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AN L^p A PRIORI ESTIMATE FOR THE TRICOMI EQUATION IN THE UPPER HALF SPACE

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Abstract. We establish an L^p a priori estimate for the Tricomi equation. Our main tool is Mihlin's multiplier theorem combined with well-known estimates of the Newtonian potential.

0. Introduction

The purpose of this paper is to establish an L^p a priori estimate for the Tricomi equation in the upper half-space. The Tricomi equation arises in transonic gas dynamics, and is a typical model equation of changing type. It has been extensively investigated from the various viewpoints. The Tricomi equation can be interpreted as an elliptic equation which degenerates on the boundary, which is our viewpoint in this paper. We can formulate the Dirichlet boundary value problem in the upper half-space as follows.

$$\frac{\partial^2 u}{\partial y^2} + y\,\Delta_x u = f, \qquad \text{in } R^n_+, \tag{0.1}$$

$$u(0,x) = \phi(x), \qquad \text{on } \partial R^n_+, \tag{0.2}$$

where $R^n_+ = \{(y,x) : y > 0,\ x \in R^{n-1}\}$, and Δ_x is the Laplacian in the x variable. An L^2 a priori estimate can be obtained very easily. Following the presentation in [8], we suppose that $u \in C_0^\infty(\overline{R^n_+})$, and multiply the equation (0.1) by $\dfrac{\partial^2 u}{\partial y^2}$. Then, we integrate over R^n_+ to obtain

$$\begin{aligned}\int_{R^{n-1}}\int_0^\infty \left(\frac{\partial^2 u}{\partial y^2}\right)^2 dy\,dx + \int_{R^{n-1}}\int_0^\infty y\left|\nabla_x \frac{\partial u}{\partial y}\right|^2 dy\,dx \\ = \frac{1}{2}\int_{R^{n-1}} \left|\nabla_x \phi\right|^2 dx + \int_{R^{n-1}}\int_0^\infty f\,\frac{\partial^2 u}{\partial y^2}\,dy\,dx,\end{aligned} \tag{0.3}$$

where ∇_x stands for the gradient in $x \in R^{n-1}$. This yields

$$\begin{aligned}\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^2(R^n_+)} + \left\|\sqrt{y}\ \nabla_x \frac{\partial u}{\partial y}\right\|_{L^2(R^n_+)} + \left\|y\Delta_x u\right\|_{L^2(R^n_+)} \\ \le M\big(\|f\|_{L^2(R^n_+)} + \|\nabla_x \phi\|_{L^2(R^{n-1})}\big),\end{aligned} \tag{0.4}$$

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where M is a positive constant independent of u. This estimate is also covered by the result of [15]. Since (0.3) is an equality, the inequality (0.4) is an optimal L^2 estimate. It is a natural question to ask whether a similar L^p estimate is valid, which motivates the present work.

In this paper, we establish a similar estimate in the L^p setting, $1 < p < \infty$. Our result is the following.

Theorem 0.1. *Let $1 < p < \infty$, and let $0 < L < \infty$. Suppose $u \in C_0^\infty(\overline{R_+^n})$ satisfies (0.1), (0.2), and supp $u \subset \{(y,x) : y < L,\ x \in R^{n-1}\}$. Then, there is a positive constant M_L independent of u such that*

$$(0.5)\qquad \left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^p(R_+^n)} + \left\|\sqrt{y}\ \nabla_x \frac{\partial u}{\partial y}\right\|_{L^p(R_+^n)} + \left\|y\Delta_x u\right\|_{L^p(R_+^n)} \le M_L\left(\|f\|_{L^p(R_+^n)} + \|\phi\|_{B_{pp}^{4/3-2/(3p)}(R^{n-1})}\right).$$

Here $B_{pp}^r(R^k)$ is the Besov space; see [1], [13], and [14]. When $p = 2$, we recover (0.4) with modification that M depends on L, and that the L^2 norm of ϕ is added to the right-hand side. The above particular estimate is not covered by any of the vast known results concerned with degenerate elliptic equations. For extensive references, readers are referred to [7], [9], [10], and [14]. It is obvious that the above simple procedure for the L^2 estimate does not work for L^p, $p \ne 2$. As in the case of regular elliptic equations, the method of potential theory is a possible approach. In fact, the fundamental solutions of certain equations of mixed type were discussed in [5]. By setting $y = z^{2/3}$, (0.1) with $f \equiv 0$ reduces to

$$(0.6)\qquad \frac{\partial^2 u}{\partial z^2} + \Delta_x u + \frac{1}{3z}\frac{\partial u}{\partial z} = 0.$$

The fundamental solutions of (0.6) were analyzed in [3]. Parametrices of more general differential operators were constructed in [2], [11], and [12]. In particular, the problems discussed in [11] and [12] are closely related to our problem. In [11], a parametrix for the second order equation of Tricomi type with the Dirichlet boundary condition was constructed by means of Fourier integral operators, and some L^2 estimates were obtained via the parametrix. This was extended to more general operators in [12]. But this elaborate device does not provide any short cut for L^p estimates. Since the equation (0.1) is of a specially simple form, we can bypass a parametrix, and directly set up integral representation of u through the Fourier transform in x, and the variation of constants formula. Our approach is quite elementary, and similar to that of [5]. However, our analysis of the integral operators is different. The above mentioned results are not useful for our analysis. We analyze the singular integral operators which involve the Airy functions by borrowing known results on the Newtonian potential for the Laplacian. Our basic tool is the following version of Mihlin's multiplier theorem.

Lemma 0.2. *Let $m(\xi)$ be the symbol of a singular integral operator $\mathcal{T}$ in R^k. Suppose that $m(\xi) \in C^\infty(R^k \setminus \{0\})$, and*

$$(0.7)\qquad |\xi|^{|\alpha|}\left|\frac{\partial^\alpha m(\xi)}{\partial \xi^\alpha}\right| \le M, \qquad \text{for all } \xi \ne 0,\ 0 \le |\alpha| \le 1 + [k]/2,$$

for some positive constant M. Then, $\mathcal{T}$ is a bounded linear operator from $L^p(R^k)$ into itself for $1 < p < \infty$, and its operator norm depends only on M, k and p.

The proof of this fact can be found in [1], [8], [13], and [14].

After some preliminaries in the next two sections, we present the proof of Theorem 0.1 in section 3.

Throughout this paper, we employ the following notation.

Z_+ represents the set of all nonnegative integers. For $\alpha = (\alpha_1, \cdots, \alpha_k) \in Z_+^k$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_k$, and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}.$$

R_+ is the set of all positive real numbers, and $R_+^n = \{(y,x) : y \in R_+,\ x \in R^{n-1}\}$, for $n \geq 2$. $C_0^\infty(\overline{R_+^n})$ is the restriction of $C_0^\infty(R^n)$ to $\overline{R_+^n}$.

1. Some properties of the Laplacian

In this section we will present representation formula for solutions of the Laplace equation and their operator estimates. Let us fix any $0 < c < 1$, and define for each $f \in C_0^\infty(\overline{R_+^n})$

$$(\Lambda_1 f)(y,x) = \mathcal{F}_\xi^{-1}\left(e^{-y|\xi|}\int_{cy}^{y} e^{t|\xi|}\,|\xi|\hat{f}(t,\xi)\,dt\right), \tag{1.1}$$

and

$$(\Lambda_2 f)(y,x) = \mathcal{F}_\xi^{-1}\left(e^{y|\xi|}\int_{y}^{y/c} e^{-t|\xi|}\,|\xi|\hat{f}(t,\xi)\,dt\right), \tag{1.2}$$

where

$$\hat{f}(t,\xi) = (\mathcal{F}_x f)(t,\xi) = \int_{R^{n-1}} f(t,x)\,e^{-ix\cdot\xi}\,dx,$$

and $\mathcal{F}_\xi^{-1}$ denotes the Fourier inversion in R^{n-1}.

