ASYMPTOTICS TOWARD THE PLANAR RAREFACTION WAVE FOR VISCOUS CONSERVATION LAW IN TWO SPACE DIMENSIONS

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ABSTRACT. This paper is concerned with the asymptotic behavior of the solution toward the planar rarefaction wave $r(\frac{x}{t})$ connecting u_+ and u_- for the scalar viscous conservation law in two space dimensions. We assume that the initial data $u_0(x, y)$ tends to constant states $u_{\pm}$ as $x \rightarrow \pm\infty$, respectively. Then, the convergence rate to $r(\frac{x}{t})$ of the solution $u(t, x, y)$ is investigated without the smallness conditions of $|u_+ - u_-|$ and the initial disturbance. The proof is given by elementary L^2 -energy method.

1. INTRODUCTION

We consider the Cauchy problem for the scalar viscous conservation law in two space dimensions:

$$(1.1) \quad u_t + f(u)_x + g(u)_y = \mu \Delta u, \quad (t, x, y) \in R_+ \times R^2,$$

$$(1.2) \quad u(0, x, y) = u_0(x, y),$$

where f and g are smooth functions, and μ is a positive constant. We assume that f is convex, i.e.,

$$(1.3) \quad f''(u) \geq \alpha > 0 \quad \text{for } u \in R,$$

and that the initial data is asymptotically constant:

$$(1.4) \quad u_0(x, y) \rightarrow u_{\pm} \quad \text{as } x \rightarrow \pm\infty \quad \text{for any fixed } y \in R,$$

where $u_{\pm}$ are constants satisfying $u_- < u_+$. The asymptotic behavior as $t \rightarrow \infty$ of the solution is closely related to that of the Riemann problem for the corresponding hyperbolic conservation law in one space dimension:

$$(1.5) \quad r_t + f(r)_x = 0, \quad (t, x) \in (-1, \infty) \times R,$$

$$(1.6) \quad r(-1, x) = r_0^R(x) \equiv \begin{cases} u_- & \text{for } x < 0, \\ u_+ & \text{for } x > 0. \end{cases}$$

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The entropy solution $r(t, x)$ of (1.5), (1.6) is given by

$$(1.7) \quad r(t, x) = \begin{cases} u_- & \text{for } x < f'(u_-)(t+1), \\ (f')^{-1}(\frac{x}{t+1}) & \text{for } f'(u_-)(t+1) \leq x \leq f'(u_+)(t+1), \\ u_+ & \text{for } f'(u_+)(t+1) < x. \end{cases}$$

The function $(t, x, y) \rightarrow r(t, x)$ is called the planar rarefaction wave. In a one dimensional case, the asymptotic behaviors of solutions were originally investigated by Il'in and Oleinik [3]. Harabetian [1] obtained the convergence rate toward the rarefaction wave. Hattori and Nishihara [2] showed more precise behaviors of the solution for the Burgers equation, employing the Hopf-Cole transformation. See also [5], [6], [7], [8], [10].

In a two dimensional case, Xin [9] has first investigated the stability of the planar rarefaction wave. Ito [4] has recently shown the convergence rate toward the planar rarefaction wave. In both papers, the smallness of initial disturbance is essentially assumed. In [4], the rarefaction wave is also assumed to be weak.

Our main purpose in this paper is to show that the solution $u(t, x, y)$ asymptotically behaves as $r(t, x)$ with the same rate as that in [4] without smallness conditions, which improves their results.

Denote $R_+^2 = \{(x, y) \in R^2; x > 0\}$, $R_-^2 = \{(x, y) \in R^2; x < 0\}$ and $D = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. Then, our main theorem is as follows.

Theorem 1. *Suppose that $u_0(x, y) - u_{\pm} \in L^2(R_{\pm}^2) \cap L^1(R_{\pm}^2)$ and $D^\alpha u_0(x, y) \in H^1(R^2)$, $|\alpha| = 1$. Then the problem (1.1), (1.2) has a unique global solution $u(t, x, y)$ satisfying*

$$(1.8) \quad \sup_{y \in R} \|u(t, \cdot, y) - r(t, \cdot)\|_{L^2(R_x)} \leq C(1+t)^{-\frac{1}{4}} \log(2+t),$$

where C is a positive constant depending on u_0 .

Our plan in this paper is as follows. In the next section, we construct a smooth rarefaction wave, which is different from that in [4], and reformulate our problem. In the last two sections, we give the proofs of theorems for the reformulated problems.

2. SMOOTH APPROXIMATION AND PRELIMINARIES

We first introduce the function $\tilde{w}(t, x)$ as a solution to the problem:

$$(2.1) \quad \tilde{w}_t + \tilde{w}\tilde{w}_x = \mu\tilde{w}_{xx}, \quad (t, x) \in (-1, \infty) \times R,$$

$$(2.2) \quad \tilde{w}(-1, x) = \tilde{r}_0^R(x) \equiv f'(r_0^R(x)).$$

The Hopf-Cole transformation gives the information of the properties of $\tilde{w}$. Using $\tilde{w}(t, x)$, we define “the smooth rarefaction wave” $w(t, x)$ as

$$(2.3) \quad w(t, x) = (f')^{-1}(\tilde{w}(t, x))|_{t \geq 0}.$$

According to (1.3), $w(t, x)$ satisfies

$$(2.4) \quad w_t + f(w)_x = \mu w_{xx} + \mu \frac{f'''(w)}{f''(w)} w_x^2, \quad (t, x) \in R_+ \times R,$$

$$(2.5) \quad w(0, x) = w_0(x) \equiv f'(\tilde{w}(0, x)).$$

The properties of the smooth rarefaction wave $w(t, x)$ are stated in the following lemma. From now on, we denote several constants by C or c without confusion.

