_s (cf. [B-T], pg. 53-54).

Let $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{n}^{\pm}$ the nilpotent Lie algebra spanned by the positive and negative roots with respect to B. Then $Lie(f)$ preserves the triangular decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-.$$

Furthermore,

$$det(Lie(f)|\mathfrak{t}) = ch(det(\psi(f)))$$

as $\mathfrak{t} = X_{\bullet}(T) \otimes k$. Since the permutation induced by $Lie(f)$ on the positive elements of the Chevalley basis is the same as that on the negative elements, we have

$$det(Lie(f)|\mathfrak{n}^+) \cdot det(Lie(f)|\mathfrak{n}^-) = 1.$$

This completes the proof. □

Recall that the invariant differential forms of top degree on G over an extension L of k form a one-dimensional L-vector space, which is the dual of $\wedge^{top}\mathfrak{g}_L$. We will refer to an element of this space as an **invariant differential** on G.

Corollary 2.3. *If ω is an invariant differential on G over k_s, and f is any automorphism of G over k_s, then $f^*(\omega) = ch(det(\psi(f)))\omega$.*

3. The Split Group

Let G_0 be a split group over k, whose root datum is isomorphic to Ψ. Such a group exists by [B-T], and we may choose an isomorphism

$$\varphi : G \longrightarrow G_0 \tag{3.1}$$

defined over k_s.

For each $\sigma \in \Gamma$, the element

$$f(\sigma) = \varphi^{-1} \circ \sigma(\varphi) \tag{3.2}$$

defines an automorphism of G over k_s. The map $f : \Gamma \longrightarrow Aut_{k_s}(G)$ is a 1-cocycle, whose class in $H^1(\Gamma, Aut_{k_s}(G))$ is independent of the choice of φ. The map $\sigma \mapsto \psi(f(\sigma))$ is then a 1-cocycle with values in $Aut(\Psi)$. Composing this with

$$det : Aut(\Psi) \longrightarrow \mathbb{Z}^\times, \tag{3.3}$$

we get a group homomorphism

$$\begin{aligned} \Gamma &\longrightarrow \mathbb{Z}^\times, \\ \sigma &\mapsto det(\psi(f(\sigma))). \end{aligned} \tag{3.4}$$

Lemma 3.5.

$$det(\psi(f(\sigma))) = d(G)(\sigma) \in \mathbb{Z}^\times.$$

Proof. By (1.8) and Lemma 2.1, it suffices to prove this for G quasi-split over k. Hence we can assume that T and B are defined over k. Let $T_0 \subset B_0$ be a maximal torus of G_0 contained in a Borel subgroup, with T_0 and B_0 defined over k. Twisting by an inner automorphism of G_0 if necessary, we can suppose that the isomorphism φ in (3.1) maps T and B to T_0 and B_0 respectively. Then using φ, we can identify $G(k_s)$, $T(k_s)$ and $B(k_s)$ with $G_0(k_s)$, $T_0(k_s)$ and $B_0(k_s)$ respectively. Now suppose that $G(k)$ is the fixed-point set of the Γ-action $g \mapsto \sigma(g)$ on $G(k_s) = G_0(k_s)$. Then $G_0(k_s)$ is the fixed-point set of the Γ-action $g \mapsto f(\sigma)(\sigma(g)) = \rho(\sigma)(g)$. Now the action of $\psi(\rho(\sigma))$ on $X^\bullet(T) = X^\bullet(T_0)$ is trivial, since G_0 is split. Hence, for any $\chi \in X^\bullet(T)$, we have

$$\begin{aligned} \psi(f(\sigma))\chi &= \psi(\sigma)^{-1}\psi(\sigma)\psi(f(\sigma))\chi \\ &= \psi(\sigma)^{-1}\psi(\rho(\sigma))\chi \\ &= \psi(\sigma)^{-1}\chi. \end{aligned}$$

Hence the action of $\psi(f(\sigma))$ on $X^\bullet(T)$ is the same as that of $\psi(\sigma)^{-1}$. This implies the result. ◻

Proposition 3.6. *Let ω_0 be an invariant differential on G_0 over k, and let $\omega = \varphi^*(\omega_0)$ on G over k_s. Then for all $\sigma \in \Gamma$,*

$$\sigma(\omega) = \delta(G)(\sigma) \cdot \omega$$

where $\delta(G)$ is the character of Γ with values in $\mu_2(k)$ defined by (1.10).

