MINIMAL LATTICE-SUBSPACES

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ABSTRACT. In this paper the existence of minimal lattice-subspaces of a vector lattice E containing a subset B of E_+ is studied (a lattice-subspace of E is a subspace of E which is a vector lattice in the induced ordering). It is proved that if there exists a Lebesgue linear topology τ on E and E_+ is τ -closed (especially if E is a Banach lattice with order continuous norm), then minimal lattice-subspaces with τ -closed positive cone exist (Theorem 2.5).

In the sequel it is supposed that $B = \{x_1, x_2, \dots, x_n\}$ is a finite subset of $C_+(\Omega)$, where Ω is a compact, Hausdorff topological space, the functions x_i are linearly independent and the existence of finite-dimensional minimal lattice-subspaces is studied. To this end we define the function $\beta(t) = \frac{r(t)}{\|r(t)\|_1}$ where $r(t) = (x_1(t), x_2(t), \dots, x_n(t))$. If $R(\beta)$ is the range of β and K the convex hull of the closure of $R(\beta)$, it is proved:

- (i) There exists an m -dimensional minimal lattice-subspace containing B if and only if K is a polytope of $\mathbb{R}^n$ with m vertices (Theorem 3.20).
- (ii) The sublattice generated by B is an m -dimensional subspace if and only if the set $R(\beta)$ contains exactly m points (Theorem 3.7).

This study defines an algorithm which determines whether a finite-dimensional minimal lattice-subspace (sublattice) exists and also determines these subspaces.

1. INTRODUCTION

It is known that $C[0, 1]$ is a universal Banach space in the sense that every separable Banach space is isometric to a closed subspace of $C[0, 1]$. In [11] it is shown that each separable Banach lattice is order-isomorphic to a closed lattice-subspace of $C[0, 1]$; therefore $C[0, 1]$ is also a universal Banach lattice. Since the sublattices of $C[0, 1]$ are not enough for this representation, the lattice-subspaces seems to be the right class of subspaces for studying Banach lattices.

The structure of lattice-subspaces has not been systematically studied. In [7] it is shown that a subspace X of a vector lattice is a lattice-subspace if and only if there exists a positive projection from the vector sublattice generated by X onto X . In [10] and [11] the existence of positive bases in lattice-subspaces is studied. A survey of lattice-subspaces and positive projections, as well as some new results, is proved in [1]. In [12] the finite-dimensional lattice-subspaces of $C(\Omega)$ are studied.

In the present paper the existence of minimal lattice-subspaces of a vector lattice E which contains a subset B of E_+ is studied. In the theory of Banach lattices (and

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in applications) we are interested in a lattice-subspace of E containing B which is as “close” as possible to the linear subspace $[B]$ generated by B .

Such a subspace is the sublattice $S(B)$ generated by B (note that $S(B)$ is the minimum sublattice containing B and also that $S(B) = [B]^\vee - [B]^\vee$ where $[B]^\vee$ is the set of finite supremum of the elements of $[B]$) but $S(B)$ is in general a “big” subspace which is “very far” from $[B]$. In Example 3.18 $[B]$ is 3-dimensional, $S(B)$ is dense in $C(\Omega)$ but a 4-dimensional lattice-subspace containing B exists. In Example 3.21 it is shown that a minimum lattice-subspace containing B does not always exist.

An important question is “how far” a minimal lattice-subspace is from $[B]$. Motivated by this question we study the existence of finite-dimensional minimal lattice-subspaces. Especially we suppose that $B = \{x_1, x_2, \dots, x_n\}$ is a subset of $C_+(\Omega)$, the vectors x_i are linearly independent and we study the existence of finite-dimensional minimal lattice-subspaces of $C(\Omega)$ containing B . In the framework of this problem we study also the question whether $S(B)$ is a finite-dimensional subspace.

To study this problem we define the function $\beta(t) = \frac{r(t)}{\|r(t)\|_1}$ where $r(t) = (x_1(t), x_2(t), \dots, x_n(t))$. This function defines a curve in the simplex Δ_n of $\mathbb{R}_+^n$ which we call basic curve of the functions x_i and is very important for our study.

In Theorem 3.7 it is proved that $S(B)$ is finite-dimensional if and only if the range $R(\beta)$ of β is finite and a positive basis of $S(B)$ is also determined. Hence we can determine whether $S(B)$ is finite-dimensional because it is very easy to check if $R(\beta)$ is finite or not. By the property that $S(B) = [B]^\vee - [B]^\vee$ we cannot conclude whether $S(B)$ is finite-dimensional and also we cannot determine a positive basis of $S(B)$.

In Theorem 3.10 it is proved that if the convex hull K of the closure of $R(\beta)$ is a polytope with m vertices, then an m -dimensional minimal lattice-subspace Y exists and a positive basis of Y is given. The determination of the basis of Y is based on the determination of the vertices of K .

In general it is difficult to study whether K is a polytope or not and determine its vertices. In Corollary 3.15 it is proved that if K is a polytope, $\beta(t_0)$ a vertex of K and t_0 an interior point of a curve c of Ω , then the derivative at t_0 (whenever it exists) of the restriction of β on c is equal to zero. If for example $\Omega \subseteq \mathbb{R}^l$ and the function β is defined on the whole set Ω , then the partial derivatives of β at t_0 are equal to zero whenever t_0 is an interior point of Ω and the derivatives at t_0 of the restriction of β on the parametric curves of $\partial(\Omega)$ are equal to zero, if $t_0 \in \partial(\Omega)$. Hence t_0 can be obtained as a solution of a system of equations.

This property helps us to determine a set of possible vertices of K , i.e., a subset G of $\mathbb{R}^n$ which contains the vertices of K , whenever K is a polytope. After the determination of G it is easier to study if K is a polytope or not (see Algorithm 3.17 and Example 3.18).

An interesting remark on the structure of the lattice-subspaces is also that a minimal lattice-subspace containing B is not necessarily a subspace of $S(B)$, Example 3.21.

Recently lattice-subspaces have been employed in economics [2], [3].

Let E be a (partially) ordered vector space with positive cone E_+ and X a subspace of E . The cone $X \cap E_+$ will be called the *induced cone* of X , and the ordering defined in X by this cone the *induced ordering*. We will denote by X_+ the

induced cone of X , i.e., $X_+ = X \cap E_+$. An *ordered subspace* of E is a subspace of E ordered by the induced cone. A *lattice-subspace* of E is an ordered subspace of E which is also a vector lattice (Riesz space).

Let X be a lattice-subspace of E . Then, for each $x, y \in X$ we will denote by $x \nabla y$ (resp. $x \triangle y$) the supremum (resp. infimum) of $\{x, y\}$ in X . It is clear that

$$x \vee y \leq x \nabla y \quad \text{and} \quad x \triangle y \leq x \wedge y$$

whenever $x \vee y, x \wedge y$ exist. If E is a vector lattice and $x \nabla y = x \vee y$ for any $x, y \in X$ then X is a sublattice (Riesz subspace) of E . Let E be an ordered Banach space with positive cone E_+ . A sequence $\{e_n\}$ is a *positive basis* of E if $\{e_n\}$ is a (Schauder) basis of E and $E_+ = \{x = \sum_{i=1}^{\infty} \lambda_i e_i \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\}$. A positive basis $\{e_n\}$ of E is unique (in the sense of a positive multiple). The following result (see [1] or [12]) is very important for the study of finite-dimensional lattice-subspaces. It can be proved either elementary or as a partial result of the Choquet-Kentall Theorem.

Theorem 1.1. *A finite-dimensional ordered vector space E is a vector lattice if and only if E has a positive basis.*

For notation and terminology not defined here we refer to [4, 6, 9].

2. MINIMAL LATTICE-SUBSPACES

Let E be a vector lattice and $B \subseteq E_+, B \neq \emptyset$. Let L be the set of lattice-subspaces of E , each of which contains B . If $X \in L$ and for any $Y \in L$ it holds:

$$Y \subseteq X \Rightarrow Y = X,$$

then we will say that X is a *minimal lattice-subspace* of E containing B .

If E is a vector lattice, then the sublattice generated by B is the minimum sublattice containing B .

As we will show later (Example 3.21) even if $E = \mathbb{R}^m$ a minimum lattice-subspace of E containing B does not always exist. So we state the following question:

Problem 2.1. *Does a minimal lattice subspace of E containing B exist?*

Let P be a cone of a linear space F (i.e., P is a convex subset of F , $\lambda x \in P$ for each $x \in P$ and $\lambda \in \mathbb{R}_+$ and $P \cap (-P) = \{0\}$). Suppose that $x, y \in P$. If there exists $z \in P$ with the properties: $z - x, z - y \in P$ and for each $w \in P, w - x, w - y \in P$ imply that $w - z \in P$, then we will say that z is the supremum of $\{x, y\}$ in P and we will denote

$$z = \sup_P \{x, y\}.$$

The infimum of $\{x, y\}$ in P is defined analogously. If for each $x, y \in P$, $z = \sup_P \{x, y\}$ exists, then $\inf_P \{x, y\}$ also exists.