Lemma 1 (Hattori and Nishihara [2]). *The smooth rarefaction wave $w(t, x)$ given by (2.3) satisfies the following properties:*

- (i) $|w(t, x) - u_{\pm}| \leq C \exp(-c|x|^2)$,
- (ii) $w_x(t, x) > 0$,
- (iii) $\|w_x(t, \cdot)\|_{L^p(R)} \leq (1+t)^{-1+\frac{1}{p}}$, $\|w_{xx}(t, \cdot)\|_{L^p(R)} \leq (1+t)^{-1}$,
- (iv) $\|w(t, \cdot) - r(t, \cdot)\|_{L^p(R)} \leq C(1+t)^{-\frac{p-1}{2p}}$.

Since there is a “forcing term” $\frac{f'''(w)}{f''(w)}w_x^2$ in the equation (2.4), we further introduce the smooth rarefaction wave $U(t, x)$ approximate to w , which satisfies

$$(2.6) \quad U_t + f(U)_x = U_{xx}, \quad (t, x) \in R_+ \times R,$$

$$(2.7) \quad U(0, x) = U_0(x) \equiv f'(\tilde{w}(0, x)).$$

The monotonicity in x of $U(t, x)$ was obtained by Xin [9], which is important in the a priori estimates in §4.

Lemma 2 (Xin [9]). *Suppose that $U_0(x)$ is monotonically increasing:*

$$(2.8) \quad \frac{d}{dt}U_0(x) > 0, \quad x \in R.$$

Then, the solution $U(t, x)$ of (2.6), (2.7) satisfies

$$(2.9) \quad \frac{d}{dx}U(t, x) > 0, \quad (t, x) \in R_+ \times R.$$

Thus, setting

$$\begin{aligned} u(t, x, y) - r(t, x) &= \{w(t, x) - r(t, x)\} + \{U(t, x) - w(t, x)\} + \{u(t, x, y) - U(t, x)\} \\ &\equiv \{w(t, x) - r(t, x)\} + v(t, x) + V(t, x, y), \end{aligned}$$

we have reached two reformulated problems:

$$(2.10) \quad v_t + \{f(w+v) - f(w)\}_x = \mu v_{xx} - \mu \frac{f'''(w)}{f''(w)}w_x^2, \quad (t, x) \in R_+ \times R,$$

$$(2.11) \quad v(0, x) = U_0(x) - w(0, x) \equiv v_0(x),$$

and

$$(2.12) \quad V_t + \{f(U+V) - f(U)\}_x + g(U+V)_y = \mu \Delta V, \quad (t, x, y) \in R_+ \times R^2,$$

$$(2.13) \quad V(0, x, y) = u_0(x, y) - U_0(x) \equiv V_0(x, y).$$

The perturbations v and V satisfy the following theorems, respectively.

Theorem 2 (Decay estimate). *Suppose that $v_0 \in H^2(R) \cap L^1(R)$. Then the problem (2.10), (2.11) has a unique global solution $v(t, x)$ satisfying*

$$v \in C^0([0, \infty); H^2(R)) \cap C^0([0, \infty); L^1(R)),$$

$$v_x \in L^2(0, T; H^2(R)),$$

and

$$(2.14) \quad \|v(t, \cdot)\|_{L^2(R)} \leq C(1+t)^{-\frac{1}{4}} \log(2+t).$$

Theorem 3 (Decay estimate). *Suppose that $V_0 \in H^2(R^2) \cap L^1(R^2)$. Then, the problem (2.12), (2.13) has a unique global solution $V(t, x, y)$ satisfying*

$$V \in C^0([0, \infty); H^2(R)), \quad \nabla V \in L^2(0, \infty; H^2(R)),$$

and

$$(2.15) \quad \sup_{y \in R} \|V(t, \cdot, y)\|_{L^2(R_x)} \leq C(1+t)^{-\frac{3}{4}}.$$

Theorem 2, Theorem 3 and Lemma 1 (iv) yield the desired estimate (1.8). In the next two sections, we devote ourselves to the proofs of Theorems 2 and 3, respectively.

3. DECAY ESTIMATES FOR THE PERTURBATION v

We begin with the Cauchy problem

$$(3.1) \quad v_t + \{f(w+v) - f(w)\}_x = \mu v_{xx} - \mu \frac{f'''(w)}{f''(w)} w_x^2, \quad (t, x) \in R_+ \times R,$$

$$(3.2) \quad v(0, x) = U(0, x) - w(0, x) \equiv v_0(x).$$

We shall show that the problem (3.1), (3.2) has a unique global solution in the solution space $X(0, \infty)$, where

$$X_M(0, T) = \left\{ \psi \left| \begin{array}{l} \psi \in C^0([0, T]; H^2(R)), \quad \psi_x \in L^2(0, T; H^2(R)) \\ \text{and} \quad \sup_{[0, T]} \|\psi(t, \cdot)\|_{H^2} \leq M \end{array} \right. \right\}.$$

In what follows, we often abbreviate the domain R of $H^2(R)$, etc.