Proof. We have $\sigma(\omega) = ch(det(\psi(f(\sigma))))\omega$ by Corollary 2.3. By the previous lemma,

$$det(\psi(f(\sigma))) = d(G)(\sigma) \in \mathbb{Z}^\times$$

So we have

$$ch(det(\psi(f(\sigma))) = \delta(G)(\sigma) \in k^\times/k^{\times 2}.$$

□

Corollary 3.7. *Let $D \in k^\times/k^{\times 2}$ represent the class of $\delta(G)$. Then $\omega/\sqrt{D}$ is an invariant differential on G over k.*

Proof. Indeed, $\sigma(\sqrt{D}) = \delta(G)(\sigma)\sqrt{D}$, so the differential $\omega/\sqrt{D}$ is fixed by Γ. Note that when $char(k) = 2$, D is in $k^{\times 2}$ and so $\sqrt{D} \in k^\times$. □

4. The Artin Conductor of M

We now assume that k is a local, non-archimedean field, with ring of integers A and uniformizer π. We let $q = \#(A/\pi A)$, and normalize the valuation on $k^\times$ so that $v(\pi) = 1$, and the absolute value so that $|\alpha| = q^{-v(\alpha)}$. We adopt the convention that $|0| = 0$.

Let V be a continuous finite dimensional complex representation of Γ. We define the **Artin conductor** $a(V) \geq 0$ in $\mathbb{Z}$ as follows. Let L be the fixed field of the kernel of the map $\Gamma \longrightarrow GL(V)$; let $\Delta = Gal(L/k)$, which is a finite group, and let

$$\Delta \supset \Delta_0 \supset \Delta_1 \supset ...$$

be the decreasing ramification filtration of Δ. Then $\Delta_0 = I$ is the inertia subgroup and Δ_1 the wild inertia subgroup. Let $g_i = \#\Delta_i$. Then [Se3, pg. 99-101],

$$a(V) = \sum_{i \geq 0} \frac{g_i}{g_0} dim(V/V^{\Delta_i}). \tag{4.1}$$

We have $a(V) = dim(V/V^I) + b(V)$, where $b(V)$ is a measure of the wild ramification of V.

If V is a quadratic character $\chi : \Gamma \longrightarrow \mathbb{Z}^\times$, we can refine the integer $a(V)$ slightly, as follows. Let K be the étale k-algebra of dimension 2 corresponding to χ, and let $A_K \subset K$ be the subring of elements integral over A. Then A_K is a free A-module of rank 2. Writing $A_K = A + A\alpha$, we may define $D = D(\alpha) = Tr(\alpha)^2 - 4\mathbb{N}(\alpha)$ in A. Then D is non-zero, and [M-H]

$$a(V) = a(\chi) = v(D). \tag{4.2}$$

If $A_K = A + A\alpha'$, then $D' \equiv D \ (mod\, A^{\times 2})$. Hence we get a class D_V in $A/A^{\times 2}$ of valuation $a(V)$; this is the desired refinement.

We define the **conductor** of the motive $M = \bigoplus_{d \geq 1} V_d(1-d)$ of G by the formula:

$$a(M) = \sum_{d \geq 1} (2d-1)a(V_d). \tag{4.3}$$

Then $a(M) \geq 0$, with equality if $M = M^I$ is unramified.

Proposition 4.4. *The conductor $a(M)$ of M and the conductor $a(detE)$ of the quadratic character $det(E) = d(G) : \Gamma \longrightarrow \mathbb{Z}^\times$ satisfy*

$$a(M) \equiv a(detE) \quad (mod\ 2).$$

Proof. Clearly,

$$a(M) \equiv \sum_{d \geq 1} a(V_d) = a(V) \quad (mod\ 2).$$

By Proposition 1.6, $V \cong E$ as $\mathbb{Q}[\Gamma]$-modules, so $a(V) = a(E)$. Finally, since E is defined over $\mathbb{R}$, a result of Serre [Se2, pg. 698] gives the congruence

$$a(E) \equiv a(detE) \quad (mod\ 2).$$

□

This result allows us to refine the conductor $a(M)$ as in (4.2). Since $detE$ is a quadratic character, there is a class D in $A/A^{\times 2}$ with

$$v(D) = a(detE).$$

Moreover, we have

$$\sigma(\sqrt{D}) = \delta(G)(\sigma) \cdot \sqrt{D}$$

for all $\sigma \in \Gamma$, where $\delta(G) : \Gamma \longrightarrow \mu_2(k)$. We define the refinement:

$$D_M = D\pi^{a(M)-a(detE)} \in A/A^{\times 2}. \tag{4.5}$$

Corollary 4.6. *The class D_M in $A/A^{\times 2}$ satisfies*

$$v(D_M) = a(M), \text{ the Artin conductor of } M, \text{ and}$$
$$\sigma(\sqrt{D_M}) = \delta(G)(\sigma) \cdot \sqrt{D_M}$$

for all $\sigma \in \Gamma$.

