A PRODUCT FORMULA FOR SPHERICAL REPRESENTATIONS OF A GROUP OF AUTOMORPHISMS OF A HOMOGENEOUS TREE, II

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ABSTRACT. Let $G = \text{Aut}(T)$ be the group of automorphisms of a homogeneous tree T and let π be the tensor product of two spherical irreducible unitary representations of G . We complete the explicit decomposition of π commenced in part I of this paper, by describing the discrete series representations of G which appear as subrepresentations of π .

1. INTRODUCTION AND NOTATION

Let G be the group of automorphisms of a homogeneous tree T . We fix a vertex o of T , and let $K = \{g \in G : go = o\}$. We consider the tensor product π of two spherical irreducible unitary representations of $G = \text{Aut}(T)$. As with any continuous unitary representation of a type I group, π can be written in an essentially unique way as a direct integral $\int_{\hat{G}} \sigma \, dm(\sigma)$. The representation space H_π of π can be decomposed as an orthogonal direct sum

$$(1.1) \quad H_\pi = H_1 + H_2,$$

of π -invariant subspaces H_1 and H_2 , where H_1 is the closed linear span in H_π of the set of vectors $\pi(g)\xi$, where $g \in G$ and where ξ is K -invariant. The H_1 component π_1 of π was completely described in [1]. The H_2 component π_2 of π must be a direct sum

$$(1.2) \quad \pi_2 = \sum_k m_k \sigma_k,$$

over the distinct discrete series representations σ_k of G , where $m_k \in \{0, 1, \dots, \infty\}$ for each k . In the present paper we describe π_2 explicitly.

The discrete series representations of G were first described by Ol'shanskii [4]. The book [2] provides a clear exposition and re-working of [4]. Let us give a quick summary here.

A finite subtree $\mathfrak{x}$ of T is called *complete* if, for each vertex x of $\mathfrak{x}$, either all $q+1$ neighbors of x in T lie in $\mathfrak{x}$ or exactly one neighbor of x in T lies in $\mathfrak{x}$. According to these two possibilities, we say that x is an *interior point* or *boundary point* of $\mathfrak{x}$, and write $x \in \text{Int}(\mathfrak{x})$ and $x \in \partial\mathfrak{x}$, respectively. Given such a subtree $\mathfrak{x}$, let

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$K(\mathfrak{x}) = \{g \in G : gx = x \text{ for all } x \in \mathfrak{x}\}$ and $\tilde{K}(\mathfrak{x}) = \{g \in G : g\mathfrak{x} = \mathfrak{x}\}$. If π is an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$, let

$$\mathcal{H}_{\mathfrak{x}} = \{\xi \in \mathcal{H} : \pi(g)\xi = \xi \text{ for all } g \in K(\mathfrak{x})\},$$

and let

$$P_{\mathfrak{x}} = \frac{1}{m(K(\mathfrak{x}))} \int_{K(\mathfrak{x})} \pi(g) dg,$$

the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{\mathfrak{x}}$. Here m and dg refer to the left (and right) Haar measure on G , normalized by requiring that $m(K) = 1$. It turns out that $P_{\mathfrak{x}} \neq 0$ for some $\mathfrak{x}$, and if we choose $\mathfrak{x}$ so that the number $|\mathfrak{x}|$ of vertices of $\mathfrak{x}$ is minimal, then $\mathfrak{x}$ is called a *minimal subtree* for π . Note that if $g \in G$, then $K(g\mathfrak{x}) = gK(\mathfrak{x})g^{-1}$ and $P_{g\mathfrak{x}} = \pi(g)P_{\mathfrak{x}}\pi(g^{-1})$. If $\mathfrak{x}$ is a minimal subtree for π , then the other minimal subtrees $\mathfrak{x}'$ for π are precisely the subtrees $g\mathfrak{x}$, i.e., the subtrees in the G -orbit $[\mathfrak{x}] = \{g\mathfrak{x} : g \in G\}$ of $\mathfrak{x}$ [2, Cor. III.3.4].