Proposition 1 (Local existence). *Suppose that $v_0 \in H^2(R)$. For any $M > 0$, there exists a positive constant T_0 depending on M such that if $\|v_0\|_{H^2} \leq M$, then the problem (3.1), (3.2) has a unique solution $v(t, x) \in X_{2M}(0, T_0)$.*

Proposition 1 can be proved in a standard way. So we omit the proof. Next, we show a priori estimates of v .

Proposition 2 (A priori estimate). *Suppose that v is a solution of (3.1), (3.2) in $X_M(0, T)$ for positive constants T and M . Then there exists a positive constant C_0 such that*

$$(3.3) \quad \|v(t)\|_{H^2}^2 + \int_0^t \int_R w_x v^2 dx d\tau + \int_0^t \|v_x(\tau)\|_{H^2}^2 d\tau \leq C_0(\|v_0\|_{H^2}^2 + 1).$$

Proof. Multiplying (3.1) by v , we have

$$(3.4) \quad \frac{1}{2} \frac{d}{dt} \int_R v^2 dx + \int_R v \{f(w+v) - f(w)\}_x dx + \mu \int_R v_x^2 dx = - \int_R v \frac{f'''(w)}{f''(w)} w_x^2 dx.$$

The second term of (3.4) is estimated by the following:

$$(3.5) \quad \begin{aligned} & \int_R v \{f(w+v) - f(w)\}_x dx = - \int_R v_x \{f(w+v) - f(w)\} dx \\ & = \int_R \left[- \left(\int_w^{w+v} f(y) dy - f(w)v \right)_x + \{f(w+v) - f(w) - f'(w)v\} w_x \right] dx \\ & \geq \frac{\alpha}{2} \int_R w_x v^2 dx. \end{aligned}$$

The right hand side is estimated as follows:

$$\left| \int_R v \frac{f'''(w)}{f''(w)} w_x^2 dx \right| \leq C \int_R w_x^2 |v| dx \leq \frac{\alpha}{4} \int_R w_x |v|^2 dx + C \int_R w_x^3 dx.$$

Integrating (3.4) over $[0, t]$ and using Lemma 1 (iii), we get

$$(3.6) \quad \|v(t)\|^2 + \int_0^t \int_R w_x v^2 dx d\tau + \int_0^t \|v_x(\tau)\|^2 d\tau \leq C_0(\|v_0\|^2 + 1).$$

Here and later, by $\|\cdot\|$ we denote the L^2 -norm in R or R^2 without confusions. Next, we derive the higher order estimates. Multiplying (3.1) by $(-v_{xx})$, we have

$$(3.7) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_R v_x^2 dx - \int_R v_{xx} \{f(w+v) - f(w)\}_x dx + \mu \int_R v_{xx}^2 dx \\ &= \mu \int_R v_{xx} \frac{f'''(w)}{f''(w)} w_x^2 dx. \end{aligned}$$

The right-hand side is estimated as

$$\left| \int_R v_{xx} \frac{f'''(w)}{f''(w)} w_x^2 dx \right| \leq \frac{1}{4} \|v_{xx}\|^2 + C \|w_x\|_{L^4}^4.$$

The second term of (3.7) is estimated as

$$\left| \int_R v_{xx} \{f(w+v) - f(w)\}_x dx \right| \leq \frac{\mu}{4} \|v_{xx}\|^2 + C \left\{ \|v_x(t)\|^2 + \int_R w_x v^2 dx \right\}.$$

Here, the maximum principle for a parabolic equation has been employed. Hence, we have

$$(3.8) \quad \|v_x(t)\|^2 + \int_0^t \|v_{xx}(\tau)\|^2 d\tau \leq C_0(\|v_0\|_{H^1}^2 + 1).$$

Differentiating (3.1) twice in x , and multiplying it by v_{xx} , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v_{xx}(t)\|^2 + \int_R v_{xx} \{f(w+v) - f(w)\}_{xxx} dx + \mu \|v_{xxx}(t)\|^2 \\ &= -\mu \int_R v_{xx} \left(\frac{f'''(w)}{f''(w)} w_x^2 \right)_{xx} dx, \end{aligned}$$

which yields

$$(3.9) \quad \|v_{xx}(t)\|^2 + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau \leq C_0(\|v_0\|_{H^2}^2 + 1).$$

Thus, the proof of Proposition 3 is complete. $\square$

Combining Proposition 1 with Proposition 2, we obtain the global result.

Theorem 4 (Global existence). *Suppose that $v_0(x) \in H^2(R)$. Then the problem (3.1), (3.2) has a unique global solution $v(t, x)$ satisfying*

$$v \in C^0([0, \infty); H^2(R)), \quad v_x \in L^2(0, \infty; H^2(R)),$$

and the estimate (3.3).

In order to obtain the decay order of v , we further assume that $v_0 \in L^1(R)$.

Lemma 3. *Suppose that $v_0 \in L^1(R) \cap H^2(R)$. Then the solution $v(t, x)$ also satisfies*

$$(3.10) \quad \|v(t)\|_{L^1} \leq \|v_0\|_{L^1} + C_1 \log(1+t),$$

where C_1 is a constant depending on $|u_+ - u_-|$.