5. The Haar Measure $|\omega_G|$

We continue to assume that k is local and non-archimedean. Let G_0 be the split form of G over k, and let $\underline{G}_0$ be a Chevalley model for G_0 over A. Let ω_0 be an invariant differential on $\underline{G}_0$ over A with non-zero reduction ($mod\ \pi$). Then ω_0 is determined up to multiplication by an element of $A^\times$.

Let $\varphi : G \longrightarrow G_0$ be an isomorphism over k_s, and define

$$\omega = \varphi^*(\omega_0) \tag{5.1}$$

on G over k_s. By the above remarks, and Corollary 2.3, ω is determined up to multiplication by $A^\times$, independent of the choice of φ.

Let D_M in $A/A^{\times 2}$ be defined by (4.5). By Proposition 3.6, and Corollaries 3.7 and 4.6, the invariant differential

$$\omega_G = \omega/\sqrt{D_M} = \varphi^*(\omega_0)/\sqrt{D_M} \tag{5.2}$$

on G is defined over k, and is well-determined up to multiplication by an element of $A^\times$.

Since $|\alpha| = 1$ for all $\alpha \in A^\times$, the Haar measure

$$|\omega_G| \text{ on } G(k) \tag{5.3}$$

is well-defined, independent of the choices of ω_0 and φ. This completes the definition of $|\omega_G|$.

6. Properties of $|\omega_G|$

We have the following properties of the Haar measure $|\omega_G|$ on $G(k)$, when we vary the group G or the local field k.

Proposition 6.1. *1) If $G = G_1 \times G_2$, then $|\omega_G| = |\omega_{G_1}| \otimes |\omega_{G_2}|$ on $G(k) = G_1(k) \times G_2(k)$.*

2) If $\varphi : G \longrightarrow G'$ is an inner twisting, defined over k_s, then $\varphi^*|\omega_{G'}| = |\omega_G|$ on $G(k)$.

3) If $f : G \longrightarrow G'$ is a central isogeny, defined over k, and N_f is the rank of the finite flat group scheme $ker f$, then

$$f^*|\omega_{G'}| = |N_f| \cdot |\omega_G| \text{ on } G(k).$$

4) If K is a finite separable extension of k, G_K is a connected reductive group over K, and $G = Res_{K/k}(G_K)$ is the restriction of scalars to k, then $|\omega_{G_K}|_K = |\omega_G|$ on $G_K(K) = G(k)$.

Remarks. In part (2), the pull-back φ^* on Haar measures is defined in [L, pg. 69]. In part (3), the groups $G(k)$ and $G(k')$ are locally isomorphic provided N_f is invertible in k. If $N_f = 0$ in k, we define $f^*|\omega_{G'}|$ to be zero, so that (3) holds trivially.

Proof. Parts (1) and (2) are simple consequences of the definitions, as $M_G = M_{G_1} \oplus M_{G_2}$ in (1) and $M_G = M_{G'}$ in (2).

For part (3), the equality of motives allows one to reduce to the case when G and G' are split over k. Let $T \subset B \subset G$ be chosen over k, and let $T' = f(T) \subset B' = f(B)$ in G'. The central isogeny f then induces an injection:

$$X_\bullet(T) \longrightarrow X_\bullet(T')$$

which maps $\Delta_\bullet$ to $\Delta'_\bullet$ and has cokernel of order N_f.

By [Sp, pg. 7], we can define the groups G and G', as well as the central isogeny f over $\mathbb{Z}$: $f_\mathbb{Z} : G_\mathbb{Z} \longrightarrow G'_\mathbb{Z}$ from the isogeny of the root data. Then $Lie(f_\mathbb{Z})$ is an isomorphism on the non-zero root spaces, and induces an injection $Lie(T_\mathbb{Z}) \longrightarrow Lie(T'_\mathbb{Z})$ with kernel of order N_f. If ω_G and $\omega_{G'}$ are bases for the invariant differential over $\mathbb{Z}$, we then have

$$f_\mathbb{Z}^*(\omega_{G'}) = \pm N_f \cdot \omega_G.$$

The result then follows by specializing to k.

