ON THE NUMBER OF TERMS IN THE MIDDLE OF ALMOST SPLIT SEQUENCES OVER TAME ALGEBRAS

J. A. DE LA PEÑA AND M. TAKANE

ABSTRACT. Let A be a finite dimensional tame algebra over an algebraically closed field k . It has been conjectured that any almost split sequence $0 \rightarrow X \rightarrow \bigoplus_{i=1}^n Y_i \rightarrow Z \rightarrow 0$ with Y_i indecomposable modules has $n \leq 5$ and in case $n = 5$, then exactly one of the Y_i is a projective-injective module. In this work we show this conjecture in case all the Y_i are directing modules, that is, there are no cycles of non-zero, non-iso maps $Y_i = M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_s = Y_i$ between indecomposable A -modules. In case, Y_1 and Y_2 are isomorphic, we show that $n \leq 3$ and give precise information on the structure of A .

Let A be a finite dimensional algebra over an algebraically closed field k . We denote by mod_A the category of finite dimensional left A -modules (an object in mod_A is simply called a module).

For a non-projective indecomposable module X , there exist an indecomposable non-injective module $\tau_A X$ called the Auslander-Reiten translate and an almost split sequence $0 \rightarrow \tau_A X \rightarrow E \rightarrow X \rightarrow 0$ (see [2], [18]). Since their introduction, almost split sequences have played a central role in the representation theory of algebras (see for example [2]).

For an almost split sequence $0 \rightarrow \tau_A X \rightarrow E \rightarrow X \rightarrow 0$, consider the indecomposable decomposition $E = \bigoplus_{i=1}^{s(X)} Y_i$. There has been considerable attention paid to the relation between properties of the algebra A and the values $s(X)$ for different modules X (and of course between properties of X and the value $s(X)$). Among other interesting results we recall that if A is representation finite, then $s(X) \leq 4$, for every indecomposable non-projective module X [3], [7] (see also [10] and [12]).

It has been conjectured by S. Brenner that, for A a tame algebra, $s(X) \leq 5$ for every indecomposable non-projective module X . This is known to hold for many examples, in particular for the important case of hereditary tame algebras.

To state the main results of this work, we recall some concepts. A *cycle* in mod_A is a sequence $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} X_s = X_0$ of non-zero non-isomorphism maps between indecomposable modules; the cycle is said to be *finite* if $f_i \notin \text{rad}_A^\infty(X_{i-1}, X_i)$, for all $i = 1, \dots, s$. An indecomposable module X is said to be *directing* if it does not belong to a cycle in mod_A . The algebra A is said to be *cycle-finite* if all cycles in mod_A are finite. We recall that a cycle finite algebra is tame [1].

The *Coxeter matrix* ϕ_A of A and its *spectral radius* $\rho(\phi_A) = \max \{ \|\lambda\| : \lambda \text{ eigenvalue of } \phi_A \}$ are important invariants (see for example [5], [14], [17], [20]). In case

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A is a triangular algebra, we denote by

$$\tilde{\rho}(A) = \max \{ \rho(\phi_B) : B = A/AeA \text{ for some idempotent } e \in A \}.$$

Theorem 1. *Assume that A is a triangular algebra such that*

$$\tilde{\rho}(A) \leq \frac{t - 2 + \sqrt{t^2 - 4t}}{2}$$

for some natural number $t \geq 4$. Let $0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence in mod_A such that Y_i is directing ($i = 1, \dots, s(X)$). Then $s(X) \leq t + 1$.

Theorem 2. *Assume A is a tame algebra and let $0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence in mod_A such that Y_i is directing ($i = 1, \dots, s(X)$). Then the following holds:*

- (a) $s(X) \leq 5$. Moreover, if $s(X) = 5$, then for some $j \in \{1, \dots, s(X)\}$, the module Y_j is projective and injective.
- (b) If $s(X) \geq 3$, and $Y_1 \cong Y_2$, then $s(X) = 3$ and the module Y_3 is projective and injective.

Theorem 3. *Assume A is a cycle-finite algebra. Then for any almost split sequence $0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ we have $s(X) \leq 5$. If $s(X) = 5$, there is some Y_j ($1 \leq j \leq s$) which is projective and injective.*

We prove the theorems in section 2 after some preliminary considerations. We gratefully acknowledge support of CONACYT and DGAPA, UNAM.

1. CYCLES AND ALMOST SPLIT SEQUENCES

1.1. Let $H = k\Delta$ be the path algebra of a quiver Δ without oriented cycles (see [6]). A *tilting module* T in mod_H satisfies: $\text{Ext}_H^1(T, T) = 0$ and there is an exact sequence $0 \rightarrow H \rightarrow T' \rightarrow T'' \rightarrow 0$ with $T', T'' \in \text{add } T$ (see [18]). For a tilting module ${}_H T$, the algebra $B = \text{End}_H(T)$ is called a *tilted algebra* of type Δ .