Proposition 1.1. *Λ_1 and Λ_2 can be uniquely extended as bounded linear operators from $L^p(R_+^n)$ into $L^p(R_+^n)$, $1 < p < \infty$.*

Proof. Choose any $f \in C_0^\infty(\overline{R_+^n})$, and extend f to R^n such that $\tilde{f}(-y,x) = -\tilde{f}(y,x)$, for all $(y,x) \in R^n$, $y \neq 0$. Let $E(y,x)$ be the Newtonian potential in R^n, and set

$$u = E * (\Delta_x \tilde{f}), \tag{1.3}$$

where $*$ stands for the convolution in R^n, and Δ_x is the Laplacian in $x \in R^{n-1}$. Then, u satisfies

$$\Delta u(y,x) = \Delta_x \tilde{f}(y,x), \qquad \text{in } R^n, \tag{1.4}$$

$$u(0,x) = 0 \qquad \text{on } \partial R_+^n. \tag{1.5}$$

By virtue of the inequality proved in [6, pp.230-235], it holds that

$$\|u\|_{L^p(R^n)} \leq M\|\tilde{f}\|_{L^p(R^n)} \leq M\|f\|_{L^p(R_+^n)}, \tag{1.6}$$

where M denotes positive constants independent of f. Furthermore, we find that

$$u \in C^\infty(\overline{R_+^n}) \cap C^1(R^n), \tag{1.7}$$

$$u(y,x) \to 0, \qquad \text{as } |x| + y \to \infty, \tag{1.8}$$

since $\Delta_x \tilde{f}$ has compact support and belongs to $L^\infty(R^n) \cap C^\infty(\overline{R^n_+})$. Next we consider $v(y,x)$ whose Fourier transform in x for $y \geq 0$ is defined by

$$\begin{aligned}(1.9)\qquad \hat{v}(y,\xi) =&\frac{1}{2}e^{-y|\xi|}\int_0^y |\xi| e^{t|\xi|}\,\hat{f}(t,\xi)\,dt\\ &+\frac{1}{2}e^{y|\xi|}\int_y^\infty |\xi|e^{-t|\xi|}\hat{f}(t,\xi)\,dt\\ &-\frac{1}{2}e^{-y|\xi|}\int_0^\infty |\xi|e^{-t|\xi|}\hat{f}(t,\xi)\,dt\\ &=\hat{v}_1(y,\xi)+\hat{v}_2(y,\xi)+\hat{v}_3(y,\xi),\end{aligned}$$

where the above three integral terms are denoted by $\hat{v}_j,\ j=1,2,3$. Then, it holds that

$$(1.10)\qquad \Delta v = \Delta_x f \qquad \text{in } R^n_+,$$

$$(1.11)\qquad v(0,x)=0 \qquad \text{on } \partial R^n_+.$$

It is easy to find that

$$(1.12)\qquad \xi^\alpha(\hat{v}_1(y,\xi)+\hat{v}_2(y,\xi)) \in C\big([0,\infty);L^1(R^{n-1})\big), \qquad \text{for each } \alpha \in Z^{n-1}_+,$$

and that

$$(1.13)\qquad \Delta\big(v_1(y,x)+v_2(y,x)\big)=\Delta_x f(y,x) \qquad \text{in } R^n_+.$$

Hence it follows that

$$(1.14)\qquad v_1+v_2 \in C^\infty(\overline{R^n_+}).$$

Suppose that L is a positive number such that

$$supp\, f \subset \{(y,x)\,:\,y<L\}.$$

Then, for $y>L$, we have

$$\begin{aligned}(1.15)\quad |v_1(y,x)+v_2(y,x)| =&|v_1(y,x)| \leq M\int_{R^{n-1}}\left|e^{-y|\xi|}\int_0^L |\xi|e^{t|\xi|}\hat{f}(t,\xi)\,dt\right|d\xi\\ &\leq M\big\|e^{-(y-L)|\xi|}\big\|_{L^2(R^{n-1})}\int_0^L \big\|\,|\xi|\hat{f}(t,\xi)\big\|_{L^2(R^{n-1})}\,dt\end{aligned}$$

where M is a positive constant independent of (y,x). Thus, it follows that

$$(1.16)\qquad \big\|(v_1+v_2)(y)\big\|_{L^\infty(R^{n-1})} \to 0, \qquad \text{as } y\to\infty.$$

Next fix any $N>0$. For each $j=1,\cdots,n-1$, it holds that

$$\begin{aligned}(1.17)\quad \big|x_j\big(v_1(y,x)+v_2(y,x)\big)\big| \leq& M\int_{R^{n-1}}\left|\frac{\partial}{\partial\xi_j}\big(\hat{v}_1(y,\xi)+\hat{v}_2(y,\xi)\big)\right|d\xi\\ &\leq M_N, \qquad \text{for all } x\in R^{n-1} \text{ and } 0\leq y\leq N,\end{aligned}$$

for some constant M_N. It follows from (1.16) and (1.17) that

$$(1.18)\qquad |v_1(y,x)+v_2(y,x)| \to 0 \qquad \text{as } |x|+y\to\infty.$$

By a similar argument, we also find that

$$(1.19)\qquad v_3(y,x)\in C^\infty(\overline{R^n_+}),$$

and

$$(1.20)\qquad v_3(y,x)\to 0, \qquad \text{as } |x|+y\to\infty.$$

By the maximum principle for the Laplacian, we conclude that

$$u(y,x) = v(y,x) \qquad \text{for all } (y,x) \in \overline{R^n_+}. \tag{1.21}$$

By noting the following inequality for each $\alpha \in Z_+^{n-1}$,

$$\left|\xi\right|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} |\xi| e^{-(t+y)|\xi|} \right| \le \frac{M_\alpha}{t+y}, \qquad \text{for all } t+y>0,\ \xi \neq 0, \tag{1.22}$$

with some positive constant M_α, we can apply Lemma 0.2 to $\hat{v}_3$ to derive

$$\|v_3(y)\|_{L^p(R^{n-1})} \le M \int_0^\infty \frac{1}{t+y} \|f(t)\|_{L^p(R^{n-1})}\, dt \tag{1.23}$$

for all $y>0$. By the well-known estimate of the Hilbert integral [13, p.271], we have

$$\|v_3\|_{L^p(R^n_+)} \le M \|f\|_{L^p(R^n_+)}, \tag{1.24}$$

where M is a positive constant independent of f.

Next choose any $g \in C_0^\infty(\overline{R^n_+})$, and extend g to R^n such that $\tilde{g}(-y,x) = \tilde{g}(y,x)$, for all $(y,x) \in R^n$. Then, we note that $\dfrac{\partial \tilde{g}}{\partial y}(-y,x) = -\dfrac{\partial \tilde{g}}{\partial y}(y,x)$, for all $(y,x) \in R^n$, $y \neq 0$, and set for $j = 1, \cdots, n-1$,

$$w_j = E * \frac{\partial^2 \tilde{g}}{\partial y \partial x_j}. \tag{1.25}$$

Then, w_j satisfies

$$\Delta w_j = \frac{\partial^2 \tilde{g}}{\partial y \partial x_j} \qquad \text{in } R^n, \tag{1.26}$$

$$w_j(0,x) = 0, \qquad \text{on } \partial R^n_+. \tag{1.27}$$

As above, we have

$$\|w_j\|_{L^p(R^n)} \le M \|\tilde{g}\|_{L^p(R^n)} \le M \|g\|_{L^p(R^n_+)}, \tag{1.28}$$

where M denotes positive constants independent of g. We also find that

$$w_j \in C^\infty(\overline{R^n_+}) \cap C^1(R^n), \tag{1.29}$$

$$w_j(y,x) \to 0, \qquad \text{as } |x| + y \to \infty. \tag{1.30}$$

Analogously to (1.9), we define, for $y \ge 0$,

$$\begin{aligned} \hat{\psi}^j(y,\xi) =& \frac{i}{2} e^{-y|\xi|} \int_0^y \xi_j\, e^{t|\xi|} \hat{g}(t,\xi)\, dt \\ & - \frac{i}{2} e^{y|\xi|} \int_y^\infty \xi_j\, e^{-t|\xi|} \hat{g}(t,\xi)\, dt \\ & + \frac{i}{2} e^{-y|\xi|} \int_0^\infty \xi_j\, e^{-t|\xi|} \hat{g}(t,\xi)\, dt \\ & = \hat{\psi}^j_1(y,\xi) + \hat{\psi}^j_2(y,\xi) + \hat{\psi}^j_3(y,\xi), \end{aligned} \tag{1.31}$$

