K -THEORY OF PROJECTIVE STIEFEL MANIFOLDS

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ABSTRACT. Using the Hodgkin spectral sequence we calculate $K^*(X_{m,k})$, the complex K -theory of the projective Stiefel manifold $X_{m,k}$, for mk even. For mk odd, we are only able to calculate $K^0(X_{m,k})$, but this is sufficient to determine the order of the complexified Hopf bundle over $X_{m,k}$.

1. INTRODUCTION

In this paper we extend the calculation of $K^*(X_{m,k})$, the complex K -theory ring of projective Stiefel manifolds, begun by Antoniano, Gitler, Ucci and Zvengrowski in [2]. Prior to the appearance of [2], $K^*(X_{m,k})$ had been calculated for the real projective spaces $X_{m,1}$ ([1]) and projective orthogonal groups ([9]).

The work of Hodgkin [10], Roux [12] and, subsequently, Antoniano, Gitler, Ucci and Zvengrowski [2] emphasized the importance of the Hodgkin spectral sequence in calculating $K^*(G/H)$ where G is a compact Lie group with $\pi_1(G)$ torsion free. Hodgkin laid the foundations for later work on the K -theory of homogeneous spaces and showed how to calculate $K^*(G/H)$ in various cases of interest. In [12] Roux dealt with the case of Stiefel manifolds viewed as $Spin(m)/Spin(m-k)$ (see also [7]). For the projective Stiefel manifolds $X_{4n,2s-1}$, the case treated in [2], the subgroup H of $Spin(4n)$ is isomorphic to the group $Z/2 \times Spin(4n-2s+1)$.

In the general case considered in the present article $X_{m,k} = Spin(m)/H$, where H contains $Spin(m-k)$ as a subgroup of index two but in general is not a product. To calculate $K^*(X_{m,k})$ the representation ring RH is computed (§4, Theorem 2) together with the restriction homomorphism $RSpin(m) \rightarrow RH$. From this it can be deduced that the Hodgkin spectral sequence collapses (§7, Theorems 4 and 5) when mk is even.

An important application is that the order of the complexified Hopf bundle associated to the double covering $V_{m,k} \rightarrow X_{m,k}$ may be calculated for all m and k (this order has been obtained in [2] only for $m = 4n$). Consequently, non-immersion results for projective Stiefel manifolds may be deduced as in [3], as well as alternative (somewhat shorter) proofs for the non-parallelizability results of [2].

2. PROJECTIVE STIEFEL MANIFOLDS AS HOMOGENEOUS SPACES

The projective Stiefel manifold $X_{m,k}$ is the quotient of the Stiefel manifold $V_{m,k}$ (of orthonormal k -frames in R^m) by the involution which takes a k -frame $\{v_1, \dots, v_k\}$ to its opposite $\{-v_1, \dots, -v_k\}$. For $1 \leq k < m$, which we henceforth assume, the special orthogonal group $SO(m)$ acts transitively on $X_{m,k}$ with

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isotropy subgroup $K_{m,k}$ consisting of all matrices of $SO(m)$ of the form $\pm I \oplus A$, where $I = I_k$ is the $(k \times k)$ identity matrix and $A \in O(m-k)$.

Let $p : Spin(m) \rightarrow SO(m)$ be the 2-fold covering map. Then

$$X_{m,k} = Spin(m)/H_{m,k},$$

where $H_{m,k} = p^{-1}(K_{m,k})$ and $Spin(m-k)$ a subgroup of index 2 of $H_{m,k}$, is generated in the Clifford algebra by the last $(m-k)$ vectors of the canonical basis $\{e_1, \dots, e_m\}$ of R^m . We have the commutative diagram

$$\begin{array}{ccccccc} Spin(m-k-1) & \longrightarrow & H_{m,k+1} & \longrightarrow & Spin(m)d & \longrightarrow & X_{m,k+1} \\ \downarrow & & \downarrow & & & & \downarrow \\ Spin(m-k) & \longrightarrow & H_{m,k} & \longrightarrow & Spin(m) & \longrightarrow & X_{m,k}d \\ \downarrow p & & \downarrow p & & \downarrow p & & \\ SO(m-k) & \longrightarrow & K_{m,k} & \longrightarrow & SO(m) & \longrightarrow & X_{m,k} \end{array}$$

The manifolds $X_{m,k}$ and $X_{m,k+1}$ are related as follows: if $Spin(m)/H_{m,k+1}$ is the corresponding expression for $X_{m,k+1}$, then $H_{m,k+1} \subset H_{m,k}$ and $Spin(m-k) \cap H_{m,k+1} = Spin(m-k-1)$.

The following description of $H_{m,k}$ will be useful. Let ω be an element of $H_{m,k}$ that is not contained in $Spin(m-k)$, so that $H_{m,k} = Spin(m-k) \sqcup \omega Spin(m-k)$. If at least one of m or k is even, ω can be chosen to lie in the center of $H_{m,k}$. In the Clifford algebra, we choose $\omega = e_1 \cdots e_m$ if m is even, and $\omega = e_1 \cdots e_k$ if k is even and m odd. Note that $\omega^2 = \pm 1$, depending on the values of m and $k \pmod 4$. In general

$$(e_1 \cdots e_r)^2 = \begin{cases} +1 & \text{if } r \equiv 0, 3 \pmod 4, \\ -1 & \text{if } r \equiv 1, 2 \pmod 4. \end{cases}$$

Let Ω be the subgroup of $H_{m,k}$ generated by ω . If $\omega^2 = +1$, then $\Omega = Z/2$ and $H_{m,k} \approx \Omega \times Spin(m-k)$. If both m and k are odd, ω cannot be chosen to lie in the center of $H_{m,k}$, and in this case we will choose $\omega = e_1 \cdots e_{k+1}$. Then ω lies in the center of $H_{m,k+1}$, and $\omega^2 = \pm 1$.

3. THE COMPLEX REPRESENTATION RING $RSpin(m)$

Next we summarize the necessary facts on the complex representation ring of $Spin(m)$ from [4] and [11].

Let $m = 2n$ or $2n+1$, and write T^n for the n -fold product $S^1 \times \cdots \times S^1$, where S^1 stands as usual for the circle group in the complex plane.

Let z_i be the character given by projection of T^n onto the i th factor. Then

$$RT^n = Z[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}].$$

Let $\tau : T^n \rightarrow Spin(m)$ be the homomorphism given by

$$\tau(e^{i\theta_1}, \dots, e^{i\theta_n}) = (\cos \theta_1 + e_1 e_2 \sin \theta_1) \cdots (\cos \theta_n + e_{2n-1} e_{2n} \sin \theta_n)$$

(product in the Clifford algebra). Then τ maps T^n onto the maximal torus $T(n)$ of $Spin(m)$. Clearly $\tau(z_1, \dots, z_n) = \pm 1$ if and only if each $z_i = \pm 1$, so τ covers its image 2^{n-1} times.

In $RSpin(m)[t]$ we define

$$\Lambda[t] = \sum_{i=0}^m t^i \Lambda^i = (1+t)^\epsilon \prod_{i=1}^n (1 + tz_i^2)(1 + tz_i^{-2})$$

(where $\epsilon = 0$ for $m = 2n$ and $\epsilon = 1$ for $m = 2n + 1$), and

$$\Delta_n[t] = \prod_{i=1}^n (z_i + tz_i^{-1}).$$

$\Lambda[t]$ is the character of the total exterior power representation which factors through the double cover $Spin(m) \rightarrow SO(m)$. As for $\Delta_n[t]$, putting $t = 1$, we have the spin-representation Δ_n in $RSpin(2n + 1)$ and the sum of the half-spin representations $\Delta_n^+ + \Delta_n^- = \Delta_n$ in $RSpin(2n)$ with

$$2^\epsilon \Delta_n^2 = 2^\epsilon \prod_{i=1}^n (z_i + z_i^{-1})^2 = 2^\epsilon \prod_{i=1}^n (1 + z_i^2)(1 + z_i^{-2}) = \Lambda[1].$$

Putting $t = -1$, we have $\chi = \Delta_n^+ - \Delta_n^-$ in $RSpin(2n)$.

Let $\epsilon : RSpin(m) \rightarrow Z$ be the augmentation which assigns to each representation its dimension, so that $\epsilon(\Lambda^i) = \binom{m}{i}$, $\epsilon(\Delta_n) = 2^n$, $\epsilon(\Delta_n^\pm) = 2^{n-1}$. Later on, when working with the Koszul resolution, it will be preferable to replace the exterior powers by the augmentation zero K -Pontrjagin classes

$$\pi[t] = \sum_{i \geq 0} t^i \pi_i = \prod_{i=1}^n (1 + t(z_i - z_i^{-1})^2).$$

Note that t^n is the highest power of t occurring in this product; so $\pi_i = 0$ for $i > n$.

One has

$$\Lambda[t] = (1+t)^m \prod_{i=1}^n \left(1 + (z_i - z_i^{-1})^2 \frac{t}{(1+t)^2} \right) = (1+t)^m \pi \left[\frac{t}{(1+t)^2} \right].$$

Let $\delta_n = \Delta_n - 2^n$ and $\delta_n^\pm = \Delta_n^\pm - 2^{n-1}$. We have (see [4] and [11])

Theorem 1. *With the above notation,*

(i) $RSpin(2n + 1)$ is the polynomial ring $Z[\Lambda^1, \dots, \Lambda^{n-1}, \Delta_n]$ and is also the polynomial ring $Z[\pi_1, \dots, \pi_{n-1}, \delta_n]$. One has

$$\Delta_n^2 = \Lambda^n + \Lambda^{n-1} + \dots + \Lambda^1 + 1 = \pi_n + 4\pi_{n-1} + 16\pi_{n-2} + \dots + 2^{2n}.$$

(ii) $RSpin(2n)$ is the polynomial ring $Z[\Lambda^1, \dots, \Lambda^{n-2}, \Delta_n^+, \Delta_n^-]$ and is also the polynomial ring $Z[\pi_1, \dots, \pi_{n-2}, \delta_n^+, \delta_n^-] = Z[\pi_1, \dots, \pi_{n-2}, \chi, \delta_n^+]$. One has

$$\Delta_n^2 = \pi_n + 4\pi_{n-1} + 16\pi_{n-2} + \dots + 2^{2n}, \Delta_n^+ \Delta_n^- = \Lambda^{n-1} + \Lambda^{n-3} + \Lambda^{n-5} + \dots$$

and $\chi^2 = \pi_n$.

(iii) Restriction $Res : RSpin(2n + 1) \rightarrow RSpin(2n)$ takes $\Lambda[t]$ to $(1+t)\Lambda[t]$, Δ_n to $\Delta_n^+ + \Delta_n^-$, and $\pi[t]$ to $\pi[t]$.

(iv) Restriction $Res : RSpin(2n) \rightarrow RSpin(2n - 1)$ takes $\Lambda[t]$ to $(1+t)\Lambda[t]$, $\Delta_n^\pm$ to Δ_{n-1} , and $\pi[t]$ to $\pi[t]$.

4. THE RING $RH_{m,k}$

Henceforth we shall write H instead of $H_{m,k}$ when m and k are clear from the context.

