A QUANTUM OCTONION ALGEBRA

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To the memory of Alberto Izquierdo

ABSTRACT. Using the natural irreducible 8-dimensional representation and the two spin representations of the quantum group $U_q(D_4)$ of D_4 , we construct a quantum analogue of the split octonions and study its properties. We prove that the quantum octonion algebra satisfies the q -Principle of Local Triality and has a nondegenerate bilinear form which satisfies a q -version of the composition property. By its construction, the quantum octonion algebra is a nonassociative algebra with a Yang-Baxter operator action coming from the R -matrix of $U_q(D_4)$. The product in the quantum octonions is a $U_q(D_4)$ -module homomorphism. Using that, we prove identities for the quantum octonions, and as a consequence, obtain at $q = 1$ new “representation theory” proofs for very well-known identities satisfied by the octonions. In the process of constructing the quantum octonions we introduce an algebra which is a q -analogue of the 8-dimensional para-Hurwitz algebra.

INTRODUCTION

Using the representation theory of the quantum group $U_q(D_4)$ of D_4 , we construct a quantum analogue $\mathbb{O}_q$ of the split octonions and study its properties.

A unital algebra over a field with a nondegenerate bilinear form $(|)$ of maximal Witt index which admits composition,

$$(x \cdot y | x \cdot y) = (x | x)(y | y),$$

must be the field, two copies of the field, the split quaternions, or the split octonions. There is a natural q -version of the composition property that the algebra $\mathbb{O}_q$ of quantum octonions is shown to satisfy (see Prop. 4.12 below). We also prove that the quantum octonion algebra $\mathbb{O}_q$ satisfies the “ q -Principle of Local Triality” (Prop. 3.12). Inside the quantum octonions are two nonisomorphic 4-dimensional subalgebras, which are q -deformations of the split quaternions. One of them is unital, and both of them give gl_2 when considered as algebras under the commutator product $[x, y] = x \cdot y - y \cdot x$.

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By its construction, $\mathbb{O}_q$ is a nonassociative algebra with a Yang-Baxter operator action coming from the R-matrix of $U_q(D_4)$. Associative algebras with a Yang-Baxter operator (or r -algebras) arise in Manin's work on noncommutative geometry and include such important examples as Weyl and Clifford algebras, quantum groups, and certain universal enveloping algebras (see for example [B]).

In the process of constructing $\mathbb{O}_q$ we define an algebra $\mathbb{P}_q$, which is a q -analogue of the 8-dimensional para-Hurwitz algebra. The quantum para-Hurwitz algebra $\mathbb{P}_q$ is shown to satisfy certain identities which become familiar properties of the para-Hurwitz algebra at $q = 1$. The para-Hurwitz algebra is a (non-unital) 8-dimensional algebra with a non-degenerate associative bilinear form admitting composition. The algebra $\mathbb{P}_q$ exhibits related properties (see Props. 5.1, 5.3).

While this paper was in preparation, we received a preprint of [Br], which constructs a quantized octonion algebra using the representation theory of $U_q(sl_2)$. Although both Bremner's quantized octonions and our quantum octonions reduce to the octonions at $q = 1$, they are different algebras. The quantized octonion algebra in [Br] is constructed from defining a multiplication on the sum of irreducible $U_q(sl_2)$ -modules of dimensions 1 and 7. As a result, it carries a $U_q(sl_2)$ -module structure and has a unit element. The quantum octonion algebra constructed in this paper using $U_q(D_4)$ has a unit element only for the special values $q = 1, -1$.

An advantage to the $U_q(D_4)$ approach is that it allows properties such as the ones mentioned above to be derived from its representation theory. Using the fact that the product in $\mathbb{O}_q$ is a $U_q(D_4)$ -module homomorphism, we prove identities for $\mathbb{O}_q$ (see Section 4) and as a consequence obtain at $q = 1$ new "representation theory" proofs for very well-known identities satisfied by the octonions that had been established previously by other methods. The $U_q(D_4)$ approach also affords connections with fixed points of graph automorphisms - although as we show in Section 7, the fixed point subalgebras of $U_q(D_4)$ are not the quantum groups $U_q(G_2)$ and $U_q(B_3)$, because those quantum groups do not have Hopf algebra embeddings into $U_q(D_4)$. In the final section of this paper we explore connections with quantum Clifford algebras. In particular, we obtain a $U_q(D_4)$ -isomorphism between the quantum Clifford algebra $C_q(8)$ and the endomorphism algebra $\text{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$ and discuss its relation to the work of Ding and Frenkel [DF] (compare also [KPS]).

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1. PRELIMINARIES ON QUANTUM GROUPS

The quantum group $U_q(\mathfrak{g})$. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system Φ , and simple roots $\Pi = \{\alpha_i \mid i = 1, \dots, r\} \subset \Phi$ relative to $\mathfrak{h}$. Let $\mathfrak{A} = (a_{\alpha, \beta})_{\alpha, \beta \in \Pi}$ be the corresponding Cartan matrix. Thus,

$$a_{\alpha, \beta} = 2(\beta, \alpha)/(\alpha, \alpha),$$

and there exist relatively prime positive integers,

$$(1.1) \quad d_\alpha = \frac{(\alpha, \alpha)}{2},$$

so the matrix $(d_\alpha a_{\alpha,\beta})$ is symmetric. These integers are given explicitly by

$$(1.2) \quad \begin{array}{ll} A_r, D_r, E_r \ (r = 6, 7, 8) & d_{\alpha_i} = 1 \text{ for all } 1 \leq i \leq r, \\ B_r & d_{\alpha_i} = 2 \text{ for all } 1 \leq i \leq r-1, \text{ and } d_{\alpha_r} = 1, \\ C_r & d_{\alpha_i} = 1 \text{ for all } 1 \leq i \leq r-1, \text{ and } d_{\alpha_r} = 2, \\ F_4 & d_{\alpha_i} = 2, i = 1, 2, \text{ and } d_{\alpha_i} = 1, i = 3, 4, \\ G_2 & d_{\alpha_1} = 1 \text{ and } d_{\alpha_2} = 3. \end{array}$$

Let $\mathbf{K}$ be a field of characteristic not 2 or 3. Fix $q \in \mathbf{K}$ such that q is not a root of unity, and such that $q^{\frac{1}{2}} \in \mathbf{K}$. For $m, n, d \in \mathbf{Z}_{>0}$, let

$$(1.3) \quad [m]_d = \frac{q^{md} - q^{-md}}{q^d - q^{-d}},$$

and set

$$(1.4) \quad [m]_d! = \prod_{j=1}^m [j]_d$$

and $[0]_d! = 1$. Let

$$(1.5) \quad \begin{bmatrix} m \\ n \end{bmatrix}_d = \frac{[m]_d!}{[n]_d! [m-n]_d!}.$$

Definition 1.6. The quantum group $U = U_q(\mathfrak{g})$ is the unital associative algebra over $\mathbf{K}$ generated by elements $E_\alpha, F_\alpha, K_\alpha$, and K_α^{-1} (for all $\alpha \in \Pi$) and subject to the relations,

$$\begin{aligned} (Q1) \quad & K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha, \quad K_\alpha K_\beta = K_\beta K_\alpha, \\ (Q2) \quad & K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta, \\ (Q3) \quad & K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta, \\ (Q4) \quad & E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q^{d_\alpha} - q^{-d_\alpha}}, \\ (Q5) \quad & \sum_{s=0}^{1-a_{\alpha,\beta}} (-1)^s \begin{bmatrix} 1-a_{\alpha,\beta} \\ s \end{bmatrix}_{d_\alpha} E_\alpha^{1-a_{\alpha,\beta}-s} E_\beta E_\alpha^s = 0, \\ (Q6) \quad & \sum_{s=0}^{1-a_{\alpha,\beta}} (-1)^s \begin{bmatrix} 1-a_{\alpha,\beta} \\ s \end{bmatrix}_{d_\alpha} F_\alpha^{1-a_{\alpha,\beta}-s} F_\beta F_\alpha^s = 0. \end{aligned}$$

Corresponding to each $\lambda = \sum_{\alpha \in \Pi} m_\alpha \alpha$ in the root lattice $\mathbf{Z}\Phi$ there is an element

$$K_\lambda = \prod_{\alpha \in \Pi} K_\alpha^{m_\alpha}$$

in $U_q(\mathfrak{g})$, and $K_\lambda K_\mu = K_{\lambda+\mu}$ for all $\lambda, \mu \in \mathbf{Z}\Phi$. Using relations (Q2),(Q3), we have

$$(1.7) \quad K_\lambda E_\beta K_\lambda^{-1} = q^{(\lambda,\beta)} E_\beta \quad \text{and} \quad K_\lambda F_\beta K_\lambda^{-1} = q^{-(\lambda,\beta)} F_\beta$$

for all $\lambda \in \mathbf{Z}\Phi$ and $\beta \in \Pi$.

The algebra $U_q(\mathfrak{g})$ has a noncocommutative Hopf structure with comultiplication Δ , antipode S , and counit ϵ given by

$$\begin{aligned}
 \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, \\
 \Delta(E_\alpha) &= E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, \\
 \Delta(F_\alpha) &= F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha, \\
 S(K_\alpha) &= K_\alpha^{-1}, \\
 S(E_\alpha) &= -K_\alpha^{-1}E_\alpha, \quad S(F_\alpha) = -F_\alpha K_\alpha, \\
 \epsilon(K_\alpha) &= 1, \quad \epsilon(E_\alpha) = \epsilon(F_\alpha) = 0.
 \end{aligned}
 \tag{1.8}$$

The opposite comultiplication Δ^{op} has the property that

$$\text{if } \Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}, \quad \text{then } \Delta^{\text{op}}(a) = \sum_a a_{(2)} \otimes a_{(1)}.$$

(We are adopting the commonly used Sweedler notation for components of the comultiplication.) The algebra U with the same multiplication, same identity, and same counit, but with comultiplication given by Δ^{op} and with S^{-1} as its antipode, is also a Hopf algebra. The comultiplication Δ^{op} can be regarded as the composition $\tau \circ \Delta$, where $\tau : U \otimes U \rightarrow U \otimes U$ is given by $\tau(a \otimes b) = b \otimes a$. The map $\psi : U \rightarrow U_q(\mathfrak{g})$ with $E_\alpha \mapsto F_\alpha$, $F_\alpha \mapsto E_\alpha$, $K_\alpha \mapsto K_\alpha^{-1}$ is an algebra automorphism, and a coalgebra anti-automorphism, so U is isomorphic to $U_q(\mathfrak{g})$ as a Hopf algebra (see [CP, p. 211]).

The defining property of the antipode S in any Hopf algebra is that it is an inverse to the identity map id_U with respect to the convolution product, so that for all $a \in U$,

$$\sum_a S(a_{(1)})a_{(2)} = \epsilon(a)\text{id}_U = \sum_a a_{(1)}S(a_{(2)}).$$

Representations. For any Hopf algebra U the comultiplication operator allows us to define a U -module structure on the tensor product $V \otimes W$ of two U -modules V and W , where

$$a(v \otimes w) = \sum_a a_{(1)}v \otimes a_{(2)}w.$$

The counit ϵ affords a representation of U on the one-dimensional module given by $a1 = \epsilon(a)1$. If V is a finite-dimensional U -module, then so is its dual space V^* , where the U -action is given by the antipode mapping,

$$(af)(v) = f(S(a)v),$$

for all $a \in U$, $f \in V^*$, and $v \in V$.

A finite-dimensional module M for $U = U_q(\mathfrak{g})$ is the sum of weight spaces, $M = \bigoplus_{\nu, \chi} M_{\nu, \chi}$, where $M_{\nu, \chi} = \{x \in M \mid K_\alpha x = \chi(\alpha)q^{(\nu, \alpha)}x \text{ for all } \alpha \in \Pi\}$, and $\chi : \mathbf{Z}\Phi \rightarrow \{\pm 1\}$ is a group homomorphism. All the modules considered in this paper will be of type I, that is, $\chi(\alpha) = 1$ for all $\alpha \in \Pi$, so we will drop χ from the notation and from our considerations.

Each finite-dimensional irreducible U -module M has a highest weight λ , which is a dominant integral weight for $\mathfrak{g}$ relative to Π , and a unique (up to scalar multiple) maximal vector v^+ so that $E_\alpha v^+ = 0$ and $K_\alpha v^+ = q^{(\lambda, \alpha)}v^+$ for all $\alpha \in \Pi$. In particular, the trivial U -module $\mathbf{K}1$ has highest weight 0. We denote by $L(\lambda)$ the irreducible U -module with highest weight λ (and $\chi \equiv 1$).

When V is a finite-dimensional U -module, we can suppose $\{x_{\nu,i}\}$ is a basis of V of weight vectors with $i = 1, \dots, \dim V_{\nu}$. Let $\{x_{\nu,i}^*\}$ be the dual basis in V^* , so that $x_{\mu,i}^*(x_{\nu,j}) = \delta_{\mu,\nu}\delta_{i,j}$. Then

$$(1.11) \quad \begin{aligned} (K_{\alpha}x_{\mu,i}^*)(x_{\nu,j}) &= x_{\mu,i}^*(S(K_{\alpha})x_{\nu,j}) = x_{\mu,i}^*(K_{\alpha}^{-1}x_{\nu,j}) \\ &= q^{-(\nu,\alpha)}x_{\mu,i}^*(x_{\nu,j}) = q^{-(\nu,\alpha)}\delta_{\mu,\nu}\delta_{i,j} = q^{-(\mu,\alpha)}\delta_{\mu,\nu}\delta_{i,j}, \end{aligned}$$

from which we see that $x_{\mu,i}^*$ has weight $-\mu$.

Suppose M and N are U -modules for a quantum group U . Define a U -module structure on $\text{Hom}_{\mathbf{K}}(M, N)$ by

$$(1.12) \quad (a\gamma)(m) = \sum_a a_{(1)}\gamma(S(a_{(2)})m).$$

(This just means that the natural map $N \otimes M^* \rightarrow \text{Hom}_{\mathbf{K}}(M, N)$ is a U -module homomorphism.)

Note that if $\gamma \in \text{Hom}_U(M, N)$, then

$$(a\gamma)(m) = \sum_a a_{(1)}\gamma(S(a_{(2)})m) = \sum_a a_{(1)}S(a_{(2)})\gamma(m) = \epsilon(a)\gamma(m),$$

so that $a\gamma = \epsilon(a)\gamma$. That says

$$\text{Hom}_U(M, N) \subseteq \text{Hom}_{\mathbf{K}}(M, N)^U = \{\gamma \in \text{Hom}_{\mathbf{K}}(M, N) \mid a\gamma = \epsilon(a)\gamma\}.$$

Conversely, if γ belongs to the invariants $\text{Hom}_{\mathbf{K}}(M, N)^U$, then we have

$$\begin{aligned} 0 &= \epsilon(K_{\alpha})\gamma = \gamma = K_{\alpha} \circ \gamma \circ K_{\alpha}^{-1}, \quad \text{and hence,} \\ 0 &= K_{\alpha} \circ \gamma - \gamma \circ K_{\alpha}; \\ 0 &= \epsilon(E_{\alpha})\gamma = E_{\alpha}\gamma = E_{\alpha} \circ \gamma - K_{\alpha} \circ \gamma \circ K_{\alpha}^{-1} \circ E_{\alpha}, \quad \text{which implies} \\ 0 &= E_{\alpha} \circ \gamma - \gamma \circ E_{\alpha}; \\ 0 &= \epsilon(F_{\alpha})\gamma = F_{\alpha}\gamma = F_{\alpha} \circ \gamma \circ K_{\alpha} - \gamma \circ F_{\alpha} \circ K_{\alpha}, \quad \text{which implies} \\ 0 &= F_{\alpha} \circ \gamma - \gamma \circ F_{\alpha} \end{aligned}$$

because of the relations in (1.8). Since these elements generate $U_q(\mathfrak{g})$, we see that $\gamma \in \text{Hom}_{\mathbf{K}}(M, N)^U$ implies $\gamma \in \text{Hom}_U(M, N)$, so the two spaces are equal:

$$(1.13) \quad \text{Hom}_U(M, N) = \text{Hom}_{\mathbf{K}}(M, N)^U.$$

Now if $M = N = L(\lambda)$, a finite-dimensional irreducible U -module of highest weight λ , then any $f \in \text{Hom}_U(M, M)$ must map the highest weight vector v^+ to a multiple of itself. Since v^+ generates M as a U -module, it follows that f is a scalar multiple of the identity. Thus, $\text{Hom}_U(M, M) = \mathbf{K}\text{id}_M$. So the space of invariants $\text{Hom}_{\mathbf{K}}(M, M)^U$, or equivalently $(M \otimes M^*)^U$, is one-dimensional.