If P is a cone of a linear space F and for each $x, y \in P$ the supremum of $\{x, y\}$ exists in P , then we will say that P is a lattice cone of F .

If $x = x_1 - x_2$ where $x_1, x_2 \in P$, then it is easy to show that $\sup\{x, 0\} = \sup_P \{x_1, x_2\} - x_2$ is the supremum of $\{x_1, x_2\}$ in $X = P - P$. Therefore the following result holds.

A cone P of a vector space F is a lattice-cone if and only if the subspace $X = P - P$, ordered by the cone P , is a vector lattice.

In the next results of this paragraph we will suppose that E is a vector lattice equipped with a linear topology τ with the properties:

- (i) E_+ is τ -closed;
- (ii) each increasing, order bounded net of E has a τ -convergent subnet (i.e., the topology τ is Lebesgue).

Property (i) implies also that τ is Hausdorff because if we suppose that $x \in E$, $x \neq 0$ and $0 \in x + V$ for each open symmetric neighborhood V of zero, then $0 \in -x + V$; therefore x and $-x$ belong to E_+ and hence $x = 0$, contradiction.

If the topology τ is order continuous (i.e., each decreasing net of E with infimum zero is τ -convergent to zero) and E is Dedekind complete, then τ satisfies (ii). If the order intervals of E are τ -compact, the statement (ii) is also satisfied (for related results see [4, Theorem 11.13]). Hence, the weak star topology of a dual Banach lattice and the weak topology of a Banach lattice with order continuous norm [4, Theorem 12.9], have property (ii).

Proposition 2.2. *Let $(P_i)_{i \in I}$ be a decreasing net of τ -closed lattice cones of E_+ (i.e., $P_i \subseteq E_+$ and $i \preceq j \Rightarrow P_i \supseteq P_j$). Then $P = \bigcap_{i \in I} P_i$ is a τ -closed lattice cone of E .*

Proof. P is a τ -closed cone of E_+ . Let $x, y \in P$. Denote by z_i the supremum of $\{x, y\}$ in P_i . For each $i, j \in I$ with $i \preceq j$ we have $P_j \subseteq P_i \subseteq E_+$; therefore,

$$x, y \leq z_i \leq z_j \leq x + y.$$

Since τ has property (ii), there exists a τ -convergent subnet of $(z_i)_{i \in I}$ which we will still denote by $(z_i)_{i \in I}$. This net is also increasing, and let $z = \lim_{i \in I} z_i$. Let $i \in I$. Then for each $j \in I$ with $i \preceq j$, we have:

$$z_j, z_j - x, z_j - y \in P_j \subseteq P_i.$$

Since the cone P_i is τ -closed, we have that

$$z, z - x, z - y \in P_i, \quad \text{for each } i \in I.$$

Therefore

$$z, z - x, z - y \in P.$$

Suppose that $w \in P$ with $w - x, w - y \in P$. Since $P \subseteq P_j$ we have that $w - z_j \in P_j \subseteq P_i$ for each $j \in I$ with $i \preceq j$. Hence $w - z \in P_i$ for each i ; therefore $w - z \in P$. So we have proved that $z = \sup_P \{x, y\}$; therefore P is a lattice cone. $\square$

Theorem 2.3. *Let $P \subseteq E_+$ be a cone and let $\Phi(P)$ be the set of τ -closed lattice cones of E_+ each of which contains P . Then $\Phi(P)$ has minimal elements.*

Proof. $\Phi(P) \neq \emptyset$ because $E_+ \in \Phi(P)$ and $\Phi(P)$, ordered by the relation " $\supseteq$ ", is a partially ordered set. Suppose that $\mathcal{F}$ is a totally ordered subset of $\Phi(P)$. Then by the previous result $Q = \bigcap_{A \in \mathcal{F}} A$ is a τ -closed lattice cone of E . By Zorn's Lemma the theorem is true. $\square$

Proposition 2.4. *Let $(X_i)_{i \in I}$ be a decreasing net of lattice-subspaces of E with τ -closed positive cones. Let $X = \bigcap_{i \in I} X_i$, $Y = X_+ - X_+$ and $Y_+ = Y \cap E_+$. Then*

- (i) $X_+ = \bigcap_{i \in I} X_i^+$.
- (ii) $Y \subseteq X$, $Y_+ = X_+$ and Y is a lattice-subspace of E with τ -closed positive cone.

Proof. (i) $X_+ = X \cap E_+ = (\bigcap_{i \in I} X_i) \cap E_+ = \bigcap_{i \in I} X_i^+$.

(ii) $Y = X_+ - X_+ \subseteq X$. $Y_+ \subseteq X \cap E_+ = X_+$. Also $X_+ = X_+ - \{0\} \subseteq Y$; therefore $X_+ \subseteq Y_+$. Hence $X_+ = Y_+$. The net $(X_i^+)_{i \in I}$ is a decreasing net of τ -closed lattice cones of E_+ ; therefore Y_+ is a τ -closed lattice cone. Hence Y is a lattice-subspace of E . $\square$

Theorem 2.5. *Let $B \subseteq E_+$ and*

$$l(B) = \{Y \subseteq E \mid Y \text{ is a lattice-subspace, } Y_+ \text{ is } \tau\text{-closed and } B \subseteq Y\}.$$

Then $l(B)$ has minimal elements.

Proof. The set $l(B)$ is nonempty because it contains E . The set $l(B)$, ordered by the relation " $\supseteq$ ", is a partially ordered set. Let $\mathcal{F}$ be a totally ordered subset of $l(B)$. By the previous proposition there exists $Y \in l(B)$ such that $Y \subseteq A$ for each $A \in \mathcal{F}$. Therefore, by Zorn's Lemma $l(B)$ has minimal elements. $\square$

Corollary 2.6. *Let E be a Banach lattice with order continuous norm and $B \subseteq E_+$. Then the set of lattice-subspaces of E with (norm) closed positive cone which contains B has minimal elements.*

3. THE FINITE-DIMENSIONAL CASE IN $C(\Omega)$

In this paper we shall denote by Ω a compact, Hausdorff topological space and by $C(\Omega)$ the Banach lattice of continuous real valued functions defined on Ω .

We will also denote by $x_1, \dots, x_n$, n fixed linearly independent positive elements of $C(\Omega)$ and by X the subspace of $C(\Omega)$ generated by $x_1, \dots, x_n$, i.e.,

$$X = [x_1, x_2, \dots, x_n].$$

In [12] necessary and sufficient conditions in order for X to be a lattice-subspace of $C(\Omega)$ are given.

In this paper we study the problem:

Problem 3.1. *Does a finite-dimensional lattice-subspace (sublattice) of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$ exist?*

For each $x \in \mathbb{R}^m$ we will denote by $x(i)$ the i -coordinate of x , by $\|x\|_1$ the norm $\|x\|_1 = \sum_{i=1}^m |x(i)|$, by $\{e_1, e_2, \dots, e_m\}$ the usual basis of $\mathbb{R}^m$ and by Δ_m the simplex (base) of $\mathbb{R}_+^m$, i.e.,

$$\Delta_m = \{x \in \mathbb{R}_+^m \mid \|x\|_1 = 1\}.$$

Also if $x \in \mathbb{R}^m, y \in \mathbb{R}^l$ we shall denote by (x, y) the vector z of $\mathbb{R}^{m+l}$ with $z(i) = x(i)$ for $i = 1, 2, \dots, m$ and $z(m+i) = y(i)$ for $i = 1, 2, \dots, l$. If A is an $m \times m$ matrix we shall denote by A^T the transpose and by A^{-1} the inverse matrix of A .

Let $y_1, y_2, \dots, y_m \in C_+(\Omega)$. Then we will call the function $v(t) = (y_1(t), y_2(t), \dots, y_m(t))$, $t \in \Omega$, the *curve* and the function $\gamma(t) = \frac{v(t)}{\|v(t)\|_1}$, $t \in \Omega$, with $v(t) \neq 0$, the *basic curve* of $y_1, y_2, \dots, y_m$. We will denote by $D(\gamma)$ the domain and by $R(\gamma)$ the range of γ . It is clear that $D(\gamma)$ is an open subset of Ω and $R(\gamma) \subseteq \Delta_m$.

In this paper we will denote by r the curve and by β the basic curve of $x_1, x_2, \dots, x_n$, i.e.,

$$r(t) = (x_1(t), x_2(t), \dots, x_n(t)), \quad t \in \Omega \quad \text{and} \quad \beta(t) = \frac{r(t)}{\|r(t)\|_1}.$$

As usual if K is a subset of a topological space F , we shall denote by $\text{int}(K)$ the interior, by $\bar{K}$ the closure and by $\partial(K)$ the boundary of K . Also whenever F is a linear topological space we shall denote by $\text{co } K$ the convex hull of K , by $\bar{\text{co}} K$ the closure of $\text{co } K$ and by $\text{ep}(K)$ the set of extreme points of K .