We next calculate RH in terms of $R\Omega$ and $RSpin(m-k)$. Recall that $m = 2n$ or $2n+1$. We let $m-k = 2c$ or $2c+1$ and $s = n-c$. Thus, for $m = 2n$, $k = 2s$ or $2s-1$, whereas, for $m = 2n+1$ we have $k = 2s$ or $2s+1$. Note that $c = [(m-k)/2]$ always.

4.1. RH for $m-k$ odd. If $m = 2n$ is even, set $\omega = e_1 e_2 \cdots e_m$ as in §2 and define $\tilde{\omega} \in T^n$ to be $(i, \cdot_{(n)}, i)$. If $m = 2n+1$ is odd (so $k = 2s$ is even), set $\omega = e_1 \cdots e_k$ as in §2 and define $\tilde{\omega} \in T^n$ to be $(i, \cdot_{(s)}, i, 1, \dots, 1)$. In each case $\tau(\tilde{\omega}) = \omega$. Let $\tilde{\Omega}$ be the subgroup generated by $\tilde{\omega}$. Then $\tilde{\Omega} \approx Z/4$. Since ω lies in the center of H , multiplication in H induces an epimorphism $\tilde{\Omega} \times Spin(2c+1) \rightarrow H$ which extends the inclusion $Spin(2c+1) \subset H$ and takes $\tilde{\omega}$ to ω , with kernel

$$K = \{(\tilde{\omega}^i, x) : \omega^i = 1\} = \begin{cases} \{(1, 1), (\tilde{\omega}^2, 1)\} & \text{if } \omega^2 = 1, \\ \{(1, 1), (\tilde{\omega}^2, -1)\} & \text{if } \omega^2 = -1. \end{cases}$$

This induces an inclusion $RH \subset R(\tilde{\Omega} \times Spin(2c+1)) = R\tilde{\Omega} \otimes RSpin(2c+1)$, and we can identify a representation in $R\tilde{\Omega} \otimes RSpin(2c+1)$ as coming from RH if it is trivial on K .

Now, $R\tilde{\Omega} = R(Z/4) = Z[\phi]/(\phi^4 = 1)$, where $\phi(\tilde{\omega}) = i$, and $R(H/Spin(2c+1)) = R(Z/2) = Z[\theta]/(\theta^2 = 1)$. The composite $\tilde{\Omega} \xrightarrow{\tau} H \rightarrow H/Spin(2c+1)$ identifies θ with ϕ^2 . When working in $R(\tilde{\Omega} \times Spin(2c+1))$ we will often drop the symbol $\otimes$ and write for example, ϕ , ρ , $\phi\rho$ instead of $\phi \otimes 1$, $1 \otimes \rho$, $\phi \otimes \rho$. Note that RH contains θ coming from $H \rightarrow H/Spin(2c+1)$ as well as the exterior powers $\Lambda^j = 1 \otimes \Lambda^j$, since $\Lambda^j(-1) = I$. As for Δ_c , we have $\Delta_c(-1) = -I$. Thus for $\omega^2 = 1$ the element $1 \otimes \Delta_c$ lies in RH and for $\omega^2 = -1$ the element $\phi \otimes \Delta_c$ lies in RH . In either case, we refer to this element as $\bar{\Delta}_c$ (note that $\bar{\Delta}_c = \phi^n \otimes \Delta_c$ when $m = 2n$, whereas $\bar{\Delta}_c = \phi^s \otimes \Delta_c$ when $m = 2n+1$ and $k = 2s$).

Proposition 1. *Let $m-k$ odd ($m-k = 2c+1$, $c \geq 0$). Then RH is the subring of $R\tilde{\Omega} \otimes RSpin(2c+1)$ generated by the elements $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$.*

Proof. We have seen above that RH contains the elements listed above. Conversely, arguing as on p.235 of [4], we prove that RH is the subring fixed by a certain automorphism of $R\tilde{\Omega} \otimes RSpin(2c+1)$. The element $(\tilde{\omega}^2, \omega^2)$ lies in the center of $\tilde{\Omega} \otimes Spin(2c+1)$, and so translation by $(\tilde{\omega}^2, \omega^2)$ gives rise to a ring involution $(\tilde{\omega}^2, \omega^2)^*$ on $R(\tilde{\Omega} \otimes Spin(2c+1))$ which takes a typical irreducible representation ρ to $\pm\rho$, the sign depending on whether $\rho(\tilde{\omega}^2, \omega^2)$ is $\pm I$. Since $(\tilde{\omega}^2, \omega^2)$ maps to 1 in H , this involution fixes RH elementwise and corresponds to $(\tilde{\omega}^2)^* \otimes (\omega^2)^*$ on $R\tilde{\Omega} \otimes RSpin(2c+1)$. The latter fixes $\theta, \Lambda^1, \dots, \Lambda^{c-1}, \bar{\Delta}_c$ and takes ϕ to $-\phi$. Hence $Fix((\tilde{\omega}^2)^* \otimes (\omega^2)^*)$ is generated by $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$. It follows that $RH = Fix((\tilde{\omega}^2)^* \otimes (\omega^2)^*)$ and therefore that RH is generated by the elements listed above. $\square$

4.2. RH for m and k both even. There is a similar result when both m and k are even. In this case $\omega = e_1 \cdots e_m$ and we define $\bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$ in RH to be $1 \otimes \Delta_c^+$ and $1 \otimes \Delta_c^-$ respectively if $\omega^2 = +1$, and $\phi \otimes \Delta_c^+$ and $\phi \otimes \Delta_c^-$ respectively if $\omega^2 = -1$.

Proposition 2. *Let m and k both be even ($m - k = 2c > 0$). Then RH is the subring of $R\tilde{\Omega} \otimes RSpin(2c)$ generated by $\theta, \Lambda^1, \dots, \Lambda^{c-2}, \bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$.*

Proof. As before, the elements listed above belong to RH . Conversely, arguing as in Proposition 1, one has that $Fix((\tilde{\omega}^2)^* \otimes (\omega^2)^*)$ is generated by $\theta, \Lambda^1, \dots, \Lambda^{c-2}, \bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$, as we wanted. $\square$

4.3. RH for m and k both odd. Let m and k be odd, so that $m - k = 2c, c > 0$. In this case $X_{m,k}$ is not orientable and $H_{m,k}$ cannot be expressed as a quotient of $\tilde{\Omega} \times Spin(2c)$. However $RH_{m,k}$ can be calculated in terms of $RSpin(2c)$ and $RH_{m,k+1}$. Since $m - (k + 1) = 2c - 1$ is odd, $RH_{m,k+1}$ is known from Proposition 1. The argument we use is based on the calculation of $RO(2n)$ at the end of [4]. Recall that ω is now $e_1 \cdots e_{k+1}$, an element in the center of $H_{m,k+1}$ (but not of $H_{m,k}$). There is a commutative diagram of restriction homomorphisms

$$\begin{array}{ccc}
 & RSpin(2c) & \\
 Res_s \nearrow & & \searrow Res \\
 RH_{m,k} & & RSpin(2c-1) \\
 Res_0 \searrow & & \nearrow Res \\
 & RH_{m,k+1} &
 \end{array}$$

Remark 1. The reason for choosing the subgroups $Spin(2c)$ and $H_{m,k+1}$ is that, if $T(c)$ and $T(c-1)$ are “the” maximal tori of $Spin(2c)$ and $Spin(2c-1)$ respectively, then $T(c)$ and $\Omega T(c-1) = \Omega \times T(c-1)$ are Cartan subgroups of $H_{m,k}$ contained in $Spin(2c)$ and $H_{m,k+1}$ respectively.

Now $H_{m,k}$ has two connected components and so possesses exactly two conjugacy classes of Cartan subgroups. From this it follows (see [4]) that the homomorphism

$$(Res_s, Res_0) : RH_{m,k} \longrightarrow RSpin(2c) \oplus RH_{m,k+1} \subset RT(c) \oplus R(\Omega T(c-1))$$

is a monomorphism. We will use the above to calculate $RH_{m,k}$ in terms of generators and relations. The generators in question are $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$, defined as follows. The element θ comes from $H_{m,k} \rightarrow H_{m,k}/Spin(2c)$, and $\Lambda^1, \dots, \Lambda^{c-1}$ are the exterior power coming from $H_{m,k} \rightarrow H_{m,k}/\pm 1$, where $H_{m,k}/\pm 1 = K_{m,k} \supset SO(2c)$. Finally, $\bar{\Delta}_c$ is the representation of $H_{m,k}$ induced from Δ_c^+ (or equivalently Δ_c^-). The elements $\theta, \Lambda^1, \dots, \Lambda^{c-1}, \bar{\Delta}_c$ of $RH_{m,k}$ restrict to $1, \Lambda^1, \dots, \Lambda^{c-1}, \Delta_c$ in $RSpin(2c)$, and to $\theta, \Lambda^1 + \theta\Lambda^0, \dots, \Lambda^{c-1} + \theta\Lambda^{c-2}$ and $(1 + \theta)\bar{\Delta}_{c-1}$ in $RH_{m,k+1}$ (Lemma 1 below).

Proposition 3. *Let m and k both be odd ($m - k = 2c > 0$). Then RH is generated by the elements $\theta, \Lambda^1, \dots, \Lambda^{c-1}, \bar{\Delta}_c$ (defined above), subject only to the relations $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$. Furthermore, $\Lambda^{2c-i} = \theta\Lambda^i$ and $\Lambda[1] = \bar{\Delta}_c^2$.*

Proof. This is divided up into various lemmas. $\square$

Lemma 1. $Res_0(\bar{\Delta}_c) = (1 + \theta)\bar{\Delta}_{c-1}$ and $Res_0(\Lambda^i) = \Lambda^i + \theta\Lambda^{i-1}$.