where $i = \sqrt{-1}$, and $\hat{\psi}^j_k$, $k = 1,2,3$, represents each integral term. It is easy to see that

$$\Delta \psi^j = \frac{\partial^2 g}{\partial y \partial x_j} \qquad \text{in } R^n_+, \tag{1.32}$$

$$\psi^j(0,x)=0 \qquad \text{on } \partial R^n_+. \tag{1.33}$$

By the same argument as above, we arrive at

$$w_j(y,x)=\psi^j(y,x) \qquad \text{for all } (y,x)\in\overline{R^n_+}, \tag{1.34}$$

and

$$\|\psi^j_3\|_{L^p(R^n_+)} \le M\|g\|_{L^p(R^n_+)}, \tag{1.35}$$

for some positive constant M independent of g. By combining (1.28), (1.34), and (1.35), we find that

$$\|\psi^j_1+\psi^j_2\|_{L^p(R^n_+)} \le M\|g\|_{L^p(R^n_+)}, \tag{1.36}$$

for each $j=1,\cdots,n-1$, for some positive constant M independent of g. By the L^p boundedness of the Riesz transforms, we can deduce

$$\begin{aligned} &\left\|\mathcal{F}^{-1}_\xi\Big(e^{-y|\xi|}\int_0^y |\xi|\,e^{t|\xi|}\hat g(t,\xi)\,dt - e^{y|\xi|}\int_y^\infty |\xi|\,e^{-t|\xi|}\hat g(t,\xi)\,dt\Big)\right\|_{L^p(R^n_+)} \\ &\quad\le M\left\|\mathcal{F}^{-1}_\xi\sum_{j=1}^{n-1}\frac{\xi_j}{|\xi|}\big(\hat\psi^j_1(y,\xi)+\hat\psi^j_2(y,\xi)\big)\right\|_{L^p(R^n_+)} \\ &\quad\le M\|g\|_{L^p(R^n_+)}, \end{aligned} \tag{1.37}$$

where M stands for positive constants independent of g. It now follows from (1.6), (1.21), (1.24), and (1.37) that

$$\left\|\mathcal{F}^{-1}_\xi\Big(e^{-y|\xi|}\int_0^y |\xi|e^{t|\xi|}\hat f(t,\xi)\,dt\Big)\right\|_{L^p(R^n_+)} \le M\|f\|_{L^p(R^n_+)}, \tag{1.38}$$

$$\left\|\mathcal{F}^{-1}_\xi\Big(e^{y|\xi|}\int_y^\infty |\xi|e^{-t|\xi|}\hat f(t,\xi)\,dt\Big)\right\|_{L^p(R^n_+)} \le M\|f\|_{L^p(R^n_+)}, \tag{1.39}$$

for some positive constant M independent of f.

Next let $0<c<1$ be fixed. We have

$$\begin{aligned} &\left\|\mathcal{F}^{-1}_\xi\Big(e^{-y|\xi|}\int_0^{cy} |\xi|e^{t|\xi|}\,\hat f(t,\xi)\,dt\Big)\right\|_{L^p(R^{n-1})} \\ &\qquad\le M\int_0^{cy}\frac{1}{y-t}\|f(t)\|_{L^p(R^{n-1})}\,dt \\ &\qquad\le \frac{M}{1-c}\,\frac{1}{y}\int_0^{cy}\|f(t)\|_{L^p(R^{n-1})}\,dt, \qquad \text{for all } y>0, \end{aligned} \tag{1.40}$$

which follows from Lemma 0.2 with help of the following inequality for each $\alpha\in Z^{n-1}_+$,

$$|\xi|^{|\alpha|}\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\big(|\xi|e^{-(y-t)|\xi|}\big)\right| \le \frac{M_\alpha}{y-t}, \tag{1.41}$$

with some positive constant M_α, for all $y>t\ge 0$, $\xi\ne 0$. By the Hardy inequality, we can infer from (1.40) that

$$\left\|\mathcal{F}^{-1}_\xi\Big(e^{-y|\xi|}\int_0^{cy} |\xi|e^{t|\xi|}\,\hat f(t,\xi)\,dt\Big)\right\|_{L^p(R^n_+)} \le M\|f\|_{L^p(R^n_+)}, \tag{1.42}$$

for some positive constant M independent of f. By the same argument, we find that

$$\begin{aligned}(1.43)\qquad &\left\|\mathcal{F}_\xi^{-1}\left(e^{y|\xi|}\int_{y/c}^{\infty}|\xi|e^{-t|\xi|}\,\hat f(t,\xi)\,dt\right)\right\|_{L^p(R^{n-1})}\\ &\le M\int_{y/c}^{\infty}\frac{1}{t-y}\|f(t)\|_{L^p(R^{n-1})}\,dt\\ &\le \frac{M}{1-c}\int_{y/c}^{\infty}\frac{1}{t}\|f(t)\|_{L^p(R^{n-1})}\,dt,\qquad \text{for all } y>0.\end{aligned}$$

For each $\phi(t)\in L^p(R_+)$, it holds that

$$\begin{aligned}(1.44)\qquad \left\|\int_{y/c}^{\infty}\frac{\phi(t)}{t}\,dt\right\|_{L^p(R_+)} &= \left\|\int_{1/c}^{\infty}\frac{\phi(y\lambda)}{\lambda}\,d\lambda\right\|_{L^p(R_+)}\\ &\le \int_{1/c}^{\infty}\frac{1}{\lambda}\|\phi(y\lambda)\|_{L^p(R_+)}\,d\lambda\\ &\le \|\phi\|_{L^p(R_+)}\int_{1/c}^{\infty}\frac{1}{\lambda^{1+1/p}}\,d\lambda \le p\,c^{1/p}\,\|\phi\|_{L^p(R_+)}.\end{aligned}$$

By applying this to (1.43), we obtain

$$(1.45)\qquad \left\|\mathcal{F}_\xi^{-1}\left(e^{y|\xi|}\int_{y/c}^{\infty}|\xi|e^{-t|\xi|}\,\hat f(t,\xi)\,dt\right)\right\|_{L^p(R^n_+)} \le M\|f\|_{L^p(R^n_+)},$$

for some positive constant M independent of f. The proof of Proposition 1.1 is complete by combining (1.38), (1.39), (1.42), and (1.45). □

Next we fix any $0<c<1$, and any real number γ to define for each $f\in C_0^\infty(\overline{R^n_+})$,

$$(1.46)\qquad (\Lambda_{1,\gamma}f)(y,x)=\mathcal{F}_\xi^{-1}\left(e^{-y|\xi|}\int_{cy}^{y}e^{t|\xi|}\,|\xi|\,\big(t/y\big)^{\gamma}\,\hat f(t,\xi)\,dt\right),$$

and

$$(1.47)\qquad (\Lambda_{2,\gamma}f)(y,x)=\mathcal{F}_\xi^{-1}\left(e^{y|\xi|}\int_{y}^{y/c}e^{-t|\xi|}\,|\xi|\,\big(t/y\big)^{\gamma}\,\hat f(t,\xi)\,dt\right).$$

Proposition 1.2. *$\Lambda_{1,\gamma}$ and $\Lambda_{2,\gamma}$ can be uniquely extended as bounded linear operators from $L^p(R^n_+)$ into $L^p(R^n_+)$, $1<p<\infty$.*

Proof. Choose any $f\in C_0^\infty(\overline{R^n_+})$, and any small $\epsilon>0$. Let us define

$$(1.48)\qquad f_0(y,x)=\begin{cases}f(y,x), & \text{for } 0\le y<\epsilon,\\ 0, & \text{otherwise}\end{cases}$$

and, for $j\ge 1$,

$$(1.49)\qquad f_j(y,x)=\begin{cases}f(y,x), & \text{for } \epsilon\,c^{-(j-1)}\le y<\epsilon\,c^{-j},\\ 0, & \text{otherwise}\end{cases}$$

so that

$$(1.50)\qquad f=\sum_{j=0}^{N(\epsilon)}f_j,\qquad \text{for some } 0\le N(\epsilon)<\infty,$$

and

$$\|f\|^p_{L^p(R^n_+)} = \sum_{j=0}^{N(\epsilon)} \|f_j\|^p_{L^p(R^n_+)}. \tag{1.51}$$