R-matrices. The quantum group $U = U_q(\mathfrak{g})$ has a triangular decomposition $U = U^-U^0U^+$, where U^+ (resp. U^-) is the subalgebra generated by the E_{α} (resp. F_{α}), and U^0 is the subalgebra generated by the $K_{\alpha}^{\pm 1}$, for all $\alpha \in \Pi$. Then $U^+ = \sum_{\mu} U_{\mu}^+$, where $U_{\mu}^+ = \{a \in U^+ \mid K_{\alpha}aK_{\alpha}^{-1} = q^{(\mu,\alpha)}a\}$. It is easy to see, using the automorphism ψ above, that U^- has the decomposition $U^- = \sum_{\mu} U_{-\mu}^-$, and $U_{\mu}^+ \neq (0)$ if and only if $U_{-\mu}^- \neq (0)$. There is a nondegenerate bilinear form $(\ , \)$ on

U (see [Jn, Lemma 6.16, Prop. 6.21]) which satisfies

$$(1.14) \quad \begin{aligned} & \text{(a) } (b, a) = q^{(2\rho, \mu - \nu)}(a, b) \text{ for all } a \in U_{-\nu}^- U^0 U_{\mu}^+ \text{ and } b \in U_{-\mu}^- U^0 U_{\nu}^+, \\ & \quad \text{where } \rho \text{ is the half-sum of the positive roots of } \mathfrak{g}. \\ & \text{(b) } (F_{\alpha}, E_{\beta}) = -\delta_{\alpha, \beta} (q_{\alpha} - q_{\alpha}^{-1})^{-1}. \end{aligned}$$

Choose a basis $a_1^{\mu}, a_2^{\mu}, \dots, a_{r(\mu)}^{\mu}$ of U_{μ}^+ and a corresponding dual basis $b_1^{\mu}, b_2^{\mu}, \dots, b_{r(\mu)}^{\mu}$ of $U_{-\mu}^-$ so that $(b_i^{\mu}, a_j^{\mu}) = \delta_{i,j}$. Set

$$\Theta_{\mu} = \sum_{i=1}^{r(\mu)} b_i^{\mu} \otimes a_i^{\mu}.$$

In particular, $\Theta_{\alpha} = -(q_{\alpha} - q_{\alpha}^{-1})F_{\alpha} \otimes E_{\alpha}$ for all $\alpha \in \Pi$.

Suppose $\pi : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$ and $\pi' : U_q(\mathfrak{g}) \rightarrow \text{End}(V')$ are two finite-dimensional $U_q(\mathfrak{g})$ -modules. Since the set of weights of V is finite, there are only finitely many $\mu \in \mathbf{Z}\Phi$ which are differences of weights of V . Therefore, all but finitely many Θ_{μ} act as zero on V and V' , and $\Theta_{V, V'} = (\pi \otimes \pi')(\Theta)$ for $\Theta = \sum_{\mu} \Theta_{\mu}$ is a well-defined mapping on $V \otimes V'$. By suitably ordering the basis of $V \otimes V'$, we see that each Θ_{μ} for $\mu \neq 0$ acting on $V \otimes V'$ has a strictly upper triangular matrix relative to that basis. Since $\Theta_0 = 1 \otimes 1$, the transformation $\Theta_{V, V'}$ is unipotent.

Consider the map $f = f_{V, V'}$, which is defined by $f_{V, V'} : V \otimes V' \rightarrow V \otimes V'$, $f(v \otimes w) = q^{-(\lambda, \mu)} v \otimes w$ for all $v \in V_{\lambda}$ and $w \in V'_{\mu}$.

Proposition 1.15 (Compare [Jn, Thm. 7.3]). *Let V and V' be finite-dimensional $U_q(\mathfrak{g})$ -modules, and let $\sigma = \sigma_{V', V}$ be defined by*

$$\begin{aligned} \sigma : V' \otimes V &\rightarrow V \otimes V', \\ w \otimes v &\mapsto v \otimes w. \end{aligned}$$

Then the mapping $\check{R}_{V', V} : V' \otimes V \rightarrow V \otimes V'$ given by $\check{R}_{V', V} = \Theta \circ f \circ \sigma$ is a $U_q(\mathfrak{g})$ -module isomorphism.

There is an alternate expression for Θ_{μ} which is more explicit and can be best understood using the braid group action. The braid group $\mathcal{B}_{\mathfrak{g}}$ associated to $\mathfrak{g}$ has generators T_{α} , $\alpha \in \Pi$, and defining relations

$$(T_{\alpha} T_{\beta})^{m_{\alpha, \beta}} = (T_{\beta} T_{\alpha})^{m_{\alpha, \beta}},$$

where $m_{\alpha, \beta} = 2, 3, 4, 6$ if $a_{\alpha, \beta} a_{\beta, \alpha} = 0, 1, 2, 3$ respectively. It acts by algebra automorphisms on $U_q(\mathfrak{g})$ according to the following rules (see [Jn, Sec. 8.14]):

$$(1.16) \quad \begin{aligned} T_{\alpha}(K_{\beta}) &= K_{s_{\alpha}\beta}, & T_{\alpha}(E_{\alpha}) &= -F_{\alpha} K_{\alpha}, & T_{\alpha}(F_{\alpha}) &= -K_{\alpha}^{-1} E_{\alpha}, \\ T_{\alpha}(E_{\beta}) &= \sum_{t=0}^{-a_{\alpha, \beta}} (-1)^t q_{\alpha}^{-t} (E_{\alpha})^{(-a_{\alpha, \beta}-t)} E_{\beta} (E_{\alpha})^{(t)}, & \alpha &\neq \beta, \\ T_{\alpha}(F_{\beta}) &= \sum_{t=0}^{-a_{\alpha, \beta}} (-1)^t q_{\alpha}^t (F_{\alpha})^{(t)} F_{\beta} (F_{\alpha})^{(-a_{\alpha, \beta}-t)}, & \alpha &\neq \beta, \end{aligned}$$

where s_{α} is the simple reflection corresponding to α in the Weyl group W of $\mathfrak{g}$, and

$$(E_{\alpha})^{(n)} = \frac{E_{\alpha}^n}{[n]_{d_{\alpha}}!}, \quad (F_{\alpha})^{(n)} = \frac{F_{\alpha}^n}{[n]_{d_{\alpha}}!}$$

(the factorials are as defined in (1.4)).

Let $s_i = s_{\alpha_i}$ be the reflection in the hyperplane perpendicular to the simple root α_i , and fix a reduced decomposition $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$ of the longest element of W . Every positive root occurs precisely once in the sequence

$$(1.17) \quad \beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_m = s_{i_1} s_{i_2} \cdots s_{i_{m-1}}(\alpha_{i_m}).$$

The elements

$$(1.18) \quad \begin{aligned} E_{\beta_t} &= T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{t-1}}} (E_{\alpha_{i_t}}), \\ F_{\beta_t} &= T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{t-1}}} (F_{\alpha_{i_t}}), \end{aligned}$$

for $t = 1, \dots, m$, are called the positive (resp. negative) root vectors of $U_q(\mathfrak{g})$. For any m -tuple $\underline{\ell} = (\ell_1, \dots, \ell_m)$ of nonnegative integers let

$$\begin{aligned} E^{\underline{\ell}} &= E_{\beta_m}^{\ell_m} E_{\beta_{m-1}}^{\ell_{m-1}} \cdots E_{\beta_1}^{\ell_1} \\ F^{\underline{\ell}} &= F_{\beta_m}^{\ell_m} F_{\beta_{m-1}}^{\ell_{m-1}} \cdots F_{\beta_1}^{\ell_1}. \end{aligned}$$

Then the elements $E^{\underline{\ell}}$ (resp. $F^{\underline{\ell}}$) determine a basis of U^+ (resp. U^-). (See for example, [Jn, Thm. 8.24].)

Consider for $t = 1, \dots, m$ the sum

$$\Theta^{[t]} = \sum_{j \geq 0} (-1)^j q_{\alpha}^{-j(j-1)/2} \frac{(q_{\alpha} - q_{\alpha}^{-1})^j}{[j]_{d_{\alpha}}!} F_{\beta_t}^j \otimes E_{\beta_t}^j,$$

where $\alpha = \alpha_{i_t}$, and E_{β_t} and F_{β_t} are as in (1.18). This lies in the direct product of all the $U_{\mu}^{-} \otimes U_{\mu}^{+}$. Then (as discussed in [Jn, Sec. 8.30]), Θ_{μ} is the $(U_{\mu}^{-} \otimes U_{\mu}^{+})$ -component of the product

$$\Theta^{[m]} \Theta^{[m-1]} \cdots \Theta^{[2]} \Theta^{[1]}.$$

That component involves only finitely many summands.

2. 8-DIMENSIONAL REPRESENTATIONS OF $U_q(D_4)$

In this section we specialize to the case that $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra of type D_4 . We can identify $\mathfrak{g}$ with the special orthogonal Lie algebra $so(8)$. The set of roots of $\mathfrak{g}$ is given by $\{\pm \epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq 4\}$, where $\epsilon_i, i = 1, \dots, 4$, is an orthonormal basis of $\mathbf{R}^4$, and the simple roots may be taken to be $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_3 = \epsilon_3 - \epsilon_4$, $\alpha_4 = \epsilon_3 + \epsilon_4$. The corresponding fundamental weights have the following expressions:

$$(2.1) \quad \begin{aligned} \omega_1 &= \alpha_1 + \alpha_2 + (1/2)\alpha_3 + (1/2)\alpha_4 = \epsilon_1, \\ \omega_2 &= \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = \epsilon_1 + \epsilon_2, \\ \omega_3 &= (1/2)\alpha_1 + \alpha_2 + \alpha_3 + (1/2)\alpha_4 = (1/2)(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4), \\ \omega_4 &= (1/2)\alpha_1 + \alpha_2 + (1/2)\alpha_3 + \alpha_4 = (1/2)(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4), \end{aligned}$$

and they satisfy $(\omega_i, \alpha_j) = \delta_{i,j}$ for all $i, j = 1, \dots, 4$.

The Lie algebra $\mathfrak{g}$ has three 8-dimensional representations—its natural representation and two spin representations with highest weights ω_1, ω_3 , and ω_4 respectively, and so does its quantum counterpart $U_q(D_4)$. (Note we are writing $U_q(D_4)$ rather than $U_q(\mathfrak{g})$ to emphasize the role that D_4 is playing here.) In particular, for the

natural representation V of $U_q(D_4)$, there is a basis $\{x_\mu \mid \mu = \pm\epsilon_i, i = 1, \dots, 4\}$ (compare [Jn, Chap. 5]) such that

$$(2.2) \quad \begin{aligned} K_\alpha x_\mu &= q^{(\mu, \alpha)} x_\mu, \\ E_\alpha x_\mu &= \begin{cases} 0 & \text{if } (\mu, \alpha) \neq -1, \\ x_{\mu+\alpha} & \text{if } (\mu, \alpha) = -1, \end{cases} \\ F_\alpha x_\mu &= \begin{cases} 0 & \text{if } (\mu, \alpha) \neq 1, \\ x_{\mu-\alpha} & \text{if } (\mu, \alpha) = 1. \end{cases} \end{aligned}$$

For D_4 , the group of graph automorphisms of its Dynkin diagram can be identified with the symmetric group S_3 of permutations on $\{1, 2, 3, 4\}$ fixing 2. Thus, $\phi \in S_3$ permutes the simple roots $\phi(\alpha_i) = \alpha_{\phi i}$ and fixes α_2 . Each graph automorphism ϕ induces an automorphism, which we also denote by ϕ , on $U_q(D_4)$ defined by

$$E_{\alpha_i} \mapsto E_{\alpha_{\phi i}} \quad F_{\alpha_i} \mapsto F_{\alpha_{\phi i}} \quad K_{\alpha_i} \mapsto K_{\alpha_{\phi i}}.$$

Corresponding to ϕ there is a new representation $\pi_\phi : U_q(D_4) \rightarrow \text{End}(V)$ of $U_q(D_4)$ on its natural representation V such that

$$\pi_\phi(a)x = \phi(a)x$$

for all $a \in U_q(D_4)$ and $x \in V$. We denote the $U_q(D_4)$ -module with this action by V_ϕ . Then we have

$$(2.3) \quad \begin{aligned} \pi_\phi(K_\alpha)x_\mu &= K_{\phi\alpha}x_\mu = q^{(\mu, \phi\alpha)}x_\mu = q^{(\phi^{-1}\mu, \alpha)}x_\mu, \\ \pi_\phi(E_\alpha)x_\mu &= E_{\phi\alpha}x_\mu = \begin{cases} 0 & \text{if } (\phi^{-1}\mu, \alpha) = (\mu, \phi\alpha) \neq -1, \\ x_{\mu+\phi\alpha} & \text{if } (\phi^{-1}\mu, \alpha) = (\mu, \phi\alpha) = -1, \end{cases} \\ \pi_\phi(F_\alpha)x_\mu &= F_{\phi\alpha}x_\mu = \begin{cases} 0 & \text{if } (\phi^{-1}\mu, \alpha) = (\mu, \phi\alpha) \neq 1, \\ x_{\mu-\phi\alpha} & \text{if } (\phi^{-1}\mu, \alpha) = (\mu, \phi\alpha) = 1. \end{cases} \end{aligned}$$

From these formulas the following is apparent:

Proposition 2.4. *V_ϕ is an irreducible $U_q(D_4)$ -module with highest weight $\phi^{-1}\omega_1$. In particular, we have the following highest weights:*

- (i) ω_1 for $\phi = \text{id}$ or $(3\ 4)$,
- (ii) ω_3 for $\phi = (1\ 3)$ or $(1\ 4\ 3)$,
- (iii) ω_4 for $\phi = (1\ 4)$ or $(1\ 3\ 4)$.

The operators E_α, F_α map elements in the basis $\{x_\mu\}$ to other basis elements or 0, and each vector x_μ is a weight vector of weight $\phi^{-1}\mu$. For the natural representation V , the basis elements are $x_{\pm\epsilon_i}, i = 1, \dots, 4$, which we abbreviate as $x_{\pm i}$. We can view the module V_ϕ as displayed below, where the index i on an arrow indicates that F_{α_i} maps the higher vector down to the lower vector and E_{α_i} maps the lower

vector up to the higher one. The action of all F_{α_j} 's and E_{α_j} 's not shown is 0.

$$\begin{array}{c}
 V_\phi \\
 \\
 x_1 \circ \\
 \downarrow \phi^{-1} 1 \\
 x_2 \circ \\
 \downarrow 2 \\
 x_3 \circ \\
 \begin{array}{cc}
 \phi^{-1} 3 \swarrow & \searrow \phi^{-1} 4 \\
 x_4 \circ & \circ x_{-4} \\
 \phi^{-1} 4 \searrow & \swarrow \phi^{-1} 3
 \end{array} \\
 x_{-3} \circ \\
 \downarrow 2 \\
 x_{-2} \circ \\
 \downarrow \phi^{-1} 1 \\
 x_{-1} \circ
 \end{array}
 \quad (2.5)$$

In particular, taking the identity, $\theta = (1 \ 4 \ 3)$, and $\theta^2 = (1 \ 3 \ 4)$, we have the following pictures for the natural representation V , and the spin representations V_- and V_+ :

$$\begin{array}{ccc}
 (2.6) & V & V_- & V_+ \\
 \\
 \begin{array}{c}
 x_1 \circ \\
 \downarrow 1 \\
 x_2 \circ \\
 \downarrow 2 \\
 x_3 \circ \\
 \begin{array}{cc}
 3 \swarrow & \searrow 4 \\
 x_4 \circ & \circ x_{-4} \\
 4 \searrow & \swarrow 3
 \end{array} \\
 x_{-3} \circ \\
 \downarrow 2 \\
 x_{-2} \circ \\
 \downarrow 1 \\
 x_{-1} \circ
 \end{array} &
 \begin{array}{c}
 x_1 \circ \\
 \downarrow 3 \\
 x_2 \circ \\
 \downarrow 2 \\
 x_3 \circ \\
 \begin{array}{cc}
 4 \swarrow & \searrow 1 \\
 x_4 \circ & \circ x_{-4} \\
 1 \searrow & \swarrow 4
 \end{array} \\
 x_{-3} \circ \\
 \downarrow 2 \\
 x_{-2} \circ \\
 \downarrow 3 \\
 x_{-1} \circ
 \end{array} &
 \begin{array}{c}
 x_1 \circ \\
 \downarrow 4 \\
 x_2 \circ \\
 \downarrow 2 \\
 x_3 \circ \\
 \begin{array}{cc}
 1 \swarrow & \searrow 3 \\
 x_4 \circ & \circ x_{-4} \\
 3 \searrow & \swarrow 1
 \end{array} \\
 x_{-3} \circ \\
 \downarrow 2 \\
 x_{-2} \circ \\
 \downarrow 4 \\
 x_{-1} \circ
 \end{array}
 \end{array}$$

There is a unique (up to scalar multiple) $U_q(D_4)$ -module homomorphism $*$: $V_- \otimes V_+ \rightarrow V$, which we can compute directly using the expressions for the co-multiplication in (1.8). We display the result of this computation in Table 2.7. To make the table more symmetric, we scale the mapping by $q^{\frac{3}{2}}$.