Proof. By definition $Res_s(\bar{\Delta}_c) = \Delta_c^+ + \Delta_c^-$, so the restriction of $\bar{\Delta}_c$ to $Spin(2c-1)$ is $2\Delta_{c-1}$. From the definition of $\bar{\Delta}_c$ as an induced representation one has that, on $\Omega T(c-1)$, trace $\bar{\Delta}_c$ is zero. But $(1 + \theta)\bar{\Delta}_{c-1} \in RH_{m,k+1}$ also restricts to $2\Delta_{c-1}$ on $RSpin(2c-1)$ and has trace zero on $\Omega T(c-1)$. Hence, $Res_0(\bar{\Delta}_c) = (1 + \theta)\bar{\Delta}_{c-1}$. As for Λ^i , we have that, on $\Omega T(c-1)$, the trace of Λ^i is the i th

symmetric polynomial on $z_1^2, z_1^{-2}, \dots, z_{c-1}^2, z_{c-1}^{-2}, 1$ plus the $(i-1)$ st symmetric polynomial on $z_1^2\theta, z_1^{-2}\theta, \dots, z_{c-1}^2\theta, z_{c-1}^{-2}\theta, \theta$. Hence $\text{Res}_0(\Lambda^i) = \Lambda^i + \theta\Lambda^{i-1}$. $\square$

Corollary 1. *In $RH_{m,k}$ one has $\theta\bar{\Delta}_c = \bar{\Delta}_c$ and $\Lambda^{2c-i} = \theta\Lambda^i$.*

Proof. The first equation follows from the fact that $\text{Res}_0(\theta\bar{\Delta}_c) = \theta(1+\theta)\bar{\Delta}_{c-1} = \text{Res}_0(\bar{\Delta}_c)$ (since $\theta^2 = 1$ implies $\theta(1+\theta) = 1+\theta$) and $\text{Res}_s(\theta\bar{\Delta}_c) = \text{Res}_s(\bar{\Delta}_c)$. The second equation follows from the fact that $\text{Res}_s(\theta\Lambda^i) = \Lambda^i \approx \Lambda^{2c-i} = \text{Res}_s(\Lambda^{2c-i})$ and $\text{Res}_0(\theta\Lambda^i) = \theta(\Lambda^i + \theta\Lambda^{i-1}) = \theta\Lambda^i + \Lambda^{i-1} \approx \theta\Lambda^{(2c-1)-i} + \Lambda^{(2c-1)-(i-1)} \approx \Lambda^{2c-i} + \theta\Lambda^{2c-i-1} = \text{Res}_0(\Lambda^{2c-i})$, so that $\theta\Lambda^i \approx \Lambda^{2c-i}$. $\square$

We next identify $\text{Res}_s(RH_{m,k})$ and $\text{Res}_0(RH_{m,k})$ as fixed rings of certain automorphisms of $RSpin(2c)$ and $RH_{m,k}$. To this end, consider the elements ω in the center of $RH_{m,k+1}$ and $\alpha = e_{k+1} \cdots e_m$ in the center of $Spin(2c)$. Then ω normalizes $Spin(2c)$ and α normalizes $H_{m,k+1}$, since α anticommutes with ω .

Let $C_\omega : RSpin(2c) \rightarrow RSpin(2c)$ be the automorphism induced by conjugation by ω , and similarly for $C_\alpha : RH_{m,k+1} \rightarrow RH_{m,k+1}$. Clearly C_ω fixes anything in the image of Res_s , since $C_\omega : RH_{m,k} \rightarrow RH_{m,k}$ is the identity. Hence C_ω fixes $\Lambda^1, \dots, \Lambda^{c-1}$. As for $\bar{\Delta}_c^+$ and $\bar{\Delta}_c^-$, we note that ω normalizes the maximal torus $T(c)$ of $Spin(2c)$, and so gives rise to $C_\omega : RT(c) \rightarrow RT(c)$. It is easy to see that C_ω inverts just one z_j (namely z_c) and so exchanges Δ_c^+ and Δ_c^- .

Turning now to $C_\alpha : RH_{m,k+1} \rightarrow RH_{m,k+1}$, we see that C_α fixes the elements $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $(1+\theta)\bar{\Delta}_{c-1}$ in $\text{Res}_0(RH_{m,k})$. Since $RH_{m,k+1}$ is generated by $\theta, \Lambda^1, \dots, \Lambda^{c-2}$ and $\bar{\Delta}_{c-1}$ we need only look at $C_\alpha(\bar{\Delta}_{c-1})$.

Lemma 2. $C_\alpha(\bar{\Delta}_{c-1}) = \theta\bar{\Delta}_{c-1}$.

Proof. Since α belongs to the center of $Spin(2c)$, it follows that on $Spin(2c-1)$ we have $C_\alpha(\bar{\Delta}_{c-1}) = \bar{\Delta}_{c-1} = \theta\bar{\Delta}_{c-1}$. And, for $h \in Spin(2c-1)$, we have

$$\begin{aligned} C_\alpha(\bar{\Delta}_{c-1})(\omega h) &= \bar{\Delta}_{c-1}(\alpha^{-1}\omega h\alpha) = \bar{\Delta}_{c-1}(-\omega h) = \bar{\Delta}_{c-1}(-1)\bar{\Delta}_{c-1}(\omega h) \\ &= -\bar{\Delta}_{c-1}(\omega h) = \theta(\omega h)\bar{\Delta}_{c-1}(\omega h), \end{aligned}$$

proving the lemma. $\square$

Let A be the subring of $RH_{m,k}$ generated by $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$. Our objective is to prove (Lemma 4) that $RH_{m,k} = A$.

Lemma 3. $\text{Res}_s(RH_{m,k}) = \text{Res}_s(A)$ and $\text{Res}_0(RH_{m,k}) = \text{Res}_0(A)$.

Proof. First, $\text{Res}_s(A) \subseteq \text{Res}_s(RH_{m,k}) \subseteq \text{Fix}(C_\omega)$. Conversely, $RSpin(2c) = Z[\Lambda^1, \dots, \Lambda^{c-2}, \Delta_c^+, \Delta_c^-]$, and C_ω fixes $\Lambda^1, \dots, \Lambda^{c-2}$ and swaps Δ_c^+ and Δ_c^- . Hence $\text{Fix}(C_\omega)$ is generated by $\Lambda^1, \dots, \Lambda^{c-2}, \Delta_c^+, \Delta_c^-$ and $\Delta_c^+ + \Delta_c^-$, i.e., by $\Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$. But the latter all belong to $\text{Res}_s(A)$. Hence $\text{Res}_s(RH_{m,k}) = \text{Res}_s(A)$, as required.

Next note that, as a $Z[\theta, \Lambda^1, \dots, \Lambda^{c-1}]$ -module, $RH_{m,k+1}$ is freely generated by 1 and $\bar{\Delta}_{c-1}$ since $\bar{\Delta}_{c-1}^2$ lies in $Z[\theta, \Lambda^1, \dots, \Lambda^{c-1}]$ (cf. §3, Theorem 1(i)). Since C_α fixes $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and sends $\bar{\Delta}_{c-1}$ to $\theta\bar{\Delta}_{c-1}$ (Lemma 2), it follows that $\text{Fix}(C_\alpha)$ is generated by $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $(1+\theta)\bar{\Delta}_{c-1}$. Since θ and $\Lambda^i + \theta\Lambda^{i-1}$ both lie in $\text{Res}_0(RH_{m,k})$, so does Λ^i . Hence $\text{Res}_0(RH) = \text{Res}_0(A)$. $\square$

Lemma 4. $A = RH_{m,k}$

Proof. Let $x \in RH_{m,k}$. By Lemma 3 there exists $y \in A$ such that $Res_s(x) = Res_s(y)$. So we may assume that $Res_s(x) = 0$. Now consider $Res_0(x)$. It belongs to $Res_0(RH_{m,k})$ which by Lemmas 1 and 3 is generated by $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $(1+\theta)\bar{\Delta}_{c-1}$. Since $((1+\theta)\bar{\Delta}_{c-1})^2 \in Z[\theta, \Lambda^1, \dots, \Lambda^{c-1}]$, we may write $Res_0(x)$ in the form $a + b\theta + (1+\theta)d\bar{\Delta}_{c-1}$, where $a, b, d \in Z[\Lambda^1, \dots, \Lambda^{c-1}]$. But $ResRes_0(x) = ResRes_s(x) = 0$ (where in both cases Res stands for “restriction to $Spin(2c-1)$ ”). Hence $a + b + 2d\bar{\Delta}_{c-1} = 0$ in $RSpin(2c-1)$. Now $RSpin(2c-1) = Z[\Lambda^1, \dots, \Lambda^{c-2}, \bar{\Delta}_{c-1}]$ is freely generated over $Z[\Lambda^1, \dots, \Lambda^{c-1}]$ by 1 and $\bar{\Delta}_{c-1}$. It follows that $a + b = 0$ and $d = 0$. Thus, $Res_0(x) = (1-\theta)a$. Now, $Res_0 : RH_{m,k} \longrightarrow RH_{m,k+1}$ sends $\Lambda^1, \dots, \Lambda^{c-1}$ to $\Lambda^1 + \theta, \dots, \Lambda^{c-1} + \theta\Lambda^{c-2}$ and hence sends $(1-\theta)\Lambda^1, \dots$ to $(1-\theta)(\Lambda^1 + \theta), \dots$, which is to say $(1-\theta)(\Lambda^1 - 1), \dots$, since $\theta^2 = 1$. Thus Res_0 sends $(1-\theta)\tilde{a}$ (where $\tilde{a} = \tilde{a}(\Lambda^1, \dots, \Lambda^{c-1})$ in $RH_{m,k}$) into $(1-\theta)\tilde{a}(\Lambda^1 - 1, \dots, \Lambda^{c-1} - \Lambda^{c-2})$. Now $\tilde{a}$ may be chosen so that $Res_0((1-\theta)\tilde{a}) = (1-\theta)a$. But clearly $Res_s((1-\theta)\tilde{a}) = 0$. Recalling that $Res_s(x) = 0$, we conclude that $x = (1-\theta)\tilde{a}$, since both restrict to the same elements in $RSpin(2c)$ and $RH_{m,k+1}$. Thus $RH_{m,k} = A$, since $(1-\theta)\tilde{a} \in A$. $\square$

Finally we determine the relations in RH .

Lemma 5. *All relations among $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$ in RH are consequences of $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$.*

Proof. We know that $\theta^2 = 1$, $\theta\bar{\Delta}_c = \bar{\Delta}_c$ and $\theta\Lambda^i = \Lambda^{2c-i}$ do hold in RH (Corollary 1). Suppose we have a relation among $\theta, \Lambda^1, \dots, \Lambda^{c-1}$, and $\bar{\Delta}_c$. Using $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$ we may write this as $f(\bar{\Delta}_c) + a\theta = 0$, where $f \in Z[\Lambda^1, \dots, \Lambda^{c-1}][x]$ and $a \in Z[\Lambda^1, \dots, \Lambda^{c-1}]$. Then $0 = Res_s(0) = f(\bar{\Delta}_c) + a$. But $\Lambda^1, \dots, \Lambda^{c-1}, \bar{\Delta}_c$ are algebraically independent in $RSpin(2c)$. Thus $f(x) = -a$ in $Z[\Lambda^1, \dots, \Lambda^{c-1}][x]$. So, the relation is $(1-\theta)a = 0$. But then $0 = Res_0(0) = (1-\theta)Res_0(a)$. Now a is a polynomial in $\Lambda^1, \dots, \Lambda^{c-1}$ and as above

$$(1-\theta)Res_0(a) = (1-\theta)a(\Lambda^1 - 1, \dots, \Lambda^{c-1} - \Lambda^{c-2}).$$

By Proposition 1, we have $a(\Lambda^1 - 1, \dots, \Lambda^{c-1} - \Lambda^{c-2}) = 0$. Since $\Lambda^1, \dots, \Lambda^{c-1}$ are polynomial generators of $Z[\Lambda^1, \dots, \Lambda^{c-1}]$, as are $\Lambda^1 - 1, \dots, \Lambda^{c-1} - \Lambda^{c-2}$, we may conclude that $a = 0$. So, all relations follow from $\theta^2 = 1$ and $\theta\bar{\Delta}_c = \bar{\Delta}_c$. This proves Lemma 5.