We recall that an indecomposable B -module X is *sincere* if $\text{Hom}_B(P, X) \neq 0$ for every projective B -module P . If X is a sincere directing indecomposable B -module, then B is a tilted algebra [18, p. 375].

1.2. The *Auslander-Reiten quiver* Γ_A of A has as vertices representatives of the iso-classes of indecomposable modules; there are as many arrows $X \rightarrow Y$ in Γ_A as $\dim_k \text{rad}_A(X, Y) / \text{rad}_A^2(X, Y)$. A path $Y_0 \rightarrow Y_1 \rightarrow \dots \rightarrow Y_s$ in Γ_A is said to be *sectional* if $\tau_A Y_{i+1} \neq Y_{i-1}$ for all $i = 1, \dots, s-1$. In case $Y_0 = Y_s$, we have a *sectional cycle* if $\tau_A Y_{i+1} \neq Y_{i-1}$ for all $i = 1, \dots, s \bmod s$. A sectional path in Γ_A contains no sectional cycle [4].

A component $\mathcal{C}$ of Γ_A is *directing* if it is formed by directing modules.

1.3. Let $\eta: 0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence in mod_A and let $B(\eta)$ be a quotient A/AeA , with e idempotent in A , of minimal dimension such that η is formed by $B(\eta)$ -modules. By [18], one of the modules $\tau_A X$, X or Y_i ($1 \leq i \leq s(X)$) is sincere as $B(\eta)$ -module.

Lemma. Let $\eta: 0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence in mod_A such that Y_i is directing as $B(\eta)$ -module, for $i = 1, \dots, s(X)$. Then

- (a) X and $\tau_A X$ are directing as $B(\eta)$ -modules;
- (b) $B(\eta)$ is a tilted algebra.

Proof. Consider the almost split sequence $\eta: 0 \rightarrow \tau_A X \xrightarrow{(h_i)} \bigoplus_{i=1}^{s(X)} Y_i \xrightarrow{(g_i)} X \rightarrow 0$.

Let $X = Z_0 \xrightarrow{f_1} Z_1 \rightarrow \dots \rightarrow Z_{s-1} \xrightarrow{f_s} Z_s = X$ be a cycle in $\text{mod}_{B(\eta)}$. The map $f_s: Z_{s-1} \rightarrow Z_s = X$ factorizes through the sink map $(g_i)_i: \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X$ and for some $j \in \{1, \dots, s(X)\}$ we get a non-zero map $f'_s: Z_{s-1} \rightarrow Y_j$. Therefore we get a cycle

$$Y_j \xrightarrow{g_j} X = Z_0 \xrightarrow{f_1} Z_1 \rightarrow \dots \rightarrow Z_{s-1} \xrightarrow{f'_s} Y_j.$$

A contradiction. Hence X (and similarly $\tau_A X$) is directing. The algebra $B(\eta)$ is tilted by (1.1). $\square$

1.4 Proposition. Let X be a directing indecomposable non-projective module and

$\eta: 0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be the corresponding almost split sequence.

Assume $s(X) \geq 3$. Then one of the following conditions holds:

- (a) $B(\eta)$ is a tilted algebra and X belongs to a directing component of $\Gamma_{B(\eta)}$;
- (b) there exists a sectional path $\tau_A X = Z_0 \rightarrow Z_1 \rightarrow \dots \rightarrow Z_s$ in $\Gamma_{B(\eta)}$ with Z_s an injective module.

Proof. We shall denote $B = B(\eta)$. Assume first that Y_i is directing as B -module for $i = 1, \dots, s(X)$. By (1.3), B is a tilted algebra. By [9], we know the structure of Γ_B : there is a postprojective, a preinjective and a connecting component (some of these components may coincide) and components of type $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/(n)$, possibly with inserted ray modules or coinserted coray modules. Since $s(X) \geq 3$, then X belongs to a postprojective, preinjective or connecting component of Γ_B , all of which are directing. Hence (a) holds.

Assume Y_1 is not directing. Let $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \rightarrow \dots \xrightarrow{f_s} Z_s = Y_1$ be a cycle in mod_B . By (1.2), we may assume that one of the following situations occurs:

- (1) $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \rightarrow \dots \xrightarrow{f_r} Z_r \xrightarrow{f_{r+1}} Z_{r+1} \rightarrow \dots \rightarrow Y_1$ such that $Z_0 \xrightarrow{f_1} Z_1 \rightarrow \dots \xrightarrow{f_r} Z_r$ is a sectional path in Γ_B and $\tau_B Z_{r+1} = Z_{r-1}$;
- (2) for all $i \in \mathbb{N}$, there is a sectional path in Γ_B , $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \rightarrow \dots \xrightarrow{f_i} Z_i$ and a path $Z_i \rightarrow Z'_{i+1} \rightarrow \dots \rightarrow Y_1$ in mod_B .