We also define

$$g_j(y,x) = \big(\Lambda_{1,\gamma} f_j\big)(y,x), \qquad \text{for } j = 0, 1, \cdots, N(\epsilon), \tag{1.52}$$

so that

$$supp\ g_j \subset \{(y,x) \,:\, \epsilon\, c^{-(j-1)} \le y \le \epsilon\, c^{-(j+1)}\}, \tag{1.53}$$

and

$$\Lambda_{1,\gamma} f = \sum_{j=0}^{N(\epsilon)} g_j. \tag{1.54}$$

For $j \ge 1$, it holds that

$$\big\|y^\gamma f_j(y,x)\big\|_{L^p(R^n_+)} \le c^{-|\gamma|} \left(\epsilon\, c^{-j}\right)^\gamma \big\|f_j(y,x)\big\|_{L^p(R^n_+)}, \tag{1.55}$$

$$c^{|\gamma|} \left(\epsilon\, c^{-j}\right)^\gamma \big\|g_j(y,x)\big\|_{L^p(R^n_+)} \le \big\|y^\gamma g_j(y,x)\big\|_{L^p(R^n_+)}. \tag{1.56}$$

By virtue of Proposition 1.1, we find that, for $j \ge 1$,

$$\big\|y^\gamma g_j(y,x)\big\|_{L^p(R^n_+)} \le M \big\|y^\gamma f_j(y,x)\big\|_{L^p(R^n_+)}, \tag{1.57}$$

and thus,

$$\big\|g_j\big\|_{L^p(R^n_+)} \le M \big\|f_j\big\|_{L^p(R_+)}, \tag{1.58}$$

for some positive constant M independent of ϵ, j, and f. Next (1.53) implies that

$$\Big|\sum_{j=1}^{N(\epsilon)} g_j(y,x)\Big|^p \le 2^p \sum_{j=1}^{N(\epsilon)} \big|g_j(y,x)\big|^p, \qquad \text{for all } (y,x) \in R^n_+. \tag{1.59}$$

It follows that

$$\Big\|\sum_{j=1}^{N(\epsilon)} g_j\Big\|^p_{L^p(R^n_+)} \le M \sum_{j=1}^{N(\epsilon)} \big\|f_j\big\|^p_{L^p(R^n_+)} \le M \big\|f\big\|^p_{L^p(R^n_+)}, \tag{1.60}$$

for some positive constant M independent of ϵ and f. Meanwhile, it is obvious that, as $\epsilon \to 0$,

$$\sum_{j=1}^{N(\epsilon)} f_j \to f, \qquad \text{in } L^2(R^n_+), \tag{1.61}$$

and consequently,

$$\sum_{j=1}^{N(\epsilon)} g_j \to \Lambda_{1,\gamma} f, \tag{1.62}$$

in the sense of distribution over R^n_+. Hence, we finally arrive at

$$\big\|\Lambda_{1,\gamma} f\big\|_{L^p(R^n_+)} \le M \big\|f\big\|_{L^p(R^n_+)}, \tag{1.63}$$

for some positive constant independent of f. $\Lambda_{2,\gamma}$ can be handled in the same manner. $\square$

The following fact is well-known; see [1].

Lemma 1.3. *Let* $0 < L < \infty$, $G_L = \{(y,x) \in R^n_+ \, : \, y < L\}$, *and* $1 < p < \infty$. *If we define, for* $h(x) \in C_0^\infty(R^{n-1})$ *and* $1/p < s < \infty$,

$$\hat{\psi}(y,\xi) = |\xi|^s \, e^{-y|\xi|} \hat{h}(\xi), \tag{1.64}$$

then it holds that

$$\big\|\psi\big\|_{L^p(G_L)} \le M_L \big\|h\big\|_{B_{pp}^{s-\frac{1}{p}}(R^{n-1})}, \tag{1.65}$$

for some positive constant M_L *independent of* h.

2. Review of the Airy functions

Let us first review some basic facts on the Airy functions. The Airy function of the first kind is denoted by $A(z)$, and can be given by

$$A(z) = \frac{1}{2\pi} \, e^{-\frac{2}{3}z^{3/2}} \int_{-\infty}^{\infty} e^{-\xi^2\sqrt{z}} e^{i\xi^3/3} \, d\xi, \tag{2.1}$$

which is valid for $z \neq 0$, $|arg(z)| < \pi$; see [8]. The Airy function of the second kind is written as $B(z)$, and can be given by

$$B(z) = e^{i\pi/6} A(ze^{i2\pi/3}) + e^{-i\pi/6} A(ze^{-i2\pi/3}), \tag{2.2}$$

and thus, for all $y > 0$,

$$B(y) = \frac{1}{\pi} e^{\frac{2}{3}y^{3/2}} \int_{-\infty}^{\infty} e^{-\xi^2\sqrt{y}/2} \, cos\big(\frac{\sqrt{3}}{2}\xi^2\sqrt{y} - \frac{\pi}{6}\big) e^{i\xi^3/3} \, d\xi. \tag{2.3}$$

Then, $A(z)$ and $B(z)$ are two linearly independent solutions of

$$\frac{d^2 w}{dz^2} - zw = 0, \tag{2.4}$$

and their Wronskian is

$$W(A(z), B(z)) = \frac{1}{\pi}, \qquad \text{for all } \ z. \tag{2.5}$$

It is known that $A(z)$ and $B(z)$ are entire functions of z. Some other properties can be also found in [4].

For later use, we define

$$Q_A(y) = A(y) e^{\frac{2}{3}y^{3/2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\xi^2\sqrt{y}} \, e^{i\xi^3/3} \, d\xi, \tag{2.6}$$

and

$$Q_B(y) = B(y) e^{-\frac{2}{3}y^{3/2}} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\xi^2\sqrt{y}/2} \, cos\big(\frac{\sqrt{3}}{2}\xi^2\sqrt{y} - \frac{\pi}{6}\big) \, e^{i\xi^3/3} \, d\xi. \tag{2.7}$$

Lemma 2.1. *For each* $\alpha \in Z_+$, *it holds that*

$$\left| y^\alpha \, \frac{\partial^\alpha Q_A(y)}{\partial y^\alpha} \right| + \left| y^\alpha \, \frac{\partial^\alpha Q_B(y)}{\partial y^\alpha} \right| \le M_\alpha, \qquad \textit{for all } y > 0, \tag{2.8}$$

for some positive constant M_α.

Proof. By direct differentiation of (2.7), we have

$$y^\alpha \, \frac{\partial^\alpha Q_B(y)}{\partial y^\alpha} = \int_{-\infty}^{\infty} e^{-\xi^2\sqrt{y}/2}\Big(P_1(\xi^2\sqrt{y})\, cos\big(\frac{\sqrt{3}}{2}\xi^2\sqrt{y} - \frac{\pi}{6}\big) + P_2(\xi^2\sqrt{y})\, sin\big(\frac{\sqrt{3}}{2}\xi^2\sqrt{y} - \frac{\pi}{6}\big)\Big) e^{i\xi^3/3}\, d\xi \tag{2.9}$$

where $P_1(t)$ and $P_2(t)$ are polynomials in t. Hence, it is easy to see that, for all $y \geq 1$,

$$\left| y^\alpha \, \frac{\partial^\alpha Q_B(y)}{\partial y^\alpha} \right| \leq M_\alpha, \qquad \text{for each } \alpha \in Z_+. \tag{2.10}$$

This inequality is also valid for $0 < y < 1$, because $B(z)$ is an entire function of z. The argument for Q_A is the same as above. □

The inequality (2.8) yields the following.

Lemma 2.2. *For each $\alpha \in Z_+^k$, we have*

$$|\xi|^{|\alpha|} \left| \frac{\partial^\alpha Q_A(y|\xi|^{2/3})}{\partial \xi^\alpha} \right| + |\xi|^{|\alpha|} \left| \frac{\partial^\alpha Q_B(y|\xi|^{2/3})}{\partial \xi^\alpha} \right| \leq M_\alpha, \tag{2.11}$$

for all $(y, \xi) \in R_+ \times R^k$, $\xi \neq 0$.