This also concludes the proof of Proposition 3. $\square$

We collect the above results on the structure of RH in Theorem 2 below. In the case mk even, it will be convenient to introduce Pontrjagin classes in RH analogous to the ones in $RSpin(m)$. The easiest way to do this is to note that, in Proposition 1, the generators $\Lambda^1, \dots, \Lambda^{c-1}, \theta$, and $\bar{\Delta}_c$ may be replaced by $\pi_1, \dots, \pi_{c-1}$, $y = \theta - 1$, and δ_c and similarly in Proposition 2, as in §3. When m and k are odd, Pontrjagin classes are not available; this is one of the reasons why this case turns out to be much thornier than the rest; but since we will use the Koszul resolution later on for all cases, we use the augmentation zero classes $\lambda_i = \Lambda^i - \binom{2c}{i}$, $i = 1, \dots, 2c$, when mk is odd.

Theorem 2. *(i) For $m - k = 2c + 1, c > 0$ ($k = 2s$ or $k = 2s - 1$), RH is generated by $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$ (or equivalently by $y = \theta - 1, \pi_1, \dots, \pi_{c-1}$, and δ_c) subject only to the relation $\theta^2 - 1$ (or equivalently $y^2 + 2y = 0$). The element π_c is given*

in terms of the generators by

$$\begin{aligned}\bar{\Delta}_c^2 &= \theta^s \sum_{i=0}^c 2^{2(c-i)} \pi_i & \text{if } m \text{ is odd,} \\ \bar{\Delta}_c^2 &= \theta^n \sum_{i=0}^c 2^{2(c-i)} \pi_i & \text{if } m \text{ is even.}\end{aligned}$$

(ii) For m and k both even ($m - k = 2c > 0, m = 2n$ and $k = 2s$), RH is generated by $\Lambda^1, \dots, \Lambda^{c-2}, \bar{\Delta}_c^+, \bar{\Delta}_c^-$ and θ (or equivalently by $\pi_1, \dots, \pi_{c-2}, \delta_c^+, \delta_c^-$ and y) subject only to the relation $\theta^2 = 1$ (equivalently $y^2 + 2y = 0$). The elements π_c and π_{c-1} are given in terms of the generators by

$$\begin{aligned}\bar{\Delta}_c^2 &= \theta^n \sum_{i=0}^c 2^{2(c-i)} \pi_i, & \bar{\Delta}_c^+ \bar{\Delta}_c^- &= \theta^n \sum_{i=0}^{c-1} 2^{2(c-1-i)} \pi_i, \\ \chi^2 &= (\delta_c^+ - \delta_c^-)^2 = \theta^n \pi_c.\end{aligned}$$

(iii) For m and k both odd ($m - k = 2c > 0$) RH is generated by $\theta, \Lambda^1, \dots, \Lambda^{c-1}$ and $\bar{\Delta}_c$ (equivalently by $y, \lambda_1, \dots, \lambda_{c-1}$, and δ_c) subject only to the relations $\theta^2 = 1$ and $\theta \bar{\Delta}_c = \bar{\Delta}_c$ (equivalently by $y^2 + 2y = 0$ and $\delta_c y = -2^c y$). The element Λ^c is given in terms of the generators by

$$\bar{\Delta}_c^2 = (1 + \theta)(1 + \Lambda^1 + \dots + \Lambda^{c-1}) + \Lambda^c.$$

Proof. The theorem follows almost immediately from Propositions 1, 2 and 3. $\square$

Remark 2. In addition to $y^2 + 2y = 0$, notice the following formulae, which follow from the definition of y and θ : $y^j = (-2)^{j-1}y$ for $j > 0$, $y\theta^j = y(1+y)^j = (-1)^j y$, for $j \geq 0$, $(1+y)^2 = 1$, $(1+\theta)^s = 2^{s-1}(1+\theta)$, $\theta(1+\theta) = 1+\theta$ and $y(1+\theta) = 0$. These will be used many times henceforth and without specific mention.

5. THE RESTRICTION $Res : RSpin(m) \longrightarrow RH$

5.1. Restriction for $m - k$ odd. Let $\tilde{\Omega}$ be as in 4.1. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{\Omega} \times Spin(2c+1) & \longrightarrow & H \subset Spin(m) \\ \uparrow i & & \uparrow j \\ \tilde{\Omega} \times T(c) & \xrightarrow{\mu} & T(n) \\ \uparrow Id \times \tau & & \uparrow \tau \\ \tilde{\Omega} \times T^c & \xrightarrow{\mu} & T_n \end{array}$$

where μ stands for multiplication and i, j are inclusions. Since ω lies in the center of H , every element of H is conjugate to an element of $\tilde{\Omega}T(c)$. Hence it suffices to calculate the homomorphism $RSpin(m) \rightarrow R\tilde{\Omega} \otimes RT(c)$ induced by μ , or equivalently to calculate $\mu^\# : RT^n \longrightarrow R\tilde{\Omega} \otimes RT^c$.

Case 1. k even, m odd ($m = 2n + 1, k = 2s, s + c = n$).

In this case the restriction is $Res: RSpin(2n+1) \rightarrow RH_{2n+1,2s}$. Hence

$$Res: Z[\pi_1, \dots, \pi_{n-1}, \delta_n] \rightarrow Z[\bar{\pi}_1, \dots, \bar{\pi}_{c-1}, \delta_c, y]/(y^2 + 2y)$$

Here, $\bar{\pi}[t]$ denotes the $\pi[t]$ corresponding to $RH[t]$. Observing, as in 4.1, that $Z/4 \approx \tilde{\Omega} \subset T^n$ is generated by $\tilde{\omega} = (i, \cdot_{(s)}, i, 1 \cdots, 1)$ and that a generic element $\xi \in T^c$ has the form $\xi = (1, \dots, 1, e^{i\theta_{s+1}}, \dots, e^{i\theta_n})$, it is easy to see that

$$\mu^\#(z_j) = \begin{cases} \phi, & 1 \leq j \leq s, \\ z_j, & s+1 \leq j \leq n. \end{cases}$$

Hence, defining $\pi'[t] = \mu^\# \pi[t]$, we have

$$\begin{aligned} \pi'[t] &= \mu^\# \prod_{j=1}^n (1 + t(z_j - z_j^{-1})^2) \\ (1) \quad &= \prod_{j=1}^s (1 + t(\phi - \phi^{-1})^2) \prod_{j=s+1}^n (1 + t(z_j - z_j^{-1})^2) \\ &= (1 + 2ty)^s \bar{\pi}[t] \quad (\text{since } (\phi - \phi^{-1})^2 = 2\theta - 2 = 2y). \end{aligned}$$

It follows that

$$(2) \quad \bar{\pi}[t] = (1 + 2ty)^{-s} \pi'[t],$$

so that $\bar{\pi}_i$ is a linear combination of $\pi'_1, \dots, \pi'_i$ with coefficients in $Z[y]$, for $i = 1, \dots, c$.

Next, consider $\Delta_n \in RSpin(2n+1)$. Since $\Delta_n = \prod_{j=1}^n (z_j + z_j^{-1})$, we have

$$\begin{aligned} j^\#(\Delta_n) &= \prod_{j=1}^s (\phi + \phi^{-1}) \prod_{j=s+1}^n (z_j + z_j^{-1}) \\ &= \begin{cases} \phi(1 + \theta)^s \Delta_c & \text{if } s \text{ odd} \\ (1 + \theta)^s \Delta_c & \text{if } s \text{ even} \end{cases} = (1 + \theta)^s \bar{\Delta}_c. \end{aligned}$$

So, we obtain

$$(3) \quad Res(\Delta_n) = 2^{s-1}(2 + y)\bar{\Delta}_c.$$

From (1) and (3), we have

Proposition 4. For $m = 2n + 1$ and $k = 2s$, $Res: RSpin(2n+1) \rightarrow RH$ is given by

$$\begin{aligned} Res(\pi[t]) &= \pi'[t] = (1 + 2ty)^s \bar{\pi}[t], \\ Res(\delta_n) &= 2^{s-1}(2 + y)\delta_c + 2^{n-1}y. \end{aligned}$$

Case 2. k odd, m even ($m = 2n, k = 2s - 1$).

In this case the restriction is $Res: RSpin(2n) \rightarrow RH_{2n,2s-1}$. Hence

$$Res: Z[\pi_1, \dots, \pi_{n-2}, \delta_n^+, \chi_n] \rightarrow Z[\bar{\pi}_1, \dots, \bar{\pi}_{c-1}, \delta_c, y]/(y^2 + 2y).$$

This time $\tilde{\omega} = (i, \cdot_{(n)}, i)$ and $\xi = (1, \dots, 1, e^{i\theta_{s+1}}, \dots, e^{i\theta_n})$ imply

$$\mu^\#(z_j) = \begin{cases} \phi, & 1 \leq j \leq s, \\ \phi z_j, & s+1 \leq j \leq n. \end{cases}$$

Hence

$$\begin{aligned}
\pi'(t) &= \mu^\#(\pi[t]) = \mu^\#\left(\prod_{j=1}^n [1 + t(z_j - z_j^{-1})^2]\right) \\
&= \prod_{j=1}^s [1 + t(\phi - \phi^{-1})^2] \prod_{j=s+1}^n [1 + t(\phi z_j - (\phi z_j)^{-1})^2] \\
&= (1 + 2ty)^s \prod_{j=s+1}^n [1 + 2t(\theta - 1) + t\theta(z_j - z_j^{-1})^2] \\
&= (1 + 2ty)^n \prod_{j=s+1}^n \left[1 + (z_j - z_j^{-1})^2 \frac{t\theta}{(1 + 2ty)}\right].
\end{aligned}$$

Therefore,

$$(4) \quad \pi'[t] = (1 + 2ty)^n \bar{\pi}[u],$$

where $u = u(t) = t(1+y)/(1+2ty)$. Since $(1+2ty)(1+2uy) = 1$ one has $u(u(t)) = 1$, so that

$$(5) \quad \bar{\pi}[t] = (1 + 2ty)^n \pi'[u],$$

from which we conclude as before that $\bar{\pi}_i$ is a linear combination of $\pi'_1, \dots, \pi'_i$ with coefficients in $Z[y]$, $i = 1, \dots, c$.