Proposition 3.2 ([12, Proposition 2.2]). *Let Y be a lattice-subspace of $C(\Omega)$ with a positive basis $\{b_1, b_2, \dots, b_n\}$. Then Y is a sublattice of $C(\Omega)$ if and only if the sets $b_i^{-1}(0, +\infty) = \{t \in \Omega \mid b_i(t) > 0\}$, $i = 1, 2, \dots, n$, are pairwise disjoint.*

Theorem 3.3 ([12, Theorem 3.6]). *The statements (i) and (ii) are equivalent:*

- (i) X is a lattice-subspace of $C(\Omega)$.
- (ii) *There exist n linearly independent vectors $P_1, P_2, \dots, P_n$ of $\mathbb{R}^n$, belonging to the closure of the range of β such that for each $t \in D(\beta)$ the vector $\beta(t)$ is a convex combination of the vectors $P_1, P_2, \dots, P_n$.*

If the statement (ii) is true, A is the $n \times n$ matrix whose i th column is the vector P_i and $b_1, b_2, \dots, b_n$ are the functions defined by the formula

$$(1) \quad (b_1, b_2, \dots, b_n)^T = A^{-1}(x_1, x_2, \dots, x_n)^T,$$

then $\{b_1, b_2, \dots, b_n\}$ is a positive basis of X .

Lemma 3.4. *The functions $y_i \in C_+(\Omega)$, $i = 1, 2, \dots, m$, are linearly independent if and only if the space generated by the range of the basic curve γ of y_i , $i = 1, 2, \dots, m$, is $\mathbb{R}^m$.*

Proof. Let W be the subspace of $\mathbb{R}^m$ generated by $R(\gamma)$. Then W is also generated by the range of the curve v of y_i , $i = 1, 2, \dots, m$. Let $\{u_i = v(t_i) \mid i = 1, 2, \dots, l\}$ be a basis of W . Then $l \leq m$.

Suppose that the functions y_i are linearly independent. Then

$$v(t) = \sum_{i=1}^l \xi_i(t) u_i, \quad \text{for each } t \in \Omega;$$

therefore

$$(2) \quad y_j(t) = \sum_{i=1}^l \xi_i(t) u_i(j), \quad j = 1, 2, \dots, m,$$

where $u_i(j)$ is the j -coordinate of u_i . For each t , the vector $(\xi_1(t), \xi_2(t), \dots, \xi_l(t))$ is the unique solution of the system (2); therefore the functions ξ_i as linear combinations of the functions y_i belong to $C(\Omega)$. By (2) we have also that

$$y_i \in L = [\xi_1, \xi_2, \dots, \xi_l], \quad \text{for each } i;$$

therefore $m \leq \dim L \leq l$. Hence $m = l$ and $W = \mathbb{R}^m$.

To prove the converse, suppose that $l = m$ and

$$\sum_{i=1}^m \lambda_i y_i = 0.$$

Then

$$\sum_{i=1}^m \lambda_i y_i(t_j) = 0 \quad \text{for each } j = 1, 2, \dots, m.$$

Since the vectors $v(t_i)$, $i = 1, 2, \dots, m$, are linearly independent, the system has the unique solution $\lambda_i = 0$ for each i ; therefore the functions y_i are linearly independent. $\square$

Sublattices.

Theorem 3.5. *Let $R(\beta) = \{P_1, P_2, \dots, P_n\}$. (By the previous lemma the vectors P_i are linearly independent and by Theorem 3.3 X is a lattice-subspace.) Let $\{b_1, b_2, \dots, b_n\}$ be the positive basis of X defined by (1) and let $I_i = b_i^{-1}(0, +\infty)$, for each i .*

Then the following statements hold:

- (i) X is a sublattice of $C(\Omega)$.
- (ii) $I_i = \beta^{-1}(P_i)$ for each i and $D(\beta) = \bigcup_{i=1}^n I_i$.
- (iii) *If y_i , $i = 1, 2, \dots, m$, are linearly independent elements of X_+ and γ is the basic curve of y_i , $i = 1, 2, \dots, m$, then there exists $\Phi \subseteq \{1, 2, \dots, n\}$ such that*
 - (a) $D(\gamma) = \bigcup_{i \in \Phi} I_i$,
 - (b) *the function γ is constant on I_i for each $i \in \Phi$,*
 - (c) $m \leq l \leq n$, where l is the cardinal number of $R(\gamma)$.

Proof. Let $z = \sum_{i=1}^n x_i$ and $B_i = \beta^{-1}(P_i)$, $i = 1, 2, \dots, n$. Then the sets B_i are pairwise disjoint and $D(\beta) = \bigcup_{i=1}^n B_i$. By (1) we have that

$$\frac{1}{z(t)} (b_1(t), b_2(t), \dots, b_n(t))^T = A^{-1}(\beta(t))^T.$$

Since $A^{-1} \cdot A = I$, the dot-product of the j -row of A^{-1} and the vector P_i is equal to 1 if $i = j$ and 0 whenever $i \neq j$; therefore

$$A^{-1}(\beta(t))^T = (e_i)^T \quad \text{for each } t \in B_i,$$

where $\{e_1, e_2, \dots, e_n\}$ is the usual basis of $\mathbb{R}^n$. Therefore

$$\frac{1}{z(t)} (b_1(t), b_2(t), \dots, b_n(t)) = e_i \quad \text{for each } t \in B_i.$$

Hence for each $t \in B_i$ it holds:

$$z(t) = b_i(t) > 0 \quad \text{and} \quad b_j(t) = 0 \quad \text{for each } j \neq i.$$

So

$$B_i \subseteq I_i \quad \text{and} \quad B_i \cap I_j = \emptyset \quad \text{for each } j \neq i.$$

Suppose that $t \in I_i \setminus B_i$. Since $D(\beta) = \bigcup_{k=1}^n B_k$, $t \in B_j$ for exactly one $j \neq i$. Hence $I_i \cap B_j \neq \emptyset$, contradiction. Hence $B_i = I_i$ for each i , and by Theorem 3.2, X is a sublattice. We have also shown the statement (ii).

The basic curve γ is

$$\gamma(t) = \frac{1}{y(t)} (y_1(t), y_2(t), \dots, y_m(t))$$

where $y = \sum_{i=1}^m y_i$. Let

$$y_j = \sum_{i=1}^n \mu_{ji} b_i, \quad j = 1, 2, \dots, m.$$

Then $y = \sum_{i=1}^n \mu_i b_i$ where $\mu_i = \sum_{j=1}^m \mu_{ji}$ for each i . Let $\Phi = \{i \mid \mu_i > 0\}$. Then it is clear that

$$D(\gamma) = \bigcup_{i \in \Phi} I_i.$$

If $i \in \Phi$ and $t \in I_i$, then

$$\gamma(t) = \frac{1}{\mu_i} (\mu_{1i}, \mu_{2i}, \dots, \mu_{mi}) = Q_i;$$

hence γ is constant on I_i . Therefore

$$R(\gamma) = \{Q_i \mid i \in \Phi\}.$$

Since Φ is a subset of $\{1, 2, \dots, n\}$, we have that $l \leq n$ and by Lemma 3.4, $m \leq l$. $\square$

Theorem 3.6. *The following statements are equivalent:*

- (i) X is a sublattice of $C(\Omega)$.
- (ii) $R(\beta) = \{P_1, P_2, \dots, P_n\}$.

Proof. Let X be a sublattice of $C(\Omega)$ and let $\{b_1, b_2, \dots, b_n\}$ be a positive basis of X . Let $x_j = \sum_{i=1}^n \lambda_{ji} b_i$. Then $z = \sum_{j=1}^n x_j = \sum_{i=1}^n \lambda_i b_i$ where $\lambda_i = \sum_{j=1}^n \lambda_{ji}$. Then the sets

$$I_i = b_i^{-1}(0, +\infty), \quad i = 1, 2, \dots, n,$$

are pairwise disjoint by Proposition 3.2. Hence for each $t \in I_k$ we have $x_i(t) = \lambda_{ik} b_k(t)$ and $x(t) = \lambda_k b_k(t)$, and therefore

$$\beta(t) = \frac{1}{\lambda_k} (\lambda_{1k}, \lambda_{2k}, \dots, \lambda_{nk}) = P_k.$$

Also $D(\beta) = \bigcup_{i=1}^n I_i$ because $t \in D(\beta)$ iff $z(t) > 0$ iff $b_i(t) > 0$ for at least one i . Hence

$$R(\beta) = \{P_1, P_2, \dots, P_n\};$$

therefore the theorem is true. $\square$

Theorem 3.7. *Let Z be the sublattice of $C(\Omega)$ generated by $x_1, x_2, \dots, x_n$ and let $m \in \mathbb{N}$. Then the statements (i) and (ii) are equivalent:*

- (i) $\dim(Z) = m$.
- (ii) $R(\beta) = \{P_1, P_2, \dots, P_m\}$.

If the statement (ii) is true, then Z is constructed as follows:

- (a) Enumerate $R(\beta)$ so that its n first vectors are linearly independent. (Such an enumeration exists by Lemma 3.4.) Denote again by P_i , $i = 1, 2, \dots, m$, the new enumeration and let $I_i = \beta^{-1}(P_i)$, $i = 1, 2, \dots, m$.
- (b) Define the functions

$$x_{n+k}(t) = a_k(t) \|r(t)\|_1, \quad t \in \Omega, \quad k = 1, 2, \dots, m - n,$$

where a_k is the characteristic function of I_{n+k} .