TABLE 2.7

*	x_1	x_2	x_3	x_{-4}	x_4	x_{-3}	x_{-2}	x_{-1}
x_1	0	0	0	$q^{-\frac{3}{2}}x_1$	0	$q^{-\frac{3}{2}}x_2$	$q^{-\frac{3}{2}}x_3$	$q^{-\frac{3}{2}}x_{-4}$
x_2	0	0	$-q^{-\frac{1}{2}}x_1$	0	$-q^{-\frac{1}{2}}x_2$	0	$q^{-\frac{3}{2}}x_4$	$q^{-\frac{3}{2}}x_{-3}$
x_3	0	$q^{\frac{1}{2}}x_1$	0	0	$-q^{-\frac{1}{2}}x_3$	$-q^{-\frac{1}{2}}x_4$	0	$q^{-\frac{3}{2}}x_{-2}$
x_4	$-q^{\frac{3}{2}}x_1$	0	0	0	$-q^{-\frac{1}{2}}x_{-4}$	$-q^{-\frac{1}{2}}x_{-3}$	$-q^{-\frac{1}{2}}x_{-2}$	0
x_{-4}	0	$q^{\frac{1}{2}}x_2$	$q^{\frac{1}{2}}x_3$	$q^{\frac{1}{2}}x_4$	0	0	0	$q^{-\frac{3}{2}}x_{-1}$
x_{-3}	$-q^{\frac{3}{2}}x_2$	0	$q^{\frac{1}{2}}x_{-4}$	$q^{\frac{1}{2}}x_{-3}$	0	0	$-q^{-\frac{1}{2}}x_{-1}$	0
x_{-2}	$-q^{\frac{3}{2}}x_3$	$-q^{\frac{3}{2}}x_{-4}$	0	$q^{\frac{1}{2}}x_{-2}$	0	$q^{\frac{1}{2}}x_{-1}$	0	0
x_{-1}	$-q^{\frac{3}{2}}x_4$	$-q^{\frac{3}{2}}x_{-3}$	$-q^{\frac{3}{2}}x_{-2}$	0	$-q^{\frac{3}{2}}x_{-1}$	0	0	0

It can be seen readily from (1.8) that the relations

$$(2.8) \quad (\phi \otimes \phi)\Delta(a) = \Delta(\phi(a)), \quad \phi(S(a)) = S(\phi(a))$$

hold for all $a \in U_q(D_4)$ and all graph automorphisms ϕ . Therefore, by identifying V_- with V_θ and V_+ with V_{θ^2} we obtain from

$$a(x * y) = \sum_a \theta(a_{(1)})x * \theta^2(a_{(2)})y$$

that the following relations hold:

$$(2.9) \quad \begin{aligned} \pi_\theta(a)(x * y) &= \sum_a \theta^2(a_{(1)})x * a_{(2)}y, \\ \pi_{\theta^2}(a)(x * y) &= \sum_a a_{(1)}x * \theta(a_{(2)})y. \end{aligned}$$

They allow us to conclude

Proposition 2.10. *The product $*$: $V_- \otimes V_+ \rightarrow V$ displayed in Table 2.7 is a $U_q(D_4)$ -module homomorphism, and it gives $U_q(D_4)$ -module homomorphisms*

$$* : V_+ \otimes V \rightarrow V_-, \quad * : V \otimes V_- \rightarrow V_+.$$

Suppose now that $\zeta = (1\ 3)$ and $\eta = (1\ 4)$. Specializing (2.5) with ϕ equal to these permutations gives

$$(2.11) \quad \begin{array}{ccc} & V_\zeta & V_\eta \\ & \begin{array}{c} x_1 \circ \\ \downarrow 3 \\ x_2 \circ \\ \downarrow 2 \\ x_3 \circ \\ 1 \swarrow \quad \searrow 4 \\ x_4 \circ \quad \circ x_{-4} \\ 4 \searrow \quad \swarrow 1 \end{array} & \begin{array}{c} x_1 \circ \\ \downarrow 4 \\ x_2 \circ \\ \downarrow 2 \\ x_3 \circ \\ 3 \swarrow \quad \searrow 1 \\ x_4 \circ \quad \circ x_{-4} \\ 1 \searrow \quad \swarrow 3 \end{array} \\ & \begin{array}{c} x_{-3} \circ \\ \downarrow 2 \\ x_{-2} \circ \\ \downarrow 3 \\ x_{-1} \circ \end{array} & \begin{array}{c} x_{-3} \circ \\ \downarrow 2 \\ x_{-2} \circ \\ \downarrow 4 \\ x_{-1} \circ \end{array} \end{array}$$

As these diagrams indicate, the transformation specified by

$$(2.12) \quad \begin{aligned} j : V &\rightarrow V, \\ x_i &\mapsto -x_i, \quad i \neq \pm 4, \\ x_4 &\mapsto -x_{-4}, \\ x_{-4} &\mapsto -x_4 \end{aligned}$$

determines explicit $U_q(D_4)$ -isomorphisms,

$$(2.13) \quad j : V_- \rightarrow V_\zeta \quad j : V_+ \rightarrow V_\eta.$$

This together with Proposition 2.10 enables us to conclude the following:

Proposition 2.14. *The map $\cdot : V_\zeta \otimes V_\eta \rightarrow V$ given by $x \otimes y \mapsto x \cdot y = j(x) * j(y)$ is a $U_q(D_4)$ -module homomorphism.*

Eigenvalues of $\check{R}_{V,V}$. Suppose now that $V = L(\omega_1)$, the natural representation of $U_q(D_4)$. The module $V^{\otimes 2}$ is completely reducible: $V^{\otimes 2} = L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$. By determining the eigenvalues of $\check{R}_{V,V}$ on the corresponding maximal vectors, we obtain the eigenvalues of $\check{R}_{V,V}$ on the summands.

First, on the maximal vector $x_1 \otimes x_1$ of the summand $L(2\omega_1)$, $\check{R}_{V,V}(x_1 \otimes x_1) = q^{-(\epsilon_1, \epsilon_1)} x_1 \otimes x_1 = q^{-1}(x_1 \otimes x_1)$. The vector $x_2 \otimes x_1 - q^{-1}x_1 \otimes x_2$ is a maximal vector for $L(\omega_2)$. Then

$$\begin{aligned} \check{R}_{V,V}(x_2 \otimes x_1 - q^{-1}x_1 \otimes x_2) &= q^{-(\epsilon_1, \epsilon_2)} \Theta(x_1 \otimes x_2 - q^{-1}x_2 \otimes x_1) \\ &= x_1 \otimes x_2 - q^{-1}x_2 \otimes x_1 - (q - q^{-1})x_2 \otimes x_1 \\ &= x_1 \otimes x_2 - qx_2 \otimes x_1 = -q(x_2 \otimes x_1 - q^{-1}x_1 \otimes x_2), \end{aligned}$$

so that $\check{R}_{V,V}$ acts as $-q$ on $L(\omega_2)$. We will argue in the remark following Proposition 2.19 below that $\check{R}_{V,V}$ acts on $L(0)$ as q^7 . Thus, the eigenvalues of $\check{R}_{V,V}$ are given by

$$(2.15) \quad \begin{array}{ll} q^{-1}, & L(2\omega_1), \\ -q, & L(\omega_2), \\ q^7, & L(0), \end{array}$$

(compare [D], [R]). Setting $\mathcal{S}^2(V) = L(2\omega_1) \oplus L(0)$, we see that $\check{R}_{V,V} + \text{id}_V$ maps $V^{\otimes 2}$ onto $\mathcal{S}^2(V)$, the *quantum symmetric space*. There are analogous decompositions for the two spin representations of $U_q(D_4)$ obtained by replacing ω_1 by ω_3 and ω_4 .

For each graph automorphism ϕ , we consider $\check{R} = \check{R}_{V_\phi, V_\phi}$ on $V_\phi \otimes V_\phi$. Note the bilinear form $(\cdot, \cdot)$ on $U_q(D_4)$ in [Jn, (6.12)] is $(\phi \otimes \phi)$ -invariant, so

$$\Theta_{\phi(\mu)} = \sum_i \phi(b_i^\mu) \otimes \phi(a_i^\mu) = (\phi \otimes \phi)(\Theta_\mu).$$

Now

$$(2.16) \quad \begin{aligned} \check{R}(v \otimes w) &= (\Theta \circ f \circ \sigma)(v \otimes w) = q^{-(\phi^{-1}\lambda, \phi^{-1}\nu)} \Theta(w \otimes v) \\ &= q^{-(\lambda, \nu)} \Theta(w \otimes v) = \check{R}_{V,V}(v \otimes w), \end{aligned}$$

whenever w is in V_λ and $v \in V_\nu$. Consequently, the actions of the R-matrices agree. It follows that

$$(2.17) \quad \mathcal{S}^2(V) = \mathcal{S}^2(V_\phi)$$

for all graph automorphisms ϕ .

A “test” element of $\mathcal{S}^2(V)$. In the next two sections we will compute various $U_q(\mathbf{D}_4)$ -module homomorphisms and prove that they are equal by evaluating them on a particular element of $\mathcal{S}^2(V)$. A good “test” element for these calculations is $x_1 \otimes x_{-1} + x_{-1} \otimes x_1$, but first we need to demonstrate that it does in fact belong to $\mathcal{S}^2(V)$. Since $\check{R}_{V,V} + \text{gid}_V$ maps $V^{\otimes 2}$ onto $\mathcal{S}^2(V)$ for $V = L(\omega_1)$, the natural representation, it suffices to show that the image of $x_1 \otimes x_{-1}$ under $\check{R}_{V,V} + \text{gid}_V$ is $x_1 \otimes x_{-1} + x_{-1} \otimes x_1$. Using the fact that $E_\gamma x_1 = 0$ for all positive roots γ , we obtain

$$\begin{aligned} \check{R}_{V,V}(x_1 \otimes x_{-1}) &= (\Theta \circ f \circ \sigma)(x_1 \otimes x_{-1}) \\ &= q\Theta(x_{-1} \otimes x_1) = q(x_{-1} \otimes x_1). \end{aligned}$$

Consequently,

$$(\check{R}_{V,V} + \text{gid}_V)(x_1 \otimes x_{-1}) = qx_{-1} \otimes x_1 + qx_1 \otimes x_{-1},$$

which implies the desired conclusion

$$x_{-1} \otimes x_1 + x_1 \otimes x_{-1} \in \mathcal{S}^2(V).$$

Bilinear forms. For the natural representation V of $U = U_q(\mathbf{D}_4)$, the dual space V^* has weights $\pm\epsilon_i$, $i = 1, \dots, 4$, by (1.11). The vector x_{-1}^* is a maximal vector since it corresponds to the unique dominant weight, ϵ_1 , and V^* is isomorphic to V . We have seen in (1.13) that the space of invariants $(V \otimes V^*)^U$ is one-dimensional. Since $V \cong V^*$, we have that the space of invariants $(V \otimes V)^U$ is one-dimensional. Thus, in $V \otimes V$ there is a unique copy of the trivial module, and hence a unique (up to scalar multiple) $U_q(\mathbf{D}_4)$ -module homomorphism

$$(|)V \otimes V \rightarrow \mathbf{K}.$$

The condition of $(|)$ being a $U_q(\mathbf{D}_4)$ -module homomorphism is equivalent to

$$(2.18) \quad (ax|y) = (x|S(a)y)$$

for all $a \in U$ and $x, y \in V$ (see [Jn, p.122]). There is an analogous bilinear form giving the unique (up to multiples) $U_q(\mathbf{D}_4)$ -module homomorphism, $V_\phi \otimes V_\phi \rightarrow \mathbf{K}$. All these bilinear forms can be regarded as $U_q(\mathbf{D}_4)$ -module maps from the symmetric tensors $\mathcal{S}^2(V_\phi) \rightarrow \mathbf{K}$, since the trivial module is an irreducible summand of $\mathcal{S}^2(V_\phi)$ in each case.

Now for $a \in U_q(D_4)$,

$$\begin{aligned}
 \sum_a (\pi_\phi(a_{(1)})x | \pi_\phi(a_{(2)})y) &= \sum_a (\phi(a_{(1)})x | \phi(a_{(2)})y) \\
 &= \sum_a (x | S(\phi(a_{(1)}))\phi(a_{(2)})y) \\
 (2.19) \quad &= \sum_a (x | \phi(S(a_{(1)})a_{(2)})y) \\
 &= (x | \phi\left(\sum_a S(a_{(1)})a_{(2)}\right)y) \\
 &= (x | \phi(\epsilon(a)1)y) = \epsilon(a)(x | y).
 \end{aligned}$$

(We have used in the third line the fact that ϕ and S commute (2.8).) Hence, by the calculation in (2.19), and by direct computation of $(|)$ on $V \otimes V$, we have

Proposition 2.20. *The $U_q(D_4)$ -module homomorphism $(|) : V \otimes V \rightarrow \mathbf{K}$ determines a $U_q(D_4)$ -module homomorphism $(|) : V_\phi \otimes V_\phi \rightarrow \mathbf{K}$ for any graph automorphism ϕ . It is given by*

$$(x_i | x_j) = \begin{cases} 0 & \text{if } j \neq -i, \\ (-1)^{i+1} \frac{q^{-(4-i)}}{q^3 + q^{-3}} & \text{if } i > 0 \text{ and } j = -i, \\ (-1)^{i+1} \frac{q^{i+4}}{q^3 + q^{-3}} & \text{if } i < 0 \text{ and } j = -i. \end{cases}$$

We are free to scale the bilinear form $(|)$ on V_ϕ by any nonzero element in $\mathbf{K}$. In writing down the values of $(x_i | x_{-i})$ in Proposition 2.20, we have chosen a particular scaling with an eye towards results (such as Proposition 4.2) to follow. It is easy to see, using the basis elements x_i , that the following holds.

Proposition 2.21. *The transformation j is an isometry $((x | y) = (j(x) | j(y)))$ for all $x, y \in V$ relative to the bilinear form in Proposition 2.20.*

Remark. Observe that $\check{R}_{V,V} \circ (|) = \xi(|)$, where ξ is the eigenvalue of $\check{R}_{V,V}$ on $L(0)$. Since $\check{R}_{V,V}(x_1 \otimes x_{-1}) = q(x_{-1} \otimes x_1)$, as our calculations for the “test element” show, we have $\xi(x_1 | x_{-1}) = q(x_{-1} | x_1)$. Therefore, $\xi q^{-3} = q q^3$ or $\xi = q^7$, as asserted in (2.15).

For D_4 , the half-sum of the positive roots is given by $\rho = 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 3\alpha_4$, so $K_{2\rho}^{-1} = K_{\alpha_1}^{-6} K_{\alpha_2}^{-10} K_{\alpha_3}^{-6} K_{\alpha_4}^{-6}$. Therefore, on the natural representation V , $K_{2\rho}^{-1}$ has a diagonal matrix relative to the basis $\{x_1, x_2, x_3, x_4, x_{-4}, x_{-3}, x_{-2}, x_{-1}\}$ with corresponding diagonal entries given by $\{q^{-6}, q^{-4}, q^{-2}, 1, 1, q^2, q^4, q^6\}$. It is easy to verify directly that

$$(2.22) \quad (x | y) = (y | K_{2\rho}^{-1} x).$$

3. OCTONIONS AND QUANTUM OCTONIONS

Suppose $\zeta = (1 \ 3)$, $\eta = (1 \ 4)$, and V_ϕ is the representation of $U_q(D_4)$ coming from the graph automorphism ϕ . Then

$$(3.1) \quad \dim_{\mathbf{K}} \text{Hom}_{U_q(D_4)}(\mathcal{S}^2(V_\zeta) \otimes \mathcal{S}^2(V_\eta), \mathbf{K}) = 1,$$

and we can realize these homomorphisms explicitly as multiples of the mapping

$$(3.2) \quad \mathcal{S}^2(V_\zeta) \otimes \mathcal{S}^2(V_\eta) \subset V_\zeta \otimes V_\zeta \otimes V_\eta \otimes V_\eta \rightarrow \mathbf{K},$$

$$\sum_{k,\ell} u_k \otimes v_k \otimes w_\ell \otimes y_\ell \mapsto \sum_{k,\ell} (u_k|v_k)(w_\ell|y_\ell).$$

On the other hand, the mapping

$$(3.3) \quad \mathcal{S}^2(V_\zeta) \otimes \mathcal{S}^2(V_\eta) \subset V_\zeta \otimes V_\zeta \otimes V_\eta \otimes V_\eta \rightarrow V_\zeta \otimes V_\eta \otimes V_\zeta \otimes V_\eta \rightarrow V \otimes V \rightarrow \mathbf{K},$$

$$\sum_{k,\ell} u_k \otimes v_k \otimes w_\ell \otimes y_\ell \mapsto \sum_{k,\ell} u_k \otimes \check{R}_{V_\zeta, V_\eta}(v_k \otimes w_\ell) \otimes y_\ell$$

$$\mapsto \sum_{k,\ell} \sum_i u_k \cdot w_\ell^{(i)} \otimes v_k^{(i)} \cdot y_\ell \mapsto \sum_{k,\ell} \sum_i (u_k \cdot w_\ell^{(i)} | v_k^{(i)} \cdot y_\ell),$$

is a $U_q(D_4)$ -module homomorphism, (where “ $\cdot$ ” is the product in Proposition 2.14 and $\check{R}_{V_\zeta, V_\eta}(v_k \otimes w_\ell) = \sum_i w_\ell^{(i)} \otimes v_k^{(i)}$ is the braiding morphism). Thus, it must be a multiple of the mapping in (3.2).