If some Z_i is injective ($1 \leq i \leq r$ in case (1) or $1 \leq i$ in case (2)), then (b) holds. We assume that no Z_i is injective in order to get a contradiction.

First observe that situation (1) cannot happen. Otherwise, we get a cycle

$$Y_1 \xrightarrow{h_1} X \rightarrow \tau_B^- Y_1 \rightarrow \tau_B^- Z_1 \rightarrow \dots \tau_B^- Z_{r-1} = Z_{r+1} \rightarrow \dots \rightarrow Y_1 \xrightarrow{h_1} X,$$

where $h_1: Y_1 \rightarrow X$ is an irreducible map. Contradicting that X is a directing A -module.

Let n be the number of iso-classes of simple A -modules and consider a sectional path $Y_1 = Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2 \rightarrow \cdots \xrightarrow{f_n} Z_n$ in Γ_B as given in situation (2). Moreover, there is a path $Z_n \rightarrow Z'_{n+1} \rightarrow \cdots \rightarrow Y_1$ in mod_B . We shall prove that $\bigoplus_{i=0}^n Z_i$ is a partial cotilting module, which yields the desired contradiction. Indeed, since the number of summands of the partial cotilting module is not bigger than n , then we get $Z_i \cong Z_j$ for some $j > i$. By (1.2), the cycle $Z_i \rightarrow \cdots \rightarrow Z_{j-1} \rightarrow Z_j$ is not sectional and hence $Z_{j-1} \cong \tau_B Z_{i+1}$ or $Z_{j-2} \cong \tau_B Z_i$ which yields a cycle through X as above.

Let us first show that $i \dim_B Z_i \leq 1$, $i = 0, \dots, n$. Otherwise $i \dim_B Z_i > 1$ and there are an indecomposable projective B -module P and a map $0 \neq g \in \text{Hom}_B(\tau_B^- Z_i, P)$ (see [18]). Since $\bigoplus_{i=1}^{s(X)} Y_i$ is a sincere B -module, there are some $j \in \{1, \dots, s(X)\}$ and a map $0 \neq g' \in \text{Hom}_B(P, Y_j)$. We get a cycle

$$X \rightarrow \tau_B^- Y_j \rightarrow \cdots \rightarrow \tau_B^- Z_i \xrightarrow{g} P \xrightarrow{g'} Y_j \xrightarrow{h_j} X,$$

again a contradiction.

Let $i, j \in \{0, \dots, n\}$; we show that $\text{Ext}_B^1(Z_i, Z_j) = 0$. Otherwise, there is a map $0 \neq g \in \text{Hom}_B(\tau_B^- Z_j, Z_i)$. Then we get a cycle

$$\begin{aligned} X \rightarrow \tau_B^- Y_j \rightarrow \cdots \rightarrow \tau_B^- Z_j \xrightarrow{g} Z_i \xrightarrow{f_{i+1}} Z_{i+1} \rightarrow \cdots \\ \cdots \rightarrow Z_n \rightarrow \cdots \rightarrow Y_1 \xrightarrow{h_i} X, \end{aligned}$$

and a contradiction. This shows that $\bigoplus_{i=0}^n Z_i$ is a partial cotilting module, which completes the proof. $\square$

1.5. We say that X is a *predecessor* of Y in Γ_A (and Y a *successor* of X) if there is a path $X = Z_0 \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_s = Y$ in Γ_A .

Proposition. Let $\eta: 0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence such that $Y_1 \cong Y_2$. Then

- (a) If $\dim_k \tau_A X < \dim_k Y_1$, then $\tau_A^{-n} X$ and $\tau_A^{-n} Y_i$ ($1 \leq i \leq s(X)$) are well defined for all $n \geq 0$. Moreover, X has no injective successors and does not belong to any oriented cycle in Γ_A .
- (b) If $\dim_k \tau_A X > \dim_k Y_1$, $\tau_A^n X$ and $\tau_A^n Y_1$ are well defined for all $n \geq 0$. Moreover, $\tau_A X$ has no projective predecessors and does not belong to any oriented cycle in Γ_A .

Proof. (a): Let $\eta: 0 \rightarrow \tau_A X \xrightarrow{(h_i)} \bigoplus_{i=1}^{s(X)} Y_i \xrightarrow{(g_i)} X \rightarrow 0$ be an almost split sequence with $Y_1 \cong Y_2$ and assume that h_1 is a monomorphism. Then for all $1 \leq i \leq s(X)$, $\dim_k Y_i < \dim_k X$. In particular, no Y_i is injective. Since the almost split sequence starting at Y_1 has X at least twice as a summand of its middle term, then the above argument implies that X is not injective. Moreover, $\dim_k \tau_A X < \dim_k X < \dim_k \tau_A^- X$. We proceed in this way to show that $\tau_A^{-n} X$ and $\tau_A^{-n} Y_i$ ($1 \leq i \leq s(X)$) are well defined for all $n \geq 0$.