When π has a minimal subtree consisting of only one vertex, it is spherical. In the next smallest case, $\mathfrak{x}$ consists of two neighboring vertices (and the edge joining them). Up to equivalence, there are precisely two irreducible unitary representations of G with such a minimal subtree; they are called the *special* representations of G (see [2, §III.2]). All irreducible unitary representations of G with minimal subtree $\mathfrak{x}$ satisfying $|\mathfrak{x}| > 2$ are equivalent to certain induced representations; one starts with an irreducible representation σ_0 of the finite group $\text{Aut}(\mathfrak{x})$ on a (finite dimensional) Hilbert space $\mathcal{H}_{\sigma_0}$ such that, for each complete subtree $\eta \subsetneq \mathfrak{x}$, $\sigma_0(g)\xi = \xi$ for all $g \in K_{\mathfrak{x}}(\eta) = \{g \in \text{Aut}(\mathfrak{x}) : gy = y \text{ for all } y \in \eta\}$ holds for no nonzero $\xi \in \mathcal{H}_{\sigma_0}$. Let σ be the representation of $\tilde{K}(\mathfrak{x})$ obtained as the composition of the natural surjection $\tilde{K}(\mathfrak{x}) \rightarrow \text{Aut}(\mathfrak{x})$ (the kernel of which is $K(\mathfrak{x})$) and σ_0 . The set of all equivalence classes of representations σ of $\tilde{K}(\mathfrak{x})$ obtained in this way is denoted by $(\tilde{K}(\mathfrak{x}))_0^{\wedge}$. One then forms the induced representation $\text{Ind}(\sigma)$ of G . It is shown in [2, Theorem III.3.14] that $\text{Ind}(\sigma)$ is irreducible, and that $\sigma \mapsto \text{Ind}(\sigma)$ induces a bijection from $(\tilde{K}(\mathfrak{x}))_0^{\wedge}$ to the set of equivalence classes of irreducible unitary representations of G with minimal subtree $\mathfrak{x}$.

The principal result of this paper is the following:

Theorem. *Let π be the tensor product of two spherical irreducible unitary representations of G . Then the irreducible unitary representations of G appearing in the direct sum (1.2) occur with multiplicity 1, and have minimal trees of one of the types in Figure 1. Moreover,*

- (a) *both special representations occur;*
- (b) *exactly two representations occur with the minimal subtree of Figure 1(b), unless $q = 2$, when there is only one;*
- (c) *for each $r \geq 1$, exactly two representations occur with the minimal subtree of Figure 1(c_r).*

In (b) and (c), the specific representations occurring are described in Sections 3 and 4 below.

Notice that the decomposition (1.2) does not depend on the particular parameters s_1, s_2 , which may correspond to either the principal or complementary series.

Our theorem should be compared with the corresponding result for $SL(2, \mathbb{R})$. While here a rather thin part of the discrete series occurs in (1.2), in the $SL(2, \mathbb{R})$ case half of the discrete series occurs in the corresponding decomposition (see [5]).

FIGURE 1.

In order to cover both the principal and complementary series simultaneously, let us introduce some more notation. Recall that (assuming $s^2 \neq q, 1/q$), $J_s : \mathcal{K}(\Omega) \rightarrow \mathcal{K}(\Omega)$ is a linear bijection and that $J_s \circ \pi^s(g) = \pi^{s^{-1}}(g) \circ J_s$ for all $g \in G$. For $x \in T$, $\xi_x \in \mathcal{K}(\Omega)$ was defined before Lemma 5.3 in [1]. We shall again here need the fact that

$$(1.3) \quad J_s \xi_x = j_n(s) \xi_x \quad \text{if } n = d(o, x),$$

where $j_0(s) = 1$ and $j_n(s) = s^{2(n-1)}(qs^2 - 1)/(q - s^2)$ for $s \geq 1$. If $s^2 \in (1/q, q)$, the inner product $\langle \cdot, \cdot \rangle_s$ on $\mathcal{K}(\Omega)$ is defined by $\langle f_1, f_2 \rangle_s = \langle f_1, J_s(f_2) \rangle$, where $\langle f_1, f_2 \rangle = \int_{\Omega} f_1 \bar{f}_2 \, d\nu_o$. Let I denote the identity operator on $\mathcal{K}(\Omega)$, and set

$$\tilde{J}_s = \begin{cases} J_s & \text{if } s^2 \in (1/q, q), \\ I & \text{if } |s| = 1. \end{cases}$$

Let $\langle \cdot, \cdot \rangle_s = \langle \cdot, \cdot \rangle$ if $|s| = 1$. Then $\langle f_1, f_2 \rangle_s = \langle f_1, \tilde{J}_s f_2 \rangle$ for all $f_1, f_2 \in \mathcal{K}(\Omega)$ if s is a parameter corresponding to either the principal or complementary series. Moreover, $\tilde{J}_s \circ \pi^s(g) = \pi^{\tilde{s}}(g) \circ \tilde{J}_s$ for all $g \in G$, where

$$\tilde{s} = \begin{cases} s^{-1} & \text{if } s^2 \in (1/q, q), \\ s & \text{if } |s| = 1. \end{cases}$$