We now consider what would happen if $q = 1$, and compute the multiple in that case. Here the braiding morphism may be taken to be $v \otimes w \mapsto w \otimes v$, since that is a module map in the $q = 1$ case. We evaluate the mappings at $e \otimes e \otimes e \otimes e$, where $e = x_4 - x_{-4}$. Now by Table 2.7, we have

$$(3.4) \quad e^2 = j(e) * j(e) = (x_4 - x_{-4}) * (x_4 - x_{-4})$$

$$= q^{\frac{1}{2}} x_4 - q^{-\frac{1}{2}} x_{-4},$$

which shows that $e^2 = e$ at $q = 1$. By Proposition 2.19,

$$(3.5) \quad (e|e) = -(x_4|x_{-4}) - (x_{-4}|x_4) = \frac{2}{q^3 + q^{-3}},$$

which gives $(e|e)_1 = 1$ at $q = 1$. (Here and throughout the paper we use $(|)_1$ to denote the bilinear form at $q = 1$.) Therefore, $(e|e)_1(e|e)_1 = 1$ and $(e^2|e^2)_1 = (e|e)_1 = 1$. Thus, the two maps in (3.2) and (3.3) are equal at $q = 1$. By specializing $\sum_k u_k \otimes v_k = x \otimes x$ and $\sum_\ell w_\ell \otimes y_\ell = y \otimes y$ (which are symmetric elements at $q = 1$), we have as a consequence,

$$(3.6) \quad (x \cdot y | x \cdot y)_1 = (x|x)_1 (y|y)_1.$$

Using the multiplication table (Table 2.7) for $x * y$ and the formula $x \cdot y = j(x) * j(y)$, we can verify that $e = x_4 - x_{-4}$ is the unit element relative to the product “ $\cdot$ ” when $q = 1$. (For further discussion, see the beginning of Section 5.) Thus, the product $\cdot : V_\zeta \otimes V_\eta \rightarrow V$ for $q = 1$ gives an 8-dimensional unital composition algebra over $\mathbf{K}$ whose underlying vector space is $V = V_\zeta = V_\eta$. Such an algebra must be the octonions. Thus, we have

Proposition 3.7. *Let V be the 8-dimensional vector space over $\mathbf{K}$ with basis $\{x_{\pm i} \mid i = 1, \dots, 4\}$. Then V with the product $\cdot$ defined by $x \cdot y = j(x) * j(y)$, where j is as in (2.12) and $*$ is as in Table 2.7, is the algebra of octonions when $q = 1$.*

Because of this result we are motivated to make the following definition.

Quantum Octonions.

Definition 3.8. Let V be the 8-dimensional vector space over $\mathbf{K}$ with basis $\{x_{\pm i} \mid i = 1, \dots, 4\}$. Then V with the product “ $\cdot$ ” defined by

$$x \cdot y = j(x) * j(y),$$

where j is as in (2.12) and $*$ is given by Table 2.7, is the algebra of *quantum octonions*. We denote this algebra by $\mathbb{O}_q = (V, \cdot)$.

The multiplication table for the quantum octonions is obtained from Table 2.7 by interchanging the entries in the columns and rows labelled by x_{-4} and x_4 . As a result, we have

TABLE 3.9

$\cdot$	x_1	x_2	x_3	x_{-4}	x_4	x_{-3}	x_{-2}	x_{-1}
x_1	0	0	0	0	$q^{-\frac{3}{2}}x_1$	$q^{-\frac{3}{2}}x_2$	$q^{-\frac{3}{2}}x_3$	$q^{-\frac{3}{2}}x_{-4}$
x_2	0	0	$-q^{-\frac{1}{2}}x_1$	$-q^{-\frac{1}{2}}x_2$	0	0	$q^{-\frac{3}{2}}x_4$	$q^{-\frac{3}{2}}x_{-3}$
x_3	0	$q^{\frac{1}{2}}x_1$	0	$-q^{-\frac{1}{2}}x_3$	0	$-q^{-\frac{1}{2}}x_4$	0	$q^{-\frac{3}{2}}x_{-2}$
x_4	0	$q^{\frac{1}{2}}x_2$	$q^{\frac{1}{2}}x_3$	0	$q^{\frac{1}{2}}x_4$	0	0	$q^{-\frac{3}{2}}x_{-1}$
x_{-4}	$-q^{\frac{3}{2}}x_1$	0	0	$-q^{-\frac{1}{2}}x_{-4}$	0	$-q^{-\frac{1}{2}}x_{-3}$	$-q^{-\frac{1}{2}}x_{-2}$	0
x_{-3}	$-q^{\frac{3}{2}}x_2$	0	$q^{\frac{1}{2}}x_{-4}$	0	$q^{\frac{1}{2}}x_{-3}$	0	$-q^{-\frac{1}{2}}x_{-1}$	0
x_{-2}	$-q^{\frac{3}{2}}x_3$	$-q^{\frac{3}{2}}x_{-4}$	0	0	$q^{\frac{1}{2}}x_{-2}$	$q^{\frac{1}{2}}x_{-1}$	0	0
x_{-1}	$-q^{\frac{3}{2}}x_4$	$-q^{\frac{3}{2}}x_{-3}$	$-q^{\frac{3}{2}}x_{-2}$	$-q^{\frac{3}{2}}x_{-1}$	0	0	0	0

The product map: $p : V_\zeta \otimes V_\eta \rightarrow V$, $p(x \otimes y) = x \cdot y$, of the quantum octonion algebra is the composition $p = * \circ (j \otimes j)$ of two $U_q(\mathbf{D}_4)$ -module maps, so is itself a $U_q(\mathbf{D}_4)$ -module homomorphism.

Let $\pi : U_q(\mathbf{D}_4) \rightarrow \text{End}(V)$ be the natural representation of $U_q(\mathbf{D}_4)$. Then we have

$$\begin{aligned}
 \pi(a)(x \cdot y) &= a \cdot p(x \otimes y) = p(\Delta(a)(x \otimes y)) \\
 &= p\left(\sum_a \zeta(a_{(1)})x \otimes \eta(a_{(2)})y\right) \\
 (3.10) \quad &= \sum_a j(\zeta(a_{(1)})x) * j(\eta(a_{(2)})y) \\
 &= \sum_a \left(\zeta(a_{(1)})x\right) \cdot \left(\eta(a_{(2)})y\right).
 \end{aligned}$$

Now when $q = 1$ we can identify $U_q(\mathbf{D}_4)$ with the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ of type $\mathbf{D}_4$. The comultiplication specializes in this case to the usual comultiplication on $U(\mathfrak{g})$, which has $\Delta(a) = a \otimes 1 + 1 \otimes a$ for all $a \in \mathfrak{g}$. In particular, for elements $a \in \mathfrak{g}$, (3.10) becomes

$$(3.11) \quad \pi(a)(x \cdot y) = (\zeta(a)x) \cdot y + x \cdot (\eta(a)y).$$

Equation (3.11) is commonly referred to as the *Principle of Local Triality* (see for example, [J, p. 8] or [S, p. 88]). As a result, we have

Proposition 3.12. *The quantum octonion algebra $\mathbb{O}_q = (V, \cdot)$ satisfies the q -Principle of Local Triality,*

$$\pi(a)(x \cdot y) = \sum_a \left(\zeta(a_{(1)})x \right) \cdot \left(\eta(a_{(2)})y \right)$$

for all $a \in U_q(D_4)$ and $x, y \in V$, where ζ and η are the graph automorphisms $\zeta = (1\ 3)$ and $\eta = (1\ 4)$ of $U_q(D_4)$, and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ is the comultiplication in $U_q(D_4)$.

A quantum para-Hurwitz algebra. At $q = 1$, the map $j : V \rightarrow V$ sending $x_i \mapsto -x_{-i}$ for $i \neq \pm 4$, $x_4 \mapsto -x_{-4}$, and $x_{-4} \mapsto -x_4$ agrees with the standard involution $x \rightarrow \bar{x}$ on the octonions, so that

$$(3.13) \quad x * y = j(x) \cdot j(y) = \bar{x} \cdot \bar{y}.$$

Thus, the product “ $*$ ” displayed in Table 2.7 gives what is called the *para-Hurwitz algebra* when $q = 1$. (Results on para-Hurwitz algebras can be found in [EM], [EP], [OO1], [OO2].)

Definition 3.14. We say that the algebra $\mathbb{P}_q = (V, *)$ with the multiplication $*$ given by Table 2.7 is the *quantum para-Hurwitz algebra*. It satisfies

$$x * y = j(x) \cdot j(y),$$

where $\mathbb{O}_q = (V, \cdot)$ is the quantum octonion algebra and j is the transformation on V defined in (2.12).

Our definitions have been predicated on using the mapping $\cdot : V_\zeta \otimes V_\eta \rightarrow V$. We can ask what would have happened if we had used the projection $V_\eta \otimes V_\zeta \rightarrow V$ instead. To answer this question, it is helpful to observe that the mapping j determines a $U_q(D_4)$ -module isomorphism $j : V_{\zeta\eta\zeta} = V_{(3\ 4)} \rightarrow V$, which is readily apparent from comparing the following picture of $V_{(3\ 4)}$ with the diagram for V in (2.6):

$$\begin{array}{ccc}
 & & V_{(3\ 4)} \\
 & & x_1 \circ \\
 & & \downarrow 1 \\
 & & x_2 \circ \\
 & & \downarrow 2 \\
 & & x_3 \circ \\
 4 \swarrow & & \searrow 3 \\
 x_4 \circ & & \circ x_{-4} \\
 3 \searrow & & \swarrow 4 \\
 & & x_{-3} \circ \\
 & & \downarrow 2 \\
 & & x_{-2} \circ \\
 & & \downarrow 1 \\
 & & x_{-1} \circ
 \end{array}$$

Suppose as before that $p : V_\zeta \otimes V_\eta \rightarrow V$ is the product $p(x \otimes y) = x \cdot y$ in the quantum octonion algebra. Consider the mapping $j \circ p \circ (j \otimes j) : V_\eta \otimes V_\zeta \rightarrow V$. We demonstrate that this is a $U_q(D_4)$ -module map. Bear in mind in these calculations

that $j : V_\zeta \rightarrow V_- = V_{\zeta\eta}$ and $j : V_\eta \rightarrow V_+ = V_{\eta\zeta}$ for $\zeta = (1\ 3)$ and $\eta = (1\ 4)$. Then for $a \in U_q(D_4)$ we have

$$\begin{aligned} j \circ p \circ (j \otimes j)(a(x \otimes y)) &= j \circ p \circ (j \otimes j) \left(\sum_a \eta(a_{(1)})x \otimes \zeta(a_{(2)})y \right) \\ &= j \circ p \left(\sum_a j(\eta(a_{(1)})x) \otimes j(\zeta(a_{(2)})y) \right) \\ &= j \circ p \left(\sum_a \eta\zeta(a_{(1)})j(x) \otimes \zeta\eta(a_{(2)})j(y) \right) \\ &= j \circ p \left(\sum_a \zeta^2\eta\zeta(a_{(1)})j(x) \otimes \eta^2\zeta\eta(a_{(2)})j(y) \right) \end{aligned}$$

Now, using the fact that $\eta\zeta\eta = \zeta\eta\zeta$ and that $(\phi \otimes \phi)\Delta = \Delta\phi$ for all graph automorphisms ϕ together with (3.10), we see that the last expression can be rewritten as

$$\begin{aligned} j \circ p \left((\zeta \otimes \eta)(\Delta(\zeta\eta\zeta(a)))(j(x) \otimes j(y)) \right) \\ = j \left(\zeta\eta\zeta(a)p(j(x) \otimes j(y)) \right) \quad (p \text{ is a } U_q(D_4)\text{-homomorphism}) \\ = a(j \circ p \circ (j \otimes j))(x \otimes y). \end{aligned}$$

Thus, $j \circ p \circ (j \otimes j) : V_\eta \otimes V_\zeta \rightarrow V$ is a $U_q(D_4)$ -module homomorphism. Since the space $\text{Hom}_{U_q(D_4)}(V_\eta \otimes V_\zeta, V)$ is one-dimensional, we may assume the product $p' : V_\eta \otimes V_\zeta \rightarrow V$ is given by the map we have just found, so that $p' = j \circ p \circ (j \otimes j)$. Alternately, $j \circ p' = p \circ (j \otimes j)$. This says that had we defined the quantum octonions using the product p' rather than p so that $x \cdot' y = p'(x \otimes y)$, the two algebras would be isomorphic via the map j .

4. PROPERTIES OF QUANTUM OCTONIONS

We use $U_q(D_4)$ -module homomorphisms to deduce various properties of the quantum octonions. At $q = 1$ these properties specialize to well-known identities satisfied by the octonions, which can be found for example in [ZSSS]. Since the vector spaces V , V_ζ , and V_η are the same, we will denote the identity map on them simply by “id” in what follows.

Proposition 4.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{S}^2(V_\zeta) \otimes V_\eta & \xrightarrow{(|) \otimes \text{id}} & V_\eta \\ \text{id} \otimes p \downarrow & & \downarrow \text{id} \\ V_\zeta \otimes V & \xrightarrow{p \circ (j \otimes \text{id})} & V_\eta, \end{array}$$

so that

$$(4.2) \quad \sum_k j(u_k) \cdot (v_k \cdot y) = \sum_k (u_k | v_k) y$$

for all $\sum_k u_k \otimes v_k \in \mathcal{S}^2(\mathbb{O}_q)$, $y \in \mathbb{O}_q$. In particular, at $q = 1$ this reduces to the well-known identity satisfied by the octonion algebra $\mathbb{O}$:

$$(4.3) \quad \bar{x} \cdot (x \cdot y) = (x|x)_1 y.$$

Proof. Since $p : V_\zeta \otimes V_\eta \rightarrow V$ is a $U_q(D_4)$ -module homomorphism, we have that $\pi(a) \circ p = p((\zeta \otimes \eta)\Delta(a))$, which implies that $\pi(\eta(a)) \circ p = p((\zeta \otimes \eta)\Delta(\eta a)) = p((\zeta \eta \otimes \text{id})\Delta(a))$. This says that $p : V_{\zeta\eta} \otimes V \rightarrow V_\eta$ is also a $U_q(D_4)$ -module homomorphism. Since $j : V_\zeta \rightarrow V_{\zeta\eta}$ is a $U_q(D_4)$ -module homomorphism, it follows that $(p \circ (j \otimes \text{id})) \circ (\text{id} \otimes p) \in \text{Hom}_{U_q(D_4)}(\mathcal{S}^2(V_\zeta) \otimes V_\eta, \mathbf{K})$. As that space of homomorphisms has dimension one, the maps in the diagram must be proportional.