Assume that Y is a minimal injective successor of X . Then we find a sectional path $\tau_A^s X = Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} Z_t = Y$, for some $s \geq 0$. Since there is an irreducible monomorphism $Z_1 \rightarrow \tau_A^- Z_0$, then Z_1 is not injective and there is an irreducible monomorphism $Z_2 \rightarrow \tau_A^- Z_1$. Continuing this way, we find a monomorphism $Z_t \rightarrow \tau_A^- Z_{t-1}$, which is a contradiction.

Let Γ be the right stable component of Γ_A containing X . Then Γ contains all the successors of X and it has no periodic modules (since $\dim_k \tau_A^{-n} X$ grows with $n \geq 0$ and hence X is not periodic). If X belongs to an oriented cycle in Γ_A , then the cycle belongs to Γ and by [10], Γ has trivial valuation, a contradiction.

(b): With the notation above, assume that h_1 is an epimorphism. In particular, Y_1 is not projective. Since the almost split sequence ending at Y_1 has $\tau_A X$ at least twice as a summand of its middle term, then $\dim_k \tau_A X < \dim_k \tau_A Y_1$ and $\tau_A X$ is not projective. By (a), $\dim_k \tau_A^2 X > \dim_k \tau_A Y_1$. We proceed by induction to show that $\tau_A^n X$ and $\tau_A^n Y_1$ are well defined for all $n \geq 0$.

If, for some $3 \leq j \leq s(X)$, the module Y_j is not projective, then $\dim_k \tau_A^2 X > \dim_k \tau_A Y_j$. By induction, $\tau_A^n Y_j$ is well defined. The second part of the statement follows from dual arguments to those used in (a). $\square$

1.6. The following results of S. Liu are important in the proof of our theorems.

Proposition [9]. Let $0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence in mod_A . Then

- (a) Assume there is a sectional path $\tau_A X = Z_0 \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_t$ in Γ_A with Z_t injective. Assume moreover that $s(X) > 4$ or that $s(X) = 4$ but no Y_i is injective ($i = 1, \dots, s(X)$). Then $\tau_A X$ has no projective predecessor in Γ_A and $\tau_A X$ does not belong to a finite cycle in mod_A .
- (b) If $\tau_A X$ belongs to a finite cycle in mod_A , then $s(X) \leq 4$ and in case $s(X) = 4$, there is a Y_i ($1 \leq i \leq s(X)$), which is projective.

1.7. The following result generalizes [3] quoted in the introduction.

Theorem [9]. Let $\eta: 0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence such that $\tau_A X$ has a projective predecessor and X has an injective successor in Γ_A . Then $s(X) \leq 4$ and in case $s(X) = 4$, there is a Y_i ($1 \leq i \leq 4$) which is projective and injective.

2. PROOF OF THE MAIN RESULTS

2.1. *Proof of the Theorem 1.* Let $0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ be an almost split sequence in mod_A and assume that Y_i is directing ($1 \leq i \leq s(X)$). Assume that $\tilde{\rho}(A) \leq \frac{t-1+\sqrt{t^2-4t}}{2}$ for a natural number $t \geq 4$. We assume that $s = s(X) \geq 5$.

Let $B = B(\eta)$. By the proof of (1.4), we may assume that B is a tilted algebra and X belongs to a directing component $\mathcal{C}$ of Γ_B .

By (1.7), we may assume that X has no injective successors in $\mathcal{C}$. Suppose that Y_s is injective in mod_B . Since $\tau_A X \rightarrow Y_s$ is a sectional map, by (1.6), we get that $\tau_A X$ has no projective predecessors in $\mathcal{C}$. Moreover, no Y_i , $1 \leq i \leq s-1$, is

projective. Indeed, if Y_1 were projective, then

$$\dim_k Y_1 + \dim_k Y_s > \dim_k \tau_A X + \dim_k X = \sum_{i=1}^s \dim_k Y_i,$$

a contradiction. By duality, we may assume that one of the following situations holds:

- (1) $0 \rightarrow \tau_B^{m+1} X \rightarrow \bigoplus_{i=1}^s \tau_B^m Y_i \rightarrow \tau_B^m X \rightarrow 0$ is a well-defined almost split sequence in mod_B , for all $m \geq 0$;
- (2) Y_s is projective and injective and $0 \rightarrow \tau_B^{m+1} X \rightarrow \bigoplus_{i=1}^{s-1} \tau_B^m Y_i \rightarrow \tau_B^m X \rightarrow 0$ is a well-defined almost split sequence in mod_B , for all $m \geq 1$.

In case (1) we shall prove that $s \leq t$; in case (2), the same proof yields $s - 1 \leq t$ and hence the result.