Proof. The L^1 -estimate (3.10) of v can be proved by the same method as that in [4]. So we omit the proof. $\square$

Theorem 5 (Decay estimate). *Suppose that $v_0 \in H^2(R) \cap L^1(R)$. Then, for any $0 < \varepsilon < \frac{1}{2}$, the solution $v(t, x)$ of (3.1), (3.2) satisfies*

$$(3.11) \quad \begin{aligned} & (1+t)^{k+\frac{1}{2}+\varepsilon} \|\partial_x^k v(t)\|^2 \\ & + \int_0^t (1+\tau)^{k+\frac{1}{2}+\varepsilon} \left(\int_R w_x |\partial_x^k v(\tau)|^2 dx + \|\partial_x^k v_x(\tau)\|^2 \right) d\tau \\ & \leq C I_k (1+t)^\varepsilon \rho_k(t), \quad k = 0, 1, \end{aligned}$$

$$(3.12) \quad \begin{aligned} & (1+t)^{2+\varepsilon} \|\partial_x^2 v(t)\|^2 \\ & + \int_0^t (1+\tau)^{2+\varepsilon} \left(\int_R w_x |\partial_x^2 v(\tau)|^2 dx + \|\partial_x^2 v_x(\tau)\|^2 \right) d\tau \\ & \leq C I_2 (1+t)^\varepsilon \rho_2(t), \end{aligned}$$

where

$$I_k = (\|v_0\|_{L^1} + \|v_0\|_{H^k} + 1)^2, \quad k = 0, 1, \quad \rho_0 = \log^2(2+t), \quad \rho_1 = \log^{10}(2+t),$$

and

$$I_2 = (\|v_0\|_{L^1} + \|v_0\|_{H^2} + 1)^{\frac{70}{3}}, \quad \rho_2 = \log^6(2+t).$$

Remark. *The estimate (3.11) with $k = 0$ shows (2.14) in Theorem 2.*

Proof. The proof is similar to one in Ito [4]. However, the smooth rarefaction wave $w(t, x)$ in [4] is different from ours and its estimates are done for the linearized equation around $w(t, x)$. Hence, we give the outline of the proof.

First, we show (3.11) with $k = 0$. From (3.4) and Lemma 1 (iii), we have

$$(3.13) \quad \frac{d}{dt} \|v(t)\|^2 + \int_R w_x v^2 dx + \|v_x(t)\|^2 \leq C(1+t)^{-2}.$$

Multiplying (3.13) by $(1+t)^{\frac{1}{2}+\varepsilon}$, we have

$$(3.14) \quad \begin{aligned} & \frac{d}{dt} \{ (1+t)^{\frac{1}{2}+\varepsilon} \|v(t)\|^2 \} + (1+t)^{\frac{1}{2}+\varepsilon} \left(\int_R w_x v^2 dx + \|v_x(t)\|^2 \right) \\ & \leq C \{ (1+t)^{-\frac{1}{2}+\varepsilon} \|v(t)\|^2 + (1+t)^{-\frac{3}{2}+\varepsilon} \}. \end{aligned}$$

By the Gagliardo-Nirenberg inequality

$$(3.15) \quad \|v(t)\|^2 \leq C \|v(t)\|_{L^1(R)}^{\frac{4}{3}} \|v_x(t)\|^{\frac{2}{3}},$$

we obtain

$$\begin{aligned}
& \frac{d}{dt} \{ (1+t)^{\frac{1}{2}+\varepsilon} \|v(t)\|^2 \} + (1+t)^{\frac{1}{2}+\varepsilon} \left(\int_R w_x v^2 dx + \|v_x(t)\|^2 \right) \\
& \leq C \{ (1+t)^{-\frac{1}{2}+\varepsilon} \|v(t)\|_{L^1}^{\frac{4}{3}} \|v_x(t)\|^{\frac{2}{3}} + (1+t)^{-\frac{3}{2}+\varepsilon} \} \\
& \leq \frac{1}{2} (1+t)^{\frac{1}{2}+\varepsilon} \|v_x(t)\|^2 + C \{ (1+t)^{-1+\varepsilon} \|v(t)\|_{L^1}^2 + (1+t)^{-\frac{3}{2}+\varepsilon} \} \\
& \leq \frac{1}{2} (1+t)^{\frac{1}{2}+\varepsilon} \|v_x(t)\|^2 + C \{ (1+t)^{-1+\varepsilon} (\|v_0\|_{L^1}^2 + C_1 \log^2(1+t)) + (1+t)^{-\frac{3}{2}+\varepsilon} \};
\end{aligned}$$

that is,

$$\begin{aligned}
(3.16) \quad & \frac{d}{dt} \{ (1+t)^{\frac{1}{2}+\varepsilon} \|v(t)\|^2 \} + (1+t)^{\frac{1}{2}+\varepsilon} \left(\int_R w_x v^2 dx + \|v_x(t)\|^2 \right) \\
& \leq C \{ (1+t)^{-1+\varepsilon} (\|v_0\|_{L^1}^2 + C_1 \log^2(1+t)) + (1+t)^{-\frac{3}{2}+\varepsilon} \}.
\end{aligned}$$

Integrating (3.16) over $[0, t]$ in t , we obtain (3.11) with $k = 0$.