To find the proportionality constant, let us consider the actions using the test element:

$$(x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1 \xrightarrow{(|) \otimes \text{id}} x_1$$

and

$$\begin{aligned} p \circ (j \otimes \text{id}) \circ (\text{id} \otimes p) & \left((x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1 \right) \\ &= p \circ (j \otimes \text{id}) \left(-q^{\frac{3}{2}} x_1 \otimes x_4 \right) = q^{\frac{3}{2}} x_1 \cdot x_4 = x_1. \end{aligned}$$

Therefore, the two maps are equal as claimed. At $q = 1$ we may substitute $x \otimes x \in \mathcal{S}^2(\mathbb{O})$ and $y \in \mathbb{O}$ into the above identity to obtain the corresponding one for the octonions. $\square$

Proposition 4.4. *The following diagram commutes:*

$$\begin{array}{ccc} V_\zeta \otimes \mathcal{S}^2(V_\eta) & \xrightarrow{\text{id} \otimes (|)} & V_\zeta \\ p \otimes \text{id} \downarrow & & \downarrow \text{id} \\ V \otimes V_\eta & \xrightarrow{p \circ (\text{id} \otimes j)} & V_\zeta, \end{array}$$

so that

$$(4.5) \quad \sum_k (y \cdot u_k) \cdot j(v_k) = \sum_k (u_k | v_k) y$$

for all $\sum_k u_k \otimes v_k \in \mathcal{S}^2(\mathbb{O}_q)$, $y \in \mathbb{O}_q$. In particular, at $q = 1$ this gives the identity

$$(4.6) \quad (y \cdot x) \cdot \bar{x} = (x|x)_1 y,$$

which is satisfied by the octonions.

Proof. Both maps are $U_q(D_4)$ -module homomorphisms and so must be multiples of each other. We compute their images on $x = x_1 \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1)$:

$$\begin{aligned} (\text{id} \otimes (|))x &= \left((x_1|x_{-1}) + (x_{-1}|x_1) \right) x_1 = x_1, \\ p \circ (\text{id} \otimes j) \circ (p \otimes \text{id})x &= p \circ (\text{id} \otimes j)(q^{-\frac{3}{2}} x_{-4} \otimes x_1) = -q^{-\frac{3}{2}} x_{-4} \cdot x_1 = x_1. \end{aligned}$$

Hence we have the desired equality. $\square$

Proposition 4.7. *The following diagram commutes:*

$$\begin{array}{ccc}
 V_\zeta \otimes V_\eta \otimes V & \xrightarrow{p \otimes \text{id}} & V \otimes V \\
 \text{id} \otimes (j \circ p \circ (\text{id} \otimes j)) \downarrow & & \downarrow (|) \\
 V_\zeta \otimes V_\zeta & \xrightarrow{(|)} & \mathbf{K}
 \end{array}$$

so that

$$(4.8) \quad (x \cdot y|z) = (x|j(y \cdot j(z))) = (j(x)|y \cdot j(z))$$

for all x, y, z in the quantum octonions. At $q = 1$ this reduces to the identity

$$(4.9) \quad (x \cdot y|z)_1 = (x|z \cdot \bar{y})_1,$$

satisfied by the octonions.

Proof. Recall we have shown that $j \circ p \circ (j \otimes j) : V_\eta \otimes V_\zeta \rightarrow V$ is a $U_q(D_4)$ -module homomorphism, which is equivalent to saying that

$$a \circ j \circ p \circ (j \otimes j) = j \circ p \circ (j \otimes j) \left((\eta \otimes \zeta) \Delta(a) \right)$$

for all $a \in U_q(D_4)$. This implies

$$\begin{aligned}
 \zeta(a) \circ j \circ p \circ (j \otimes j) &= j \circ p \circ (j \otimes j) \left((\eta \otimes \zeta) \Delta(\zeta(a)) \right) \\
 &= j \circ p \circ (j \otimes j) \left((\eta \otimes \zeta)(\zeta \otimes \zeta) \Delta(a) \right) \\
 &= j \circ p \circ (j \otimes j) \left((\eta \zeta \otimes \text{id}) \Delta(a) \right),
 \end{aligned}$$

so that $j \circ p \circ (j \otimes j) : V_{\eta\zeta} \otimes V \rightarrow V_\zeta$ is a $U_q(D_4)$ -module homomorphism. From this we deduce that

$$j \circ p \circ (\text{id} \otimes j) = j \circ p \circ (j \otimes j) \circ (j \otimes \text{id}) : V_\eta \otimes V \rightarrow V_\zeta$$

is a $U_q(D_4)$ -module homomorphism as well. Thus, the mappings in the above diagram are scalar multiples of one another. The scalar can be discovered by computing the values of them on $x_1 \otimes x_{-3} \otimes x_{-2}$:

$$(x_1 \cdot x_{-3}|x_{-2}) = q^{-\frac{3}{2}}(x_2|x_{-2}) = \frac{-q^{-\frac{3}{2}}q^{-2}}{q^3 + q^{-3}} = \frac{-q^{-\frac{7}{2}}}{q^3 + q^{-3}},$$

$$(x_1|j(x_{-3} \cdot j(x_{-2}))) = (x_1|j(q^{-\frac{1}{2}}x_{-1})) = \frac{-q^{-\frac{1}{2}}q^{-3}}{q^3 + q^{-3}} = \frac{-q^{-\frac{7}{2}}}{q^3 + q^{-3}}.$$

Hence, they agree, and we have the result. The fact that j is an isometry gives the last equality in (4.8). $\square$

Proposition 4.10. $(x \cdot y|z) = (j(y)|j(z) \cdot K_{2\rho}^{-1}x)$ for all $x, y, z \in \mathbb{O}_q = (V, \cdot)$, which at $q = 1$ gives

$$(4.11) \quad (x \cdot y|z)_1 = (y|\bar{x} \cdot z)_1$$

for the octonions.

Proof. This is an immediate consequence of the calculation

$$\begin{aligned} (x \cdot y|z) &\stackrel{(1)}{=} (x|j(y \cdot j(z))) \stackrel{(2)}{=} (j(y \cdot j(z))|K_{2\rho}^{-1}x) \\ &\stackrel{(3)}{=} (y \cdot j(z)|j(K_{2\rho}^{-1}x)) \stackrel{(3)}{=} (j(y)|j(z) \cdot K_{2\rho}^{-1}x), \end{aligned}$$

where (1) is from Proposition 4.7, (2) uses (2.22), and (3) is Proposition 4.7. $\square$

Note that $K_{2\rho}^{-1}$ acts on $\mathbb{O}_q$ as an algebra automorphism because it is group-like.

Proposition 4.12. *The following diagram commutes:*

$$\begin{array}{ccc} V_\eta \otimes \mathcal{S}^2(V_\zeta) \otimes V_\eta & \xrightarrow{\text{id} \otimes (|) \otimes \text{id}} & V_\eta \otimes V_\eta \\ (j \circ p \circ (j \otimes j)) \otimes p \downarrow & & \downarrow (|) \\ V \otimes V & \xrightarrow{(|)} & \mathbf{K}. \end{array}$$

When $q = 1$, this gives the identity

$$(4.13) \quad (x \cdot y|x \cdot z)_1 = (x|x)_1 (y|z)$$

for the octonions.

Proof. First, it is helpful to observe that

$$\begin{aligned} \text{Hom}_{U_q(\mathbb{D}_4)}(V_\eta \otimes \mathcal{S}^2(V_\zeta) \otimes V_\eta, \mathbf{K}) &\cong \text{Hom}_{U_q(\mathbb{D}_4)}(\mathcal{S}^2(V_\zeta), V_\eta \otimes V_\eta) \\ &= \text{Hom}_{U_q(\mathbb{D}_4)}(L(0), L(0)), \end{aligned}$$

so these spaces have dimension 1. We evaluate the mappings on $x = x_{-1} \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$:

$$\begin{aligned} (|) \circ (\text{id} \otimes (|) \otimes \text{id})(x) &= (x_{-1}|x_1) = \frac{q^3}{q^3 + q^{-3}}, \\ (|) \circ ((j \circ p \circ (j \otimes j)) \otimes p)(x) &= (j(x_{-1} \cdot x_1)|x_{-1} \cdot x_1) \\ &= -q^3(x_{-4}|x_4) = \frac{q^3}{q^3 + q^{-3}}. \quad \square \end{aligned}$$

Proposition 4.14. *The following diagram commutes:*

$$\begin{array}{ccc} V_\zeta \otimes \mathcal{S}^2(V_\eta) \otimes V_\zeta & \xrightarrow{\text{id} \otimes (|) \otimes \text{id}} & V_\zeta \otimes V_\zeta \\ p \otimes (j \circ p \circ (j \otimes j)) \downarrow & & \downarrow (|) \\ V \otimes V & \xrightarrow{(|)} & \mathbf{K} \end{array}$$

In particular, at $q = 1$ this gives the following identity satisfied by the octonions:

$$(4.15) \quad (y \cdot x|z \cdot x)_1 = (y|z)_1 (x|x)_1.$$

Proof. The proof is virtually identical to that of Proposition 4.12. We check the maps on $x = x_{-1} \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$:

$$\begin{aligned} (|) \circ (p \otimes (j \circ p \circ (j \otimes j)))(x) &= (x_{-1} \cdot x_1|j(x_{-1} \cdot x_1)) = (j(x_{-1} \cdot x_1)|x_{-1} \cdot x_1) \\ &= -q^3(x_{-4}|x_4) = \frac{q^3}{q^3 + q^{-3}}, \end{aligned}$$

which equals $(|) \circ (\text{id} \otimes (|) \otimes \text{id})(x)$. $\square$

r -algebras. The notion of an r -algebra (that is, an algebra equipped with a Yang-Baxter operator) arises in Manin's work [M] on noncommutative geometry. Noteworthy examples of r -algebras are Weyl and Clifford algebras, noncommutative tori, certain universal enveloping algebras, and quantum groups (see, for example, Baez [B]).

Definition 4.16. Suppose A is an algebra over a field $\mathbf{F}$ with multiplication $p : A \otimes A \rightarrow A$. Assume $R \in \text{End}(A^{\otimes 2})$ is a Yang-Baxter operator, i.e. $R_{1,2}R_{2,3}R_{1,2} = R_{2,3}R_{1,2}R_{2,3}$, where $R_{1,2} = R \otimes \text{id} \in \text{End}(A^{\otimes 3})$ and $R_{2,3} = \text{id} \otimes R \in \text{End}(A^{\otimes 3})$. Then A is said to be an r -algebra if

$$(4.17) \quad \begin{aligned} R \circ (p \otimes \text{id}_A) &= (\text{id}_A \otimes p) \circ R_{1,2} \circ R_{2,3}, \\ R \circ (\text{id}_A \otimes p) &= (p \otimes \text{id}_A) \circ R_{2,3} \circ R_{1,2}. \end{aligned}$$

When A has a unit element 1, the Yang-Baxter operator R is also required to satisfy $R(1 \otimes x) = x \otimes 1$ and $R(x \otimes 1) = 1 \otimes x$ for all $x \in A$.

Our quantum octonions satisfy r -algebra properties similar to those in (4.17) with respect to the R -matrix of $U_q(\mathbf{D}_4)$.

Proposition 4.18. (a) In $\text{Hom}_{U_q(\mathbf{D}_4)}(V_\zeta \otimes V_\eta \otimes V, V \otimes V)$ we have

$$\check{R}_{V,V} \circ (p \otimes \text{id}) = (\text{id} \otimes p) \circ \check{R}_{V_\zeta,V} \circ \check{R}_{V_\eta,V}.$$

(b) In $\text{Hom}_{U_q(\mathbf{D}_4)}(V \otimes V_\zeta \otimes V_\eta, V \otimes V)$ we have

$$\check{R}_{V,V} \circ (\text{id} \otimes p) = (p \otimes \text{id}) \circ \check{R}_{V,V_\eta} \circ \check{R}_{V,V_\zeta}.$$

Proof. Suppose that $\pi_1 : V \otimes V \rightarrow L(2\omega_1)$, $\pi_2 : V \otimes V \rightarrow L(\omega_2)$, and $\pi_3 : V \otimes V \rightarrow L(0)$ are the projections of $V \otimes V$ onto its irreducible summands. Let T_ℓ and T_r denote the transformations on the left and right in (a). We compute T_ℓ and T_r on several elements of $V_\zeta \otimes V_\eta \otimes V$ which have been chosen so that Θ acts as the identity. We use the fact that $\eta(\epsilon_1) = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$, $\zeta(\epsilon_1) = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)$, $\eta(\epsilon_2) = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$, and $\zeta(\epsilon_3) = \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)$.

We begin by evaluating the action of T_ℓ and T_r on $x_{-1} \otimes x_1 \otimes x_{-1}$:

$$\left(\check{R}_{V,V} \circ (p \otimes \text{id}) \right) (x_{-1} \otimes x_1 \otimes x_{-1}) = -q^{\frac{3}{2}} \check{R}_{V,V}(x_4 \otimes x_{-1}) = -q^{\frac{3}{2}}(x_{-1} \otimes x_4),$$

$$\begin{aligned} & \left((\text{id} \otimes p) \circ \check{R}_{V_\zeta,V} \circ \check{R}_{V_\eta,V} \right) (x_{-1} \otimes x_1 \otimes x_{-1}) \\ &= q^{\frac{1}{2}} \left((\text{id} \otimes p) \circ \check{R}_{V_\zeta,V} \right) (x_{-1} \otimes x_{-1} \otimes x_1) \\ &= (\text{id} \otimes p)(x_{-1} \otimes x_{-1} \otimes x_1) \\ &= -q^{\frac{3}{2}}(x_{-1} \otimes x_4). \end{aligned}$$

Now $(x_{-1}|x_4) = 0$ implies that $x_{-1} \otimes x_4$ lies in $L(2\omega_1) \oplus L(\omega_2)$. The vector $x_{-1} \otimes x_4$ is not an eigenvector of $\check{R}_{V,V}$, because $\check{R}_{V,V}(x_4 \otimes x_{-1}) = x_{-1} \otimes x_4$. Therefore, $\pi_1(x_{-1} \otimes x_4) \neq 0$ and $\pi_2(x_{-1} \otimes x_4) \neq 0$. Since $\dim \text{Hom}_{U_q(\mathbf{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(2\omega_1)) = 1 = \dim \text{Hom}_{U_q(\mathbf{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(\omega_2))$, we see from the above calculation that $\pi_1 T_\ell = \pi_1 T_r$ and $\pi_2 T_\ell = \pi_2 T_r$.

Consider the action of T_ℓ and T_r on $x_3 \otimes x_2 \otimes x_{-1}$:

$$\left(\check{R}_{V,V} \circ (p \otimes \text{id}) \right) (x_3 \otimes x_2 \otimes x_{-1}) = q^{\frac{1}{2}} \check{R}_{V,V}(x_1 \otimes x_{-1}) = q^{\frac{3}{2}}(x_{-1} \otimes x_1),$$

$$\begin{aligned}
& \left((\text{id} \otimes p) \circ \check{R}_{V_\zeta, V} \circ \check{R}_{V_\eta, V} \right) (x_3 \otimes x_2 \otimes x_{-1}) \\
&= q^{\frac{1}{2}} \left((\text{id} \otimes p) \circ \check{R}_{V_\zeta, V} \right) (x_3 \otimes x_{-1} \otimes x_2) \\
&= q(\text{id} \otimes p)(x_{-1} \otimes x_3 \otimes x_2) \\
&= q^{\frac{3}{2}}(x_{-1} \otimes x_1).
\end{aligned}$$

The projection π_3 comes from the form $(|)$, and so it is nonzero when it acts on $x_{-1} \otimes x_1$. Consequently, applying π_3 to both sides of $T_\ell(x_3 \otimes x_2 \otimes x_{-1}) = T_r(x_3 \otimes x_2 \otimes x_{-1})$ and using $\dim \text{Hom}_{U_q(\text{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(0)) = 1$, we obtain $\pi_3 T_\ell = \pi_3 T_r$. Therefore, $T_\ell = (\pi_1 + \pi_2 + \pi_3)T_\ell = (\pi_1 + \pi_2 + \pi_3)T_r = T_r$. The calculations for part (b) are almost identical and are left as an exercise for the reader. $\square$

5. PROPERTIES OF THE QUANTUM PARA-HURWITZ ALGEBRA

Recall that the quantum para-Hurwitz algebra $\mathbb{P}_q = (V, *)$ is obtained from identifying the underlying vector spaces of V_θ , V_{θ^2} and V , where V is the natural representation of $U_q(\text{D}_4)$ and θ is the graph automorphism $\theta = (1\ 4\ 3)$, and using the multiplication $p^* : V_\theta \otimes V_{\theta^2} \rightarrow V$, $x \otimes y \mapsto x * y = j(x) \cdot j(y)$. The resulting product is displayed in Table 2.7. As we saw in the last section, the quantum octonion algebra enjoys very nice composition properties relative to the bilinear form $(|)$. It is known that the para-Hurwitz algebra $\mathbb{P}$ (that is the $q = 1$ version of $\mathbb{P}_q$), which is obtained from the octonions using the product $x * y = \bar{x} \cdot \bar{y}$, has an associative bilinear form admitting composition. Here we establish similar results for the bilinear form on $\mathbb{P}_q = (V, *)$

Proposition 5.1. *For the quantum para-Hurwitz algebra $\mathbb{P}_q = (V, *)$,*

$$(x * y|z) = (x|y * z) \quad \text{and} \quad (x * y|z) = (y|z * K_{2\rho}^{-1}x)$$

for all $x, y, z \in \mathbb{P}_q$. At $q = 1$ these relations reduce to well-known properties of the para-Hurwitz algebra: $(x * y|z)_1 = (x|y * z)_1$ and $(x * y|z)_1 = (y|z * x)_1$.