- (c) $Z = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m]$.

Proof. Suppose that (ii) is true and the assumptions (a), (b) are satisfied. We shall show that (c) is true. It is clear that $m \geq n$. The sets I_i are open subsets of $D(\beta)$ because the sets $\{P_i\}$ are open subsets of $R(\beta)$. Also $D(\beta) = \bigcup_{i=1}^m I_i$. Since $D(\beta)$ is an open subset of Ω , the sets I_i are open, nonempty subsets of Ω . Also $\partial(I_i) \cap I_j = \emptyset$. Hence $\partial(I_i) \subseteq \Omega \setminus D(\beta)$; therefore $\|r(t)\|_1 = 0$ for each $t \in \partial(I_i)$. This implies that the functions x_{n+k} are continuous; therefore $x_{n+k} \in C_+(\Omega)$ for each k .

Let v be the curve and γ the basic curve of x_i , $i = 1, 2, \dots, m$. Then by the definition of x_{n+k} we have that

$$v(t) = (r(t), 0) \quad \text{for each } t \in \bigcup_{i=1}^n I_i$$

and

$$v(t) = (r(t), \|r(t)\|_1 e_{i-n}) \quad \text{if } t \in I_i, i > n.$$

Let $t \in I_i$. Then

$$\gamma(t) = (\beta(t), 0) = (P_i, 0) = Q_i, \quad \text{if } i \leq n$$

and

$$\gamma(t) = \frac{1}{2} (\beta(t), e_{i-n}) = \frac{1}{2} (P_i, e_{i-n}) = Q_i, \quad \text{for each } i = n+1, \dots, m.$$

Since $D(\gamma) = D(\beta) = \bigcup_{i=1}^m I_i$, we have that

$$R(\gamma) = \{Q_i \mid i = 1, 2, \dots, m\}.$$

The vectors Q_i , $i = 1, 2, \dots, m$, are linearly independent. Hence the functions x_i , $i = 1, 2, \dots, m$, are also linearly independent; therefore the subspace Y generated by x_i , $i = 1, 2, \dots, m$, is an m -dimensional sublattice of $C(\Omega)$ by the previous theorem. Therefore $Z \subseteq Y$. Since x_i , $i = 1, 2, \dots, n$, are linearly independent elements of Z_+ and the cardinal number of $R(\beta)$ is m , by the statement (iii) of Theorem 3.5 we have that $m \leq \dim Z$. Therefore $\dim Z = m$; hence $Z = Y$.

Suppose now that the statement (i) is true. Then x_i , $i = 1, 2, \dots, n$, are linearly independent elements of Z_+ ; therefore by Theorem 3.5, there exist a nonempty subset Φ of $\{1, 2, \dots, m\}$ and nonempty, pairwise disjoint open subsets I_i , $i \in \Phi$, of Ω such that $D(\beta) = \bigcup_{i \in \Phi} I_i$ and β is constant on each I_i . Hence $R(\beta) = \{P_1, P_2, \dots, P_l\}$ where l is the cardinal number of Φ . By the same theorem we have also that $n \leq l \leq m$. As we have proved before, we can construct an l -dimensional sublattice Y of Ω containing $x_1, x_2, \dots, x_n$; therefore $Z \subseteq Y$ and $m \leq l$. Hence $l = m$ and therefore the statement (ii) is true. $\square$

Lattice-subspaces. A subset K of $\mathbb{R}^l$ is a *polytope* if K is the convex hull of a finite subset of $\mathbb{R}^l$. The extreme points of K are called vertices of K .

Theorem 3.8. Let Y be an l -dimensional lattice-subspace of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$. Suppose that $\{b_1, b_2, \dots, b_l\}$ is a positive basis of Y ,

$$x_i = \sum_{j=1}^l \lambda_{ij} b_j, \quad i = 1, 2, \dots, n,$$

$$\sigma_i = \sum_{j=1}^n \lambda_{ji}, \quad i = 1, 2, \dots, l,$$

$$\Phi = \{i \mid \sigma_i \neq 0\},$$

$$P_i = \frac{1}{\sigma_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi,$$

and K is the convex hull of $\overline{R(\beta)}$. Then

- (i) $P_i \in \overline{R(\beta)}$ for each $i \in \Phi$.
- (ii) K is a polytope with vertices $P_{i_1}, P_{i_2}, \dots, P_{i_m}$ where $n \leq m \leq l$ and $i_\nu \in \Phi$ for each $\nu = 1, 2, \dots, m$.

Proof. Let $x_{n+1}, \dots, x_l \in Y_+$ such that

$$Y = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_l].$$

Let

$$x_i = \sum_{j=1}^l \lambda_{ij} b_j, \quad i = 1, 2, \dots, l,$$

$$s_i = \sum_{j=1}^l \lambda_{ji}, \quad i = 1, 2, \dots, l,$$

and $v(t) = (x_1(t), x_2(t), \dots, x_l(t))$, $t \in \Omega$. Then $\|v(t)\|_1 = \sum_{i=1}^l s_i b_i$ and the function

$$\gamma(t) = \frac{v(t)}{\|v(t)\|_1}, \quad \|v(t)\|_1 \neq 0,$$

is the basic curve of $x_1, x_2, \dots, x_l$. By [12, Proposition 2.3], for each $i = 1, 2, \dots, l$ there exists a sequence $(\omega_{i\nu})$ of Ω such that

$$\lim_{\nu \rightarrow \infty} \frac{b_j(\omega_{i\nu})}{b_i(\omega_{i\nu})} = 0, \quad \text{for each } j \neq i.$$

Then

$$\lim_{\nu \rightarrow \infty} \frac{x_j(\omega_{i\nu})}{\|v(\omega_{i\nu})\|_1} = \lim_{\nu \rightarrow \infty} \left(\frac{\sum_{k=1}^l \lambda_{jk} \frac{b_k}{b_i}}{\sum_{k=1}^l s_k \frac{b_k}{b_i}} \right) (\omega_{i\nu}) = \frac{\lambda_{ji}}{s_i},$$

therefore

$$(3) \quad \lim_{\nu \rightarrow \infty} \gamma(\omega_{i\nu}) = \frac{1}{s_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{li}) = M_i.$$

Let A be the $l \times l$ matrix with columns the vectors M_i , $i = 1, 2, \dots, l$. Then using the expansion of x_i relative to the positive basis of Y we get

$$(4) \quad (x_1, x_2, \dots, x_l)^T = A(s_1 b_1, s_2 b_2, \dots, s_l b_l)^T.$$

Since $\{x_1, x_2, \dots, x_l\}$ is also a basis of Y , we have that $\text{rank } A = l$; therefore the vectors M_i , $i = 1, 2, \dots, l$, are linearly independent. Let

$$(5) \quad \gamma(t) = \sum_{i=1}^l \xi_i(t) M_i$$

be the expansion of $\gamma(t)$ relative to the basis $\{M_1, M_2, \dots, M_l\}$ of $\mathbb{R}^l$. Then

$$(\gamma(t))^T = A(\xi_1(t), \xi_2(t), \dots, \xi_l(t))^T$$

and by (4) we get

$$(\xi_1(t), \xi_2(t), \dots, \xi_l(t)) = \frac{1}{\|v(t)\|_1} (s_1 b_1(t), s_2 b_2(t), \dots, s_l b_l(t)).$$

Hence $\xi_i(t) \in \mathbb{R}_+$ and $\sum_{i=1}^l \xi_i(t) = 1$. Therefore $\gamma(t)$ is a convex combination of $M_1, M_2, \dots, M_l$. Therefore

$$R(\gamma) \subseteq \text{co}\{M_1, M_2, \dots, M_l\}.$$

Let $P(x) = (x(1), x(2), \dots, x(n))$, $x \in \mathbb{R}^l$, be the natural projection of $\mathbb{R}^l$ onto $\mathbb{R}^n$. Then

$$(6) \quad P\left(\frac{s_i}{\sigma_i} M_i\right) = P_i, \quad \text{for each } i \in \Phi.$$

If $i \notin \Phi$, then $P(M_i) = 0$, because $\sigma_i = 0$ and therefore $\lambda_{ki} = 0$ for each $k = 1, 2, \dots, n$. Also

$$\beta(t) = \frac{\|v(t)\|_1}{\|r(t)\|_1} P(\gamma(t)), \quad \text{for each } t \in D(\beta) \subseteq D(\gamma);$$

therefore by (5) we get

$$\beta(t) = \sum_{i \in \Phi} \frac{\|v(t)\|_1}{\|r(t)\|_1} \xi_i(t) \frac{\sigma_i}{s_i} P_i.$$