Lemma 2.3. *For each $\alpha \in Z_+$, it holds that*

$$\left| y^\alpha \, \frac{\partial^{\alpha+1} Q_A(y)}{\partial y^{\alpha+1}} \right| + \left| y^\alpha \, \frac{\partial^{\alpha+1} Q_B(y)}{\partial y^{\alpha+1}} \right| \leq M_\alpha, \qquad \textit{for all } \; y > 0. \tag{2.12}$$

Proof. For $y \geq 1$, (2.12) is a consequence of (2.8). For $0 < y < 1$, we directly differentiate $Q_A(y)$ and $Q_B(y)$, and use the fact that $A(z)$ and $B(z)$ are entire in z. □

Lemma 2.4. *For each $\alpha \in Z_+^k$, we have*

$$|\xi|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial}{\partial y} \Big(\frac{Q_A(y|\xi|^{2/3})}{|\xi|^{2/3}} \Big) \right| + |\xi|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial}{\partial y} \Big(\frac{Q_B(y|\xi|^{2/3})}{|\xi|^{2/3}} \Big) \right| \leq M_\alpha, \tag{2.13}$$

and

$$|\xi|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \Big(\frac{1}{y|\xi|^{-1/3}} \, \frac{\partial}{\partial \xi_j} Q_A(y|\xi|^{2/3}) \Big) \right| + |\xi|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \Big(\frac{1}{y|\xi|^{-1/3}} \, \frac{\partial}{\partial \xi_j} Q_B(y|\xi|^{2/3}) \Big) \right| \leq M_\alpha, \tag{2.14}$$

for all $(y, \xi) \in R_+ \times R^k$, $\xi \neq 0$, $j = 1, \cdots, k$.

This follows immediately from (2.12).

Next we define

$$J_A(y) = A(y)\, y^{1/4}\, e^{\frac{2}{3}y^{3/2}} = \frac{1}{2\pi}\, y^{1/4} \int_{-\infty}^{\infty} e^{-\xi^2\sqrt{y}}\, e^{i\xi^3/3}\, d\xi, \tag{2.15}$$

and

(2.16)

$$J_B(y) = B(y)\, y^{1/4}\, e^{-\frac{2}{3}y^{3/2}} = \frac{1}{\pi}\, y^{1/4} \int_{-\infty}^{\infty} e^{-\xi^2\sqrt{y}/2}\, cos\big(\frac{\sqrt{3}}{2}\xi^2\sqrt{y} - \frac{\pi}{6}\big)\, e^{i\xi^3/3}\, d\xi.$$

It is easy to see that, for each $\alpha \in Z_+$,

$$\left| y^\alpha \frac{\partial^\alpha J_A(y)}{\partial y^\alpha} \right| + \left| y^\alpha \frac{\partial^\alpha J_B(y)}{\partial y^\alpha} \right| \le M_\alpha, \qquad \text{for all } y > 0, \tag{2.17}$$

which yields the following estimates.

Lemma 2.5. *For each $\alpha \in Z_+^k$, we have*

$$\left|\xi\right|^{|\alpha|} \left| \frac{\partial^\alpha J_A(y|\xi|^{2/3})}{\partial \xi^\alpha} \right| + \left|\xi\right|^{|\alpha|} \left| \frac{\partial^\alpha J_B(y|\xi|^{2/3})}{\partial \xi^\alpha} \right| \le M_\alpha, \tag{2.18}$$

and

$$\left|\xi\right|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial}{\partial y} J_A(y|\xi|^{2/3}) \right| + \left|\xi\right|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial}{\partial y} J_B(y|\xi|^{2/3}) \right| \le \frac{M_\alpha}{y}, \tag{2.19}$$

for all $(y, \xi) \in R_+ \times R^k$, $\xi \neq 0$.

Proof. (2.18) follows directly from (2.17). If we set $H_A(y) = y \dfrac{\partial J_A(y)}{\partial y}$ and $H_B(y) = y \dfrac{\partial J_B(y)}{\partial y}$, then

$$\left| y^\alpha \frac{\partial^\alpha H_A(y)}{\partial y^\alpha} \right| + \left| y^\alpha \frac{\partial^\alpha H_B(y)}{\partial y^\alpha} \right| \le M_\alpha, \qquad \text{for all } y > 0. \tag{2.20}$$

Since $y \dfrac{\partial}{\partial y} J_A(y|\xi|^{2/3}) = H_A(y|\xi|^{2/3})$ and $y \dfrac{\partial}{\partial y} J_B(y|\xi|^{2/3}) = H_B(y|\xi|^{2/3})$, (2.19) follows from (2.20). □

3. Proof of Theorem 0.1

Choose any $f \in C_0^\infty(\overline{R_+^n})$, $\phi \in C_0^\infty(R^{n-1})$, and define for $j = 1, \cdots, n-1$,

$$\begin{aligned} \hat{v}^j(y,\xi) = &- \pi\, A(y|\xi|^{2/3}) \int_0^y i\,\xi_j |\xi|^{-2/3} B(t|\xi|^{2/3})\, \hat{f}(t,\xi)\, dt \\ &- \pi\, B(y|\xi|^{2/3}) \int_y^\infty i\,\xi_j |\xi|^{-2/3} A(t|\xi|^{2/3})\, \hat{f}(t,\xi)\, dt \\ &+ \sqrt{3}\,\pi\, A(y|\xi|^{2/3}) \int_0^\infty i\,\xi_j |\xi|^{-2/3} A(t|\xi|^{2/3}) \hat{f}(t,\xi)\, dt \\ &+ 3^{2/3}\Gamma(2/3) A(y|\xi|^{2/3})\, i\,\xi_j\, \hat{\phi}(\xi), \end{aligned} \tag{3.1}$$

where $i = \sqrt{-1}$, and $\Gamma(\cdot)$ stands for the Gamma function. Then, $v^j(y,x)$ satisfies

$$\frac{\partial^2 v^j}{\partial y^2} + y\, \Delta_x v^j = \frac{\partial f}{\partial x_j} \qquad \text{in } R_+^n, \tag{3.2}$$

and

$$v^j(0,x) = \frac{\partial \phi(x)}{\partial x_j} \qquad \text{on } \partial R_+^n. \tag{3.3}$$

We can rewrite (3.1) as

(3.4)

$$\begin{aligned}
\hat{v}^j(y,\xi) &= -\pi\, Q_A(y|\xi|^{2/3})e^{-\frac{2}{3}y^{3/2}|\xi|}\int_0^y i\,\xi_j\,|\xi|^{-2/3}e^{\frac{2}{3}t^{3/2}|\xi|}Q_B(t|\xi|^{2/3})\,\hat{f}(t,\xi)\,dt\\
&- \pi\, Q_B(y|\xi|^{2/3})e^{\frac{2}{3}y^{3/2}|\xi|}\int_y^\infty i\,\xi_j\,|\xi|^{-2/3}e^{-\frac{2}{3}t^{3/2}|\xi|}Q_A(t|\xi|^{2/3})\,\hat{f}(t,\xi)\,dt\\
&+ \sqrt{3}\,\pi\, Q_A(y|\xi|^{2/3})e^{-\frac{2}{3}y^{3/2}|\xi|}\int_0^\infty i\,\xi_j\,|\xi|^{-2/3}e^{-\frac{2}{3}t^{3/2}|\xi|}Q_A(t|\xi|^{2/3})\hat{f}(t,\xi)\,dt\\
&+ 3^{2/3}\Gamma(2/3)e^{-\frac{2}{3}y^{3/2}|\xi|}\,Q_A(y|\xi|^{2/3})\,i\,\xi_j\,\hat{\phi}(\xi)\\
&= \hat{v}^j_1(y,\xi)+\hat{v}^j_2(y,\xi)+\hat{v}^j_3(y,\xi)+\hat{v}^j_4(y,\xi),
\end{aligned}$$

where the three integral terms are denoted by $\hat{v}^j_k,\ k=1,2,3,$ and the last term by $\hat{v}^j_4$. By virtue of (2.11), (2.13), and the fact that $f\in C_0^\infty(\overline{R^n_+}),\ \phi\in C_0^\infty(R^{n-1})$, we first find that

$$\xi^\alpha\hat{v}(y,\xi)\in C\big([0,\infty);L^1(R^{n-1})\big),\qquad \text{for each }\ \alpha\in Z_+^{n-1}. \tag{3.5}$$