For part (4), we have

$$M_G = Ind_{\Gamma_K}^{\Gamma}(M_{G_K})$$

where Γ_K is the subgroup of Γ fixing K. Let $\varepsilon_{K/k}$ be the sign character of the permutation representation of Γ on $\Gamma/\Gamma_K = Hom(K, k_s)$; let $D_{K/k} \in A/A^{\times 2}$ be associated to the quadratic character $\varepsilon_{K/k}$, and let $f_{K/k}$ be the degree of the residue class extension in K/k.

If ω_K is an invariant differential on G_K over K, then the exterior product

$$\omega = \frac{\bigwedge_{\sigma \in \Gamma/\Gamma_K} \omega_K^\sigma}{(\sqrt{D_{K/k}})^{dim(G_K)}} \tag{6.2}$$

is an invariant differential on G defined over k. Note that $G(k_s) = \prod_{\sigma \in \Gamma/\Gamma_K} G_K^\sigma(k_s)$. Now, suppose $\{X_1, ..., X_n\}$ is a basis of $\mathfrak{g}_K$, the Lie algebra of G_K, such that $\omega_K(X_1 \wedge ... \wedge X_n) = 1$. Let $\{\theta_1, ..., \theta_r\}$ be a basis of the free A-module A_K, the ring of integers of K. Then $\{\theta_i X_j : 1 \le i \le r, 1 \le j \le n\}$ is a basis of $\mathfrak{g}$, the Lie algebra of G, and by a direct computation, one sees that:

$$\bigwedge_{\sigma} \omega_K^{\sigma}(\bigwedge_{i,j} \theta_i X_j) = D_{K/k}^{\frac{dim(G_K)}{2}}.$$

Hence, $|\omega_K|_K = |\omega|$ as Haar measures on $G_K(K) = G(k)$. This is compatible with scaling ω_K by $\beta \in K^{\times}$, as $|\beta|_K = |\mathbb{N}_{K/k}(\beta)|$.

Now write $M_K = \bigoplus U_d(1-d)$ and $M = \bigoplus V_d(1-d)$, with $V_d = Ind(U_d)$. By [Se3, pg. 101],

$$a(V_d) = f_{K/k} \cdot a(U_d) + dimU_d \cdot a(\varepsilon_{K/k}). \tag{6.3}$$

Since $\sum (2d-1) dimU_d = dim(G_K)$, we have

$$a(M) = f_{K/k} \cdot a(M_K) + dim(G_K) \cdot a(\varepsilon_{K/k}). \tag{6.4}$$

The corresponding result for the refinements D_{M_K} of $a(M_K)$ in $A_K/A_K^{\times 2}$ and D_M of $a(M)$ in $A/A^{\times 2}$ is then

$$D_M \equiv \mathbb{N}_{K/k}(D_{M_K}) \cdot D_{K/k}^{dim(G_K)}. \tag{6.5}$$

Now if $G_{0,K}$ is a split form of G_K, $\varphi_K : G_K \longrightarrow G_{0,K}$ an isomorphism over k_s, and $\omega_{0,K}$ an invariant differential on $G_{0,K}$ with good reduction, then by definition

$$\omega_{G_K} = \frac{\varphi_K^*(\omega_{0,K})}{\sqrt{D_{M_K}}}.$$

As observed above, the form on G over k which gives the same Haar measure on $G(k) = G_K(K)$ as ω_{G_K} is given by

$$\omega = \frac{\bigwedge \varphi_K^*(\omega_{0,K})^{\sigma}}{\sqrt{\mathbb{N}_{K/k}(D_{M_K}) \cdot D_{K/k}^{dim(G_K)}}} = \frac{\varphi^*(\omega_0)}{\sqrt{D_M}} = \omega_G.$$

This completes the proof. □

7. Comparison with Bruhat-Tits Theory

First, we assume that G is quasi-split over k. In [Gr, §4], a Haar measure $|\omega'_G|$ was defined on $G(k)$. The definition used the theory of special points in the building of G, and models over A. If G is split, then $|\omega'_G| = |\omega_G|$ by definition. It seems likely that this is true in general. The key case, when G is absolutely quasi-simple and simply connected, was treated by Prasad [P]. We deduce what we can from his results here.