Proof. The proof amounts to using the fact that j is an isometry together with the fact that j commutes with $K_{2\rho}^{-1}$. So we have

$$\begin{aligned}
(x * y|z) &= (j(x) \cdot j(y)|z) \\
&= \begin{cases} (x|j(y) \cdot j(z)) = (x|y * z), \\ (y|j(z) \cdot K_{2\rho}^{-1}j(x)) = (y|j(z) \cdot j(K_{2\rho}^{-1}x)) = (y|z * K_{2\rho}^{-1}x). \end{cases} \quad \square
\end{aligned}$$

Proposition 5.2. *Ignoring the $U_q(\text{D}_4)$ -module structures, we have*

$$\begin{array}{ccc}
V \otimes \mathcal{S}^2(V) \otimes V & \xrightarrow{\text{id} \otimes (|) \otimes \text{id}} & V \otimes V \\
(j \circ p^* \circ (j \otimes j)) \otimes p^* \downarrow & & \downarrow (|) \\
V \otimes V & \xrightarrow{(|)} & \mathbf{K}
\end{array}$$

and

$$\begin{array}{ccc}
 V \otimes \mathcal{S}^2(V) \otimes V & \xrightarrow{\text{id} \otimes (|) \otimes \text{id}} & V \otimes V \\
 p^* \otimes (j \circ p^* \circ (j \otimes j)) \downarrow & & \downarrow (|) \\
 V \otimes V & \xrightarrow{(|)} & \mathbf{K}
 \end{array}$$

Proof. These diagrams commute when p is used in place of p^* . Now when $p^* = p \circ (j \otimes j)$ is taken, we have

$$\begin{aligned}
 (|) \circ (j \circ p^* \circ (j \otimes j)) \otimes p^* &= (|) \circ ((j \circ p) \otimes (p \circ (j \otimes j))) \\
 &\stackrel{(1)}{=} (|) \circ p \otimes (j \circ p \circ (j \otimes j)) \\
 &= (|) \circ \text{id} \otimes (|) \otimes \text{id}
 \end{aligned}$$

where (1) follows from the fact that j is an isometry. The other diagram can be shown to commute in a similar way. $\square$

Proposition 5.3. *The following diagram commutes and shows at $q = 1$ that the relation $(x \cdot y) \cdot x = (x|x)_1 y = x \cdot (y \cdot x)$ holds for the para-Hurwitz algebra:*

$$\begin{array}{ccccc}
 V_{\theta^2} \otimes V & \xleftarrow{\text{id} \otimes (j \circ p^* \circ (j \otimes j))} & \mathcal{S}^2(V_{\theta^2}) \otimes V_{\theta} = \mathcal{S}^2(V) \otimes V = \mathcal{S}^2(V_{\theta}) \otimes V_{\theta^2} & \xrightarrow{\text{id} \otimes p^*} & V_{\theta} \otimes V \\
 \downarrow p^* & & (|) \otimes \text{id} \downarrow & & j \circ p^* \circ (j \otimes j) \downarrow \\
 V_{\theta} & = & V & = & V_{\theta^2}
 \end{array}$$

Proof. Again it suffices to check the maps on $x = (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$, and for it we have

$$\begin{aligned}
 (j \circ p^* \circ (j \otimes j)) \circ (\text{id} \otimes p^*)(x) &= j \circ p^* \circ (j \otimes j)(x_1 \otimes (-q^{\frac{3}{2}})x_4) \\
 &= -j(x_1) = x_1 = (|) \otimes \text{id}(x),
 \end{aligned}$$

while

$$p^* \circ (\text{id} \otimes (j \circ p^* \circ (j \otimes j)))(x) = p^*(x_1 \otimes q^{\frac{3}{2}}x_{-4}) = x_1 = (|) \otimes \text{id}(x). \quad \square$$

6. IDEMPOTENTS AND DERIVATIONS OF QUANTUM OCTONIONS

It is apparent from Table 3.9 that $e_1 = q^{-\frac{1}{2}}x_4$ and $e_2 = -q^{\frac{1}{2}}x_{-4}$ are orthogonal idempotents in the quantum octonion algebra $\mathbb{O}_q = (V, \cdot)$. Their sum $e = e_1 + e_2$ acts as a left and right identity element on the span of $e_1, e_2, x_j, x_{-j}, j = 2, 3$. Moreover, the relations $e \cdot x_1 = q^2x_1$, $x_1 \cdot e = q^{-2}x_1$, $e \cdot x_{-1} = q^{-2}x_{-1}$, and $x_{-1} \cdot e = q^2x_{-1}$ hold in $\mathbb{O}_q$. Hence, at $q = 1$ or -1 , the element e is an identity element. Relative to the basis $e_1, e_2, x_{\pm j}, j = 1, 2, 3$, the left and right multiplication operators $L_{e_1}, R_{e_1}, L_{e_2}, R_{e_2}$ can be simultaneously diagonalized. The operators L_{e_1}, R_{e_1} determine a Peirce decomposition of $\mathbb{O}_q$,

(6.1)

$$\mathbb{O}_q = \oplus_{(\gamma, \delta)} (\mathbb{O}_q)_{\gamma, \delta} \quad \text{where} \quad (\mathbb{O}_q)_{\gamma, \delta} = \{v \in \mathbb{O}_q \mid e_1 \cdot v = \gamma v, \text{ and } v \cdot e_1 = \delta v\}.$$

Now $(\mathbb{O}_q)_{0,q^{-2}} = \mathbf{K}x_1$, $(\mathbb{O}_q)_{1,0} = \mathbf{K}x_2 + \mathbf{K}x_3$, $(\mathbb{O}_q)_{1,1} = \mathbf{K}e_1$, $(\mathbb{O}_q)_{0,0} = \mathbf{K}e_2$, $(\mathbb{O}_q)_{0,1} = \mathbf{K}x_{-2} + \mathbf{K}x_{-3}$, and $(\mathbb{O}_q)_{q^{-2},0} = \mathbf{K}x_{-1}$. The idempotent e_2 gives a similar Peirce decomposition.

Suppose d is a derivation of $\mathbb{O}_q$ such that $d(e_1) = 0 = d(e_2)$. It is easy to see that d must map each Peirce space $(\mathbb{O}_q)_{\gamma,\delta}$ into itself. Thus, we may suppose that

$$(6.2) \quad \begin{aligned} d(x_1) &= b_1x_1, & d(x_{-1}) &= b_{-1}x_{-1}, \\ d(x_2) &= c_{2,2}x_2 + c_{3,2}x_3, & d(x_3) &= c_{2,3}x_2 + c_{3,3}x_3, \\ d(x_{-2}) &= d_{2,2}x_{-2} + d_{3,2}x_{-3}, & d(x_{-3}) &= d_{2,3}x_{-2} + d_{3,3}x_{-3}. \end{aligned}$$

Applying d to the relation $x_1 \cdot x_{-1} = q^{-\frac{3}{2}}x_{-4} = -q^{-2}e_2$ (or to the relation $x_{-1} \cdot x_1 = -q^{\frac{3}{2}}x_4 = -q^2e_1$) shows that $b_{-1} = -b_1$. Now from $x_k \cdot x_k = 0$, we deduce that $c_{j,k} = 0$ for $j \neq k$, and similarly from $x_{-k} \cdot x_{-k} = 0$ it follows that $d_{j,k} = 0$. There are relations $x_1 \cdot x_{-j} = \xi x_k$, $x_{-j} \cdot x_1 = \xi x_k$, $x_{-1} \cdot x_j = \xi x_{-k}$, and $x_j \cdot x_{-1} = \xi x_{-k}$ (where $j \neq k$ and ξ in each one indicates some appropriate scalar that can be found in Table 3.9), and applying d to those equations gives $b_1 + d_{j,j} = c_{k,k}$ and $b_{-1} + c_{j,j} = d_{k,k}$. The second is equivalent to the first since $b_{-1} = -b_1$. From $x_{\pm j} \cdot x_{\pm k} = \xi x_{\pm 1}$ we see that $c_{2,2} + c_{3,3} = b_1$ and $d_{2,2} + d_{3,3} = -b_1$. Finally, applying d to $x_j \cdot x_{-j} = \xi e_1$ or $x_{-j} \cdot x_j = \xi e_2$, we obtain $d_{j,j} = -c_{j,j}$ for $j = 2, 3$. Every nonzero product between basis elements in the quantum octonions is one of these types, or it is one involving the idempotents e_1, e_2 , so we have determined all the possible relations. Consequently, we can conclude that for any derivation d such that $d(e_1) = 0 = d(e_2)$, there are scalars $b = b_1$ and $c = c_{2,2}$ so that the following hold:

$$(6.3) \quad \begin{aligned} d(e_1) &= 0 = d(e_2), \\ d(x_1) &= bx_1, & d(x_{-1}) &= -bx_{-1}, \\ d(x_2) &= cx_2, & d(x_{-2}) &= -cx_{-2}, \\ d(x_3) &= (b-c)x_3, & d(x_{-3}) &= -(b-c)x_{-3}. \end{aligned}$$

The same calculations show that any transformation defined by (6.3) with b, c arbitrary scalars is a derivation of $\mathbb{O}_q$.

Now suppose that d is an arbitrary derivation of $\mathbb{O}_q$, and let $d(e_1) = a_0e_1 + a'_0e_2 + \sum_{j=-3, j \neq 0}^3 a_jx_j$ and $d(e_2) = b_0e_1 + b'_0e_2 + \sum_{j=-3, j \neq 0}^3 b_jx_j$. Computing $d(e_1) \cdot e_1 + e_1 \cdot d(e_1) = d(e_1)$, we obtain

$$2a_0e_1 + q^{-2}a_1x_1 + a_2x_2 + a_3x_3 + q^{-2}a_{-1}x_{-1} + a_{-2}x_{-2} + a_{-3}x_{-3} = d(e_1),$$

which shows that $a_0 = a'_0 = a_{-1} = a_1 = 0$. A similar calculation with the relation $d(e_2^2) = d(e_2)$ gives $b_0 = b'_0 = b_{-1} = b_1 = 0$. From $d(e_1) \cdot e_2 + e_1 \cdot d(e_2) = 0$ it follows that $b_2 = -a_2$ and $b_3 = -a_3$, and from $d(e_2) \cdot e_1 + e_2 \cdot d(e_1) = 0$, the relations $b_{-2} = -a_{-2}$, $b_{-3} = -a_{-3}$ can be determined.

Now $e_1 \cdot d(x_1) = -d(e_1) \cdot x_1$ shows that the coefficients of x_2 and x_3 in $d(x_1)$ are $q^{\frac{3}{2}}a_{-3}$ and $q^{\frac{3}{2}}a_{-2}$ respectively; while $d(x_1) \cdot e_2 = -x_1 \cdot d(e_2)$ says that those coefficients are $-q^{-\frac{3}{2}}b_{-3} = q^{-\frac{3}{2}}a_{-3}$ and $-q^{-\frac{3}{2}}b_{-2} = q^{-\frac{3}{2}}a_{-2}$. Thus, when q is not a cube root of unity, we obtain that $a_{-2} = 0 = a_{-3}$. Likewise the relations $a_2 = 0 = a_3$ can be found from analyzing $d(x_{-1}) \cdot e_1 = -x_{-1}d(e_1)$ and $e_2 \cdot d(x_{-1}) = -d(e_2) \cdot x_{-1}$. Consequently, when q is not a cube root of unity, every derivation of $\mathbb{O}_q$ must annihilate e_1 and e_2 , and so must be as in (6.3) for suitable scalars b, c . To summarize, we have

Theorem 6.4. *The derivation algebra of the quantum octonion algebra $\mathbb{O}_q = (V, \cdot)$ is two-dimensional. Every derivation d of $\mathbb{O}_q$ is given by (6.3) for some scalars $b, c \in \mathbf{K}$.*

Simplicity of the quantum octonions. The idempotents also provide a proof of the following.

Theorem 6.5. *The quantum octonion algebra $\mathbb{O}_q = (V, \cdot)$ is simple.*

Proof. Let I be a nonzero ideal of $\mathbb{O}_q$. Since $e_1 I \subseteq I$ and $I e_1 \subseteq I$, it follows that

$$I = \bigoplus_{\gamma, \delta} I_{\gamma, \delta},$$

where $I_{\gamma, \delta} = \{x \in I \mid e_1 \cdot x = \gamma x \text{ and } x \cdot e_1 = \delta x\}$. Assume first that $x_i \in I$ for some $i = \{\pm 1, \pm 2, \pm 3, \pm 4\}$. If we multiply x_i by x_{-i} when $i = \pm 1, \pm 2$, or ± 3 , we see that e_1 or e_2 belongs to I . Therefore, either $e_1 \mathbb{O}_q + \mathbb{O}_q e_1 \subseteq I$ or $e_2 \mathbb{O}_q + \mathbb{O}_q e_2 \subseteq I$, and as a consequence, $I = \mathbb{O}_q$. It remains to consider the possibility that some nonzero linear combination $ax_2 + bx_3$ or $ax_{-2} + bx_{-3}$ is in I . Multiplying by one of the elements x_2, x_3, x_{-2} , or x_{-3} , we obtain that x_1 or x_{-1} is in I , and that implies $I = \mathbb{O}_q$ as before. Consequently, $\mathbb{O}_q$ is simple. $\square$

7. QUANTUM QUATERNIONS

The subalgebra $\mathbb{H}_q$ of $\mathbb{O}_q$ generated by $x = x_2$ and $y = x_{-2}$ is 4-dimensional and has x, y, e_1, e_2 as a basis, where $e_1 = q^{-\frac{1}{2}}x_4$ and $e_2 = -q^{\frac{1}{2}}x_{-4}$ are the orthogonal idempotents of Section 6. The sum $e = e_1 + e_2$ is the identity element of $\mathbb{H}_q$, and multiplication in this algebra is given by Table 7.1. It is easy to see that at $q = 1$ this algebra is isomorphic to the algebra $M_2(\mathbf{K})$ of 2×2 matrices over $\mathbf{K}$ via the mapping $e_1 \mapsto E_{1,1}$, $e_2 \mapsto E_{2,2}$, $x \mapsto E_{1,2}$, $y \mapsto E_{2,1}$, which sends basis elements to standard matrix units. Since the split quaternion algebra is isomorphic to $M_2(\mathbf{K})$, this algebra reduces to the quaternions at $q = 1$. Another way to see this is to observe that the bilinear form on $\mathbb{O}_q$, when restricted to $\mathbb{H}_q$, satisfies (3.6) at $q = 1$. Thus, at $q = 1$ we have a 4-dimensional unital algebra with a nondegenerate bilinear form admitting composition, and so it must be the quaternions.

TABLE 7.1

$\cdot$	e_1	e_2	x	y
e_1	e_1	0	x	0
e_2	0	e_2	0	y
x	0	x	0	$q^{-1}e_1$
y	y	0	qe_2	0

The algebra $\mathbb{H}_q$ is neither associative nor flexible if $q \neq \pm 1$, since

$$(x \cdot y) \cdot x = q^{-1}x \neq qx = x \cdot (y \cdot x).$$

However, it is *Lie admissible*. Indeed, if $h = q^{-1}e_1 - qe_2$, then

$$[h, x] = (q + q^{-1})x, \quad [h, y] = -(q + q^{-1})y, \quad \text{and} \quad [x, y] = h.$$

Therefore, the elements $x, \frac{2}{q+q^{-1}}y, \frac{2}{q+q^{-1}}h$ determine a standard basis for sl_2 . Since $e = e_1 + e_2$ is the identity, we see that $\mathbb{H}_q$ is the Lie algebra gl_2 under the commutator product $[a, b] = ab - ba$.

The subalgebra generated by x_3 and x_{-3} is isomorphic to the algebra $\mathbb{H}_q$. However, that is not true if we consider the subalgebra $\mathbb{H}'_q$ generated by x_1 and x_{-1} . Let $u = x_{-1}$ and $v = x_1$. Then multiplication in $\mathbb{H}'_q$ is given by Table 7.2.