Next, we have (in $RSpin(2n)$) Δ_n^+ and Δ_n^- . If we write, as in §3, $\Delta_n[t] = \prod_{j=1}^n (z_j + tz_j^{-1})$, we have $\Delta_n[t] = \Delta_n^+ + t\Delta_n^-$ for $t = \pm 1$. Hence, for $t = \pm 1$,

$$\begin{aligned}
j^\#(\Delta_n[t]) &= \prod_{j=1}^s (\phi + t\phi^{-1}) \prod_{j=s+1}^n (\phi z_j + t\phi^{-1}z_j^{-1}) \\
&= (1 + t\theta)^s \phi^n \prod_{j=s+1}^n (z_j + t\theta z_j^{-1}) \\
&= \phi^n (1 + t\theta)^s \Delta_c[t\theta] = 2^{s-1} (1 + t\theta) \phi^n \Delta_c[t\theta],
\end{aligned}$$

from which it follows that

$$(6) \quad Res(\Delta_n^+) = 2^{s-1} \bar{\Delta}_c \text{ and } Res(\Delta_n^-) = 2^{s-1} \theta \bar{\Delta}_c.$$

From (4) and (6) we have

Proposition 5. For $m = 2n$ and $k = 2s - 1$, $Res: RSpin(2n) \rightarrow RH$ is given by

$$\begin{aligned}
Res(\pi[t]) &= \pi'[t] = (1 + 2ty)^n \bar{\pi}[t(1+y)/(1+2ty)], \\
Res(\delta_n^+) &= 2^{s-1} \delta_c, \\
Res(\chi_n) &= -2^{s-1} \delta_c y - 2^{n-1} y.
\end{aligned}$$

5.2. Restriction for m and k both even. Next $m = 2n$, $k = 2s$ and as before $c = n - s$. The restriction is given by $Res: RSpin(2n) \rightarrow RH_{2n, 2s}$. Hence

$$Res: Z[\pi_1, \dots, \pi_{n-2}, \delta_n^+, \chi_n] \rightarrow Z[\bar{\pi}_1, \dots, \bar{\pi}_{c-2}, \delta_c^+, \delta_c, y]/(y^2 + 2y).$$

As in 5.1, case 2, we have

$$(7) \quad \pi'[t] = (1 + 2ty)^n \bar{\pi}[u],$$

$$(8) \quad Res(\Delta_n^+) = 2^{s-1} \bar{\Delta}_c \text{ and } Res(\Delta_n^-) = 2^{s-1} \theta \bar{\Delta}_c$$

(here $\bar{\Delta}_c = \bar{\Delta}_c^+ + \bar{\Delta}_c^-$). Hence, from (7) and (8) we have

Proposition 6. *For $m = 2n$ and $k = 2s$, $Res:RSpin(2n) \rightarrow RH$ is given by*

$$\begin{aligned} Res(\pi[t]) &= \pi'[t] = (1 + 2ty)^n \bar{\pi}[t(1+y)/(1+2ty)], \\ Res(\delta_n^+) &= 2^{s-1} \delta_c, \quad (\delta_c = \delta_c^+ + \delta_c^-), \\ Res(\chi_n) &= -2^{s-1} \delta_c y - 2^{n-1} y. \end{aligned}$$

5.3. Restriction for m and k both odd. Here $m = 2n + 1$, $k = 2s + 1$. We write $\bar{\Lambda}[t]$ for the corresponding $\Lambda[t]$ to $RH_{m,k}[t]$. The restriction is given by $Res:RSpin(2n+1) \rightarrow RH_{2n+1,2s+1}$. Hence

$$Res : Z[\Lambda^1, \dots, \Lambda^{n-1}, \Delta_n] \rightarrow Z[\bar{\Lambda}^1, \dots, \bar{\Lambda}^{c-1}, \bar{\Delta}_c, \theta]/(\theta^2 - 1, \theta \bar{\Delta}_c - \bar{\Delta}_c).$$

We may identify $Res(x)$, $x \in RSpin(2n+1)$, by calculating the restriction of x to $RSpin(2c)$ and to $RH_{m,k+1}$. In the case of $RH_{m,k+1}$, going back to 5.1, case 1, where $Res:RSpin(2n+1) \rightarrow RH_{m,k+1}$, we have (note $k+1 = 2(s+1)$) that

$$\begin{aligned} Res(\Lambda[t]) &= \mu^\#(1+t) \prod_{i=1}^n (1 + tz_i^2)(1 + tz_i^{-2}) \\ &= (1+t) \prod_{j=1}^{s+1} (1+t\theta)(1+t\theta^{-1}) \prod_{j=s+2}^n (1 + tz_j^2)(1 + tz_j^{-2}) \\ &= (1+t\theta)^{2s+2} \hat{\Lambda}[t] \end{aligned}$$

($\hat{\Lambda}[t]$ denotes the $\Lambda[t]$ corresponding to $RH_{m,k+1}$) and $Res(\Delta_n) = 2^s(\theta+1)\bar{\Delta}_{c-1}$.

Now, $\Lambda[t]$ restricts to $(1+t)^{2s+1}\Lambda[t]$ in $RSpin(2c)[t]$, and from 4.3 we have that $Res_0(\Lambda[t]) = (1+t\theta)\hat{\Lambda}[t]$ and $Res_s(\hat{\Lambda}[t]) = \Lambda[t] \in RSpin(2c)[t]$. Hence

$$(9) \quad Res(\Lambda[t]) = \Lambda'[t] = (1+t\theta)^k \bar{\Lambda}[t].$$

The element $\Delta_n \in RSpin(2n+1)$ restricts to $2^s \Delta_c \in RSpin(2c)$. But $Res_s(\bar{\Delta}_c) = \Delta_c$ and $Res_0(\bar{\Delta}_c) = (1+\theta)\bar{\Delta}_{c-1}$. Hence,

$$(10) \quad Res(\Delta_n) = 2^s \bar{\Delta}_c.$$

From (9) and (10) we have

Proposition 7. *For $m = 2n + 1$, $k = 2s + 1$, $Res:RSpin(2n+1) \rightarrow RH$ is given by*

$$\begin{aligned} Res(\Lambda[t]) &= \Lambda'[t] = (1+t\theta)^k \bar{\Lambda}[t], \\ Res(\Delta_n) &= 2^s \bar{\Delta}_c. \end{aligned}$$

6. THE HODGKIN SPECTRAL SEQUENCE

6.1. The theorem. Let G be a compact, connected Lie group with $\pi_1(G)$ torsion-free and H a closed subgroup of G . The Hodgkin spectral sequence calculates the complex K -theory of G/H in terms of RG and RH . The result is the following (cf. [10], [12]).

Theorem 3. *Given G and H as above, there is a strongly convergent sequence $E_r(G/H)$ with the following three properties:*

- (i) *As an algebra, $E_2^p(G/H) = Tor_{RG}^p(RH; Z)$,*
- (ii) *The differential $d_r : E_r^{p-r} \rightarrow E_r^p$ is zero for r even,*

(iii) $E_{\infty}^*(G/H)$ is the graded algebra associated to a negative filtration of $K^*(G/H)$ compatible with its multiplication

$$F^p K^* \otimes F^q K^* \rightarrow F^{p+q} K^*,$$

$$F^p K^* = F_{-p} K^* = \tilde{F}^{2p} K^0 \oplus \tilde{F}^{2p+1} K.$$

Notes: 1. RG, RH and $R\{1\} = Z$ are trivially $Z/2$ -graded by $R^0 = R$ and $R^1 = 0$.

2. If $Tor_{RG}^*(RH, Z)$ is generated as an algebra by its elements of degree ≤ 2 , then the Hodgkin spectral sequence collapses. Hence in this case we have two isomorphisms

$$E_2^{0,0} = RH \otimes_{RG} Z \xrightarrow{\sim} E_{\infty}^{0,0} = F_0 K^0,$$

$$E_2^{-1,0} = Tor_{RG}^{-1}(RH; Z) \xrightarrow{\sim} E_{\infty}^{-1,0} = F_{-1} K^{-1}.$$

6.2. Some consequences. Let G and H be as above, and let $Res: RG \rightarrow RH$ be the homomorphism induced by inclusion.

As before, $\epsilon_G: RG \rightarrow R\{1\} = Z$ is the augmentation which assigns to each representation its dimension. We will suppose that $RG = Z[\gamma_1, \dots, \gamma_n]$ with $\epsilon_G(\gamma_i) = 0$, $i = 1, \dots, n$ (if $\epsilon_G(\gamma_i) \neq 0$ we take $\tilde{\gamma}_i = \gamma_i - \epsilon_G(\gamma_i)$), and RH is generated by $h_1, \dots, h_m$ with $\epsilon_H(h_j) = 0$, $j = 1, \dots, m$.

Let $\Gamma = Z[\gamma_1, \dots, \gamma_r]$, $1 \leq r \leq m$. We state the following straightforward applications of the change of rings theorem and Koszul resolution (cf. [5] and [12]):

1. If RH is a free (or more generally flat) Γ -module and $Res(\gamma_i) = h_i$, $i = 1, \dots, r$, it follows that $Tor_{RG}^*(RH; Z) = Tor_A^*(B; Z)$, where

$$A = RG/(\gamma_1, \dots, \gamma_r), \quad B = RH/(h_1, \dots, h_r)$$

and the A -module structure of B (which we call Res/A , or Res by a small abuse of notation) is induced by Res .

2. If $\tau_1, \dots, \tau_s \in A$ and $(Res/A)(\tau_i) = 0$, $1 \leq i \leq s$, then, setting $A' = A/(\tau_1, \dots, \tau_s)$, B is an A' -module via Res/A and

$$Tor_A^*(B; Z) \approx \Lambda_Z^*[t_1, \dots, t_s] \otimes Tor_{A'}^*(B; Z),$$

where $\Lambda_Z^*[t_1, \dots, t_s]$ denotes the graded exterior algebra with generators t_i of degree 1,

3. If RH is a trivial RG -module, then

$$Tor_{RG}^*(RH; Z) \approx \Lambda_{RG}^*[t_1, \dots, t_n],$$

where $t_1, \dots, t_n$ are of degree 1.

7. THE COMPLEX K -THEORY OF $X_{m,k}$ FOR mk EVEN

Now that RH and the restriction $Res: RSpin(m) \rightarrow RH$ are known, we are in position to determine $K^*(X_{m,k})$ for mk even. To do this we must determine the structure of the ring B defined in §6, as well as the restriction $A \rightarrow B$, so as to be able to apply the Koszul resolution.

7.1. Structure of B . From §5 relations (1), (2), (4), (5) and (7) (cases where mk is even), we see that $\pi'_1, \dots, \pi'_c$ can be taken as generators for $RH_{m,k}$ instead of $\bar{\pi}_1, \dots, \bar{\pi}_c$. Since $\text{Res}(\pi_i) = \pi'_i, i = 1, \dots, c$ and RH is a free $Z[\pi_1, \dots, \pi_c]$ -module it follows from 6.2 (1) that

$$(11) \quad \text{Tor}_{R\text{Spin}(m)}^*(RH; Z) \approx \text{Tor}_A^*(B; Z),$$

where

$$A = R\text{Spin}(m)/(\pi_1, \dots, \pi_c), \quad B = RH/(\pi'_1, \dots, \pi'_c)$$

(unfortunately, for m and k both odd RH is not a free $Z[\lambda_1, \dots, \lambda_c]$ -module).

The homomorphism Res induces an algebra homomorphism $\text{Res}/A : A \rightarrow B$ that makes B an A -algebra. While A is still a polynomial algebra (on c fewer generators), the structure of B is more complicated as we now see.

From §5, (1), (5) and (7) we have

$$(12) \quad \bar{\pi}[t] = \begin{cases} (1 + 2ty)^{-s} \pi'[t] & (m = 2n + 1), \\ (1 + 2ty)^n \pi'[u] & (m = 2n), \end{cases}$$

where $u = t(1 + y)/(1 + 2ty)$.

Now $\bar{\pi}_i = 0$ for $i > c$, and in B , we have $\pi'_i = 0$ for $1 \leq i \leq c$ ($\pi'_0 = 1$). So, for $1 \leq i \leq c$, in B we have:

$$(13) \quad \bar{\pi}[t] = \begin{cases} T_c(1 + 2ty)^{-s} & (m = 2n + 1), \\ T_c(1 + 2ty)^n & (m = 2n), \end{cases}$$

where T_c is truncation defined by $T_c(\sum_{i \geq 0} a_i t^i) = \sum_{i=0}^c a_i t^i$. Thus,

$$(14) \quad \bar{\pi}_i = \begin{cases} (-1)^{i-1} \binom{n}{i} 2^{2i-1} y, & 1 \leq i \leq c, \\ -\binom{s+i-1}{i} 2^{2i-1} y, & \end{cases}$$

recalling that $y^2 = -2y$ and the well known identity $\binom{-s}{i} = (-1)^i \binom{s+i-1}{i}$ (cf. [8]).