Since the Haar measure $|\omega'_G|$ is also defined using an invariant differential ω'_G on G over k, we have

$$|\omega'_G| = \lambda_G |\omega_G| \tag{7.1}$$

with λ_G in the subgroup $q^{\mathbb{Z}}$ of $\mathbb{R}_+^{\times}$.

Proposition 7.2. *We have $\lambda_G = 1$ if G is unramified over k. Furthermore,*

1) $\lambda_{G_1 \times G_2} = \lambda_{G_1} \lambda_{G_2}$.
2) $\lambda_G = \lambda_{G'}$ *if G and G' are separably isogeneous over k.*
3) $\lambda_{G_K} = \lambda_G$ *if $G = Res_{K/k}(G_K)$.*

Proof. If G is unramified, $a(M) = 0$, and D_M is in $A^\times/A^{\times 2}$. Also, ω'_G is defined using a hyperspecial point in the building of G, which is a special vertex in the building over the maximal unramified extension in k_s. Hence $\omega_G = \varphi^*(\omega_0)/\sqrt{D_M}$ is a unit multiple of ω'_G, and $\lambda_G = 1$.

Properties $(1) - (4)$ of Proposition 6.1 hold for $|\omega'_G|$, which implies properties $(1) - (3)$ in the proposition. □

Corollary 7.3. *If $char(k) = 0$, then $|\omega_G| = |\omega'_G|$.*

If $char(k) = p$, then $|\omega_G| = |\omega'_G|$ if G is a torus with Galois splitting field of degree prime to p, or if G is semi-simple with fundamental group of order prime to p.

Proof. If the characteristic of k is zero, any central isogeny is separable. By Proposition 7.2, it suffices to prove the equality $|\omega_G| = |\omega'_G|$ for G semi-simple, simply-connected, and for G a torus. Indeed, G is isogeneous to the product of the simply-connected cover of its derived group, and its connected center.

If G is semi-simple and simply-connected, then G is isomorphic to a product $\prod Res_{K_i/k}(G_i)$, with each G_i absolutely quasi-simple over K_i. Again by Proposition 7.2, it suffices to prove the equality for G absolutely quasi-simple. This is the content of Theorem 1.6 of Prasad [P].

If G is a torus, there is an integer n such that $G^n \times \prod Res_{K_i/k}\mathbb{G}_m$ is isogeneous to $\prod Res_{K_j/k}\mathbb{G}_m$ by a Theorem of Ono [O, Thm 1.5.1, pg. 114]. Since the result is true for $\mathbb{G}_m$, it is true for G^n. So $\lambda_G^n = 1$; since λ is positive, we also have: $\lambda_G = 1$.

If the characteristic of k is p, and G is a torus with Galois splitting field of degree prime to p, then the same Theorem of Ono alluded to above says that the isogeny from $G^n \times \prod Res_{K_i/k}\mathbb{G}_m$ to $\prod Res_{K_j/k}\mathbb{G}_m$ can be chosen to be separable. Hence the same argument as above works to give the result.

If G is semi-simple with fundamental group of order prime to p, the isogeny $\tilde{G} \longrightarrow G$ from the simply-connected cover is separable. So it suffices to check the result for $\tilde{G}$. By the above argument, we may assume that $G = \tilde{G}$ is absolutely quasi-simple, where the result follows from Prasad [P]. □

Now if G is not necessarily quasi-split, choose an inner twisting $\varphi : G \longrightarrow G_{qs}$, where G_{qs} is the quasi-split inner form of G. In [Gr], the measure $|\omega'_G|$ on $G(k)$ was defined to be $\varphi^*|\omega'_{G_{qs}}|$. Then we have

Corollary 7.4. *If $char(k) = 0$, then $|\omega_G| = |\omega'_G|$. Furthermore, let $J \subset G(k)$ be an Iwahori subgroup. Then,*

$$\int_J |\omega_G| = q^{-N} \cdot det(1 - Fw_G|E(1)^I)$$

with $N = \sum(d-1)dimV_d^I$, F the geometric Frobenius in Γ/I with eigenvalue q^{-1} on $\mathbb{Q}(1)$, and w_G the element of the Weyl group W^I associated to the inner twisting $\varphi : G \longrightarrow G_{qs}$ over the maximal unramified extension of k.