Since $\beta(t)$ and P_i belong to the simplex Δ_n of $\mathbb{R}_+^n$, we have that $\beta(t)$ is a convex combination of the vectors P_i , $i \in \Phi$; hence

$$R(\beta) \subseteq \text{co}\{P_i \mid i \in \Phi\} = L.$$

Since Φ is finite, the set L is closed; hence $\overline{R(\beta)} \subseteq L$. We shall show that $P_i \in \overline{R(\beta)}$, for each $i \in \Phi$. By (3) and (6) we have that $P(\frac{s_i}{\sigma_i} \gamma(\omega_{i\nu})) \rightarrow P_i$. Since $P_i \neq 0$, we have that $P(\gamma(\omega_{i\nu})) \neq 0$, for each ν . Therefore $r(\omega_{i\nu}) = \|v(\omega_{i\nu})\|_1 P(\gamma(\omega_{i\nu})) \neq 0$; hence $\omega_{i\nu} \in D(\beta)$, for each ν . Similarly with the proof of (3) we can show that $P_i = \lim \beta(\omega_{i\nu})$. Hence $P_i \in \overline{R(\beta)}$; therefore $K = L$. Also $\text{ep}(K) \subseteq \{P_i \mid i \in \Phi\}$. Hence

$$\text{ep}(K) = \{P_{i_1}, P_{i_2}, \dots, P_{i_m}\}$$

where $i_\nu \in \Phi$ for $\nu = 1, 2, \dots, m$; therefore

$$K = \text{co}\{P_{i_1}, P_{i_2}, \dots, P_{i_m}\}.$$

By Lemma 3.4, the subspace generated by $R(\beta)$, and therefore also by K , is the space $\mathbb{R}^n$. Hence $\text{ep}(K)$ contains at least n vectors; therefore $n \leq m \leq l$. $\square$

Theorem 3.9 ([5, Theorem 2]). *Let $d_1, d_2, \dots, d_m \in \mathbb{R}^l$ and let the polytope $D = \text{co}\{d_1, d_2, \dots, d_m\}$. Then there exist non-negative, real-valued continuous functions $\xi_1, \xi_2, \dots, \xi_m$ defined on D such that $x = \sum_{i=1}^m \xi_i(x) d_i$ and $\sum_{i=1}^m \xi_i(x) = 1$, for each $x \in D$.*

The previous result in a more general form is given also in [8].

Theorem 3.10. *Let the set $K = \overline{\text{co } R(\beta)}$ be a polytope with vertices $P_1, P_2, \dots, P_m$. Suppose that the n first vertices $P_1, P_2, \dots, P_n$ of K are linearly independent¹. Suppose also that ξ_i , $i = 1, 2, \dots, m$, are positive continuous real-valued functions defined on $D(\beta)$ such that $\sum_{i=1}^m \xi_i(t) = 1$ and $\beta(t) = \sum_{i=1}^m \xi_i(t) P_i$, for each $t \in D(\beta)$.*

¹A such enumeration of the vertices of K exists by Lemma 3.4.

Let x_{n+i} , $i = 1, 2, \dots, m - n$, be the functions $x_{n+i}(t) = \xi_{n+i}(t) \|r(t)\|_1$ for each $t \in D(\beta)$ and $x_{n+i}(t) = 0$ if $t \notin D(\beta)$. Then

$$Y = [x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m]$$

is a minimal lattice-subspace of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$ and $\dim Y = m$.

A positive basis $\{b_1, b_2, \dots, b_m\}$ of Y is given by the formula

$$(b_1, b_2, \dots, b_m)^T = A^{-1} (x_1, x_2, \dots, x_m)^T$$

where A is the $m \times m$ matrix with columns the vectors R_i , $i = 1, 2, \dots, m$, defined below, in (8).

Proof. We shall show that Y is a lattice-subspace of $C(\Omega)$. Let $v(t) = (x_1(t), x_2(t), \dots, x_m(t))$, $\gamma(t) = \frac{v(t)}{\|v(t)\|_1}$ and $l = m - n$. Then

$$\begin{aligned} v(t) &= (r(t), 0) + (0, \sum_{i=1}^l \xi_{n+i}(t) \|r(t)\|_1 e_i) \\ &= \|r(t)\|_1 \sum_{i=1}^m \xi_i(t) (P_i, 0) + \|r(t)\|_1 \sum_{i=1}^l \xi_{n+i}(t) (0, e_i) \\ (7) \quad &= \|r(t)\|_1 \sum_{i=1}^m \xi_i(t) M_i, \quad \text{for each } t \in D(\beta) \end{aligned}$$

where M_i are the following vectors of $\mathbb{R}^m$:

$$M_i = (P_i, 0) \quad \text{for } i = 1, 2, \dots, n$$

and

$$M_i = (P_{n+i}, e_i) \quad \text{for } i = 1, 2, \dots, l.$$

The vectors M_i are linearly independent with $\|M_i\|_1 = 1$ for $i = 1, 2, \dots, n$ and $\|M_i\|_1 = 2$ for $i = n + 1, \dots, m$. Hence $\|v(t)\|_1 = \|r(t)\|_1 g(t)$, where $g(t) = \sum_{i=1}^m \xi_i(t) \|M_i\|_1 = 1 + \sum_{i=n+1}^m \xi_i(t)$. Therefore, by (7) we have

$$(8) \quad \gamma(t) = \frac{1}{g(t)} \sum_{i=1}^m \xi_i(t) \|M_i\|_1 R_i, \quad \text{where } R_i = \frac{M_i}{\|M_i\|_1}.$$

Hence $\gamma(t)$ is a convex combination of R_i , $i = 1, 2, \dots, m$. We shall show that $R_i \in \overline{R(\gamma)}$ for each i . If $P_i = \beta(t_i)$, then $P_i = \sum_{j=1}^m \xi_j(t_i) P_j$ and by our assumption that P_i is an extreme point of K , we have that $\xi_i(t_i) = 1$ and $\xi_j(t_i) = 0$ for each $j \neq i$. Hence by (8) we have

$$\gamma(t_i) = \frac{1}{g(t_i)} \|M_i\|_1 R_i = R_i.$$

If $P_i \notin R(\beta)$, then there exists a sequence (ω_ν) of $D(\beta)$ such that

$$P_i = \lim_{\nu \rightarrow \infty} \beta(\omega_\nu).$$

Then

$$\beta(\omega_\nu) = \sum_{j=1}^m \xi_j(\omega_\nu) P_j.$$

Since $0 \leq \xi_j(\omega_\nu) \leq 1$, there exists a subsequence of (ω_ν) , which we will denote again by (ω_ν) such that

$$\lambda_j = \lim_{\nu \rightarrow \infty} \xi_j(\omega_\nu), \quad \text{for each } j = 1, 2, \dots, m.$$

Hence

$$P_i = \sum_{j=1}^m \lambda_j P_j,$$

which implies that $\lambda_i = 1$ and $\lambda_j = 0$ for each $j \neq i$, because P_i is an extreme point of K . By (8) and the definition of g we have that

$$\lim_{\nu \rightarrow \infty} \gamma(\omega_\nu) = R_i.$$

So by Theorem 3.3, Y is a lattice-subspace and a positive basis of Y is as in the formulation of the theorem.

Suppose that $Z \subseteq Y$ is a lattice-subspace containing $x_1, x_2, \dots, x_n$ and let $\dim Z = l$. Then $l \leq m$. By Theorem 3.8 the number m of vertices of K is less than or equal to l ; therefore $m = l$. Hence $Z = Y$; therefore Y is minimal. $\square$

Definition 3.11. Let C be a convex subset of a normed space E . We shall say that x_0 is a conic point of C if x_0 is an extreme point of C , $C \setminus \{x_0\} \neq \emptyset$, and there exists a real number $\rho > 0$ such that

$$x_0 + \rho \frac{x - x_0}{\|x - x_0\|} \in C, \quad \text{for each } x \in C, x \neq x_0.$$

Proposition 3.12. Let D be a convex subset of a normed space E and $x_0 \in E$. If² $d = d(x_0, D) > 0$ and $C = \text{co}(\{x_0\} \cup D)$, then x_0 is a conic point of C . (If D is bounded and closed, then C is also bounded and closed.)

Proof. Let $x \in C, x \neq x_0$. Then $x = \lambda x_0 + (1 - \lambda)y$, where $y \in D$ and $\lambda \in [0, 1]$. Hence $x - x_0 = (1 - \lambda)(y - x_0)$; therefore

$$\|x - x_0\| = (1 - \lambda) \|y - x_0\| \geq (1 - \lambda) d.$$

Also $x_0 + l(y - x_0) \in C$ for each $l \in [0, 1]$. Therefore

$$x_0 + d \frac{x - x_0}{\|x - x_0\|} = x_0 + \frac{d(1 - \lambda)}{\|x - x_0\|} (y - x_0) \in C.$$

To show that x_0 is an extreme point of C suppose that $x_0 = \frac{x_1 + x_2}{2}$ where $x_1, x_2 \in C$ and $x_1, x_2 \neq x_0$. Then $x_i = \lambda_i x_0 + (1 - \lambda_i)y_i$ with $\lambda_i \in (0, 1)$ and $y_i \in D$. Then $x_0 = \frac{1}{2 - \lambda_1 - \lambda_2} ((1 - \lambda_1)y_1 + (1 - \lambda_2)y_2) \in D$, contradiction. Hence x_0 is a conic point of C . $\square$

Example 3.13. (i) For each cone $P \neq \{0\}$ of a normed space, 0 is a conic point of P .