Substituting u for t in the second equation of (12) and using (4), we get

$$(15) \quad \pi'[t] = (1 + 2ty)^n \sum_{i \geq c} \binom{n}{i} (2uy)^i.$$

We write $\bar{f}(t)$ for $f(-t/(1+t))$. Observe that $\overline{fg} = \bar{f} \bar{g}$ and $\overline{f+g} = \bar{f} + \bar{g}$. Then, $f(2uy) = \bar{f}(2ty)$. Hence $\pi'[t] = F(2ty)$, where $F(t) = (1+t)^n \overline{T_c(1+t)^n}$.

Lemma 6. *With the above notation*

- (i) $(1+t)^s T_c(1+t)^{-s} = (1+t)^n \overline{T_c(1+t)^n}$,
- (ii) $(1+t)^s T_c(1+t)^{-s} = (-1)^c \sum_{i=0}^c \binom{n}{i} \binom{i-1}{c} t^i$.

Proof. Fix $s \geq 0$. Both results are proved by induction on c starting with $c = 0$ where both sides of the equations are $(1+t)^s$, since $c = 0$ implies $n = s$. The inductive steps in (i) and (ii) are obtained by using the elementary identities:

$$\begin{aligned} \binom{i+1}{j} &= \binom{i}{j} + \binom{i}{j-1}, & \binom{i}{j} &= (-1)^j \binom{j-i-1}{j}, \\ \binom{n}{i} \binom{i}{j} &= \binom{n}{j} \binom{n-j}{i-j} \end{aligned}$$

valid for all i, j and $n \geq 0$ (see [8]). Specifically, for (i) the inductive step is

$$\begin{aligned}
& (1+t)^{n+1} \overline{T_{c+1}(1+t)^{n+1}} - (1+t)^n \overline{T_c(1+t)^n} \\
&= (1+t)^{n+1} \overline{(T_c(1+t)^{n+1} - (1+t)T_c(1+t)^n)} \\
&= (1+t)^{n+1} \overline{\binom{n}{c+1}} t^{c+1} \quad (\text{since } tT_c(f) = T_{c+1}(tf)) \\
&= (1+t)^{n+1} \binom{n}{c+1} \left(\frac{-t}{1+t} \right)^{c+1} \\
&= (-1)^{c+1} \binom{n}{c+1} (1+t)^s t^{c+1} = (1+t)^s \binom{-s}{c+1} t^{c+1} \\
&= (1+t)^s T_{c+1}(1+t)^{-s} - (1+t)^s T_c(1+t)^{-s},
\end{aligned}$$

whereas for (ii) the inductive step is: consider

$$(-1)^{c+1} \sum_{i=0}^n \binom{n}{i} \binom{i-1}{c} t^i - (-1)^c \sum_{i=0}^n \binom{n}{i} \binom{i-1}{c} t^i.$$

Using the elementary identities above we have

$$\binom{n}{i} \binom{i-1}{c} + \binom{n+1}{i} \binom{i-1}{c+1} = (-1)^{c+1} \binom{-s}{c+1} \binom{s}{i-c-1}$$

which is equal to the coefficient of t^i in $(-1)^{c+1} \binom{-s}{c+1} (1+t)^s t^{c+1}$. $\square$

Applying Lemma 6 to the expression (15) for $\pi'[t]$ for m even and (13) for m odd, we conclude that in B , for m even or odd,

$$(16) \quad \pi'_i = (-1)^{c+1+i} 2^{2i-1} \binom{n}{i} \binom{i-1}{c} y, \quad i > c.$$

In addition to this explicit formula for π'_i , $i > c$, it is important to establish certain linear relations among these π'_i . We do this next.

Since $\bar{\pi}_i = 0, i > c$, from (12) we see that for $i > c$

$$\begin{aligned}
(17) \quad 0 &= \sum_{j \geq 0} \pi'_j \binom{n-j}{i-j} (1+y)^j (2y)^{i-j} \\
&= \binom{n}{i} (2y)^i + \theta^i \pi'_i + \sum_{j=c+1}^{i-1} \binom{n-j}{i-j} \theta^j (2y)^{i-1} \pi'_j.
\end{aligned}$$

Substituting in (13), we get

$$(18) \quad \pi'_i = 2^{2i-1} y \binom{n}{i} - \sum_{j=c+1}^{i-1} \binom{n-j}{i-j} 2^{2(i-j)} \pi'_j.$$

Now, by substituting the values of $\bar{\pi}_i$ given in (14) in the relations for δ_c^2 , χ^2 and $\delta_c^+ \delta_c^-$ from Theorem 2, and using elementary binomial identities (see [8] for example) we get the relations

$$(19) \quad \delta_c^2 + 2^{c+1} \delta_c = \begin{cases} 2^{2c-1} y [1 + (-1)^{s-1} \binom{n-1}{c}] & (m = 2n), \\ 2^{2c-1} y [1 + (-1)^{s-1} \binom{n}{c}] & (m = 2n+1), \end{cases}$$

$$(20) \quad \chi^2 = (\delta_c^+ - \delta_c^-)^2 = \theta^n \bar{\pi}_c = (-1)^{s-1} 2^{2c-1} y \binom{n}{c} \quad (m \text{ even, } k \text{ even}),$$

$$(21) \quad \delta_c^+ \delta_c^- + 2^{c-1} \delta_c = 2^{2c-3} y \left[1 + (-1)^s \binom{n-1}{c-1} \right] \quad (m \text{ even, } k \text{ even}),$$

From (18) and $\delta_c = \delta_c^+ + \delta_c^-$, it follows trivially that

$$(22) \quad \delta_c \delta_c^+ = 2^{2c-3} y \left[1 + (-1)^s \binom{n-1}{c-1} \right] - 2^{c-1} \delta_c + (\delta_c^+)^2 \quad (m \text{ even, } k \text{ even}).$$

The ring structure of B is thus given by

Proposition 8. Let $L = \left[1 + (-1)^{s-1} \binom{(m-1)/2}{c} \right]$ and $M = \left[1 + (-1)^s \binom{n-1}{c-1} \right]$.

(i) For $m - k$ odd, as an abelian group

$$B \approx Z \oplus Zy \oplus Z\delta_c \oplus Z\delta_c y$$

with multiplication given by the table

	y	δ_c
y	$-2y$	$\delta_c y$
δ_c	$\delta_c y$	$2^{c+1} \delta_c + 2^{2c-1} yL$

As a ring, $B \approx Z[\delta_c, y]/I$, where I is the ideal generated by $y^2 + 2y$ and $\delta_c^2 + 2^{c+1} \delta_c - 2^{2c-1} yL$.

(ii) For m and k both even

$$B \approx Z \oplus Zy \oplus Z\delta_c \oplus Z\delta_c^+ \oplus Z\delta_c y \oplus Z\delta_c^+ y \oplus Z(\delta_c^+)^2 \oplus Z(\delta_c^+)^2 y$$

with multiplication given by the table

	y	δ_c	δ_c^+
y	$-2y$	$y\delta_c$	$y\delta_c^+$
δ_c	$y\delta_c$	$-2^{c+1} \delta_c + 2^{2c-1} yL$	$-2^{c-1} \delta_c + 2^{2c-3} yM + (\delta_c^+)^2$
δ_c^+	$y\delta_c^+$	$-2^{c-1} \delta_c + 2^{2c-3} yM + (\delta_c^+)^2$	$(\delta_c^+)^2$

or, $B \approx Z[y, \delta_c, \delta_c^+]/J$, where J is the ideal generated by $y^2 + 2y$, $\delta_c^2 + 2^{c+1} \delta_c - 2^{2c-1} yL$, and $\delta_c \delta_c^+ + 2^{c-1} \delta_c - 2^{2c-3} yM - (\delta_c^+)^2$.

Remark 3. The remaining products, with $(\delta_c^2)^2$ and $y(\delta_c^2)$ are easily deduced in terms of the abelian group generators for B using the products given above.

7.2. Structure of A . As for A , from relation (16) (and proceeding as in [2]), we can choose a new basis for A , namely $\tau_1, \dots, \tau_{s-3}, \rho_1, \rho_2, \rho_3$, such that $\text{Res}(\tau_i) = 0$, $i = 1, \dots, s-3$, and, for $m = 2n+1$ and $k = 2s$, $\text{Res}(\rho_1) = 0$, $\text{Res}(\rho_2) = \text{Res}(\delta_n)$, $\text{Res}(\rho_3) = by$, while for $m = 2n$ and $k = 2s$ or $2s-1$, $\text{Res}(\rho_1) = \text{Res}(\chi_n)$, $\text{Res}(\rho_2) = \text{Res}(\delta_n^+)$, $\text{Res}(\rho_3) = by$.

In both cases $b = \text{g.c.d.}\{2^{2i-1} \binom{n}{i}, i = c+1, \dots, [m/2] - 2\}$, and we remind the reader that the formulae for $\text{Res}(\delta_n)$, $\text{Res}(\chi_n)$, and $\text{Res}(\delta_n^+)$ were given in §5. Hence, from 6.2(2) we have, for $m = 2n$,

$$(23) \quad \text{Tor}_A^*(B; Z) \approx \Lambda_Z^*[t_1, \dots, t_{s-3}] \otimes_Z \text{Tor}_{A'}^*(B; Z)$$

with $\dim t_i = 1$, $i = 1, \dots, s-3$, and $A' = A/(\tau_1, \dots, \tau_{s-3})$, while, for $m = 2n+1$,

$$(24) \quad \text{Tor}_A^*(B; Z) \approx \Lambda_Z^*[t_1, \dots, t_{s-2}] \otimes_Z \text{Tor}_{A''}^*(B; Z)$$

with $\dim t_i = 1$, $i = 1, \dots, s-2$, and $A'' = A/(\tau_1, \dots, \tau_{s-3}, \rho_1)$.

7.3. $K^*(X_{m,k})$ for mk even. We treat separately the three cases m even, m odd k even and, m odd k odd, noting that in all cases A , A' , and A'' are polynomial algebras so that the Koszul resolution is applicable.

7.3.1. m even. $Tor_{A'}^*(B; Z)$ is the homology of the Koszul complex $\Lambda_B^*(x_1, x_2, x_3)$, where $d(x_i) = Res(\rho_i)$, $i = 1, 2, 3$. As in [2], we take a new basis u_1, u_2, u_3 for $\Lambda_B^*(x_1, x_2, x_3)$ with $d(u_1) = 2^\alpha y$, $2^\alpha = g.c.d.\{2^{n-1}, b\}$; $d(u_2) = 2^{s-1}\delta_c$; and $d(u_3) = 0$. Hence for $k = 2s - 1$

$$(25) \quad Tor_{A'}^*(B; Z) \approx \Lambda_Z^*[z_1, z_2, z_3, z_4] \otimes Z[y, \delta_c]/I,$$

and for $k = 2s$

$$(26) \quad Tor_{A'}^*(B; Z) \approx \Lambda_Z^*[z_1, z_2, z_3, z_4] \otimes Z[y, \delta_c, \delta_c^+]/J$$

with y, δ_c and δ_c^+ of degree 0 and, z_1, z_2, z_3, z_4 of degree 1 given by, $z_1 = (y + 2)u_1$, $z_2 = -2^{n+c-2-\alpha}Lu_1 + (\delta_c + 2^{c+1})u_2$, $z_4 = -2^{s-1-r}\delta_c u_1 + 2^{\alpha-r}yu_2$ and $z_3 = u_3$.