Assume that Y_s is not projective (i.e. case (1)). Consider the Grothendieck group $K_0(B)$ and the Coxeter matrix ϕ_B as a linear transformation $\phi_B: K_0(B) \rightarrow K_0(B)$. Since B is tilted, there are a quiver Δ and a tilting $k\Delta$ -module T such that $B = \text{End}_{k\Delta}(T)$. Therefore, there is an isometry $\sigma: K_0(k\Delta) \rightarrow K_0(B)$, $[L] \mapsto [\text{Hom}_{k\Delta}(T, L)] - [\text{Ext}_{k\Delta}^1(T, L)]$, where $[Y]$ denotes the class in $K_0(B)$ of a B -module Y . Moreover, $\phi_B \sigma = \sigma \phi_\Delta$ (see [18]). In particular, $\rho(\phi_B) = \rho(\phi_\Delta)$.

Observe that Δ contains the subquiver $\Delta(s_1, \dots, s_m)$ formed by one source 0, sinks $1, \dots, m$ and s_i arrows from 0 to i such that $\sum_{i=1}^m s_i = s$. Indeed, $\Delta = \mathcal{S}^{op}$, where $\mathcal{S}$ is the slice in the component $\mathcal{C}$ formed as the full subquiver of $\mathcal{C}$ whose vertices Z are the starting points of sectional paths to X . Since $\mathcal{S}$ contains the arrows $Y_i \rightarrow X$, $i = 1, \dots, s$, we get the claim (of course, m is the number of isoclasses among the Y_i). Moreover, by [20], we have $\rho(\phi_\Delta) \geq \rho(\phi_{\Delta(s_1, \dots, s_m)})$.

On the other hand, $\rho(\phi_{\Delta(s_1, \dots, s_m)}) = \frac{b-2+\sqrt{b^2-4b}}{2}$, where $b = \sum_{i=1}^m s_i^2$.

Hence $\rho(\phi_{\Delta(s_1, \dots, s_m)}) \geq \frac{s-2+\sqrt{s^2-4s}}{2} = \rho(\phi_{\Delta_0})$, where

Altogether we get

$$\frac{t-2+\sqrt{t^2-4t}}{2} \geq \tilde{\rho}(A) \geq \rho(\phi_B) = \rho(\phi_\Delta) \geq \rho(\phi_{\Delta_0}) = \frac{s-2+\sqrt{s^2-4s}}{2}.$$

Hence $t \geq s$, as desired. $\square$

2.2. We recall that the algebra A is said to be *tame* if for every $d \in \mathbb{N}$ there are finitely many $A - k[t]$ -bimodules $M_1, \dots, M_s$ which are finitely generated free as right $k[t]$ -modules and such that every indecomposable A -module X with $\dim_k X = d$ is isomorphic to $M_i \otimes_{k[t]} S$ for some $1 \leq i \leq s$ and some simple $k[t]$ -module S .

From now on we will assume that A is a basic connected algebra of the form $A = kQ/I$ (see [6]). The following result follows an idea of von Höhne (see [12]).

Proposition. *Let $A = kQ/I$ be a tame algebra and let X be a non-projective indecomposable module such that the almost split sequence $0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^s Y_i \rightarrow X \rightarrow 0$ has directing middle terms Y_i , $i = 1, \dots, s$. Then for all vertices $i \in Q_0$ we have*

$$|\dim_k \tau_A X(i) - \dim_k X(i)| \leq 2.$$

Proof. Let $B = B(\eta)$. By (1.3), B is a tame tilted algebra. Then it easily follows that $p \dim_B X \leq 1$ and $\text{Hom}_B(X, {}_B B) = 0$. Therefore, by [18], $[\tau_A X] = [\tau_B X] = \phi_B[X]$.

Consider the Euler (non-symmetric) bilinear form $\langle -, - \rangle: K_0(B) \times K_0(B) \rightarrow \mathbb{Z}$ defined by $\langle [L], [M] \rangle = \sum_{i=0}^{\infty} (-1)^i \dim_k \text{Ext}_B^i(L, M)$ and the corresponding quadratic form χ_B . Since $\text{gl dim } B \leq 2$, then χ_B is also the Tits form of B and therefore χ_B is weakly non-negative (see for example [12]). We have $\chi_B([X]) = 1 = \chi_B([P_i])$, for the indecomposable projective module P_i corresponding to the vertex $i \in Q_0$. Then

$$\begin{aligned} 0 \leq \chi_B([X] + [P_i]) &= 2 + \langle [P_i], [X] \rangle + \langle [X], [P_i] \rangle \\ &= 2 + \dim_k X(i) - \langle [P_i], \phi_B[X] \rangle \\ &= 2 + \dim_k X(i) - \dim_k \tau_A X(i). \end{aligned}$$

Hence, $\dim_k \tau_A X(i) - \dim_k X(i) \leq 2$ and similarly we get the other inequality. $\square$

Corollary. *Let A be a tame algebra and $\mathcal{C}$ be a directing component of Γ_A . Let $X \in \mathcal{C}$ be such that $\tau_A^m X$ is well-defined for all $m \geq 0$. Then $\lim_{m \rightarrow \infty} \sqrt[m]{\dim_k \tau_A^m X} = 1$.*

2.3. Proof of the Theorem 2. Let A be a tame algebra and consider an almost split sequence $\eta: 0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ such that Y_i is directing, $1 \leq i \leq s(X)$.