(ii) Let C be a closed, convex, bounded subset of a Banach space E and let x_0 be an extreme point of C . If $C = \overline{\text{coep}}(C)$ (i.e., C is the closure of the convex hull of the extreme points of C) and $x_0 \notin D = \overline{\text{co}}(\text{ep}(C) \setminus \{x_0\})$, then $C = \text{co}(\{x_0\} \cup D)$; therefore x_0 is a conic point of C .

(iii) Each vertex of a polytope C of $\mathbb{R}^m$ is a conic point of C .

²With $d(x_0, D)$ we denote the distance from x_0 to D .

We prove below that the tangent vector of a curve of C at a conic point of C is equal to zero.

Proposition 3.14. *Let C be a closed, convex subset of a normed space E and x_0 be a conic point of C . Let $\phi : (-\epsilon, \epsilon) \rightarrow C$ be a function with $\phi(0) = x_0$ where ϵ is a positive real number. Then*

$$\phi'(0) = 0,$$

whenever the derivative $\phi'(0)$ exists.

Proof. Let $\phi'(0) = \lim_{t \rightarrow 0} \frac{\phi(t) - \phi(0)}{t} \neq 0$. Then there exists $\delta > 0$ such that $\phi(t) \neq \phi(0)$ for each $|t| < \delta$. Hence

$$\lim_{t \rightarrow 0+} \frac{\phi(t) - \phi(0)}{\|\phi(t) - \phi(0)\|} = \lim_{t \rightarrow 0+} \frac{\phi(t) - \phi(0)}{t} \cdot \lim_{t \rightarrow 0+} \frac{1}{\left\| \frac{\phi(t) - \phi(0)}{t} \right\|} = \frac{\phi'(0)}{\|\phi'(0)\|},$$

and similarly

$$\lim_{t \rightarrow 0-} \frac{\phi(t) - \phi(0)}{\|\phi(t) - \phi(0)\|} = -\frac{\phi'(0)}{\|\phi'(0)\|}.$$

Since x_0 is a conic point of C , there exists $\rho > 0$ such that

$$x_0 + \rho \frac{x - x_0}{\|x - x_0\|} \in C, \quad \text{for each } x \in C, x \neq x_0.$$

Therefore

$$\lim_{\nu \rightarrow \infty} \left(\phi(0) + \rho \frac{\phi(1/\nu) - \phi(0)}{\|\phi(1/\nu) - \phi(0)\|} \right) = x_0 + \rho \frac{\phi'(0)}{\|\phi'(0)\|} = z_1 \in C$$

and

$$\lim_{\nu \rightarrow \infty} \left(\phi(0) + \rho \frac{\phi(-1/\nu) - \phi(0)}{\|\phi(-1/\nu) - \phi(0)\|} \right) = x_0 - \rho \frac{\phi'(0)}{\|\phi'(0)\|} = z_2 \in C.$$

Hence $x_0 = \frac{1}{2}(z_1 + z_2)$, contradiction. Therefore $\phi'(0) = 0$. $\square$

Corollary 3.15. *Let the set $K = \text{co } \overline{R(\beta)}$ be a polytope of $\mathbb{R}^n$ and let $\beta(t_0)$ be a vertex of K . If ϵ is a positive real number and $g : (-\epsilon, \epsilon) \rightarrow \Omega$ is a function with $g(0) = t_0$ and $\phi(\lambda) = \beta(g(\lambda))$, then*

$$\phi'(0) = 0,$$

whenever the derivative exists.

Remark 3.16. Suppose that there exists a finite-dimensional lattice-subspace of $C(\Omega)$ containing X . Then K is a polytope of $\mathbb{R}^n$. Suppose that $\beta(t_0)$ is a vertex of K . If c is a curve of Ω and t_0 an interior point of c , then the derivative at t_0 of the restriction of β on the curve c is equal to zero.

If for example $\Omega \subseteq \mathbb{R}^l$, then the partial derivatives of β at t_0 are equal to zero whenever $t_0 \in \text{int}(\Omega)$. If $t_0 \in \partial(\Omega)$, the derivatives at t_0 of the restriction of β on the parametric curves of $\partial(\Omega)$ are equal to zero.

Algorithm 3.17. *Theorem 3.10 and Corollary 3.15 define a process which in many cases, especially when $\Omega \subseteq \mathbb{R}^l$, determines whether a finite dimensional minimal lattice-subspace exists and determines also a positive basis of these subspaces. To study this problem we study if K is a polytope or not.*

If the set $R(\beta)$ is closed, then each extreme point (vertex) P_0 of $K = \text{co } R(\beta)$ belongs to $R(\beta)$; therefore $P_0 = \beta(t_0)$. Also the geometry of the boundary of $D(\beta)$ and the differentiability of the functions x_i are very important for this study.

Let $\Omega = [a, b]$, the functions x_i are differentiable and $D(\beta) = \Omega$. Suppose that the set K is a polytope with vertices $\beta(t_i)$, $i = 1, 2, \dots, m$. Then at least $m - 2$ of t_i belong to (a, b) ; therefore the equation

$$(9) \quad \beta'(t) = 0,$$

where β' is the derivative of β , has at least $m - 2$ roots in (a, b) . Hence the vertices of K belong to the set

$$G = \{\beta(t) | t = a, t = b, \text{ or } t \text{ is a root of (9)}\}$$

which we call the set of possible vertices of K . Let $D = \text{co } G$. It is easy to show that K is a polytope if and only if D is a polytope and $R(\beta) \subseteq D$.

Hence in this case the algorithm is the following:

- (i) Determine equation (9). If this equation does not have at least $n - 2$ roots in (a, b) , then K is not a polytope.
- (ii) Determine the roots t_i of (9) in (a, b) .
- (iii) We study whether $R(\beta) \subseteq D$. So we study whether $\beta(t)$ is a convex combination of $\beta(a), \beta(b), \beta(t_i)$, for each i . If $R(\beta) \not\subseteq D$, then K is not a polytope.
- (iv) Determine the vertices of K and a positive basis of the minimal lattice-subspace, in accordance with Theorem 3.10.

We give three examples below. In (i) it is shown that a finite-dimensional minimal lattice-subspace does not always exist. In (ii) we consider three elements x_1, x_2, x_3 of $C(\Omega)$, where Ω is a square of $\mathbb{R}^2$. We show that a 4-dimensional minimal lattice-subspace Y exists and a positive basis of Y is determined. We also remark that the sublattice generated by the elements x_i is dense in $C(\Omega)$. In (iii) the functions x_i are as in (ii), but Ω is a circle of $\mathbb{R}^2$. It is shown that a finite-dimensional minimal lattice-subspace does not exist. This difference between (ii) and (iii) depends on the geometry of the boundary of Ω .

Example 3.18. (i) Let $\Omega = [0, 1]$, $x_1(t) = 1, x_2(t) = t, x_3(t) = t^2$. Then

$$\beta(t) = \left(\frac{1}{1+t+t^2}, \frac{t}{1+t+t^2}, \frac{t^2}{1+t+t^2} \right), \quad t \in [0, 1],$$

is the basic curve of x_1, x_2, x_3 and $\beta'(t) \neq 0$ for each $t \in (0, 1)$. Suppose that Y is a finite-dimensional lattice-subspace of $C(\Omega)$ containing the functions x_i . Then $\dim Y \geq 3$, and therefore by Theorem 3.8 K is a polytope of $\mathbb{R}^3$ with at least three vertices, $\beta(t_1), \beta(t_2), \beta(t_3)$. Hence $\beta'(t) = 0$ for at least one point of $(0, 1)$, contradiction.

(ii) Let $\Omega = [0, 1] \times [0, 1]$, $x_1(u, v) = 1, x_2(u, v) = u, x_3(u, v) = v$ and $X = [x_1, x_2, x_3]$. Then

$$\beta(u, v) = \left(\frac{1}{1+u+v}, \frac{u}{1+u+v}, \frac{v}{1+u+v} \right), \quad (u, v) \in \Omega,$$

is the basic curve of x_1, x_2, x_3 and let $K = \text{co } R(\beta)$. Since the range of β is not finite, the sublattice Z generated by x_1, x_2, x_3 is an infinite-dimensional subspace of $C(\Omega)$, Theorem 3.7. In this example we can also show that Z is dense in $C(\Omega)$ because Z is a sublattice of $C(\Omega)$ and Z contains the constant functions.

FIGURE 1

In order to study the existence of minimal lattice-subspaces we study whether the set K is a polytope of $\mathbb{R}^3$. To this end suppose that K is a polytope. Then by Theorem 3.8, K has at least three vertices and let $\beta(t_0)$ be a vertex of K . Then t_0 is also a vertex of Ω because in the contrary case t_0 will be an interior point of a line segment parallel to an axis of $\mathbb{R}^2$; therefore, and by the previous corollary, at least one of the partial derivatives of β at t_0 will be equal to zero, contradiction. Hence the points $P_1 = \beta(0, 0) = (1, 0, 0)$, $P_2 = \beta(1, 0) = (1/2, 1/2, 0)$, $P_3 = \beta(0, 1) = (1/2, 0, 1/2)$ and $P_4 = \beta(1, 1) = (1/3, 1/3, 1/3)$ define the set of possible vertices of K . Let $D = \text{co}\{P_1, P_2, P_3, P_4\}$. From the above remarks we have that K is a polytope if and only if $K = D$ or equivalently if $R(\beta) \subseteq D$. It is easy to show that

$$\beta(u, v) = \sum_{i=1}^4 \xi_i(u, v) P_i,$$

where $\xi_1 \in C(\Omega)$, $\xi_2(u, v) = 2\left(\frac{1-v}{1+u+v} - \xi_1(u, v)\right)$, $\xi_3(u, v) = 2\left(\frac{1-u}{1+u+v} - \xi_1(u, v)\right)$ and $\xi_4(u, v) = 3\left(\frac{u+v-1}{1+u+v} + \xi_1(u, v)\right)$.