For $k = 2s - 1$, I is the ideal generated by: $y^2 + 2y$; $\delta_c^2 + 2^{c+1}\delta_c - 2^{2c-1}yL$; $2^{s-1}\delta_c$; $2^\alpha y$; $(y + 2)z_4 - 2^{s-1-r}\delta_c z_1$; $2^r z_4$, where $r = \min\{\alpha, s - 1\}$; $z_1 y$; $z_1 z_4$; $z_2 z_4$; $z_2 \delta_c - 2^{2c-1+r-\alpha}Lz_4$; and $\delta_c z_4 + 2^{c+1}z_4 - 2^{\alpha-r}yz_2$.

For $k = 2s$, J is the ideal generated by the same elements given in I and the element $\delta_c \delta_c^+ + 2^{c-1}\delta_c - 2^{2c-3}yM - (\delta_c^+)^2$.

Therefore we have

Theorem 4. *The Hodgkin spectral sequence for $X_{2n,k}$ collapses, and so, as graded algebras,*

$$K^*(X_{2n,2s-1}) \approx \Lambda_Z^*[t_1, \dots, t_{s-3}, z_1, z_2, z_3, z_4] \otimes Z[y, \delta_c]/I,$$

$$K^*(X_{2n,2s}) \approx \Lambda_Z^*[t_1, \dots, t_{s-3}, z_1, z_2, z_3, z_4] \otimes Z[y, \delta_c, \delta_c^+]/J,$$

where I and J are the ideals generated by the above elements, except that $z_1 z_4$ and $z_2 z_4$ are replaced by $z_1 z_4 + \lambda$, $z_2 z_4 + \mu$, for some $\lambda, \mu \in Z[y, \delta_c]$ if k odd, or by $\lambda, \mu \in Z[y, \delta_c, \delta_c^+]$ if k even.

Proof. This follows from (11), (18), (23), (24), (25) (or (26) for $k = 2s$), Theorem 1, Note 2 of §6 and §6 of [2]. As for the elements $z_i z_4$, $i = 1, 2$, they are in $Tor^2 = E_2^\infty = \tilde{F}^2/\tilde{F}^0$. Since $z_i z_4 = 0$ in Tor^* , we have $z_i z_4 \in \tilde{F}^0(K^*) \subset K^0$. Hence, in K^0 , $z_i z_4 = a_i y + b_i \delta_c + c_i \delta_c y$, $a_i, b_i, c_i \in Z$, $i = 1, 2$ for $k = 2s - 1$ (similarly for k even with the extra terms added). $\square$

7.3.2. m odd, k even. $Tor_{A''}^*(B; Z)$ is the homology of the Koszul complex $\Lambda_B^*(x_1, x_2)$:

$$0 \longrightarrow B \longrightarrow B \oplus B \xrightarrow{d} B \longrightarrow 0$$

where $d(x_1) = Res(\rho_3) = by$ and $d(x_2) = Res(\delta_n) = 2^{s-1}(2 + y)\delta_c + 2^{n-1}y$. Then, in $Tor_{A''}^0(B; Z)$ we have $by = 0$ and $2^{s-1}(2 + y)\delta_c + 2^{n-1}y = 0$. Hence, by multiplying the second equation by y , it follows that $2^n y = 0$. Thus $2^\alpha y = 0$, where $2^\alpha = g.c.d.\{2^n, b\}$.

For $i \geq 1$, $H_i(\Lambda_B^*)$ may be computed by hand or using a suitable software such as Maple (Maple V Release 4, Waterloo Maple Inc., June 1996, Gröbner package). A set of generators of H_1 is, $z_1 = (-2^{n+c-r} - 2^{n-1-r}\delta_c)x_1 + (2^{c+1-r}b + 2^{-r}b\delta_c)x_2$, $r = \min\{n - 1, c + 1\}$, $z_2 = (2 + y)x_1$, and $z_3 = 2^{n-\alpha}x_1 + 2^{-\alpha}byx_2$.

From this and by calculating $H_i, i > 1$ from the Koszul resolution, it is not hard to verify that as graded algebras

$$(27) \quad \text{Tor}_{A^*}^*(B; Z) \approx \Lambda_Z^*[z_1, z_2, z_3] \otimes Z[y, \delta_c]/I$$

with z_1, z_2, z_3 of degree 1, y and δ_c of degree 0, and I the ideal generated by $y^2 + 2y; z_2 z_3; y z^2; \delta_c^2 + 2^{c+1} \delta_c - 2^{2c-1} y L; 2^\alpha y; 2^{s-1}(2+y)\delta_c + 2^{n-1} y; z_2 z_3; 2^r \delta_c z_1 + 2^{2c-2+\alpha} L y z_3; 2^r y z_1 + 2^{n-1} \delta_c z_2 + 2^\alpha (2^c y - \delta_c) z_3; -2^{n-\alpha} z_2 + (2+y) z_3; 2^\alpha z_3 - (2^{n-1} + 2^{s-1} \delta_c) z_2$.

Hence we have

Theorem 5. *The Hodgkin spectral sequence for $X_{2n+1, 2s}$ collapses, and so, as graded algebras,*

$$K^*(X_{2n+1, 2s}) \approx \Lambda_Z^*[t_1, \dots, t_{s-2}, z_1, z_2, z_3] \otimes Z[y, \delta_c]/I,$$

where I is the ideal generated by the elements listed in (27).

Remark 4. The element $y + 1 \in K^*(X_{m,k})$ can be identified with the complexified Hopf bundle $c\xi_{m,k}$ over $X_{m,k}$ (see [2]). Then it follows from the preceding results that for mk even the order of $c\xi_{m,k}$ is

$$2^{\alpha(m,k)} = g.c.d. \left\{ 2^{[(m-1)/2]}, 2^{2i-1} \binom{n}{i}, i = c+1, \dots, [(m-3)/2] \right\}.$$

8. $K^*(X_{m,k})$ FOR mk ODD

When mk is odd, various things go wrong. First, the Pontrjagin classes do not make sense, so we are obliged to work with the exterior powers $\bar{\lambda}_i = \bar{\Lambda}^i - \binom{2c}{i}$ instead (recalling that we are writing $\bar{\Lambda}[t]$ for the $\Lambda[t]$ corresponding to $RH_{m,k}[t]$ in order to distinguish the $\Lambda[t]$ in $RSpin(2n+1)[t]$). Second, there is the more serious problem that RH is no longer a free $Z[\lambda_1, \dots, \lambda_c]$ -module, so that change of rings cannot be used.

We shall therefore content ourselves with calculating $K^0(X_{m,k})$ and, in particular, the order of y and δ_c in $K^*(X_{m,k})$. The relevant complex is $\Lambda_{RH}^*[x_1, \dots, x_n]$, which is given by

$$0 \rightarrow \Lambda_{RH}^n(x_1, \dots, x_n) \rightarrow \dots \rightarrow \Lambda_{RH}^1(x_1, \dots, x_n) \rightarrow \Lambda_{RH}^0(x_1, \dots, x_n) = RH,$$

where $d(x_i) = \text{Res}(\lambda_i) = \lambda'_i$ ($0 < i < n$) and $d(x_n) = \text{Res}(\delta_n) = 2^s \delta_c$. Then, since $H_0(\Lambda_{RH}^*) = RH/\text{Im } d$, we have

$$H_0(\Lambda_{RH}^*) = Z[\lambda_1, \dots, \lambda_{c-1}, \delta_c, y]/(\lambda'_1, \dots, \lambda'_{n-1}, 2^s \delta_c, y^2 + 2y, y\delta_c + 2^c y).$$

In RH we have, recalling Lemma 1, Cor. 1, and 4.3, Cor. 1, that $\bar{\Lambda}^j = \theta \bar{\Lambda}^{2c-j}$ for all j and $\bar{\Lambda}^c = \bar{\Delta}_c^2 - (1+\theta)(1 + \bar{\Lambda}^1 + \dots + \bar{\Lambda}^{c-1})$ (Theorem 2). Also note that $y\bar{\Delta}_c = (\theta - 1)\bar{\Delta}_c = \bar{\Delta}_c - \bar{\Delta}_c = 0$.

Let $\lambda'[t] = \Lambda'[t] - (1+t)^{2n+1} = \sum_{i>0} \lambda'_i t^i$ ($\lambda'_i = \chi_i \lambda'[t]$, where χ_i stands for "coefficient of t^i " as usual). Using this and applying (9), we have

$$\lambda'[t] = (1+t\theta)^{2s+1}(\bar{\Lambda}[t] - f(t)),$$

where $f(t) = F(\theta, t)$, with (by definition)

$$F(z, t) = \frac{(1+t)^{2n+1}}{(1+zt)^{2s+1}}.$$

Thus

$$(28) \quad \chi_i(1+t\theta)^{-2s-1}\lambda'[t] = \chi_i(\bar{\Lambda}[t] - f(t)) = \bar{\Lambda}^i - \chi_i f(t), \quad i > 0.$$

Now in $H_0(\Lambda_{RH}^*)$ we have $\lambda'_i = 0$, $0 < i < n$. Since $\chi_i(1+t\theta)^{-2s-1}\lambda'[t]$ is a $Z[\theta]$ -linear combination of $\lambda'_1, \dots, \lambda'_{n-1}$, from (28) we have

$$(29) \quad \bar{\Lambda}^i - \chi_i f(t) = 0, \quad 0 < i < n.$$

But in RH , $\bar{\Lambda}^{c+i} = \theta \bar{\Lambda}^{c-i}$, $i \geq 0$. Substituting this in (29), we have, in $H_0(\Lambda_{RH}^*)$,

$$(30) \quad (\chi_{c+i} - \theta \chi_{c-i})f(t) = 0, \quad i \geq 0.$$

In particular,

$$(31) \quad y\chi_c f(t) = 0.$$

Our next objective is to replace the relations (30) and (31) by the single relation *by*, where

$$b = g.c.d. \left\{ 2^{2c} \binom{n}{c}, 2^{2i-1} \binom{n}{i}, i = c+1, \dots, n-1 \right\}.$$

For $i \geq 0$, $(\chi_{c+i} - \theta \chi_{c-i})F(z, t)$ is a polynomial, $h_i(z)$, say. Clearly, $h_i(1) = (\chi_{c+i} - \theta \chi_{c-i})(1+t)^{2c} = \binom{2c}{c+i} - \binom{2c}{c-i} = 0$. Since $h_i(1) = h_i(-1)$ modulo 2 and $\theta^2 = 1$,

$$\begin{aligned} h_i(\theta) &= \frac{1}{2}(h_i(1) + h_i(-1)) + \frac{\theta}{2}(h_i(1) - h_i(-1)) \\ &= \frac{1+\theta}{2}h_i(1) + \frac{1-\theta}{2}h_i(-1) = 0 - \frac{1}{2}yh_i(-1). \end{aligned}$$

Now $h_i(-1) = (\chi_{c+i} + \chi_{c-i})F_{s,c}$ where $F_{s,c} = (1+t)^{2c} \left(\frac{1+t}{1-t} \right)^{2s+1}$. It follows that

$$\begin{aligned} y\chi_c f(t) &= (\theta - 1)\chi_c f(t) = -(1 - \theta)\chi_c F(\theta, t) \\ &= -(\chi_c - \theta \chi_c)F(\theta, t) = -h_0(\theta) = \frac{1}{2}yh_0(-1) = y\chi_c F_{s,c}, \end{aligned}$$

and also

$$(\chi_{c+i} - \theta \chi_{c-i})f(t) = h_i(\theta) = -\frac{1}{2}yh_i(-1) = -\frac{1}{2}(\chi_{c+i} + \chi_{c-i})F_{s,c}.$$

Lemma 7. *With the above notation,*

(i) $\chi_c F_{s,c} = 2^{2c} \binom{s+c}{c}$, all $s \geq 0$, all $c \geq 0$.