(a): Assume that $s = s(X) \geq 5$.

Let $B = B(\eta)$ be a tilted algebra by (1.4) and assume that X belongs to a directing component $\mathcal{C}$ of Γ_B . As in (2.1), we may assume that one of the following situations occurs:

- (1) $0 \rightarrow \tau_B^{m+1} X \rightarrow \bigoplus_{i=1}^s \tau_B^m Y_i \rightarrow \tau_B^m X \rightarrow 0$ is well-defined for all $m \geq 0$;
- (2) Y_s is projective and injective and $0 \rightarrow \tau_B^{m+1} X \rightarrow \bigoplus_{i=1}^{s-1} \tau_B^m Y_i \rightarrow \tau_B^m X \rightarrow 0$ is well-defined for $m \geq 1$.

Assume that case (1) holds.

Consider P' the direct sum of all indecomposable projective modules in $\mathcal{C}$. Let $P' = Be'$ and $C = B/Be'B$ which is a quotient of A by the two-sided ideal generated by an idempotent element. Observe that for $m \geq 1$, $\tau_B^m X$ is a C -module. Indeed, otherwise an indecomposable direct summand P of P' would be a predecessor of $\tau_B^m X$, for some $m \geq 1$, and hence a predecessor of $\tau_B X$, a contradiction to (1.6).

Moreover, C is a tilted algebra. Indeed, consider $\mathcal{S}$ the slice in $\mathcal{C}$ formed by those $Z \in \mathcal{C}$ such that there is a sectional path from Z to X (see [18]). We claim that any indecomposable projective P in $\mathcal{C}$ belongs to $\mathcal{S}$. Indeed, P is a predecessor

of some Y_j ($1 \leq j \leq s$) because $\bigoplus_{i=1}^s Y_i$ is a sincere B -module. Consider a path $P = Z_0 \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_t = Y_j \rightarrow X$ and assume it is not sectional. Hence for some $1 \leq i \leq t-1$, $\tau_B Z_{i+1} = Z_{i-1}$. We may assume that the modules Z_i , $1 \leq i \leq t$, are not projectives. Then we get a path

$$P \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_{i-1} \rightarrow \tau_B Z_{i+2} \rightarrow \tau_B Z_{i+3} \rightarrow \cdots \rightarrow \tau_B Y_j \rightarrow \tau_B X,$$

a contradiction. Therefore, $\tau_B \mathcal{S}$ is a slice in Γ_C and C is a tilted algebra.

We shall consider $N = \tau_B X$ as C -module. Observe that $\tau_C^m N = \tau_B^{m+1} X$ is well-defined. Consider the Grothendieck group $K_0(C)$ of C and the Coxeter matrix ϕ_C as a linear transformation $\phi_C: K_0(C) \rightarrow K_0(C)$. We want to show that for $m \geq 0$, the class $[\tau_C^m N] \in K_0(C)$ satisfies $[\tau_C^m N] = \phi_C^m[N]$. By [18], it is enough to show that $p \dim_C \tau_C^m N \leq 1$ and $\text{Hom}_C(\tau_C^m N, {}_C C) = 0$ for $m \geq 0$.

Indeed, if $p \dim_C \tau_C^m N > 1$, there is a map $0 \neq g \in \text{Hom}_C(I, \tau_C^{m+1} N)$ for some indecomposable injective C -module I . As B -module, I is also injective (otherwise, $\tau_B^- I$ is a predecessor of $\tau_C^m N = \tau_B^{m+1} X$ and $\tau_B^- I \in \text{mod}_C$). Hence there is a map $0 \neq g' \in \text{Hom}_B(Y_j, I)$ for some $1 \leq j \leq s$, because $\bigoplus_{i=1}^s Y_i$ is sincere. We get a cycle,

$$Y_j \xrightarrow{g'} I \xrightarrow{g} \tau_C^{m+1} N = \tau_B^{m+2} X \rightarrow \cdots \rightarrow \tau_B X \rightarrow Y_j,$$

a contradiction. Now, assume $0 \neq g \in \text{Hom}_C(\tau_C^m N, \bar{P})$ for some indecomposable projective C -module $\bar{P}$. Since $\tau_B \mathcal{S}$ is a slice in Γ_C , then $\bar{P}$ is a predecessor of N and as B -module, $\bar{P} \in \mathcal{C}$ (use convexity of $\mathcal{C}$, see [15]). Therefore $\bar{P}$ is also projective as B -module (otherwise $\tau_B \bar{P}$ is defined and $\text{Hom}_B(P', \tau_B \bar{P}) = 0$, for any projective P' in $\mathcal{S}$; hence $\tau_B \bar{P} \in \text{mod}_C$, a contradiction). This contradicts (1.6).