If s_1, s_2 are parameters corresponding to either the principal or complementary series, let

$$\tilde{J}_{s_1, s_2} = \tilde{J}_{s_1} \otimes \tilde{J}_{s_2} : \mathcal{K}(\Omega \times \Omega) \rightarrow \mathcal{K}(\Omega \times \Omega).$$

Then

$$\tilde{J}_{s_1, s_2} \circ \pi^{s_1, s_2}(g) = \pi^{\tilde{s}_1, \tilde{s}_2}(g) \circ \tilde{J}_{s_1, s_2},$$

where $\pi^{s_1, s_2}(g) = \pi^{s_1}(g) \otimes \pi^{s_2}(g)$. Also, $\pi^{s_1, s_2}(g)$ preserves the inner product

$$\langle F_1, F_2 \rangle_{s_1, s_2} = \langle F_1, \tilde{J}_{s_1, s_2}(F_2) \rangle',$$

where $\langle F_1, F_2 \rangle' = \int_{\Omega \times \Omega} F_1 \bar{F}_2 \, d(\nu_o \times \nu_o)$. We shall also simply write $\pi(g)$ in place of $\pi^{s_1, s_2}(g)$, when s_1, s_2 are understood. The representation space $H_{\pi} = \mathcal{H}_{s_1, s_2}$ of π is the completion of $\mathcal{K}(\Omega \times \Omega)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{s_1, s_2}$.

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2. THE POSSIBLE MINIMAL SUBTREES

We start by describing, for each finite complete subtree $\mathfrak{r}$ of T ,

$$\mathcal{H}'_{\mathfrak{r}} = \{\xi \in \mathcal{H}_{s_1, s_2} : (1) P_{\mathfrak{r}}\xi = \xi, \text{ and } (2) P_{\mathfrak{h}}\xi = 0 \text{ for all } \mathfrak{h} \subsetneq \mathfrak{r}\},$$

where the $\mathfrak{h}$'s here are complete subtrees of T . Note that if $g \in G$, then $\mathcal{H}'_{g(\mathfrak{r})} = \pi(g)(\mathcal{H}'_{\mathfrak{r}})$, and so we may assume that $o \in \mathfrak{r}$. If $|\mathfrak{r}| > 2$, then $\mathfrak{r}$ has interior points, and in this case we may suppose that o is one of them.

Proposition 2.1. *Suppose that $|\mathfrak{r}| > 2$ and that $o \in \text{Int}(\mathfrak{r})$. Then $\mathcal{H}'_{\mathfrak{r}}$ is a finite dimensional subspace of $\mathcal{K}(\Omega \times \Omega)$. Assuming that $\mathcal{H}'_{\mathfrak{r}} \neq \{0\}$, then either*

- (i) $\mathfrak{r}$ is as in Figure 1(b); in this case $\mathcal{H}'_{\mathfrak{r}}$ consists of the functions

$$(2.1) \quad \sum_{\substack{i,j=0 \\ i \neq j}}^q c_{i,j} \xi_{a_i} \otimes \xi_{a_j},$$

where the coefficients $c_{i,j}$ satisfy

$$(2.2) \quad \sum_{\substack{j=0 \\ j \neq i}}^q c_{i,j} = 0 \quad \text{for each } i, \text{ and } \sum_{\substack{i=0 \\ i \neq j}}^q c_{i,j} = 0 \quad \text{for each } j,$$

or

- (ii) replacing $\mathfrak{r}$ by $g\mathfrak{r}$ for some $g \in G$, if necessary, $\mathfrak{r}$ is as in Figure 1(c_r); in this case $\mathcal{H}'_{\mathfrak{r}}$ consists of the functions

$$(2.3) \quad \sum_{i,j=1}^q c_{i,j} \xi_{a_i} \otimes \xi_{y_j} + \sum_{i,j=1}^q d_{i,j} \xi_{y_j} \otimes \xi_{a_i},$$

where the coefficients $c_{i,j}$ and $d_{i,j}$ satisfy

$$(2.4) \quad \sum_{j=1}^q c_{i,j} = 0 = \sum_{j=1}^q d_{i,j} \quad \text{for each } i, \text{ and } \sum_{i=1}^q c_{i,j} = 0 = \sum_{i=1}^q d_{i,j} \quad \text{for each } j.$$