Proof. This was established for $|\omega'_G|$ in [Gr, §4]. Note that if $G = G_{qs}$ is quasi-split, then $w_G = 1$. □

8. The Space of Haar Measures

Let P_G be the one-dimensional real vector space of invariant measures on $G(k)$, and let P_G^+ be the cone of positive Haar measures in P_G. We define, from $|\omega_G|$, the following element of P_G^+:

$$\mu_G = |\omega_G| \cdot q^{-a(M)/2}. \tag{8.1}$$

Let $\varphi : G \longrightarrow G'$ be an isomorphism over k_s. We define an $\mathbb{R}$-linear map

$$\varphi^* : P_{G'} \longrightarrow P_G \tag{8.2}$$

as follows. Let μ' be an element of $P_{G'}$, and write $\mu' = c|\omega'|$, for some invariant differential ω' on G' over k, and $c \in \mathbb{R}$. Let $d \in k^\times/k^{\times 2}$ be the class of the map:

$$\delta(G) \cdot \delta(G') : \Gamma \longrightarrow \mu_2(k).$$

It follows from Proposition 3.6 that the differential

$$\omega = \varphi^*(\omega')/\sqrt{d}$$

on G is defined over k. We then define

$$\varphi^*(\mu') = c|\omega| \cdot |d|^{\frac{1}{2}} \in P_G. \tag{8.3}$$

This is independent of the choice of ω' and d, and we have the following result.

Proposition 8.4. *The map $\varphi^* : P_{G'} \longrightarrow P_G$ is an $\mathbb{R}$-linear isomorphism, which maps $P_{G'}^+$ to P_G^+. Furthermore, $\varphi^*(\mu_{G'}) = \mu_G$.*

The isomorphism $P_{G'} \cong P_G$ is independent of the choice of the isomorphism $\varphi : G \longrightarrow G'$ over k_s.

Proof. All the statements will follow once we show that $\varphi^*(\mu_{G'}) = \mu_G$. This identity follows from a comparison of G and G' with the split group G_0 over k_s. Indeed, $\mu_{G_0} = |\omega_0|$, and for $\varphi : G \longrightarrow G_0$, we have

$$\begin{aligned}\mu_G &= |\omega_G| q^{-a(M)/2} \\ &= |\varphi^*(\omega_0)/\sqrt{D_{M_G}}| \cdot |D_{M_G}|^{\frac{1}{2}} \\ &= \varphi^*(\mu_{G_0}).\end{aligned}$$

□

9. Global Measure

In this section, we assume that k is a global field. Let ω be a non-zero invariant differential on G over k, and let $|\omega|_v$ be the associated Haar measure on $G(k_v)$, for each place v. We define the **global conductor** $f(M)$ of the motive M of G by the formula:

$$f(M) = \prod_{v \text{ finite}} q_v^{a(M/k_v)}. \tag{9.1}$$

This product is finite because for almost all v, G is unramified over k_v and $a(M/k_v) = 0$. The conductor $f(M)$ is an integer ≥ 1. If G is an inner form of a split group over k, then $f(M) = 1$.

If v is finite, let $|\omega_{G_v}|$ be the Haar measure on $G(k_v)$ defined in §5. If v is archimedean, let $|\omega_{G_v}|$ be the measure on $G(k_v)$ defined in [Gr, §11]. In the archimedean case, we can also define $|\omega_{G_v}|$ as follows. Let G_0 be the split form of G over k_v, and $G_{0,\mathbb{Z}}$ the Chevalley model for G_0 over $\mathbb{Z}$. Let ω_0 be an invariant differential on G_0 which generates the free $\mathbb{Z}$-module $Hom(\wedge^{top} Lie(G_{0,\mathbb{Z}}), \mathbb{Z})$. Then

ω_0 is determined up to sign. If $\varphi : G \longrightarrow G_0$ is an isomorphism over k_s, and K (respectively K_0) is the maximal compact subgroup of G (respectively G_0), then,

$$\omega_{G_v} = \frac{\varphi^*(\omega_0)}{i^{dim(G/K)-dim(G_0/K_0)}} \tag{9.2}$$

is defined on G over k_v, and is determined up to sign. The Haar measure $|\omega_{G_v}|$ is thus well-defined.

Proposition 9.3. *We have $|\omega|_v = |\omega_{G_v}|$ for almost all v, and the following product formula holds:*

$$\prod_v \frac{|\omega_{G_v}|}{|\omega|_v} = f(M)^{\frac{1}{2}} \text{ in } \mathbb{R}_+^\times.$$

Proof. For almost all v, G is unramified over k_v, and ω generates the A_v-module of invariant differentials on the reductive model $\underline{G}$ over A_v. At these places, $|\omega|_v = |\omega_{G_v}|$.