Next, we derive (3.11) with $k = 1$. From (3.7), we have

$$(3.17) \quad \frac{1}{2} \frac{d}{dt} \|v_x(t)\|^2 - \int_R v_{xx} \{ f(w+v) - f(w) \}_x dx + \mu \|v_{xx}(t)\|^2 \leq C(1+t)^{-3}.$$

Here

$$\begin{aligned}
& - \int_R v_{xx} \{ f(w+v) - f(w) \}_x dx \\
& = \frac{1}{2} \int_R f''(w+v) w_x v_x^2 dx + \int_R \left[\frac{1}{2} v_x^3 - v_{xx} \{ f'(w+v) - f'(w) \} w_x \right] dx.
\end{aligned}$$

Hence, due to (1.3), we have

$$\begin{aligned}
(3.18) \quad & \frac{d}{dt} \|v_x(t)\|^2 + \alpha \int_R w_x v_x^2 dx + \|v_{xx}(t)\|^2 \\
& \leq C \left\{ \int_R |v_{xx}| |v| |w_x| dx + \int_R |v_x|^3 dx + (1+t)^{-3} \right\} \\
& \leq \frac{1}{2} \|v_{xx}(t)\|^2 + C \left\{ \int_R w_x^2 v^2 dx + \|v_x(t)\|_{L^3}^3 + (1+t)^{-3} \right\}.
\end{aligned}$$

Multiplying (3.18) by $(1+t)^{\frac{3}{2}+\varepsilon}$, we have

$$\begin{aligned}
(3.19) \quad & \frac{d}{dt} \{ (1+t)^{\frac{3}{2}+\varepsilon} \|v_x(t)\|^2 \} + \alpha (1+t)^{\frac{3}{2}+\varepsilon} \int_R w_x v_x^2 dx + (1+t)^{\frac{3}{2}+\varepsilon} \|v_{xx}(t)\|^2 \\
& \leq C \left\{ (1+t)^{\frac{1}{2}+\varepsilon} \|v_x(t)\|^2 + (1+t)^{\frac{3}{2}+\varepsilon} \int_R w_x^2 v^2 dx \right. \\
& \quad \left. + (1+t)^{\frac{3}{2}+\varepsilon} \|v_x(t)\|_{L^3}^3 + (1+t)^{-\frac{3}{2}+\varepsilon} \right\}.
\end{aligned}$$

Noting that

$$\begin{aligned}
(1+t)^{\frac{3}{2}+\varepsilon} \int_R w_x^2 v^2 dx & \leq (1+t)^{\frac{3}{2}+\varepsilon} \|w_x(t)\|_{L^\infty} \int_R w_x v^2 dx \\
& \leq C(1+t)^{\frac{1}{2}+\varepsilon} \int_R w_x v^2 dx,
\end{aligned}$$

and making use of (3.11) with $k = 0$ and the Gagliardo-Nirenberg inequality

$$(3.20) \quad \|v_x(t)\|_{L^3(R)}^3 \leq C \|v_{xx}(t)\|_{L^2(R)}^{\frac{7}{4}} \|v(t)\|_{L^2(R)}^{\frac{5}{4}},$$

we obtain

$$\begin{aligned} & (1+t)^{\frac{3}{2}+\varepsilon} \|v_x^2(t)\|^2 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon} \left(\alpha \int_R w_x v_x^2 dx + \|v_{xx}(\tau)\|^2 \right) d\tau \\ & \leq C \left\{ I_0(1+t)^\varepsilon \rho_0 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon} \|v(\tau)\|_{L^2(R)}^{10} d\tau \right\} \\ & \leq C \left\{ I_0(1+t)^\varepsilon \rho_0 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon} (I_0(1+\tau)^{-\frac{1}{2}} \rho_0)^5 d\tau \right\}, \end{aligned}$$

which yields (3.11) with $k = 1$. Finally, multiply (3.9) by $(1+t)^{2+\varepsilon}$ and use (3.11). After several calculations, we can obtain the desired estimate (3.12). Though the details are omitted, we cannot multiply (3.9) by $(1+t)^{\frac{5}{2}+\varepsilon}$ in our method. Because we have the decay order $\|w_{xx}(t)\|^2 = O(t^{-2})$, not $O(t^{-\frac{5}{2}})$ (cf. Ito [4]).

Thus the proof is complete. $\square$

4. DECAY ESTIMATES FOR THE PERTURBATION V

In this section, we consider the Cauchy problem in two space dimension:

$$(4.1) \quad V_t + \{f(U+V) - f(U)\}_x + g(U+V)_y = \Delta V,$$

$$(4.2) \quad V(0, x, y) = V_0(x, y) \equiv u_0(x, y) - U_0(x).$$

The solution space is

$$\tilde{X}_M(0, T) = \left\{ \psi \left| \begin{array}{l} \psi \in C^0([0, T]; H^2(R^2)), \quad \nabla \psi \in L^2(0, T; H^2(R^2)) \\ \text{and} \quad \sup_{[0, T]} \|\psi(t, \cdot, \cdot)\|_{H^2} \leq M \end{array} \right. \right\},$$

with $T > 0$. Then we have

Proposition 3 (Local existence). *Suppose that $V_0 \in H^2(R^2)$. For any $M > 0$, there exists a positive constant T_0 depending on M such that if $\|V_0\|_{H^2} \leq M$, then the problem (4.1), (4.2) has a unique solution $V(t, x, y) \in \tilde{X}_{2M}(0, T_0)$.*

Proposition 3 can be proved in a standard way. So we omit the proof. Next, we show a priori estimates of V .