Proof. For each $v \in \partial\mathfrak{r}$ and $n \geq |v|$ (where $|v| = d(o, v)$), let

$$(2.5) \quad D_{v,n} = \{(\omega_1, \omega_2) \in \Omega_o(v) \times \Omega_o(v) : \omega_1 \neq \omega_2 \text{ and } d(o, \omega_1 \wedge \omega_2) = n\};$$

we also let

$$\Delta_v = \{(\omega_1, \omega_2) \in \Omega_o(v) \times \Omega_o(v) : \omega_1 = \omega_2\}$$

for each $v \in \partial\mathfrak{r}$. Then

$$\Omega \times \Omega = \bigcup_{v \in \partial\mathfrak{r}} \Delta_v \cup \bigcup_{\substack{v \in \partial\mathfrak{r} \\ n \geq |v|}} D_{v,n} \cup \bigcup_{\substack{v,w \in \partial\mathfrak{r} \\ v \neq w}} \Omega_o(v) \times \Omega_o(w),$$

a disjoint union. Let $F \in \mathcal{K}(\Omega \times \Omega)$ satisfy $P_{\mathfrak{r}}F = F$. As $o \in \mathfrak{r}$, this simply means that $F(g^{-1}\omega_1, g^{-1}\omega_2) = F(\omega_1, \omega_2)$ for all $g \in K(\mathfrak{r})$ and all $\omega_1, \omega_2 \in \Omega$. It follows that F is constant on each of the above sets Δ_v , $D_{v,n}$ and $\Omega_o(v) \times \Omega_o(w)$. Hence we can write

$$(2.6) \quad F = \sum_{v \in \partial\mathfrak{r}} a_v \mathbf{1}_{\Delta_v} + \sum_{\substack{v \in \partial\mathfrak{r} \\ n \geq |v|}} b_{v,n} \mathbf{1}_{D_{v,n}} + \sum_{\substack{v,w \in \partial\mathfrak{r} \\ v \neq w}} c_{v,w} \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)}.$$

Moreover, the fact that $F \in \mathcal{K}(\Omega \times \Omega)$ implies that for each $v \in \partial \mathfrak{r}$ there is an n_v such that $b_{v,n} = a_v$ for all $n \geq n_v$. Conversely, any function F of the form (2.6) which satisfies this last condition is in $\mathcal{K}(\Omega \times \Omega)$ and satisfies $P_{\mathfrak{r}}F = F$. Thus each $F \in \mathcal{K}(\Omega \times \Omega)$ satisfying $P_{\mathfrak{r}}F = F$ can be written in the form

$$(2.7) \quad F = \sum_{v \in \partial \mathfrak{r}} \left\{ \sum_{n=|v|}^M b_{v,n} \mathbf{1}_{D_{v,n}} + a_v \mathbf{1}_{D'_{v,M}} \right\} + \sum_{\substack{v, w \in \partial \mathfrak{r} \\ v \neq w}} c_{v,w} \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)}$$

for some integer $M \geq \max\{|v| : v \in \partial \mathfrak{r}\}$. Here

$$(2.8) \quad D'_{v,M} = \{(\omega_1, \omega_2) \in \Omega_o(v) \times \Omega_o(v) : \omega_1 = \omega_2, \text{ or } \omega_1 \neq \omega_2 \text{ and } d(o, \omega_1 \wedge \omega_2) > M\}.$$

Suppose $\xi \in \mathcal{H}_{s_1, s_2}$ and $P_{\mathfrak{r}}\xi = \xi$. Let us write down a sequence (F_N) in $\mathcal{K}(\Omega \times \Omega)$ such that $P_{\mathfrak{r}}F_N = F_N$ for each N and such that, for each $f \in \mathcal{K}(\Omega \times \Omega)$, $\langle F_N, f \rangle_{s_1, s_2} \rightarrow \langle \xi, f \rangle_{s_1, s_2}$ as $N \rightarrow \infty$. If $\xi \in \mathcal{H}_{s_1, s_2}$, and if $F \in \mathcal{K}(\Omega \times \Omega)$, it is convenient to use the notation

$$(2.9) \quad \langle \xi, F \rangle' = \langle \xi, \tilde{J}_{s_1, s_2}^{-1} F \rangle_{s_1, s_2}.$$