For v finite, let $\mu_{G_v} = |\omega_{G_v}|q^{-a(M/k_v)/2}$ as in (8.1). For v archimedean, let $\mu_{G_v} = |\omega_{G_v}|$. Then the product formula is equivalent to the statement

$$\prod_v \frac{\mu_{G_v}}{|\omega|_v} = 1. \tag{9.4}$$

This is independent of the choice of $\omega \neq 0$, by the usual product formula: $\prod_v |\alpha|_v = 1$, for $\alpha \in k^\times$.

We first prove (9.4) for $G = G_0$ split over k. In this case, we take ω_0 to generate the Chevalley differentials over $\mathbb{Z}$; then ω_0 is determined up to sign, and $|\omega_0|_v = |\omega_{G_v}| = \mu_{G_v}$ for all v. Hence (9.4) holds, because all the terms are 1.

Now let G be arbitrary, and choose an isomorphism $\varphi : G \longrightarrow G_0$ with the split form over k_s. Let $d \in k^\times/k^{\times 2}$ be in the class of $\delta(G)$, let ω_0 be as above, and let $\omega = \varphi^*(\omega_0)/\sqrt{d}$ over k. Then we have, for all v

$$\frac{\mu_{G_v}}{|\omega|_v} = |d|_v^{\frac{1}{2}} \cdot \frac{\mu_{(G_0)_v}}{|\omega_0|_v} = |d|_v^{\frac{1}{2}}.$$

Since $\prod_v |d|_v^{\frac{1}{2}} = 1$, the proposition is proved. □

Remarks. This gives a proof of Theorem 11.5 in [Gr, §11], when k is a number field. Indeed, we have shown in §7 that $|\omega_G| = |\omega'_G|$, where $|\omega'_G|$ is the Haar measure defined in [Gr]. Also the constant $\varepsilon(M)$ in the functional equation of the L-function of M is given by the formula:

$$\varepsilon(M) = |d_k|^{\frac{dim(G)}{2}} f(M)^{\frac{1}{2}} \tag{9.5}$$

where d_k is the discriminant of k over $\mathbb{Q}$. It also gives a proof of Theorem 11.5 when k is a function field, assuming that G has finite fundamental group of order prime to $char(k)$, and putting $|d_k| = q^{2g-2}$ as in [P].

10. Mass Formulae

We can use Proposition 9.3 to derive a number of explicit mass formulae. Let k be a totally real number field, and let G be a connected, reductive group over k, with $G(k \otimes \mathbb{R}) = \prod_{v|\infty} G(k_v)$ compact. Recall that $M = \bigoplus_{d \geq 1} V_d(1-d)$, and let

$$\Lambda(M, s) = \prod_v L_v(M, s) = \prod_{d \geq 1} \Lambda(V_d, s + 1 - d)$$

be the global L-function of the motive M, so that

$$\Lambda(M,s) = L_\infty(M,s)L(M,s) \tag{10.1}$$

where $L(M,s)$ is the usual Artin L-function of M. We have Artin's functional equation [T, pg. 18-19]

$$\Lambda(M,s) = \varepsilon(M,s)\Lambda(M^\vee, 1-s) \tag{10.2}$$

with

$$\varepsilon(M,s) = \Big(|d_k|^{dim(G)} f(M)\Big)^{\frac{1}{2}-s}. \tag{10.3}$$

In particular, taking $\Lambda(M) = \Lambda(M,0)$, we find that

$$\Lambda(M) = |d_k|^{\frac{dim(G)}{2}} f(M)^{\frac{1}{2}} \Lambda(M^\vee(1)). \tag{10.4}$$

Now let $\mathbb{A}$ be the ring of adeles of k, and let $K = G(k\otimes\mathbb{R}) \times \prod_{v \text{ finite}} K_v$ be an open compact subgroup of $G(\mathbb{A})$. The double coset space

$$\Sigma = G(k)\backslash G(\mathbb{A})/K$$

is then finite. If $\sigma \in \Sigma$, and $g \in G(\mathbb{A})$ represents the class of σ, then

$$\Gamma_\sigma = G(k) \cap gKg^{-1}$$

is a finite arithmetic subgroup of $G(k)$, of order w_σ. We define

$$Mass_K = \sum_\sigma \frac{1}{w_\sigma} \tag{10.5}$$

where the sum is taken over all σ in the double coset space Σ.