Proposition 4 (A priori estimate). *Suppose that V is a solution of (4.1), (4.2) in $\tilde{X}_M(0, T)$ for positive constants T and M . Then there exists a positive constant C_1 depending on V_0 such that*

$$(4.3) \quad \|V(t)\|_{H^2}^2 + \int_0^t \int_{R^2} U_x V^2 dx dy d\tau + \int_0^t \|\nabla V(\tau)\|_{H^2}^2 d\tau \leq C_1 \|V_0\|_{H^2}^2.$$

Proof. Multiplying (4.1) by V and integrating the resultant equation over R^2 , we have

$$(4.4) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|V(t)\|^2 + \int_{R^2} V \{f(U+V) - f(U)\}_x dx dy \\ & + \int_{R^2} V g(U+V)_y dx dy + \mu \|\nabla V(t)\|^2 = 0. \end{aligned}$$

The second and third terms are, respectively, estimated as follows:

$$\begin{aligned}
 \int_{R^2} V\{f(U+V) - f(U)\}_x dx dy &= - \int_{R^2} V_x\{f(U+V) - f(U)\} dx dy \\
 &= \int_{R^2} \left[- \left(\int_U^{U+V} f(y) dy - f(U)V \right)_x \right. \\
 &\quad \left. + \{f(U+V) - f(U) - f'(U)V\}U_x \right] dx dy \\
 &\geq \frac{\alpha}{2} \int_{R^2} U_x V^2 dx dy.
 \end{aligned}
 \tag{4.5}$$

Since U is independent of y ,

$$\begin{aligned}
 \int_{R^2} Vg(U+V)_y dx dy &= - \int_{R^2} V_y g(U+V) dx dy \\
 &= - \int_{R^2} \partial_y \left(\int_U^{U+V} g(\xi) d\xi \right) dx dy = 0.
 \end{aligned}
 \tag{4.6}$$

Using (4.5) and (4.6), we have the basic estimate

$$\|V(t)\|^2 + \int_0^t \int_{R^2} U_x V^2 dx dy d\tau + \int_0^t \|\nabla V(\tau)\|^2 d\tau \leq C\|V_0\|^2.
 \tag{4.7}$$

The estimates of the derivatives in x, y of V can be obtained similarly to those in Proposition 2. We omit the details. $\square$

The combination of Propositions 3 and 4 gives the global result.

Theorem 6 (Global existence). *Suppose that $V_0(x) \in H^2(R^2)$. Then the problem (4.1), (4.2) has a unique global solution $V(t, x, y)$ satisfying*

$$V \in C^0([0, \infty); H^2(R)), \quad \nabla V \in L^2(0, \infty; H^2(R)).$$

and the estimate (4.3).

We now show the decay estimates on V . As in Lemma 3, the following L^1 -estimate plays an important role.

Lemma 4 (Ito [4]). *Suppose further, in Theorem 6, that $V_0 \in L^1(R^2)$. Then the solution $V(t, x, y)$ also satisfies*

$$\|V(t)\|_{L^1(R^2)} \leq \|V_0\|_{L^1(R^2)}.
 \tag{4.8}$$

Applying Lemma 4, we have the following theorem.

Theorem 7 (Decay estimate). *Suppose that $V_0(x, y) \in H^2(R^2) \cap L^1(R^2)$ and let $V(t, x, y)$ be the solution of (4.1), (4.2). Then, for any $\varepsilon > 0$, there exists a constant $C > 0$ such that the following decay estimates hold:*

$$\begin{aligned}
 &(1+t)^{1+\varepsilon} \|V(t)\|^2 \\
 &+ \int_0^t (1+\tau)^{1+\varepsilon} \left(\int_{R^2} U_x |V(\tau)|^2 dx dy + \|\nabla V(\tau)\|^2 \right) d\tau \\
 &\leq C(1+t)^\varepsilon (\|V_0\|_{L^1} + \|V_0\|)^2,
 \end{aligned}
 \tag{4.9}$$

$$\begin{aligned}
(4.10) \quad & (1+t)^{\frac{15}{8}+\varepsilon} \|V_x(t)\|^2 \\
& + \int_0^t (1+\tau)^{\frac{15}{8}+\varepsilon} \left(\int_{R^2} U_x |V_x(\tau)|^2 dx dy + \|\nabla V_x(\tau)\|^2 \right) d\tau \\
& \leq C(1+t)^\varepsilon \log^4(2+t) (\|V_0\|_{L^1} + \|V_0\|_{H^1})^2,
\end{aligned}$$

$$\begin{aligned}
(4.11) \quad & (1+t)^{2+\varepsilon} \|V_y(t)\|^2 \\
& + \int_0^t (1+\tau)^{2+\varepsilon} \left(\int_{R^2} U_x |V_y(\tau)|^2 dx dy + \|\nabla V_y(\tau)\|^2 \right) d\tau \\
& \leq C(1+t)^\varepsilon (\|V_0\|_{L^1} + \|V_0\|_{H^1})^2,
\end{aligned}$$

$$\begin{aligned}
(4.12) \quad & (1+t)^{\frac{39}{16}+\varepsilon} \|V_{xx}(t)\|^2 \\
& + \int_0^t (1+\tau)^{\frac{39}{16}+\varepsilon} \left(\int_{R^2} U_x |V_{xx}(\tau)|^2 dx dy + \|\nabla V_{xx}(\tau)\|^2 \right) d\tau \\
& \leq C(1+t)^\varepsilon \log^8(2+t) (\|V_0\|_{L^1} + \|V_0\|_{H^2})^2,
\end{aligned}$$

$$\begin{aligned}
(4.13) \quad & (1+t)^{\frac{23}{8}+\varepsilon} \|V_{xy}(t)\|^2 \\
& + \int_0^t (1+\tau)^{\frac{23}{8}+\varepsilon} \left(\int_{R^2} U_x |V_{xy}(\tau)|^2 dx dy + \|\nabla V_{xy}(\tau)\|^2 \right) d\tau \\
& \leq C(1+t)^\varepsilon \log^8(2+t) (\|V_0\|_{L^1} + \|V_0\|_{H^2})^2,
\end{aligned}$$

$$\begin{aligned}
(4.14) \quad & (1+t)^{3+\varepsilon} \|V_{yy}(t)\|^2 \\
& + \int_0^t (1+\tau)^{3+\varepsilon} \left(\int_{R^2} U_x |V_{yy}(\tau)|^2 dx dy + \|\nabla V_{yy}(\tau)\|^2 \right) d\tau \\
& \leq C(1+t)^\varepsilon (\|V_0\|_{L^1} + \|V_0\|_{H^2})^2.
\end{aligned}$$