TABLE 7.2

$\cdot$	e_1	e_2	u	v
e_1	e_1	0	$q^{-2}u$	0
e_2	0	e_2	0	q^2v
u	0	q^2u	0	q^2e_1
v	$q^{-2}v$	0	$q^{-2}e_2$	0

There is no identity element in $\mathbb{H}'_q$. But at $q = 1$, the element $e_1 + e_2$ becomes an identity element, and the algebra is isomorphic to $M_2(\mathbf{K})$. The bilinear form at $q = 1$ has the composition property, so the algebra $\mathbb{H}'_q$ too might rightfully be termed a quantum quaternion algebra. The elements $h' = q^2e_1 - q^{-2}e_2, u, v$ determine a standard basis for sl_2 :

$$[h', u] = 2u, \quad [h', v] = -2v, \quad [u, v] = h'.$$

Since $[q^2e_1 + q^{-2}e_2, h'] = [q^2e_1 + q^{-2}e_2, u] = [q^2e_1 + q^{-2}e_2, v] = 0$, the algebra $\mathbb{H}'_q$ under the commutator product is isomorphic to gl_2 . It is neither associative nor flexible, because

$$((e_1 + v) \cdot e_1) \cdot (e_1 + v) = e_1 + q^{-4}v \neq e_1 + q^{-2}v = (e_1 + v) \cdot (e_1 \cdot (e_1 + v)).$$

8. CONNECTIONS WITH $U_q(B_3)$ AND $U_q(G_2)$

In this section we assume that the field $\mathbf{K}$ has characteristic zero. We consider the subalgebras of $U_q(D_4)$ that are described as the fixed points of graph automorphisms by

$$(8.1) \quad \begin{aligned} U_{\zeta, \eta} &= \{a \in U_q(D_4) \mid \zeta(a) = a = \eta(a)\}, \\ U_{\delta} &= \{a \in U_q(D_4) \mid \delta(a) = a\}, \end{aligned}$$

where $\delta = (34) = \zeta\eta\zeta$. Then $U_{\delta} \supseteq U_{\zeta, \eta}$. In the non-quantum case, the enveloping algebras $U(G_2) \subseteq U_{\zeta, \eta} \subseteq U(D_4)$ and $U(B_3) \subseteq U_{\delta} \subseteq U(D_4)$. We consider the structure of V, V_{ζ}, V_{η} as modules for the subalgebras given in (8.1).

Now for any $a \in U_{\delta}$, we have that

$$j(ax) = j(\delta(a)x) = aj(x)$$

for all $x \in V$, so that any $a \in U_{\delta}$ commutes with j . In particular, the eigenspaces of j are U_{δ} -invariant. But the eigenspaces of j are 1-dimensional (spanned by $x_4 - x_{-4}$), and 7-dimensional (spanned by $\{x_i \mid i = \pm 1, \pm 2, \pm 3\} \cup \{x_4 + x_{-4}\}$). Consider the 7-dimensional space under the action of $U_{\zeta, \eta}$. Since $F_{\alpha_2}, F_{\alpha_1} + F_{\alpha_3} + F_{\alpha_4}, E_{\alpha_2}, E_{\alpha_1} + E_{\alpha_3} + E_{\alpha_4} \in U_{\zeta, \eta}$, it is easy to check that the 7-dimensional space is an irreducible $U_{\zeta, \eta}$ -module, hence an irreducible U_{δ} -module as well. The action of $U_{\zeta, \eta}$ on V_{ζ} and V_{η} is the same as its action on V . As a result we have

Proposition 8.2. *The modules $V, V_\zeta \cong V_-$, and $V_\eta \cong V_+$ are isomorphic as modules for the subalgebra $U_{\zeta, \eta}$ in (8.1). They decompose into the sum of irreducible $U_{\zeta, \eta}$ -modules of dimensions 1 and 7, which are the eigenspaces of j .*

For $q = 1$, Proposition 8.2 amounts to saying that V, V_ζ, V_η are isomorphic $U(G_2)$ -modules which decompose into irreducible modules of dimension 1 and 7.

The elements $F_{\alpha_1}, F_{\alpha_2}, F_{\alpha_3} + F_{\alpha_4}, E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_3} + E_{\alpha_4}$ belong to U_δ , so if we regard V_ζ or V_η as a module for U_δ , then we obtain a diagram

$$\begin{array}{ccccc}
 & & & & x_4 \\
 & & & \nearrow^{F_{\alpha_1}} & \\
 x_1 & \xrightarrow{F_{\alpha_3}+F_{\alpha_4}} & x_2 & \xrightarrow{F_{\alpha_2}} & x_3 & \dots \\
 & & & \searrow_{F_{\alpha_3}+F_{\alpha_4}} & \\
 & & & & x_{-4}
 \end{array}$$

which is gotten by superimposing the two diagrams of V_ζ and V_η . Using that, it is easy to argue that both these modules remain irreducible for U_δ . Moreover, for any $a \in U_\delta$, we have

$$\begin{aligned}
 j(\zeta(a)x) &= j((\zeta\delta)(a)x) = j((\eta\zeta)(a)x) \\
 &= j(\theta^2(a)x) = \eta(a)j(x),
 \end{aligned}$$

where $\theta = (1\ 4\ 3)$. Consequently, $V_\zeta \cong V_\eta$ as U_δ -modules.

At $q = 1$ these computations show that V_ζ and V_η (the two spin modules of $U(D_4)$) remain irreducible for $U(B_3)$, and in fact they are the spin representation of that algebra. To summarize, we have

Proposition 8.3. *When regarded as modules for the subalgebra U_δ of $U_q(D_4)$, $V_\zeta \cong V_-$ and $V_\eta \cong V_+$ are irreducible and isomorphic.*

In light of these propositions it is natural to expect that these fixed point subalgebras of the graph automorphisms might be the quantum groups $U_q(G_2)$ and $U_q(B_3)$. However, that is not true. In fact the next “nonembedding” result shows that there are no Hopf algebra homomorphisms of them into $U_q(D_4)$ except for the trivial ones given by the counit.

Proposition 8.4. *When q is not a root of unity, then any Hopf algebra homomorphisms*

$$U_q(G_2) \rightarrow U_q(B_3), \quad U_q(B_3) \rightarrow U_q(D_4), \quad U_q(G_2) \rightarrow U_q(D_4)$$

are trivial ($a \mapsto \epsilon(a)1$ for all a).

The method we use to establish this result is due originally to Hayashi [H]. We begin with a little background before presenting the proof.

Suppose $U_q(\mathfrak{g})$ is a quantum group and (X, π) , $\pi : U_q(\mathfrak{g}) \rightarrow \text{End}(X)$, is a representation of $U_q(\mathfrak{g})$. This determines another representation (X, π') , $\pi' = \pi \circ S^2$, of $U_q(\mathfrak{g})$, where S is the antipode. But it is well-known [Jn, p. 56] that S^2 has the following expression:

$$S^2(a) = K_{2\rho}^{-1} a K_{2\rho} \quad \text{for all } a \in U_q(\mathfrak{g}),$$

where $\rho = \frac{1}{2} \sum_{\gamma > 0} \gamma = \sum_{\alpha \in \Pi} c_\alpha \alpha$ (the half-sum of the positive roots of $\mathfrak{g}$) and $K_{2\rho} = \prod_{\alpha \in \Pi} K_{\alpha}^{c_\alpha}$ (where Π is the set of simple roots of $\mathfrak{g}$). The two representations (X, π) and (X, π') are in fact isomorphic by the following map:

$$\phi : (X, \pi) \rightarrow (X, \pi'), \quad x \mapsto \pi(K_{2\rho}^{-1})x,$$

because

$$\begin{aligned} \phi(\pi(a)x) &= \pi(K_{2\rho}^{-1})\pi(a)x = \pi(K_{2\rho}^{-1}a)x = \pi(S^2(a)K_{2\rho}^{-1})x \\ &= \pi(S^2(a))\pi(K_{2\rho}^{-1})x = \pi'(a)\phi(x) \quad \text{for all } a \in U_q(\mathfrak{g}). \end{aligned}$$

Examples 8.5. Let $\mathfrak{g} = D_4$ and let $X = V, V_-, V_+$ (the natural or spin representations of $U_q(D_4)$). Here $\rho = 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 3\alpha_4$, so $K_{2\rho}^{-1} = K_{\alpha_1}^{-6} K_{\alpha_2}^{-10} K_{\alpha_3}^{-6} K_{\alpha_4}^{-6}$. Relative to the basis $\{x_1, x_2, x_3, x_4, x_{-4}, x_{-3}, x_{-2}, x_{-1}\}$ we have, as before,

$$(8.6) \quad \phi_{D_4} = \phi_{D_4}^- = \phi_{D_4}^+ = \text{diag}\{q^{-6}, q^{-4}, q^{-2}, 1, 1, q^2, q^4, q^6\}.$$

Next let $\mathfrak{g} = G_2$ and let X be the 7-dimensional irreducible $U_q(G_2)$ -representation. Here $\rho = 5\alpha_1 + 3\alpha_2$, so $K_{2\rho}^{-1} = K_{\alpha_1}^{-10} K_{\alpha_2}^{-6}$. With respect to the basis (see [Jn], Chap. 5)

$$(8.7) \quad \{x_{2\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}, x_{\alpha_1}, x_0, x_{-\alpha_1}, x_{-(\alpha_1+\alpha_2)}, x_{-(2\alpha_1+\alpha_2)}\},$$

we have

$$(8.8) \quad \phi_{G_2} = \text{diag}\{q^{-10}, q^{-8}, q^{-2}, 1, q^2, q^8, q^{10}\}.$$

Finally, let X be the 7-dimensional natural representation or the 8-dimensional spin representation of $U_q(B_3)$. In this case $2\rho = 5\alpha_1 + 8\alpha_2 + 9\alpha_3$, so that $K_{2\rho}^{-1} = K_{\alpha_1}^{-5} K_{\alpha_2}^{-8} K_{\alpha_3}^{-9}$. If we assume as in [Jn] that a short root satisfies $(\alpha, \alpha) = 2$, then setting $(\epsilon_i, \epsilon_i) = 2$ for $i = 1, 2, 3$, we have that the matrix of the bilinear form relative to the basis $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \alpha_3 = \epsilon_3$ of simple roots is

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

From this we can compute that $(-2\rho, \epsilon_i) = -10, -6, -2$ for $i = 1, 2, 3$, respectively. Since we know, for any weight vector x_μ , that $K_\alpha x_\mu = q^{(\mu, \alpha)} x_\mu$, we obtain that, relative to the basis

$$(8.9) \quad \{x_{\epsilon_1}, x_{\epsilon_2}, x_{\epsilon_3}, x_0, x_{-\epsilon_3}, x_{-\epsilon_2}, x_{-\epsilon_1}\},$$

for the natural representation,

$$(8.10) \quad \phi_{B_3} = \text{diag}\{q^{-10}, q^{-6}, q^{-2}, 1, q^2, q^6, q^{10}\}.$$

The spin representation of $U_q(B_3)$ has a basis

$$\{x_{\xi_1}, x_{\xi_2}, x_{\xi_3}, x_{\xi_4}, x_{-\xi_4}, x_{-\xi_3}, x_{-\xi_2}, x_{-\xi_1}\},$$

where $\xi_1 = (1/2)(\epsilon_1 + \epsilon_2 + \epsilon_3)$, $\xi_2 = (1/2)(\epsilon_1 + \epsilon_2 - \epsilon_3)$, $\xi_3 = (1/2)(\epsilon_1 - \epsilon_2 + \epsilon_3)$, and $\xi_4 = (1/2)(-\epsilon_1 + \epsilon_2 + \epsilon_3)$. The corresponding matrix of $\phi_{B_3}^+$ is given by

$$\phi_{B_3}^+ = \text{diag}\{q^{-9}, q^{-7}, q^{-3}, q^{-1}, q, q^3, q^7, q^9\}.$$

Now suppose that $f : U_q(\mathfrak{g}_1) \rightarrow U_q(\mathfrak{g}_2)$ is a Hopf algebra homomorphism between two quantum groups. Consider representations (X, π) and (X, π') for $U_q(\mathfrak{g}_2)$ as

above. Then $(X, \pi \circ f)$, $(X, \pi' \circ f)$ are representations for $U_q(\mathfrak{g}_1)$. Since $\pi' \circ f = \pi \circ S^2 \circ f = \pi \circ f \circ S^2$, it follows that $\pi' \circ f = (\pi \circ f)'$. Moreover, the map

$$\phi_{\mathfrak{g}_2} : (X, \pi) \rightarrow (X, \pi')$$

satisfies

$$\phi_{\mathfrak{g}_2}((\pi \circ f)(a)x) = \pi'(f(a))\phi_{\mathfrak{g}_2}(x) = (\pi \circ f)'(a)\phi_{\mathfrak{g}_2}(x)$$

for all $a \in U_q(\mathfrak{g}_1)$ and $x \in X$, so it is an isomorphism of $U_q(\mathfrak{g}_1)$ -modules: $\phi_{\mathfrak{g}_2} : (X, \pi \circ f) \rightarrow (X, (\pi \circ f)')$.

Proof of Proposition 8.4. Suppose $f : U_q(G_2) \rightarrow U_q(B_3)$ is a Hopf algebra homomorphism. Consider the natural representation $Y = (X, \pi)$ of $U_q(B_3)$, and the corresponding representation

$$(8.11) \quad \phi_{B_3} : (X, \pi \circ f) \rightarrow (X, (\pi \circ f)')$$

of $U_q(G_2)$ -modules. Either these are the 7-dimensional irreducible $U_q(G_2)$ -module or they are the sum of 7 trivial 1-dimensional modules. In the first case ϕ_{B_3} is a multiple of ϕ_{G_2} , so the eigenvalues of ϕ_{B_3} are $\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\}$. This is impossible since q is not a root of unity, so it must be that $\pi \circ f$ is trivial.

Analogously, consider the spin representation $Y^+ = (X^+, \pi^+)$ of $U_q(B_3)$. Then as a $U_q(G_2)$ -module, such a representation should be the sum of representations of dimension 1 or 7. Reasoning as in (8.11), we see that if a 7-dimensional $U_q(G_2)$ -module occurs, then $\phi_{B_3}^+$ restricted to the 7-dimensional submodule should be proportional to ϕ_{G_2} . This would mean that

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\} \subseteq \text{diag}\{q^{\pm 9}, q^{\pm 7}, q^{\pm 3}, q^{\pm 1}\}$$

for some scalar λ , which is impossible. Thus $\pi^+ \circ f$ is trivial.

We have seen that the natural and spin representations for $U_q(B_3)$ are trivial modules for $f(U_q(G_2))$. Thus, $f(a) - \epsilon(f(a))1$ is in the annihilator of those representations. Since any finite-dimensional irreducible $U_q(B_3)$ -representation of type I can be obtained from the tensor product of those, and since f is a Hopf homomorphism, it follows that $f(U_q(G_2))$ acts trivially on any finite-dimensional irreducible $U_q(B_3)$ -representation of type I. However, $\bigcap \text{ann}(V) = (0)$ for all such representations (see [Jn, Prop. 5.11]). Consequently, $f(a) = \epsilon(f(a))1$ for all $a \in U_q(G_2)$.

The other two cases in Proposition 8.4 are similar. For a Hopf homomorphism $f : U_q(G_2) \rightarrow U_q(D_4)$, it boils down to the fact that any finite-dimensional irreducible $U_q(D_4)$ -representation occurs in tensor products of copies of the natural or spin representations and

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\} \not\subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\}.$$

In the final case, $f : U_q(B_3) \rightarrow U_q(D_4)$, the argument amounts to the impossibility of

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 6}, \lambda q^{\pm 2}, \lambda\} \subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\},$$

and

$$\{\lambda q^{\pm 9}, \lambda q^{\pm 7}, \lambda q^{\pm 3}, \lambda q^{\pm 1}\} \subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\}. \quad \square$$

9. QUANTUM OCTONIONS AND QUANTUM CLIFFORD ALGEBRAS

In this section we investigate the representation of the quantum Clifford algebra $C_q(8)$ on $\mathbb{O}_q \oplus \mathbb{O}_q$. Our main reference for the quantum Clifford algebra is [DF], but we need to make several adjustments to make the results of [DF] conform to our work here. It is convenient to begin with the general setting of the quantum group of D_n and then to specialize to D_4 .