If μ_K is the unique Haar measure on the locally compact group $G(\mathbb{A})$ giving the open compact subgroup K volume 1, then we also have

$$Mass_K = \int_{G(k)\backslash G(\mathbb{A})} \mu_K. \tag{10.6}$$

Proposition 10.7. *Assume that G is quasi-split over k_v for all finite places v, and that $K_v = \underline{G}^0(A_v) \subset G(k_v)$ is the special open compact subgroup defined in* [Gr, §4]. *Then,*

$$Mass_K = \tau(G)\cdot\frac{1}{2^n}\cdot L(M)$$

where $\tau(G)$ is the Tamagawa number of G, n is the rank of the complex Lie group $G(k\otimes\mathbb{C})$, and $L(M) = L(M,0)$.

Remarks. Note that if l is the rank of G over k_s and d is the degree of k over $\mathbb{Q}$, then $n = ld$.

Proof. Let $\omega \neq 0$ be an invariant differential on G over k, and $|\omega|_v$ the associated Haar measure on $G(k_v)$. For v finite, if G is unramified over k_v, with reductive model $\underline{G}$ over A_v, and ω has good reduction ($mod\ \pi_v$), then,

$$\int_{\underline{G}(A_v)} L_v(M^\vee(1))|\omega|_v = 1.$$

Hence the product

$$\bigotimes_v L_v(M^\vee(1))|\omega|_v$$

defines a measure on $G(\mathbb{A})$. By definition, the Tamagawa measure $|\omega|_{\mathbb{A}}$ is given by

$$|\omega|_{\mathbb{A}} = \frac{\bigotimes_v L_v(M^\vee(1))|\omega|_v}{|d_k|^{\frac{dim(G)}{2}}\Lambda(M^\vee(1))}. \tag{10.8}$$

Note that this is well-defined since the fact that $G(k\otimes\mathbb{R})$ is compact implies that $\Lambda(M^\vee(1))$ is finite. Also, it is independent of the choice of $\omega \neq 0$. The Tamagawa number $\tau(G)$ is then defined by

$$\tau(G) = \int_{G(k)\backslash G(\mathbb{A})} |\omega|_{\mathbb{A}}. \tag{10.9}$$

On the other hand, the Haar measure μ_K on $G(\mathbb{A})$ is the product

$$\mu_K = \mu_{G(k\otimes\mathbb{R})} \otimes \prod_{v \text{ finite}} |\omega_{G_v}|L_v(M^\vee(1)) \tag{10.10}$$

where $\mu_{G(k\otimes\mathbb{R})}$ is the measure giving $G(k\otimes\mathbb{R})$ volume 1. Indeed, by Corollary 7.3, we have $|\omega_{G_v}| = |\omega'_{G_v}|$, and the latter measure is constructed such that

$$\int_{K_v} |\omega'_{G_v}|L_v(M^\vee(1)) = 1.$$

By [Gr, §7], we have

$$\mu_{G(k\otimes\mathbb{R})} \cdot 2^n \prod_{v|\infty} L_v(M)e_v(G) = \prod_{v|\infty} |\omega_{G_v}|L_v(M^\vee(1)). \tag{10.11}$$

In fact, $\prod_{v|\infty} e_v(G) = 1$ as G is quasi-split at all finite places of k (cf. [K]). Hence,

$$\mu_K = 2^{-n} \prod_{v|\infty} L_v(M)^{-1} \prod_v |\omega_{G_v}|L_v(M^\vee(1)). \tag{10.12}$$

By Proposition 9.3, we also have the formula

$$\prod_v \frac{|\omega_{G_v}|L_v(M^\vee(1))}{|\omega|_v L_v(M^\vee(1))} = f(M)^{\frac{1}{2}}.$$

Hence, we have

$$\begin{aligned}
\mu_K &= 2^{-n} \prod_{v|\infty} L_v(M)^{-1} \cdot \Lambda(M^\vee(1))|d_k|^{\frac{dim(G)}{2}} f(M)^{\frac{1}{2}}|\omega|_{\mathbb{A}} \\
&= 2^{-n} \prod_{v|\infty} L_v(M)^{-1} \cdot \Lambda(M)|\omega|_{\mathbb{A}} \\
&= 2^{-n} L(M)|\omega|_{\mathbb{A}}
\end{aligned}$$

and the mass formula follows from (10.9). □

Even if G is not quasi-split at all finite places v, one can obtain an explicit mass formula, by replacing K_v at the bad primes by an Iwahori subgroup $J_v \subset G(k_v)$, and using Corollary 7.4. We leave the details to the reader.

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