Remark. From (4.9) and (4.11), the estimate (2.15) in Theorem 3 is obtained as

$$\begin{aligned}
\sup_y \|V(t, \cdot, y)\|^2 & \leq C \|V(t, \cdot, \cdot)\| \|V_y(t, \cdot, \cdot)\| \\
& \leq C(1+t)^{-\frac{1}{2}-1} = C(1+t)^{-\frac{3}{2}}. \quad \square
\end{aligned}$$

Proof. From (4.4)–(4.6), we get

$$(4.15) \quad \frac{1}{2} \frac{d}{dt} \|V(t)\|^2 + \frac{1}{2} \alpha \int_{R^2} U_x V^2 dx dy + \mu \|\nabla V(t)\|^2 \leq 0.$$

Multiplying (4.15) by $2(1+t)^{1+\varepsilon}$, we have

$$\begin{aligned}
(4.16) \quad & \frac{d}{dt} \{(1+t)^{1+\varepsilon} \|V(t)\|^2\} + (1+t)^{1+\varepsilon} \left(\alpha \int_{R^2} U_x V^2 dx dy + 2 \|\nabla V(t)\|^2 \right) \\
& \leq (1+\varepsilon)(1+t)^\varepsilon \|V(t)\|^2.
\end{aligned}$$

By the Gagliardo-Nirenberg inequality

$$\|V(t)\|_{L^2(R^2)}^2 \leq C \|V(t)\|_{L^1(R^2)} \|\nabla V(t)\|_{L^2(R^2)},$$

we obtain

$$\begin{aligned}
 (4.17) \quad & \frac{d}{dt} \{(1+t)^{1+\varepsilon} \|V(t)\|^2\} + (1+t)^{1+\varepsilon} \left(\alpha \int_{R^2} U_x V^2 dx dy + 2 \|\nabla V(t)\|^2 \right) \\
 & \leq C(1+t)^{\frac{1+\varepsilon}{2}} \|\nabla V(t)\| (1+t)^{\frac{\varepsilon-1}{2}} \|V(t)\|_{L^1(R^2)} \\
 & \leq (1+t)^{1+\varepsilon} \|\nabla V(t)\|^2 + C(1+t)^{\varepsilon-1} \|V(t)\|_{L^1(R^2)}^2 \\
 & \leq (1+t)^{1+\varepsilon} \|\nabla V(t)\|^2 + C(1+t)^{\varepsilon-1} \|V_0\|_{L^1(R^2)}^2.
 \end{aligned}$$

Integrating (4.17) over $[0, t]$ in t , we obtain (4.9). Next, we estimate V_y and V_x . First, multiplying $\frac{\partial}{\partial y}(4.1)$ by V_y , we have

$$\begin{aligned}
 (4.18) \quad & \frac{1}{2} \frac{d}{dt} \|V_y(t)\|^2 + \int_{R^2} V_y \{f(U+V) - f(U)\}_{xy} dx dy \\
 & + \int_{R^2} V_y g(U+V)_{yy} dx dy + \mu \|\nabla V_y(t)\|^2 = 0.
 \end{aligned}$$

The integration by parts gives:

The second and third terms of (4.18)

$$= \frac{1}{2} \int_{R^2} f''(U+V) U_x V_y^2 dx dy + \frac{1}{2} \int_{R^2} \{f''(U+V) V_x V_y^2 + g''(U+V) V_y^3\} dx dy.$$

Hence,

$$\begin{aligned}
 (4.19) \quad & \frac{1}{2} \frac{d}{dt} \|V_y(t)\|^2 + \frac{\alpha}{2} \int_{R^2} U_x V_y^2 dx dy + \mu \|\nabla V_y(t)\|^2 \\
 & \leq C \int_{R^2} (|V_y|^3 + |V_x| |V_y|^2) dx dy.
 \end{aligned}$$

Since

$$\begin{aligned}
 & C \int_{R^2} |V_x| |V_y|^2 dx dy \\
 & \leq C \int_R \sup_{y \in R} |V_x(t, x, y)| \|V_y(t, x, \cdot)\|_{L^2(R_y)}^2 dx \\
 & \leq C \int_R \|V_x(t, x, \cdot)\|_{L^2(R_y)}^{\frac{1}{2}} \|V_{xy}(t, x, \cdot)\|_{L^2(R_y)}^{\frac{1}{2}} \|V_y(t, x, \cdot)\|_{L^2(R_y)}^2 dx \\
 & \leq \frac{\mu}{4} \|V_{xy}(t)\|_{L^2(R^2)}^2 + C \int_R \|V_x(t, x, \cdot)\|_{L^2(R_y)}^{\frac{2}{3}} \|V_y(t, x, \cdot)\|_{L^2(R_y)}^{\frac{8}{3}} dx \\
 & \leq \frac{\mu}{4} \|V_{xy}(t)\|_{L^2(R^2)}^2 \\
 & \quad + C \int_R \|V_x(t, x, \cdot)\|_{L^2(R_y)}^{\frac{2}{3}} \|V(t, x, \cdot)\|_{L^2(R_y)}^{\frac{4}{3}} \|V_{yy}(t, x, \cdot)\|_{L^2(R_y)}^{\frac{4}{3}} dx \\
 & \leq \frac{\mu}{4} \|\nabla V_y(t)\|_{L^2(R^2)}^2 + C \int_R \|V_x(t, x, \cdot)\|_{L^2(R_y)}^2 \|V(t, x, \cdot)\|_{L^2(R_y)}^4 dx \\
 & \leq \frac{\mu}{4} \|\nabla V_y(t)\|_{L^2(R^2)}^2 + C \sup_{x \in R} \|V(t, x, \cdot)\|_{L^2(R_y)}^4 \|V_x(t)\|_{L^2(R^2)}^2 \\
 & \leq \frac{\mu}{4} \|\nabla V_y(t)\|_{L^2(R^2)}^2 + C \|V(t)\|_{L^2(R_y)}^2 \|V_x(t)\|_{L^2(R^2)}^4
 \end{aligned}$$