Then, it follows from (3.2) that

$$v(y,x)\in C^\infty(\overline{R^n_+}). \tag{3.6}$$

Next we will show that v decays to zero at infinity. Let L be a positive number such that $\mathit{supp}\, f\subset\{(y,x)\,:\,y<L\}$. Then, for $y>L$, $v^j_2(y,x)$ vanishes, and it follows from (2.11) that

$$\begin{aligned}
&\big\|v^j_1(y)\big\|_{L^\infty(R^{n-1})}+\big\|v^j_3(y)\big\|_{L^\infty(R^{n-1})}+\big\|v^j_4(y)\big\|_{L^\infty(R^{n-1})}\\
&\quad\le M\Big(\big\|\hat{v}^j_1(y)\big\|_{L^1(R^{n-1})}+\big\|\hat{v}^j_3(y)\big\|_{L^1(R^{n-1})}+\big\|\hat{v}^j_4(y)\big\|_{L^1(R^{n-1})}\Big)\\
&\quad\le C\Big(\big\|e^{-\frac{2}{3}(y^{3/2}-L^{3/2})|\xi|}\big\|_{L^2(R^{n-1})}+\big\|e^{-\frac{2}{3}y^{3/2}|\xi|}\big\|_{L^2(R^{n-1})}\Big),
\end{aligned}\tag{3.7}$$

where C is a positive constant depending on $\hat{f}$ and $\hat{\phi}$.

It is now apparent that, for $j=1,\cdots,n-1$,

$$\big\|v^j(y)\big\|_{L^\infty(R^{n-1})}\to 0,\qquad \text{as }\ y\to\infty. \tag{3.8}$$

Next we fix any $N>0$. For $j,\,\mu=1,\cdots,n-1$, we use (2.14) to see that

$$\begin{aligned}
\big|x_\mu v^j(y,x)\big| &\le M\sum_{k=1}^4\int_{R^{n-1}}\left|\frac{\partial\hat{v}^j_k(y,\xi)}{\partial\xi_\mu}\right|d\xi\\
&\le C_N,\qquad \text{for all }\ x\in R^{n-1},\ 0\le y<N,
\end{aligned}\tag{3.9}$$

for some positive constant C_N. It follows from (3.8) and (3.9) that

$$\big|v^j(y,x)\big|\to 0,\qquad \text{as }\ |x|+y\to\infty. \tag{3.10}$$

By means of the maximum principle, we conclude that if $u\in C_0^\infty(\overline{R^n_+})$ satisfies (0.1) and (0.2) with the same f and ϕ as in (3.1), then

$$\frac{\partial u(y,x)}{\partial x_j}=v^j(y,x),\qquad \text{for all }\ (y,x)\in R^n_+,\ \ j=1,\cdots,n-1. \tag{3.11}$$

We define

$$\hat{w}_k(y,\xi) = \sum_{j=1}^{n-1} i\,\xi_j\, y\, \hat{v}_k^j(y,\xi), \qquad k = 1,2,3,4, \tag{3.12}$$

and

$$\hat{w}(y,\xi) = \hat{w}_1(y,\xi) + \hat{w}_2(y,\xi) + \hat{w}_3(y,\xi) + \hat{w}_4(y,\xi). \tag{3.13}$$

For the estimate of $w_1(y,x)$, we write, for $y > 0$,

(3.14)

$$\begin{aligned}\hat{w}_1(y,\xi) &= \pi\, y\, Q_A(y|\xi|^{2/3}) e^{-\frac{2}{3}y^{3/2}|\xi|} \int_0^{y/2} |\xi|^{4/3} e^{\frac{2}{3}t^{3/2}|\xi|} Q_B(t|\xi|^{2/3})\, \hat{f}(t,\xi)\, dt \\ &\quad + \pi\, J_A(y|\xi|^{2/3}) y^{3/4} e^{-\frac{2}{3}y^{3/2}|\xi|} \int_{y/2}^{y} |\xi|\, t^{-1/4} e^{\frac{2}{3}t^{3/2}|\xi|} J_B(t|\xi|^{2/3})\, \hat{f}(t,\xi)\, dt \\ &= \hat{w}_1^I(y,\xi) + \hat{w}_1^{II}(y,\xi),\end{aligned}$$

where the two integral terms are denoted by $\hat{w}_1^I(y,\xi)$ and $\hat{w}_1^{II}(y,\xi)$. By virtue of (2.11), and the inequality for each $\alpha \in Z_+^{n-1}$,

(3.15)

$$\big|\xi\big|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \left(|\xi|^{4/3}\, e^{-\frac{2}{3}(y^{3/2}-t^{3/2})|\xi|} \right) \right| \le \frac{M_\alpha}{y^2}, \qquad \text{for all } \xi \neq 0,\ \ 0 \le t < y/2,$$

we obtain, for $1 < p < \infty$,

$$\big\|w_1^I(y)\big\|_{L^p(R^{n-1})} \le \frac{M}{y} \int_0^{y/2} \big\|f(t)\big\|_{L^p(R^{n-1})}\, dt, \tag{3.16}$$

which yields by Hardy's inequality

$$\big\|w_1^I\big\|_{L^p(R_+^n)} \le M \big\|f\big\|_{L^p(R_+^n)}, \tag{3.17}$$

for some positive constant M independent of f.

For the estimate of w_1^{II}, we first observe by change of variables

$$\int_{R_+} \int_{R^{n-1}} \big|w_1^{II}(y,x)\big|^p\, dx\, dy = \frac{2}{3} \int_{R_+} \int_{R^{n-1}} \big|w_1^{II}(z^{2/3},x)\big|^p\, z^{-1/3}\, dx\, dz, \tag{3.18}$$

and hence, we need to write

$$\begin{aligned}\hat{w}_1^{II}(z^{2/3},\xi) z^{-1/(3p)} &= \frac{2\pi}{3} z^{\frac{1}{2}-\frac{1}{3p}}\, J_A(z^{2/3}|\xi|^{2/3}) e^{-\frac{2}{3}z|\xi|} \\ &\quad \times \int_{z/2^{3/2}}^{z} |\xi|\, e^{\frac{2}{3}s|\xi|} J_B(s^{2/3}|\xi|^{2/3}) \hat{f}(s^{2/3},\xi) s^{-\frac{1}{3p}}\, s^{\frac{1}{3p}-\frac{1}{2}}\, ds.\end{aligned} \tag{3.19}$$

Thus, it follows from Lemma 0.2, Proposition 1.2, Lemma 2.5, (3.18), and (3.19) that

$$\begin{aligned}\big\|w_1^{II}\big\|_{L^p(R_+^n)} &= \left(\frac{2}{3} \int_{R_+} \int_{R^{n-1}} \big|w_1^{II}(z^{2/3},x)\big|^p\, z^{-1/3}\, dx\, dz \right)^{1/p} \\ &\le M \left(\int_{R_+} \int_{R^{n-1}} \big|f(s^{2/3},x)\, s^{-1/(3p)}\big|^p\, dx\, ds \right)^{1/p} \\ &\le M \big\|f\big\|_{L^p(R_+^n)},\end{aligned} \tag{3.20}$$

where M is a positive constant independent of f.

Next we will estimate w_2. For $y>0$, we write

$$\hat{w}_2(y,\xi) = \pi\, y\, Q_B(y|\xi|^{2/3}) e^{\frac{2}{3}y^{3/2}|\xi|} \int_{2y}^{\infty} |\xi|^{4/3} e^{-\frac{2}{3}t^{3/2}|\xi|} Q_A(t|\xi|^{2/3})\, \hat{f}(t,\xi)\, dt \tag{3.21}$$
$$+ \pi\, J_B(y|\xi|^{2/3}) y^{3/4} e^{\frac{2}{3}y^{3/2}|\xi|} \int_{y}^{2y} |\xi|\, t^{-1/4} e^{-\frac{2}{3}t^{3/2}|\xi|} J_A(t|\xi|^{2/3})\, \hat{f}(t,\xi)\, dt$$
$$= \hat{w}_2^I(y,\xi) + \hat{w}_2^{II}(y,\xi),$$

where the two integral terms are denoted by $\hat{w}_2^I(y,\xi)$ and $\hat{w}_2^{II}(y,\xi)$. Since it holds that

$$|\xi|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \left(|\xi|^{4/3}\, e^{-\frac{2}{3}(t^{3/2}-y^{3/2})|\xi|} \right) \right| \le \frac{M_\alpha}{t^2}, \qquad \text{for all } \xi \neq 0,\ 0 \le 2y < t, \tag{3.22}$$

we obtain

$$\big\| w_2^I(y) \big\|_{L^p(R^{n-1})} \le M \int_{2y}^{\infty} \frac{1}{t} \big\| f(t) \big\|_{L^p(R^{n-1})}\, dt \tag{3.23}$$

and hence, by (1.44),

$$\big\| w_2^I \big\|_{L^p(R^n_+)} \le M \big\| f \big\|_{L^p(R^n_+)}, \tag{3.24}$$

for some positive constant independent of f.