Let Δ be a quiver such that $C = \text{End}_{k\Delta}(T)$ for a tilting module T . Consider the isometry $\sigma: K_0(k\Delta) \rightarrow K_0(C)$, $[L] \mapsto [\text{Hom}_{k\Delta}(T, L)] - [\text{Ext}_{k\Delta}^1(T, L)]$, which satisfies $\phi_C \sigma = \sigma \phi_\Delta$. Observe that T does not have preinjective direct summands (since there are no projective modules in the connecting component of Γ_C) and since N is a predecessor of a slice in Γ_C , then $N = \Sigma L$ for some preinjective $k\Delta$ -module L , with $\Sigma = \text{Hom}_{k\Delta}(T, -)$. Hence [18] implies that $\tau_C^m N = \Sigma \tau_{k\Delta}^m L$ and $[\tau_C^m N] = \sigma \phi_\Delta^m [L]$, for $m \geq 0$.

We have the following facts:

- (i) $\lim_{m \rightarrow \infty} \sqrt[m]{\dim_k \tau_C^m N}$ exists and equals $\lim_{m \rightarrow \infty} \sqrt[m]{\dim_k \tau_{k\Delta}^m L}$ (see [12] for a detailed proof).
- (ii) $\lim_{m \rightarrow \infty} \sqrt[m]{\dim_k \tau_{k\Delta}^m L} = \rho(\phi_\Delta)$ (see [5], [15]).
- (iii) Δ contains a subquiver $\Delta(s_1, \dots, s_m)$ with $s = \sum_{i=1}^m s_i$ (notation as in 2.1).

$$\text{By (2.2), } 1 = \lim_{m \rightarrow \infty} \sqrt[m]{\dim_k \tau_C^m N} = \rho(\phi_\Delta) \geq \rho(\phi_{\Delta(s_1, \dots, s_m)}) \geq \frac{s-2+\sqrt{s^2-4s}}{2}.$$

Therefore $s \leq 4$, contradicting our assumption on $s(X)$.

We get that situation (2) holds, that is, Y_s is projective and injective and for all $m \geq 0$, $0 \rightarrow \tau_B^{m+1} X \rightarrow \bigoplus_{i=1}^{s-1} \tau_B^m Y_i \rightarrow \tau_B^m X \rightarrow 0$ is well defined. Proceeding as above we get that $s-1 \leq 4$. Hence $s(X) = 5$ and Y_s is projective and injective.

(b): Assume that $s(X) \geq 3$ and that $Y_1 \cong Y_2$. Then by (1.5), we may assume that either case (1) or case (2) in part (a) of the proof holds. In case (1), the argument above yields $1 \geq \rho(\phi_{\Delta(s_1, \dots, s_m)})$, with $\sum_{i=1}^m s_i = s(X)$ and $s_1 \geq 2$, which

is impossible since $\rho(\phi_{\Delta(2,1)})$ and $\rho(\phi_{\Delta(3)})$ are both strictly bigger than 1. Hence, case (2) holds and $2 = s_1 = s(X) - 1$. $\square$

2.4. Proof of Theorem 3. Assume A is a cycle-finite algebra, hence tame. Let X be an indecomposable non-projective module and $\eta: 0 \rightarrow \tau_A X \rightarrow \bigoplus_{i=1}^{s(X)} Y_i \rightarrow X \rightarrow 0$ an almost split sequence. Assume that $s = s(X) \geq 5$. Let $B = B(\eta)$.

If X is not directing, then X belongs to a finite cycle in mod_A . By (1.6), we should have $s \leq 4$. Then X and $\tau_A X$ are directing. In case all Y_i ($1 \leq i \leq s$) are directing, Theorem 2 shows that $s = 5$.

We may assume that Y_1 is not directing. By (1.4), there is a sectional path $\tau_A X = Z_0 \rightarrow Z_1 \rightarrow \cdots \rightarrow Z_s$ in Γ_B with Z_s an injective B -module. By (1.6), $\tau_A X$ has no projective predecessor in Γ_B .