(ii) For all $i > 0$ we have $(\chi_{c+i} + \chi_{c-i})F_{s,c} = \chi_{c+i}F_{s,c+i} - \binom{2i}{i}\chi_c F_{s,c} + Z$ -linear combination of $(\chi_{c+j} + \chi_{c-j})F_{s,c}$ ($0 < j < i$).

Proof. (i) By induction on $n = s + c$ starting with $n = 0$ and using

(a) $\chi_c F_{s,c} = \chi_c F_{s-1,c} + 4\chi_{c-1}F_{s,c-1}$ (this follows from $F_{s,c} - F_{s-1,c} = 4tF_{s,c-1}$),

(b) $\chi_0 F_{s,0} = 1$, all $s \geq 0$,

(c) $\chi_c F_{0,c} = \chi_c \left(\frac{(1+t)^{2c+1}}{1-t} \right) = 2^{2c}$, $\forall c \geq 0$ (this follows using $\frac{1}{1-t} = \sum_{i \geq 0} t^i$).

(ii) We have, for $i > 0$,

$$(1+t)^{2i} = 1 + t^{2i} + \sum_{0 < j < i} \binom{2i}{j} t^j (1 + t^{2(i-j)}).$$

Hence $(\chi_{c+i} + \chi_{c-i})F_{s,c} = \chi_{c+i}(1 + t^{2i})F_{s,c}$ and the result follows. $\square$

From the lemma we deduce (respectively from (i), (ii))

1. $y\chi_c f = 2^{2c} \binom{s+c}{c} y$.
2. For $i > 0$, $\frac{1}{2}h_i(-1) = 2^{2(c+i)-1} \binom{s+c+i}{i} - \frac{1}{2} \binom{2i}{i} 2^{2c} \binom{s+c}{c} + Z$ -linear combination of $\frac{1}{2}h_j(-1)$ ($0 < j < i$).

An easy inductive argument (using Pascal's triangle) now shows that

$$\begin{aligned} & g.c.d. \left\{ 2^{2c} \binom{s+c}{c}, \frac{1}{2}h_i(-1) \ (0 < i < n-c) \right\} \\ &= g.c.d. \left\{ 2^{2c} \binom{s+c}{c}, 2^{2(c+i)-1} \binom{s+c}{c+i} \ (0 < i < n-c) \right\}. \end{aligned}$$

Writing b for this, we conclude that the relations $y\chi_c f(t)$, $(\chi_{c+i} - \theta\chi_{c-i})f(t)$ ($0 < i < n-c$) may be replaced by the single relation by .

The next item on the agenda is the relation involving $\bar{\Delta}_c^2$. In $H_0(\Lambda_{RH}^*)$ we have

$$\begin{aligned} (32) \quad \bar{\Delta}_c^2 &= (1+\theta)(1+\bar{\Lambda}^1 + \cdots + \bar{\Lambda}^{c-1}) + \bar{\Lambda}^c \\ &= (1+\theta)(1+\chi_1 + \cdots + \chi_{c-1})f(t) + \chi_c f(t). \end{aligned}$$

But

$$\begin{aligned} (1+\theta)\chi_i f(t) &= \chi_i(1+\theta)f(t) = \chi_i(1+\theta)(1+t)^{2c} \left(\frac{1+t}{1+\theta t} \right)^{2s+1} \\ &= \chi_i(1+\theta)(1+t)^{2c} \end{aligned}$$

since $\theta(1+\theta) = (1+\theta)$ and using the binomial theorem. Substituting this in the relation (32) for $\bar{\Delta}_c^2$, we have

$$\begin{aligned} (33) \quad \bar{\Delta}_c^2 &= (1+\theta)(1+\chi_1 + \cdots + \chi_{c-1})(1+t)^{2c} + \chi_c f(t) \\ &= (1+\theta) \sum_{i=0}^{c-1} \binom{2c}{c} + \chi_c f(t) = (2+y) \frac{1}{2} \left[2^{2c} - \binom{2c}{c} \right] + \chi_c f(t). \end{aligned}$$

Now

$$\begin{aligned} \chi_c f &= \chi_c F(\theta, t) = \chi_c \left(\frac{F(1, t) + F(-1, t)}{2} \right) + \theta \chi_c \left(\frac{F(1, t) - F(-1, t)}{2} \right) \\ &= \frac{1}{2} \left(\binom{2c}{c} + 2^{2c} \binom{n}{c} \right) + \frac{1}{2} \left(\binom{2c}{c} - 2^{2c} \binom{n}{c} \right) \theta \end{aligned}$$

(using Lemma 7 for $F(-1, t)$). Hence from (33) in $H_0(\Lambda_{RH}^*)$ we have

$$\bar{\Delta}_c^2 = 2^{2c-1} y \left[1 - \binom{n}{c} \right] + 2^{2c},$$

or, equivalently,

$$\delta_c^2 + 2^{c+1} \delta_c = 2^{2c-1} y \left[1 - \binom{n}{c} \right].$$

Thus,

$$K^0(X_{m,k}) = Z[\bar{\Delta}_c, y]/(y^2 + 2y, \bar{\Delta}_c y, 2^s \delta_c, by, \bar{\Delta}_c^2 - D),$$

where $D = 2^{2c} + 2^{2c-1} \left\{ 1 - \binom{n}{c} \right\} y$.

Let R be the ring $Z[y]/(y^2 + 2y, by)$. Then

$$K^0(X_{m,k}) = R[\bar{\Delta}_c]/(y\bar{\Delta}_c, 2^s \delta_c, \bar{\Delta}_c^2 - D).$$

An easy calculation shows that $R = Z \oplus Z_b y$ (Z_b being Z/b).

Consider the inclusion $R + R\bar{\Delta}_c$ in $R[\bar{\Delta}_c]$. This induces an epimorphism $R + R\bar{\Delta}_c \rightarrow K^0(X_{m,k})$ (because of the relations $y\bar{\Delta}_c, \bar{\Delta}_c^2 - D$). Now $r_0 + r_1\bar{\Delta}_c$ lies in the kernel if and only if $r_0 + r_1\bar{\Delta}_c = Fy\bar{\Delta}_c + G(2^s\bar{\Delta}_c - 2^n) + H(\bar{\Delta}_c^2 - D)$, where $r_0, r_1 \in R$ and $F, G, H \in R[\bar{\Delta}_c]$. Writing $F = F_0(\bar{\Delta}_c^2) + \bar{\Delta}_c F_1(\bar{\Delta}_c^2)$ and similarly for G (note that any $F, G \in R[\bar{\Delta}_c]$ can be written this way), we see that the expression

$$\begin{aligned} & (F_0(\bar{\Delta}_c^2) + \bar{\Delta}_c F_1(\bar{\Delta}_c^2)) \bar{\Delta}_c y + (G_0(\bar{\Delta}_c^2) + \bar{\Delta}_c G_1(\bar{\Delta}_c^2)) (2^s \bar{\Delta}_c - 2^n) \\ & - \{F_0(D)\bar{\Delta}_c y + DF_1(D)y + 2^s G_0(D)\bar{\Delta}_c - 2^n \bar{\Delta}_c G_1(D) - 2^n G_0(D) + 2^s DG_1(D)\} \end{aligned}$$

is divisible by $\bar{\Delta}_c^2 - D$. Thus, for a suitable choice of H ,

$$r_0 + r_1\bar{\Delta}_c = F_0(D)\bar{\Delta}_c y + G_0(D) \{2^s \bar{\Delta}_c - 2^n\} + G_1(D) \{2^s D - 2^n \bar{\Delta}_c\}$$

(here we used the fact that $Dy = 0$ in R , which follows easily from $by = 0$).

Now $F_0(D), G_0(D), G_1(D)$ are arbitrary elements of R . Hence the kernel above is generated over Z by $\bar{\Delta}_c y, \bar{\Delta}_c y^2, 2^s \bar{\Delta}_c - 2^n, 2^s \bar{\Delta}_c y - 2^n y, 2^s D - 2^n \bar{\Delta}_c$ and $2^s Dy - 2^n \bar{\Delta}_c y$, or, equivalently, by $\bar{\Delta}_c y, 2^s \delta_c, 2^n y, 2^s D - 2^n \bar{\Delta}_c$, that is, by $\bar{\Delta}_c y, 2^s \delta_c, 2^n y, 2^{s+2c-1} \{1 - \binom{n}{c}\} y = 2^{n+c-1} [1 - \binom{n}{c}] y$. But the last term is zero even if $s = 0$ (for then $\binom{n}{c} = 1$). Thus,

$$\begin{aligned} K^0(X_{m,k}) &= (R + R\bar{\Delta}_c) / (2^s \delta_c, \Delta_c y, 2^n y) = (R + Z\bar{\Delta}_c) / (2^s \delta_c, 2^n y) \\ &= (Z + Z_b y + Z\delta_c) / (2^s \delta_c, 2^n y) = Z + Z_{2^\alpha} y + Z_{2^s} \delta_c, \end{aligned}$$

where

$$2^\alpha = g.c.d. \left\{ 2^n; 2^{2c} \binom{n}{c}; 2^{2(c+i)-1} \binom{n}{c+i}, 0 < i < n-c \right\}.$$

Furthermore, $y^2 = -2y$, $y\delta_c = -2^c y$, and $\delta_c^2 = -2^{c+1}\delta_c + 2^{2c-1}[1 - \binom{n}{c}]y$.

We end by observing that:

1. The generators occurring in Theorems 4 and 5 may be described geometrically via the α and β constructions, as in [10] and [12].
2. Using the Gysin sequence, $K^*(X_{2n,2s})$ may be deduced from $K^*(X_{2n,2s-1})$ as in [7], but $K^*(X_{2n+1,2s+1})$ cannot be similarly deduced, as $X_{2n+1,2s+1}$ is non-orientable.
3. In the special cases $X_{2n,2n-1} = PO(2n)$ and $X_{m,1} = RP^{m-1}$ these results reduce to those of [9], [10] and [1]. For $X_{4n,2k-1}$ they agree with [2].

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