As above, we write

$$\hat{w}_2^{II}(z^{2/3},\xi) z^{-1/(3p)} = \frac{2\pi}{3} z^{\frac{1}{2}-\frac{1}{3p}}\, J_B(z^{2/3}|\xi|^{2/3}) e^{\frac{2}{3}z|\xi|} \tag{3.25}$$
$$\times \int_{z}^{2^{3/2}z} |\xi|\, e^{-\frac{2}{3}s|\xi|} J_A(s^{2/3}|\xi|^{2/3}) \hat{f}(s^{2/3},\xi) s^{-\frac{1}{3p}}\, s^{\frac{1}{3p}-\frac{1}{2}}\, ds,$$

and derive by Lemma 0.2, Proposition 1.2 and Lemma 2.5,

$$\big\| w_2^{II} \big\|_{L^p(R^n_+)} \le M \big\| f \big\|_{L^p(R^n_+)}, \tag{3.26}$$

for some constant M independent of f. We proceed to estimate w_3. It is easy to see that for each $\alpha \in Z_+^{n-1}$

$$|\xi|^{|\alpha|} \left| \frac{\partial^\alpha}{\partial \xi^\alpha} \left(|\xi|^{4/3}\, e^{-\frac{2}{3}(y^{3/2}+t^{3/2})|\xi|} \right) \right| \le \frac{M_\alpha}{t^2+y^2}, \qquad \text{for all } \xi \neq 0,\ y>0,\ t>0. \tag{3.27}$$

Hence, by virtue of Lemma 0.2 and (2.11), we obtain

$$\big\| w_3(y) \big\|_{L^p(R^{n-1})} \le M \int_0^{\infty} \frac{y}{y^2+t^2} \big\| f(t) \big\|_{L^p(R^{n-1})}\, dt \tag{3.28}$$

which, combined with an integral inequality given in [13, p.271], yields

$$\big\| w_3 \big\|_{L^p(R^n_+)} \le M \big\| f \big\|_{L^p(R^n_+)}, \tag{3.29}$$

for some positive constant M independent of f.

To estimate w_4, we write

$$\hat{w}_4(z^{2/3},\xi) z^{-\frac{1}{3p}} = -3^{2/3}\Gamma(2/3) \big(z|\xi| \big)^{\frac{2}{3}-\frac{1}{3p}} \big|\xi\big|^{\frac{4}{3}+\frac{1}{3p}} e^{-\frac{2}{3}z|\xi|} Q_A(z^{2/3}|\xi|^{2/3}) \hat{\phi}(\xi). \tag{3.30}$$

Since it holds that for each $\alpha \in Z_+^{n-1}$,

$$|\xi|^{|\alpha|}\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\left(\left(z|\xi|\right)^{\frac{2}{3}-\frac{1}{3p}}e^{-\frac{1}{3}z|\xi|}\right)\right| \le M_\alpha, \qquad \text{for all } z>0,\ \xi\neq 0, \tag{3.31}$$

it follows from Lemma 1.3 and (2.11) that

$$\|w_4\|_{L^p(G_L)} \le M_L\|\phi\|_{B_{pp}^{4/3-2/(3p)}(R^{n-1})}, \tag{3.32}$$

where $0<L<\infty$, and $G_L=\{(y,x)\in R_+^n\,:\,y<L\}$. Since $y\Delta_x u = w_1+w_2+w_3+w_4$, we combine (3.17), (3.20), (3.24), (3.26), (3.29), and (3.32) to obtain

$$\left\|\frac{\partial^2 u}{\partial y^2}\right\|_{L^p(R_+^n)} + \left\|y\Delta_x u\right\|_{L^p(R_+^n)} \le M_L\left(\|f\|_{L^p(R_+^n)} + \|\phi\|_{B_{pp}^{4/3-2/(3p)}(R^{n-1})}\right), \tag{3.33}$$

provided *supp* $u\subset\{(y,x)\,:\,y<L\}$.

It remains to estimate $\sqrt{y}\,\nabla_x\dfrac{\partial u}{\partial y}$. Let us fix any $j=1,\cdots,n-1$, and set

$$\hat\sigma_1(y,\xi) = -\pi\sqrt{y}\,\frac{\partial A(y|\xi|^{2/3})}{\partial y}\int_0^y i\,\xi_j|\xi|^{-2/3}B(t|\xi|^{2/3})\,\hat f(t,\xi)\,dt, \tag{3.34}$$

$$\hat\sigma_2(y,\xi) = -\pi\sqrt{y}\,\frac{\partial B(y|\xi|^{2/3})}{\partial y}\int_y^\infty i\,\xi_j|\xi|^{-2/3}A(t|\xi|^{2/3})\,\hat f(t,\xi)\,dt, \tag{3.35}$$

$$\hat\sigma_3(y,\xi) = \sqrt{3}\,\pi\sqrt{y}\,\frac{\partial A(y|\xi|^{2/3})}{\partial y}\int_0^\infty i\,\xi_j|\xi|^{-2/3}A(t|\xi|^{2/3})\hat f(t,\xi)\,dt, \tag{3.36}$$

$$\hat\sigma_4(y,\xi) = 3^{2/3}\Gamma(2/3)\sqrt{y}\,\frac{\partial A(y|\xi|^{2/3})}{\partial y}\,i\,\xi_j\,\hat\phi(\xi), \tag{3.37}$$

and

$$\hat\sigma(y,\xi) = \hat\sigma_1(y,\xi)+\hat\sigma_2(y,\xi)+\hat\sigma_3(y,\xi)+\hat\sigma_4(y,\xi), \tag{3.38}$$

so that

$$\sqrt{y}\,\frac{\partial^2 u(y,x)}{\partial y\,\partial x_j} = \sigma(y,x), \qquad \text{for all } (y,x)\in R_+^n. \tag{3.39}$$

To estimate $\sigma_1(y,x)$, we write

$$\begin{aligned}
&\hat\sigma_1(y,\xi) = \hat\sigma_1^{I}(y,\xi)+\hat\sigma_1^{II}(y,\xi)+\hat\sigma_1^{III}(y,\xi)\\
&= -\pi\sqrt{y}\left(\frac{\partial}{\partial y}\left\{Q_A(y|\xi|^{2/3})e^{-\frac{2}{3}y^{3/2}|\xi|}\right\}\right)\int_0^{y/2} i\xi_j|\xi|^{-2/3}e^{\frac{2}{3}t^{3/2}|\xi|}Q_B(t|\xi|^{2/3})\hat f(t,\xi)\,dt\\
&- \pi\sqrt{y}\left(\frac{\partial}{\partial y}\left\{J_A(y|\xi|^{2/3})y^{-1/4}\right\}\right)e^{-\frac{2}{3}y^{3/2}|\xi|}\\
&\qquad\times\int_{y/2}^y i\xi_j|\xi|^{-1}t^{-1/4}e^{\frac{2}{3}t^{3/2}|\xi|}J_B(t|\xi|^{2/3})\hat f(t,\xi)\,dt\\
&+ \pi y^{3/4}J_A(y|\xi|^{2/3})e^{-\frac{2}{3}y^{3/2}|\xi|}\int_{y/2}^y i\xi_j|\xi|^{-1}t^{-1/4}|\xi|e^{\frac{2}{3}t^{3/2}|\xi|}J_B(t|\xi|^{2/3})\hat f(t,\xi)\,dt.
\end{aligned} \tag{3.40}$$