Assume that no Y_i is a projective B -module, $1 \leq i \leq s$. Then $0 \rightarrow \tau_B^{m+1} X \rightarrow \bigoplus_{i=1}^s \tau_B^m Y_i \rightarrow \tau_B^m X \rightarrow 0$ is well-defined for all $m \geq 0$. Consider the sequence $\tau\eta: 0 \rightarrow \tau_B^2 X \rightarrow \bigoplus_{i=1}^s \tau_B Y_i \rightarrow \tau_B X \rightarrow 0$. If all $\tau_B Y_i$ are directing as B -modules ($1 \leq i \leq s$), then by Theorem 2 we are done. Hence by (1.4), we consider the quotient algebra $B_1 = B(\tau\eta)$ and we get a sectional path $Z'_t \rightarrow \cdots \rightarrow Z'_1 \rightarrow Z'_0 = \tau_B X$ in Γ_{B_1} with Z'_t a projective B_1 -module. Since Z'_t cannot be a projective B -module predecessor of $\tau_A X$, then B_1 is a proper quotient of B . Consider the sequence $\tau^2\eta: 0 \rightarrow \tau_B^3 X \rightarrow \bigoplus_{i=1}^s \tau_B^2 Y_i \rightarrow \tau_B^2 X \rightarrow 0$. Again either all $\tau_B^2 Y_i$ are directing as B_1 -modules and $s \leq 5$ or $B_2 = B(\tau^2\eta)$ is a proper quotient of B_1 . Repeating the process finitely many steps we end with an algebra $B_t = B(\tau^t\eta)$ where all $\tau_B^t Y_i$ are directing. Theorem 2 yields $s = 5$ and some $\tau_B^t Y_j$ is projective ($1 \leq j \leq s$), a contradiction.

We may therefore assume that Y_s is a projective B -module. By the dual argument, we may suppose that some Y_j ($1 \leq j \leq s$) is an injective B -module. As in (2.1), $j = s$. Let $Y_s = P_w$ be a projective-injective B -module. Then the irreducible maps of η look as follows:

Since $\tau_A X$ is directing in mod_A , then $\tau_A X(w) = 0$. Hence $Y_i(w) = 0$ for $1 \leq i \leq s-1$. Then $Y_s = P_w$ is a sincere B -module and $B = C[M] = \begin{pmatrix} C & M \\ 0 & k \end{pmatrix}$ is a one-point extension of an algebra C by the module $M = \text{rad } P_w$. Since P_w is injective, also $M = \tau_A X$. Hence $\text{rad } P_w$ is directing as A -module, and therefore P_w is directing as B -module. We get that B is a tilted algebra and by Theorem 2 we are done. $\square$

2.5 Examples. a) The bound in Theorems 1 and 2 is optimal. Consider A_t the algebra given by the quiver Q

and ideal I_t generated by $\sum_{i=1}^t \beta_i \alpha_i$. Then $\tilde{\rho}(A_t) = \frac{t-2+\sqrt{t^2-4}}{2}$. There is an almost split sequence as follows:

that is, $s(\tau_A^{-2}P_0) = t + 1$. The algebra A_t is tame if and only if $t \leq 3$.

b) Consider B_t the algebra given by the above quiver Q and ideal I'_t generated by $\beta_i \alpha_i - \beta_1 \alpha_1$, $i = 2, \dots, t$. Again $\tilde{\rho}(B_t) = \frac{t-2+\sqrt{t^2-4}}{2}$. There is an almost split sequence as follows:

That is, $s(I_0/S_0) = t + 1$. The algebra B_t is tame for $t \leq 4$.

c) Consider the algebra C given by the quiver

with ideal I generated by $\beta_i\alpha_i - \beta_1\alpha_1$, $i = 2, 3, 4$ and $\alpha_j\beta_i = 0$ for any $1 \leq i, j \leq 4$. Then C has a Galois covering $\tilde{C}$ given by the quiver $\tilde{Q}$

and ideal $\tilde{I}$ generated by $\beta_{ji}\alpha_{ji} - \beta_{j1}\alpha_{j1}$ for $i = 2, 3, 4$ and $j \in \mathbb{Z}$ and $\alpha_{j+1,s}\beta_{j,t} = 0$ for all $1 \leq s, t \leq 4$ and $j \in \mathbb{Z}$.

It is easy to check that $\tilde{C}$ has no hypercritical or pg -critical subcategories and hence $\tilde{C}$ is tame of polynomial growth [19]. Since the covering $\pi: \tilde{C} \rightarrow C$ is defined by the action of the free group $\mathbb{Z}$, then C is tame [13], [19]. There is an almost split sequence in mod_C of the form

and there are cycles in mod_C as follows:

$$S_i \rightarrow I_i \rightarrow I_w = P_w \rightarrow P_i \rightarrow S_i$$

for all $i = 1, \dots, 4$.

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INSTITUTO DE MATEMÁTICAS, UNAM CIUDAD UNIVERSITARIA 04510 MÉXICO, D. F. MÉXICO
E-mail address: `jap@penelope.matem.unam.mx`

E-mail address: `takane@gauss.matem.unam.mx`