Explicit form of $\check{R}_{V,V}$ for $U_q(D_n)$. The natural representation V of the quantum group $U_q(D_n)$ has a basis

$$(9.1) \quad \{x_1, x_2, \dots, x_{2n}\} = \{x_{\epsilon_1}, \dots, x_{\epsilon_n}, x_{-\epsilon_n}, \dots, x_{-\epsilon_1}\},$$

where $E_\alpha, F_\alpha, K_\alpha$ act as in (2.2). A straightforward computation shows that relative to the basis $\{x_1, x_2, \dots, x_{2n}\}$, the matrix of $\check{R} = \check{R}_{V,V}$ is given by

$$(9.2) \quad \begin{aligned} \check{R} = q^{-1} \sum_{i=1}^{2n} E_{i,i} \otimes E_{i,i} + \sum_{i \neq j, j'} E_{j,i} \otimes E_{i,j} + (q^{-1} - q) \sum_{j>i} E_{j,j} \otimes E_{i,i} \\ + q \sum_{i=1}^{2n} E_{i',i} \otimes E_{i,i'} + (q - q^{-1}) \sum_{i<j} (-q)^{\rho_{j'} - \rho_{i'}} E_{i',j} \otimes E_{i,j'}, \end{aligned}$$

where $i' = 2n + 1 - i$ and $\rho = (\rho_1, \dots, \rho_{2n}) = (n-1, n-2, \dots, 1, 0, 0, -1, \dots, 2-n, 1-n)$. The projection $(,) : V \otimes V \rightarrow K$ is given by

$$(x_i, x_j) = \delta_{i,j'} (-1)^{n-1} (-q)^{\rho_{i'}}.$$

If we modify the basis slightly by defining

$$(9.3) \quad e_i = (-1)^{n-1+\rho_{i'}} x_i, \quad e_{i'} = x_{i'}$$

for $i = 1, \dots, n$, then we obtain the usual expression for $\check{R}$:

$$(9.4) \quad \begin{aligned} \check{R} = q^{-1} \sum_{i=1}^{2n} E_{i,i} \otimes E_{i,i} + \sum_{i \neq j, j'} E_{j,i} \otimes E_{i,j} + (q^{-1} - q) \sum_{j>i} E_{j,j} \otimes E_{i,i} \\ + q \sum_{i=1}^{2n} E_{i',i} \otimes E_{i,i'} + (q - q^{-1}) \sum_{i<j} q^{(\rho_{j'} - \rho_{i'})} E_{i',j} \otimes E_{i,j'}, \end{aligned}$$

and the bilinear form is given by

$$(e_i, e_j) = \delta_{i,j'} q^{\rho_{i'}}.$$

The quantum Clifford algebra. In [DF], Ding and Frenkel define the quantum Clifford algebra $C_q(2n)$ as the quotient of the tensor algebra $T(V)$ of V by the ideal generated by $\{v \otimes w + q\check{R}'_{2,1}(v \otimes w) - (w, v)1 \mid v, w \in V\}$, where $\check{R}'$ is $\check{R}$ with q and q^{-1} interchanged, and $\check{R}'_{2,1} = \sigma \circ \check{R}' \circ \sigma$, where $\sigma(v \otimes w) = w \otimes v$. This quantum Clifford algebra can be thought of as a unital associative algebra generated by $\psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*$, with relations

$$(9.5) \quad \begin{aligned} \psi_i \psi_j &= -q^{-1} \psi_j \psi_i, & \psi_i^* \psi_j^* &= -q \psi_j^* \psi_i^* \quad (i > j), \\ \psi_i \psi_i &= 0 = \psi_i^* \psi_i^*, & \psi_i \psi_j^* &= -q^{-1} \psi_j^* \psi_i \quad (i \neq j), \\ \psi_i \psi_i^* + \psi_i^* \psi_i &= (q^{-2} - 1) \sum_{i<j} \psi_j \psi_j^* + 1. \end{aligned}$$

If we make the change of notation $Y_i = \psi_{n-i}^*, Y_i^* = \psi_{n-i}$, $i = 1, \dots, n$, then the relations in (9.5) become

$$(9.6) \quad \begin{aligned} Y_i Y_j &= -q^{-1} Y_j Y_i, & Y_i^* Y_j^* &= -q Y_j^* Y_i^* \quad (i > j), \\ Y_i Y_i &= 0 = Y_i^* Y_i^*, & Y_i Y_j^* &= -q Y_j^* Y_i \quad (i \neq j), \\ Y_i Y_i^* + Y_i^* Y_i &= (q^{-2} - 1) \sum_{i>j} Y_j Y_j^* + 1. \end{aligned}$$

In order to construct a representation of $C_q(8)$ on $\mathbb{O}_q \oplus \mathbb{O}_q$ we will need the following proposition:

Proposition 9.7. *Let I be the ideal of the tensor algebra $T(V)$ generated by $\{v \otimes w + q^{-1} \tilde{R}_{2,1}(v \otimes w) - (w, v)1 \mid v, w \in V\}$, where $\tilde{R}_{2,1} = \sigma \circ \tilde{R} \circ \sigma$. Then $C_q(2n) \cong T(V)/I$.*

Proof. Consider the following basis of V :

$$(9.8) \quad \{y_i = q^{\rho_i} e_i, \quad y_i^* = e_{i'} \mid i = 1, \dots, n\},$$

which satisfies

$$(9.9) \quad (y_i, y_j^*) = \delta_{i,j}, \quad (y_j^*, y_i) = \delta_{i,j} q^{2\rho_i}.$$

Let $g : V \otimes V \rightarrow I$ be defined by

$$(9.10) \quad g(v \otimes w) = v \otimes w + q^{-1} \tilde{R}_{2,1}(v \otimes w) - (w, v)1.$$

From the explicit form of $\tilde{R}$ on the basis $\{e_i\}$, one can check that the following hold:

$$(9.11) \quad \begin{aligned} g(y_i \otimes y_j) &= y_i \otimes y_j + q^{-1} y_j \otimes y_i \quad (i > j), \\ g(y_j^* \otimes y_i^*) &= y_j^* \otimes y_i^* + q^{-1} y_i^* \otimes y_j^* \quad (i > j), \\ g(y_i \otimes y_i) &= (1 + q^{-2}) y_i \otimes y_i, \quad g(y_i^* \otimes y_i^*) = (1 + q^{-2}) y_i^* \otimes y_i^*, \\ g(y_j^* \otimes y_i) &= y_j^* \otimes y_i + q^{-1} y_i \otimes y_j^* \quad (i \neq j), \\ g(y_i^* \otimes y_i) &= y_i^* \otimes y_i + y_i \otimes y_i^* + (1 - q^{-2}) \sum_{i>j} y_j \otimes y_j^* - 1. \end{aligned}$$

These give the relations in (9.6) in $T(V)/I$. However, we also have the relations

$$(9.12) \quad \begin{aligned} g(y_j \otimes y_i) &= q^{-1} y_i \otimes y_j + q^{-2} y_j \otimes y_i \quad (i > j), \\ g(y_i^* \otimes y_j^*) &= q^{-1} y_j^* \otimes y_i^* + q^{-2} y_i^* \otimes y_j^* \quad (i > j), \\ g(y_i \otimes y_j^*) &= q^{-1} y_j^* \otimes y_i + q^{-2} y_i \otimes y_j^* \quad (i \neq j), \\ g(y_i \otimes y_i^*) &= y_i^* \otimes y_i + q^{-2} y_i \otimes y_i^* \\ &\quad + (1 - q^{-2}) q^{2\rho_i} \left(\sum_{j=1}^n y_j \otimes y_j^* + \sum_{j>i} q^{-2\rho_j} y_j^* \otimes y_j \right) - q^{2\rho_i}. \end{aligned}$$

The first three identities don't say anything new. So to establish the proposition, it suffices to verify that the relation $g(y_i \otimes y_i^*) = 0$ holds in $C_q(2n)$, that is,

Lemma 9.13. *In $C_q(2n)$ we have*

$$q^{-2} Y_i Y_i^* + Y_i^* Y_i + (1 - q^{-2}) q^{2\rho_i} \left(\sum_{j=1}^n Y_j Y_j^* + \sum_{j>i} q^{-2\rho_j} Y_j^* Y_j \right) = q^{2\rho_i}.$$

Proof. As a shorthand notation we set $a_i = Y_i Y_i^*$, $a'_i = Y_i^* Y_i$, and $b_i = a_i + a'_i$ for $i = 1, \dots, n$. From the relations in $C_q(2n)$ we know that $b_i = (q^{-2} - 1) \sum_{j < i} a_j + 1$, so that $b_{i+1} = (q^{-2} - 1)a_i + b_i$ or

$$(9.14) \quad a_{i+1} + a'_{i+1} = q^{-2}a_i + a'_i.$$

Letting $c_i = q^{-2}a_i + a'_i$, we have

$$c_i - q^2 c_{i+1} = q^{-2}a_i + a'_i - a_{i+1} - q^2 a'_{i+1} \stackrel{1}{=} (1 - q^2)a'_{i+1},$$

where (1) follows from (9.14). Therefore,

$$(9.15) \quad \begin{aligned} c_i &= (1 - q^2)a'_{i+1} + q^2(1 - q^2)a'_{i+2} + \dots + q^{2(n-i-1)}(1 - q^2)a'_n + q^{2(n-i)}c_n \\ &= (q^{-2} - 1)q^{2\rho_i} \left(\sum_{j > i} q^{-2\rho_j} a'_j \right) + q^{2\rho_i} c_n. \end{aligned}$$

On the other hand,

$$(9.16) \quad \begin{aligned} c_n &= q^{-2}a_n + a'_n = (q^{-2} - 1)a_n + a_n + a'_n = (q^{-2} - 1)a_n + b_n \\ &= (q^{-2} - 1)a_n + (q^{-2} - 1) \left(\sum_{j < n} a_j \right) + 1 = (q^{-2} - 1) \left(\sum_{j \leq n} a_j \right) + 1. \end{aligned}$$

The left-hand side of the identity in the statement of the lemma is

$$\begin{aligned} c_i + (1 - q^{-2})q^{-2\rho_i} \left(\sum_{j=1}^n a_j \right) + (1 - q^{-2})q^{2\rho_i} \left(\sum_{j > i} q^{-2\rho_j} a'_j \right) \\ \stackrel{1}{=} c_i + q^{2\rho_i}(1 - c_n) + (1 - q^{-2})q^{2\rho_i} \left(\sum_{j > i} q^{-2\rho_j} a'_j \right) \\ \stackrel{2}{=} q^{2\rho_i} \quad (\text{the right side}), \end{aligned}$$

where (1) is (9.16) and (2) is (9.15). $\square$

The $U_q(\mathbf{D}_4)$ case. Let us return now to our quantum octonions. Consider the two “intertwining operators” Φ^0 and Φ^1 which are the $U_q(\mathbf{D}_4)$ -module homomorphisms

$$\begin{aligned} \Phi^0 : V_\eta \otimes V &\xrightarrow{j \otimes \text{id}} V_{\eta \circ \zeta} \otimes V \xrightarrow{j \circ p \circ (j \otimes j)} V_\zeta, \\ \Phi^1 : V_\zeta \otimes V &\xrightarrow{j \otimes \text{id}} V_{\zeta \circ \eta} \otimes V \xrightarrow{p} V_\eta. \end{aligned}$$

We scale these mappings by setting $\Psi^0 = q^{-\frac{3}{2}}\Phi^0$ and $\Psi^1 = q^{-\frac{3}{2}}\Phi^1$. Let $\mathfrak{S}_q = (\text{id} + q^{-1}\check{R})$, where $\check{R} = \check{R}_{V,V}$, which we know maps $V \otimes V$ onto $\mathcal{S}^2(V)$.

Lemma 9.17 (Compare [DF, Prop. 3.1.2]).

$$\begin{aligned} \Psi^1 \circ (\Psi^0 \otimes \text{id}) \circ (\text{id} \otimes \mathfrak{S}_q) &= \text{id} \otimes (,), \\ \Psi^0 \circ (\Psi^1 \otimes \text{id}) \circ (\text{id} \otimes \mathfrak{S}_q) &= \text{id} \otimes (,). \end{aligned}$$

Proof. It is enough to check the values on $x_1 \otimes x_1 \otimes x_{-1}$:

$$\begin{aligned} &\Psi^1 \circ (\Psi^0 \otimes \text{id}) \circ (\text{id} \otimes \mathfrak{S}_q)(x_1 \otimes x_1 \otimes x_{-1}) \\ &= \Psi^1 \circ (\Psi^0 \otimes \text{id})(x_1 \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1)) \\ &= q^{-\frac{3}{2}}\Psi^1(q^{-\frac{3}{2}}x_4 \otimes x_1) = q^{-3}x_1. \end{aligned}$$

But $(x_1, x_{-1}) = (q^3 + q^{-3})(x_1 | x_{-1}) = q^{-3}$, so both maps agree on $x_1 \otimes x_1 \otimes x_{-1}$. The computation for the other map is left as an exercise. $\square$

Now for any $v \in V$ define

$$(9.18) \quad \begin{aligned} \Psi(v) : V_\zeta \oplus V_\eta &\rightarrow V_\zeta \oplus V_\eta, \\ \Psi(v)(x + y) &= \Psi^0(y \otimes v) + \Psi^1(x \otimes v). \end{aligned}$$

The relations in the previous lemma say that

$$\Psi \circ (\Psi \otimes \text{id}) \circ (\text{id} \otimes \mathfrak{S}_q) = \text{id} \otimes (\cdot, \cdot),$$

so for any basis elements $e_i, e_j \in V$ and $x \in V_\zeta \oplus V_\eta$, we have

$$\begin{aligned} (e_j, e_i)x &= \Psi \circ (\Psi \otimes \text{id}) \circ (\text{id} \otimes \mathfrak{S}_q)(x \otimes e_j \otimes e_i) \\ &= \Psi \circ (\Psi \otimes \text{id})(x \otimes (e_j \otimes e_i + q^{-1}\check{R}(e_j \otimes e_i))) \\ &= \Psi(e_i)\Psi(e_j)(x) + q^{-1} \sum_{k,\ell} (\check{R} \circ \sigma)_{i,j}^{k,\ell} \Psi(e_\ell)\Psi(e_k)(x), \end{aligned}$$

where $(\check{R} \circ \sigma)_{i,j}^{k,\ell}$ is the coordinate matrix of $\check{R} \circ \sigma$ in the basis $\{e_i \otimes e_j\}$. So $\Psi(e_i)$ and $\Psi(e_j)$ satisfy

$$(e_j, e_i)\text{id} = \Psi(e_i)\Psi(e_j) + q^{-1} \sum_{k,\ell} (\check{R} \circ \sigma)_{i,j}^{k,\ell} \Psi(e_\ell)\Psi(e_k).$$

Notice now that

$$\begin{aligned} e_i \otimes e_j + q^{-1}\check{R}_{2,1}(e_i \otimes e_j) &= e_i \otimes e_j + q^{-1}\sigma \circ \check{R} \circ \sigma(e_i \otimes e_j) \\ &= e_i \otimes e_j + q^{-1} \sum_{k,\ell} (\check{R} \circ \sigma)_{i,j}^{k,\ell} e_\ell \otimes e_k, \end{aligned}$$

so the operators $\Psi(v)$ satisfy the relations of the quantum Clifford algebra. This shows that the map

$$(9.19) \quad \begin{aligned} \Psi : V &\rightarrow \text{End}(V_\zeta \oplus V_\eta), \\ v &\mapsto \Psi(v) : x + y \rightarrow q^{-3} \left(j(y \cdot j(v)) + j(x) \cdot v \right), \end{aligned}$$

extends to a representation $\Psi : C_q(8) \rightarrow \text{End}(V_\zeta \oplus V_\eta) = \text{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$.

In [DF], Ding and Frenkel show that $C_q(2n)$ is isomorphic to the classical Clifford algebra, so $C_q(8)$ is a central simple associative algebra of dimension 2^8 . The same is true of $\text{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$. As a result we have

Proposition 9.20. *The map $\Psi : C_q(8) \rightarrow \text{End}(V_\zeta \oplus V_\eta) = \text{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$ in (9.19) is an algebra isomorphism.*

It is worth noting that at $q = 1$ the map in Proposition 9.20 reduces to

$$(9.21) \quad \begin{aligned} \Psi : \mathbb{O} &\rightarrow \text{End}(\mathbb{O} \oplus \mathbb{O}), \\ v &\mapsto \Psi(v) : x + y \mapsto v \cdot \bar{y} + \bar{x} \cdot v, \end{aligned}$$

so

$$\Psi(v)\Psi(v) : x + y \mapsto \Psi(v)(v \cdot \bar{y} + \bar{x} \cdot v) = v \cdot (\bar{v} \cdot x) + (y \cdot \bar{v}) \cdot v = (v|v)_1(x + y)$$

(see Props. 2.21, 4.3). This shows that Ψ extends to the standard Clifford algebra—that is, to the quotient $T(\mathbb{O})/\langle v \otimes v - (v|v)_1 1 \rangle$ of the tensor algebra, in this case as in [KPS].

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