By Lemma 0.2, (2.11), (2.13), and an inequality similar to (3.15), we have

$$\big\|\sigma_1^I(y)\big\|_{L^p(R^{n-1})} \le \frac{M}{y}\int_0^{y/2}\big\|f(t)\big\|_{L^p(R^{n-1})}\,dt, \qquad \text{for all } y>0, \tag{3.41}$$

which yields

$$\big\|\sigma_1^I\big\|_{L^p(R^n_+)} \le M\big\|f\big\|_{L^p(R^n_+)}, \tag{3.42}$$

for some positive constant M independent of f. Again by Lemma 0.2 and Lemma 2.5, we obtain

$$\big\|\sigma_1^{II}(y)\big\|_{L^p(R^{n-1})} \le \frac{M}{y}\int_{y/2}^{y}\big\|f(t)\big\|_{L^p(R^{n-1})}\,dt, \tag{3.43}$$

and hence,

$$\big\|\sigma_1^{II}\big\|_{L^p(R^n_+)} \le M\big\|f\big\|_{L^p(R^n_+)}, \tag{3.44}$$

for some positive constant M independent of f. $\hat{\sigma}_1^{III}(y,\xi)$ has the same structure as $\hat{w}_1^{II}(y,\xi)$ above with an additional Riesz transform. Hence, we can copy (3.20) so that

$$\big\|\sigma_1^{III}\big\|_{L^p(R^n_+)} \le M\big\|f\big\|_{L^p(R^n_+)}. \tag{3.45}$$

We next write

(3.46)

$$\begin{aligned}
&\hat{\sigma}_2(y,\xi) = \hat{\sigma}_2^I(y,\xi) + \hat{\sigma}_2^{II}(y,\xi) + \hat{\sigma}_2^{III}(y,\xi)\\
&= -\pi\sqrt{y}\left(\frac{\partial}{\partial y}\big\{Q_B(y|\xi|^{2/3})e^{\frac{2}{3}y^{3/2}|\xi|}\big\}\right)\int_{2y}^{\infty} i\xi_j|\xi|^{-2/3}e^{-\frac{2}{3}t^{3/2}|\xi|}Q_A(t|\xi|^{2/3})\hat{f}(t,\xi)\,dt\\
&- \pi\sqrt{y}\left(\frac{\partial}{\partial y}\big\{J_B(y|\xi|^{2/3})y^{-1/4}\big\}\right)e^{\frac{2}{3}y^{3/2}|\xi|}\\
&\qquad\qquad \times\int_y^{2y} i\xi_j|\xi|^{-1}t^{-1/4}e^{-\frac{2}{3}t^{3/2}|\xi|}J_A(t|\xi|^{2/3})\hat{f}(t,\xi)\,dt\\
&- \pi y^{3/4}J_B(y|\xi|^{2/3})e^{\frac{2}{3}y^{3/2}|\xi|}\int_y^{2y} i\xi_j|\xi|^{-1}t^{-1/4}|\xi|e^{-\frac{2}{3}t^{3/2}|\xi|}J_A(t|\xi|^{2/3})\hat{f}(t,\xi)\,dt.
\end{aligned}$$

It follows from Lemma 0.2, (2.11) and (2.13) that

$$\big\|\sigma_2^I(y)\big\|_{L^p(R^{n-1})} \le M\int_{2y}^{\infty}\frac{1}{t}\big\|f(t)\big\|_{L^p(R^{n-1})}\,dt, \tag{3.47}$$

and thus,

$$\big\|\sigma_2^I\big\|_{L^p(R^n_+)} \le M\big\|f\big\|_{L^p(R^n_+)}, \tag{3.48}$$

for some positive constant M independent of f. By means of (2.11) and (2.19), we obtain

$$\big\|\sigma_2^{II}\big\|_{L^p(R^n_+)} \le M\big\|f\big\|_{L^p(R^n_+)}. \tag{3.49}$$

By the same argument as for w_2^{II} above, we find

$$\big\|\sigma_2^{III}\big\|_{L^p(R^n_+)} \le M\big\|f\|_{L^p(R^n_+)}. \tag{3.50}$$

For the estimate of σ_3, we rewrite (3.36) as

$$\hat{\sigma}_3(y,\xi) = \sqrt{3}\,\pi\,\sqrt{y}\left(\frac{\partial}{\partial y}\{Q_A(y|\xi|^{2/3})e^{-\frac{2}{3}y^{3/2}|\xi|}\}\right) \times \int_0^\infty i\,\xi_j|\xi|^{-2/3}e^{-\frac{2}{3}t^{3/2}|\xi|}Q_A(t|\xi|^{2/3})\hat{f}(t,\xi)\,dt. \tag{3.51}$$

By means of (2.13) and the same argument as for w_3 above, we derive

$$\|\sigma_3(y)\|_{L^p(R^{n-1})} \le M\int_0^\infty \left(\frac{y^{1/2}}{t^{3/2}+y^{3/2}} + \frac{y}{t^2+y^2}\right)\|f(t)\|_{L^p(R^{n-1})}\,dt, \tag{3.52}$$

which, together with an integral inequality given in [13, p.271], yields

$$\|\sigma_3\|_{L^p(R^n_+)} \le M\|f\|_{L^p(R^n_+)} \tag{3.53}$$

for some positive constant M independent of f.

To estimate σ_4, we rewrite (3.37) as

$$\begin{aligned}\hat{\sigma}_4(y,\xi) &= \hat{\sigma}_4^I(y,\xi) + \hat{\sigma}_4^{II}(y,\xi)\\ &= 3^{2/3}\Gamma(2/3)\sqrt{y}\,e^{-\frac{2}{3}y^{3/2}|\xi|}\frac{\partial}{\partial y}\left(\frac{Q_A(y|\xi|^{2/3})}{|\xi|^{2/3}}\right)i\xi_j|\xi|^{2/3}\hat{\phi}(\xi)\\ &\quad - 3^{2/3}\Gamma(2/3)y\,e^{-\frac{2}{3}y^{3/2}|\xi|}Q_A(y|\xi|^{2/3})i\xi_j|\xi|\hat{\phi}(\xi).\end{aligned} \tag{3.54}$$

Setting $y = z^{2/3}$, we have

$$\hat{\sigma}_4^I(z^{2/3},\xi)z^{-\frac{1}{3p}} = 3^{2/3}\Gamma(2/3)\frac{\xi_j}{|\xi|}\left(z|\xi|\right)^{\frac{1}{3}-\frac{1}{3p}}e^{-\frac{2}{3}z|\xi|}\mathcal{L}(z,\xi)i|\xi|^{\frac{4}{3}+\frac{1}{3p}}\hat{\phi}(\xi), \tag{3.55}$$

where $\mathcal{L}(z,\xi) = \dfrac{\partial}{\partial y}\dfrac{Q_A(y|\xi|^{2/3})}{|\xi|^{2/3}}$.

By virtue of Lemma 0.2, Lemma 1.3, (2.13), the L^p boundedness of the Riesz transforms, and an inequality similar to (3.31), we have

$$\|\sigma_4^I\|_{L^p(G_L)} \le M_L\|\phi\|_{B_{pp}^{4/3-2/(3p)}(R^{n-1})}, \tag{3.56}$$

where $0 < L < \infty$, and $G_L = \{(y,x) \in R^n_+ \,:\, y < L\}$.

We can estimate σ_4^{II} in the same manner so that

$$\|\sigma_4\|_{L^p(G_L)} \le M_L\|\phi\|_{B_{pp}^{4/3-2/(3p)}(R^{n-1})}, \tag{3.57}$$

where M_L is a positive constant independent of ϕ. By combining the above estimates, and choosing L such that *supp* $u \subset G_L$, we have, for $j = 1,\cdots,n-1$,

$$\left\|\sqrt{y}\,\frac{\partial^2 u}{\partial y\partial x_j}\right\|_{L^p(R^n_+)} \le M_L\left(\|f\|_{L^p(R^n_+)} + \|\phi\|_{B_{pp}^{4/3-2/(3p)}(R^{n-1})}\right). \tag{3.58}$$

Now the proof of Theorem 0.1 is complete.

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