Since $\beta(u, v)$ and the points P_i belong to the plane $x(1) + x(2) + x(3) = 1$ of $\mathbb{R}^3$ we have that $\sum_{i=1}^4 \xi_i(u, v) = 1$. If $\xi(u, v) = \frac{1-u-v}{1+u+v}$ and if we put $\xi_1 = \xi^+$, then the functions ξ_i , $i = 1, 2, 3, 4$, are positive and continuous; therefore $R(\beta) \subseteq D$. Hence K is a polytope with vertices P_i , $i = 1, 2, 3, 4$, and the three first of them are linearly independent. By Theorem 3.10,

$$Y = [x_1, x_2, x_3, x_4],$$

where $x_4(u, v) = \xi_4(u, v) \|r(u, v)\|_1 = 3(1 - u - v)^+$, is a minimal lattice-subspace containing x_1, x_2, x_3 .

A positive basis $\{b_1, b_2, b_3, b_4\}$ of Y is given by the formula

$$(b_1, b_2, b_3, b_4)^T = A^{-1} (x_1, x_2, x_3, x_4)^T,$$

where A is the 4×4 matrix with columns the vectors $R_i = \frac{M_i}{\|M_i\|_1}$, $i = 1, 2, 3, 4$, and $M_1 = (P_1, 0) = (1, 0, 0, 0)$, $M_2 = (P_2, 0) = (1/2, 1/2, 0, 0)$, $M_3 = (P_3, 0) = (1/2, 0, 1/2, 0)$, $M_4 = (P_4, e_1) = (1/3, 1/3, 1/3, 1)$.

After the computations we get

$$\begin{aligned} b_1(u, v) &= x_1 - x_2 - x_3 + \frac{1}{3}x_4 = \begin{cases} 1 - u - v & | \ u + v \leq 1, \\ 0 & | \ u + v > 1, \end{cases} \\ b_2(u, v) &= 2x_2 - \frac{2}{3}x_4 = \begin{cases} 2u & | \ u + v \leq 1, \\ 2(1 - v) & | \ u + v > 1, \end{cases} \\ b_3(u, v) &= 2x_3 - \frac{2}{3}x_4 = \begin{cases} 2v & | \ u + v \leq 1, \\ 2(1 - u) & | \ u + v > 1, \end{cases} \\ b_4(u, v) &= 2x_4 = \begin{cases} 0 & | \ u + v \leq 1, \\ 3(u + v - 1) & | \ u + v > 1 \end{cases} \quad (\text{Figure 1}). \end{aligned}$$

(iii) Let $\Omega = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 \leq 1\}$ and let x_i , $i = 1, 2, 3$, be the functions of the previous example. Suppose that K is a polytope and $\beta(t_0)$ a vertex of K . As before we have that $t_0 \in \partial(\Omega)$ and let $t_0 = (\cos \theta_0, \sin \theta_0)$. Then by the corollary we have $\phi'(\theta_0) = 0$ where $\phi(\theta) = \beta(\cos \theta, \sin \theta)$. This is a contradiction because $\phi'(\theta) \neq 0$ for each θ . Therefore a finite-dimensional lattice-subspace containing the functions x_i does not exist.

To study subspaces of $\mathbb{R}^l$, $l > 1$, suppose that $\Omega = \{1, 2, \dots, l\}$. Then $C(\Omega) = \mathbb{R}^l$,

$$x_i = (x_i(1), x_i(2), \dots, x_i(l)), \quad i = 1, 2, \dots, n,$$

are linearly independent, positive elements of $\mathbb{R}^l$ and

$$X = [x_1, x_2, \dots, x_n].$$

The curve r and the basic curve β of the vectors x_i , $i = 1, 2, \dots, n$, are the functions:

$$r(i) = (x_1(i), x_2(i), \dots, x_n(i)), \quad i = 1, 2, \dots, l,$$

and

$$\beta(i) = \frac{r(i)}{\|r(i)\|_1}, \quad \text{for each } i \text{ with } \|r(i)\|_1 \neq 0.$$

Let m be the cardinal number of $R(\beta)$. Then $m \leq l$ and by Lemma 3.4, $n \leq m$; therefore $n \leq m \leq l$. Let K be the convex hull of $R(\beta)$. Then K , as the convex hull of a finite subset of $\mathbb{R}^n$, is a polytope with d vertices. It is clear that

$$n \leq d \leq m \leq l$$

and that each vertex of K belongs to $R(\beta)$. Let

$$R(\beta) = \{P_1, P_2, \dots, P_m\}$$

be an enumeration of $R(\beta)$ such that:

- (i) the vectors P_i , $i = 1, 2, \dots, n$, are linearly independent and
- (ii) the points P_i , $i = 1, 2, \dots, d$, are the vertices of K .

As an application of Theorems 3.6, 3.3, 3.7 and 3.10 we obtain the following:

Theorem 3.19 (The case of $\mathbb{R}^l$). *Suppose that $\Omega = \{1, 2, \dots, l\}$ and that the above assumptions are satisfied. Then*

- (i) X is a sublattice of $\mathbb{R}^l$ if and only if $R(\beta)$ contains exactly n points (i.e., $m = n$).
- (ii) X is a lattice-subspace of $\mathbb{R}^l$ if and only if the polytope K has n vertices (i.e., $d = n$).
- (iii) Let $m > n$. If $I_k = \beta^{-1}(P_k)$, and

$$x_k = \sum_{i \in I_k} \|r(i)\|_1 e_i, \quad k = n+1, n+2, \dots, m,$$

then

$$Z = [x_1, \dots, x_n, x_{n+1}, \dots, x_m]$$

is the sublattice generated by $x_1, x_2, \dots, x_n$ and $\dim Z = m$.

- (iv) Let $d > n$. If $\xi_i : D(\beta) \rightarrow \mathbb{R}_+$, $i = 1, 2, \dots, d$, such that $\sum_{i=1}^d \xi_i(j) = 1$ and $\beta(j) = \sum_{i=1}^d \xi_i(j) P_i$ for each $j \in D(\beta)$, and x_{n+i} , $i = 1, 2, \dots, d-n$, are the following vectors of $\mathbb{R}^l$:

$$x_{n+i} = \sum_{j \in D(\beta)} \xi_{n+i}(j) \|r(j)\|_1 e_j,$$

then

$$Y = [x_1, \dots, x_n, x_{n+1}, \dots, x_d]$$

is a minimal lattice-subspace of $\mathbb{R}^l$ containing $x_1, x_2, \dots, x_n$ and $\dim Y = d$.

In the following result Ω is again a compact, Hausdorff, topological space.

Theorem 3.20. Let $K = \overline{\text{co } R(\beta)}$ and let L be the set of finite-dimensional minimal lattice-subspaces of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$. Then the following statements are equivalent:

- (i) K is a polytope with m vertices.
- (ii) $L \neq \emptyset$ and $\dim Y = m$, for each $Y \in L$.
- (iii) $L \neq \emptyset$.

Proof. Suppose that (i) is true. Then by Theorem 3.10, there exists $Y \in L$ with $\dim Y = m$. Suppose that $Z \in L$ and $\{b_1, b_2, \dots, b_l\}$ is a positive basis of Z . Let

$$x_i = \sum_{j=1}^l \lambda_{ij} b_j, \quad i = 1, 2, \dots, n,$$

$$\sigma_j = \sum_{i=1}^n \lambda_{ij}, \quad j = 1, 2, \dots, l,$$

$$\Phi = \{j \mid \sigma_j \neq 0\} \quad \text{and}$$

$$P_i = \frac{1}{\sigma_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi.$$

Then by Theorem 3.8 $P_i \in K$ for each $i \in \Phi$ and the vertices of K are among the points P_i , $i \in \Phi$; therefore there exist $i_1, i_2, \dots, i_m \in \Phi$ such that $P_{i_1}, P_{i_2}, \dots, P_{i_m}$

are the vertices of K . Also $n \leq m \leq l$. Let $T : Z \rightarrow \mathbb{R}^l$ such that $T(\sum_{i=1}^l \xi_i b_i) = \sum_{i=1}^l \xi_i e_i$ and let $y_i = T(x_i)$, $i = 1, 2, \dots, n$. The basic curve b of $y_1, y_2, \dots, y_n$ is:

$$b(i) = \frac{1}{\sigma_i} (\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{ni}), \quad i \in \Phi,$$

with range

$$R(b) = \{P_i \mid i \in \Phi\}.$$

So $R(b)$ is a subset of K containing the vertices of K ; therefore

$$K = \text{co } R(b).$$

Hence $\text{co } R(b)$ is a polytope with vertices $P_{i1}, P_{i2}, \dots, P_{im}$. By the previous theorem, there exists an m -dimensional lattice-subspace F of $\mathbb{R}^l$ containing $y_1, y_2, \dots, y_n$. If $G = T^{-1}(F)$, then G is a lattice-subspace of Z and therefore also of $C(\Omega)$ containing $x_1, x_2, \dots, x_n$. Since Z is minimal, we have that $G = Z$, and therefore $\dim Z = \dim F = m$. Hence we have shown that (i) $\Rightarrow$ (ii).