and

$$C \int_{R^2} |V_y|^3 dx dy \leq \frac{\mu}{4} \|\nabla V_y(t)\|_{L^2(R^2)}^2 + C \|V(t)\|_{L^2(R^2)}^2 \|V_x(t)\|_{L^2(R^2)}^2 \|V_y(t)\|_{L^2(R^2)}^2,$$

we have

$$(4.20) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|V_y(t)\|^2 + \frac{\alpha}{2} \int_{R^2} U_x V_y^2 dx dy + \frac{\mu}{2} \|\nabla V_y(t)\|^2 \\ & \leq C \|V(t)\|_{L^2(R^2)}^2 \|V_x(t)\|_{L^2(R^2)}^2 (\|V_x(t)\|_{L^2(R^2)}^2 + \|V_y(t)\|_{L^2(R^2)}^2). \end{aligned}$$

Noting that $\|V(t)\|_{L^2(R^2)}^2 \leq C(1+t)^{-1}$, we multiply (4.20) by $(1+t)^{2+\varepsilon}$ and integrating it over $[0, t]$ to obtain (4.11). Second, multiplying $\frac{\partial}{\partial x}(4.1)$ by V_x . Then, after similar calculations to the above, we have

$$(4.21) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|V_x(t)\|^2 + \frac{\alpha}{2} \int_{R^2} U_x |V_x|^2 dx dy + \frac{\mu}{2} \|\nabla V_x(t)\|^2 \\ & \leq C \|V(t)\|_{L^2(R^2)}^2 \|V_x(t)\|_{L^2(R^2)}^2 (\|V_x(t)\|_{L^2(R^2)}^2 + \|V_y(t)\|_{L^2(R^2)}^2) \\ & \quad + C \|U_x(t)\|_{L^\infty} \int_{R^2} U_x V^2 dx dy. \end{aligned}$$

Since $\|U_x(t)\|_{L^\infty(R)} \leq \|w_x(t)\|_{L^\infty(R)} + \|v_x(t)\|_{L^\infty(R)} \leq C(1+t)^{-\frac{7}{8}} \log^4(2+t)$ by virtue of Theorem 5, we can multiply (4.21) by $(1+t)^{\frac{15}{8}+\varepsilon}$, not $(1+t)^{2+\varepsilon}$, to obtain (4.10). The estimates (4.12)–(4.14) for the second derivatives of V are obtained by more complicated calculations than those for the first derivatives. We omit the details.

Thus the proof of Theorem 7 is complete. $\square$

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REFERENCES

- [1] E. Harabetian, *Rarefactions and large time behavior for parabolic equations and monotone schemes*, Comm. Math. Phys. **114** (1988), 527–536. MR **89d**:35084
- [2] Y. Hattori and K. Nishihara, *A note on the stability of the rarefaction wave of the Burgers equation*, Japan J. Indust. Appl. Math. **8** (1991), 85–96. MR **91k**:35227
- [3] A. M. Il'in and O. A. Oleinik, *Asymptotic behavior of solutions of the Cauchy problem for certain quasilinear equations for large time* (Russian), Mat. Sb. **51** (1960), 191–216. MR **22**:11222
- [4] K. Ito, *Asymptotic decay toward the planar rarefaction waves of solutions for viscous conservation laws in several space dimensions*, Math. Models Methods Appl. Sci. **6** (1996), 315–338. MR **97e**:35106
- [5] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. **3** (1986), 1–13. MR **88e**:35173
- [6] A. Matsumura and K. Nishihara, *Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas*, Comm. Math. Phys. **144** (1992), 325–335. MR **93d**:76056
- [7] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction wave of the solutions of Burgers' equation with nonlinear degenerate viscosity*, Nonlinear Analysis **23** (1994), 605–614. MR **95m**:35168
- [8] A. Szeppeszy and K. Zumbrun, *Stability of rarefaction waves in viscous media*, Arch. Rational Mech. Anal. **133** (1996), 249–298. MR **97a**:35192

- [9] Z. P. Xin, *Asymptotic stability of planar rarefaction waves for viscous conservation laws in several dimensions*, Trans. Amer. Math. Soc. **319** (1990), 805-820. MR **90j**:35138
- [10] Z. P. Xin, *Asymptotic stability of rarefaction waves for 2×2 viscous hyperbolic conservation laws—the two-mode case*, J. Differential Equations **78** (1989), 191-219. MR **90h**:35155

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