Suppose now that the statement (ii) is true. Let $Y \in L$ and $K = \overline{\text{co } R(\beta)}$. Then by Theorem 3.8, K is a polytope with k vertices and

$$n \leq k \leq m.$$

By Theorem 3.10 there exists $Z \in L$ with $\dim Z = k$. By our assumption we have that $k = m$; therefore K has m vertices. Hence (ii) $\Rightarrow$ (i).

Also (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) by Theorem 3.8. $\square$

In the following example we construct the sublattice Z generated by a four-dimensional subspace X of $\mathbb{R}^7$ as well as two minimal lattice-subspaces Y and Y' which contain X . It is remarkable that $Y \cap Y'$ is not a lattice-subspace as well as that both Y and Y' are not subspaces of Z .

Example 3.21. Let

$$\begin{aligned} x_1 &= (1, 2, 1, 0, 1, 1, 4), \\ x_2 &= (0, 1, 1, 1, 1, 0, 2), \\ x_3 &= (2, 1, 0, 1, 1, 1, 2), \\ x_4 &= (1, 0, 1, 1, 1, 0, 0), \end{aligned}$$

and let $X = [x_1, x_2, x_3, x_4]$. Let r be the curve and β the basic curve of x_i , $i = 1, 2, 3, 4$. Then $r(1) = (1, 0, 2, 1)$, $r(2) = (2, 1, 1, 0)$, $r(3) = (1, 1, 0, 1)$, $r(4) = (0, 1, 1, 1)$, $r(5) = (1, 1, 1, 1)$, $r(6) = (1, 0, 1, 0)$, $r(7) = (4, 2, 2, 0)$ and $\beta(1) = \frac{1}{4}(1, 0, 2, 1)$, $\beta(2) = \beta(7) = \frac{1}{4}(2, 1, 1, 0)$, $\beta(3) = \frac{1}{3}(1, 1, 0, 1)$, $\beta(4) = \frac{1}{3}(0, 1, 1, 1)$, $\beta(5) = \frac{1}{4}(1, 1, 1, 1)$, $\beta(6) = \frac{1}{2}(1, 0, 1, 0)$. In order to enumerate $R(\beta)$ as in Theorem 3.19 we remark the following:

- (i) The vectors $P_1 = \beta(4)$, $P_2 = \beta(1)$, $P_3 = \beta(6)$ and $P_4 = \beta(3)$ are linearly independent.
- (ii) Let $\beta(2) = P_5$. Then it is easy to show that for any proper subset Φ of $\{P_1, P_2, P_3, P_4, P_5\}$, $\text{co } \Phi \neq \text{co}\{P_1, P_2, P_3, P_4, P_5\} = K$; therefore P_i , $i = 1, 2, 3, 4, 5$, are vertices of the polytope K .
- (iii) It is easy also to show that

$$(10) \quad \beta(5) = \frac{3(1-\theta)}{8}P_1 + \theta P_2 + \frac{1-5\theta}{4}P_3 + \frac{3(1-\theta)}{8}P_4 + \theta P_5.$$

Hence for any $\theta \in [0, \frac{1}{5}]$ the vector $P_6 = \beta(5)$ is a convex combination of P_i , $i = 1, 2, 3, 4, 5$; therefore $P_6 \in K$.

Hence

$$R(\beta) = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

and in accordance with the notations of Theorem 3.19, $n = 4$, $d = 5$ and $m = 6$. Since $n < d$, X is not a lattice-subspace and therefore also X is not a sublattice of $\mathbb{R}^7$. Let Z be the sublattice of $\mathbb{R}^7$ generated by x_1, x_2, x_3, x_4 . In order to determine Z we define the sets

$$I_5 = \beta^{-1}(P_5) = \{2, 7\}, \quad I_6 = \beta^{-1}(P_6) = \{5\}$$

and the vectors

$$x_5 = \|r(2)\|_1 e_2 + \|r(7)\|_1 e_7 = 4e_2 + 8e_7$$

and

$$x_6 = \|r(5)\|_1 e_5 = 4e_5.$$

Then by the theorem

$$Z = [x_1, x_2, x_3, x_4, x_5, x_6].$$

By Theorem 3.3 a positive basis $\{b_1, b_2, b_3, b_4, b_5, b_6\}$ of Z is given by the formula

$$(b_1, b_2, b_3, b_4, b_5, b_6)^T = A^{-1} (x_1, x_2, x_3, x_4, x_5, x_6)^T,$$

where A is the 6×6 matrix with columns the vectors $\gamma(i)$, $i = 1, 2, \dots, 6$, and γ is the basic curve of the vectors x_i , $i = 1, 2, \dots, 6$. So after the computations we find that $b_1 = 4e_1$, $b_2 = 8e_2 + 16e_7$, $b_3 = 3e_3$, $b_4 = 3e_4$, $b_5 = 8e_5$ and $b_6 = 2e_6$.

To determine a minimal lattice-subspace define the vectors ξ_i , $i = 1, 2, 3, 4, 5$, of $\mathbb{R}^7$ such that

$$\sum_{i=1}^5 \xi_i(j) = 1 \quad \text{and} \quad \beta(j) = \sum_{i=1}^5 \xi_i(j) P_i, \quad \text{for each } j = 1, 2, \dots, 7.$$

$$\begin{aligned} \beta(1) = P_2 = \sum_{i=1}^5 \xi_i(1) P_i &\Rightarrow \xi_2(1) = 1 \text{ and } \xi_k(1) = 0 \text{ for } k \neq 2. \\ \beta(2) = P_5 = \sum_{i=1}^5 \xi_i(2) P_i &\Rightarrow \xi_5(2) = 1 \text{ and } \xi_k(2) = 0 \text{ for } k \neq 5. \\ \beta(3) = P_4 = \sum_{i=1}^5 \xi_i(3) P_i &\Rightarrow \xi_4(3) = 1 \text{ and } \xi_k(3) = 0 \text{ for } k \neq 4. \\ \beta(4) = P_1 = \sum_{i=1}^5 \xi_i(4) P_i &\Rightarrow \xi_1(4) = 1 \text{ and } \xi_k(4) = 0 \text{ for } k \neq 1. \\ \beta(5) = P_6 = \sum_{i=1}^5 \xi_i(5) P_i &\Rightarrow \xi_1(5) = \xi_4(5) = \frac{3(1-\theta)}{8}, \xi_2(5) = \xi_5(5) = \theta, \\ &\quad \xi_3(5) = \frac{1-5\theta}{4}, \quad \text{by (10).} \\ \beta(6) = P_3 = \sum_{i=1}^5 \xi_i(6) P_i &\Rightarrow \xi_3(6) = 1 \text{ and } \xi_k(6) = 0 \text{ for } k \neq 3. \\ \beta(7) = P_2 = \sum_{i=1}^5 \xi_i(7) P_i &\Rightarrow \xi_2(7) = 1 \text{ and } \xi_k(7) = 0 \text{ for } k \neq 2. \end{aligned}$$

Define also the vector

$$\begin{aligned} y_5 &= \sum_{j=1}^7 \xi_5(j) \|r(j)\|_1 e_j = \|r(2)\|_1 e_2 + \theta \|r(5)\|_1 e_5 \\ &= 4e_2 + 4\theta e_5, \quad \theta \in [0, 1/5]. \end{aligned}$$

Suppose that $\theta > 0$ in y_5 and that y'_5 is the vector corresponding to $\theta = 0$, i.e., $y'_5 = 4e_2$. Then the subspaces

$$Y = [x_1, x_2, x_3, x_4, y_5] \quad \text{and} \quad Y' = [x_1, x_2, x_3, x_4, y'_5]$$

are minimal lattice-subspaces containing the vectors x_i . Since the vectors $x_1, x_2, x_3, x_4, y_5, y'_5$ are linearly independent, we have $Y \neq Y'$. Also $X = Y \cap Y'$ is not a lattice-subspace. An important remark is that the vectors y_5, y'_5 do not belong to Z . To show this suppose that $y_5 \in Z$. Then $y_5 \in Z_+$, and therefore

$$y_5 = \sum_{i=1}^6 \lambda_i b_i, \quad \text{with } \lambda_i \in \mathbb{R}_+ \text{ for each } i.$$

This implies that $\lambda_2 = 1/2$ and $\lambda_2 = 0$, contradiction. Hence $y_5 \notin Z$. Also $y'_5 \notin Z$. Therefore Y, Y' are not subspaces of Z .

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