For $v, w \in \partial \mathfrak{r}$, $n \geq |v|$ and $N \geq \max\{|v| : v \in \partial \mathfrak{r}\}$, let

$$\begin{aligned} b_{v,n} &= \langle \xi, \mathbf{1}_{D_{v,n}} \rangle' / (\nu_o \times \nu_o)(D_{v,n}), \\ c_{v,w} &= \langle \xi, \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)} \rangle' / (\nu_o \times \nu_o)(\Omega_o(v) \times \Omega_o(w)), \text{ and} \\ \delta_{v,N} &= \langle \xi, \mathbf{1}_{D'_{v,N}} \rangle' / (\nu_o \times \nu_o)(D'_{v,N}). \end{aligned}$$

Form

$$(2.10) \quad F_N = \sum_{v \in \partial \mathfrak{r}} \left\{ \sum_{n=|v|}^N b_{v,n} \mathbf{1}_{D_{v,n}} + \delta_{v,N} \mathbf{1}_{D'_{v,N}} \right\} + \sum_{\substack{v, w \in \partial \mathfrak{r} \\ v \neq w}} c_{v,w} \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)}.$$

Then it is routine to check that if F is as in (2.7), then

$$\langle F_N, F \rangle' = \langle \xi, F \rangle' \quad \text{once } N \geq M.$$

It follows that $\langle F_N, f \rangle_{s_1, s_2} \rightarrow \langle \xi, f \rangle_{s_1, s_2}$ as $N \rightarrow \infty$ if $f \in \mathcal{K}(\Omega \times \Omega)$ satisfies $P_{\mathfrak{r}}f = f$. To see that this holds for arbitrary $f \in \mathcal{K}(\Omega \times \Omega)$, first note that, because $o \in \mathfrak{r}$,

$$(\pi^{s_1, s_2}(g)f)(\omega_1, \omega_2) = f(g^{-1}\omega_1, g^{-1}\omega_2) = (\pi^{\tilde{s}_1, \tilde{s}_2}(g)f)(\omega_1, \omega_2)$$

for all $g \in K(\mathfrak{r})$ and $\omega_1, \omega_2 \in \Omega$, and hence

$$(2.11) \quad \pi^{s_1, s_2}(g)(\tilde{J}_{s_1, s_2}^{-1}(f)) = \tilde{J}_{s_1, s_2}^{-1}(\pi^{\tilde{s}_1, \tilde{s}_2}(g)(f)) = \tilde{J}_{s_1, s_2}^{-1}(\pi^{s_1, s_2}(g)(f)).$$

Thus $P_{\mathfrak{r}}\tilde{J}_{s_1, s_2}^{-1}f = \tilde{J}_{s_1, s_2}^{-1}(P_{\mathfrak{r}}f)$. Hence for any $f \in \mathcal{K}(\Omega \times \Omega)$,

$$\langle \xi, f \rangle' = \langle P_{\mathfrak{r}}\xi, f \rangle' = \langle \xi, P_{\mathfrak{r}}f \rangle',$$

because $P_{\mathfrak{r}}$ is Hermitian with respect to $\langle \cdot, \cdot \rangle_{s_1, s_2}$. Note also that the projection $P_{\mathfrak{r}}$ leaves $\mathcal{K}(\Omega \times \Omega)$ invariant. For the same reasons, $\langle F_N, f \rangle' = \langle P_{\mathfrak{r}}F_N, f \rangle' = \langle F_N, P_{\mathfrak{r}}f \rangle'$. It follows that $\langle F_N, f \rangle' \rightarrow \langle \xi, f \rangle'$ as $N \rightarrow \infty$ for any $f \in \mathcal{K}(\Omega \times \Omega)$.

We next show that if $\xi \in \mathcal{H}'_{\mathfrak{r}}$, then all the coefficients $b_{v,n}$ and $\delta_{v,N}$ in (2.10) are zero. If $v \in \partial \mathfrak{r}$, there is clearly a complete subtree $\mathfrak{v} \subsetneq \mathfrak{r}$ containing o and v (because $|\mathfrak{r}| > 2$). It follows from (2.11) that, if $n \geq |v|$,

$$\pi^{s_1, s_2}(g)(\tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D_{v,n}})) = \tilde{J}_{s_1, s_2}^{-1}(\pi^{\tilde{s}_1, \tilde{s}_2}(g)(\mathbf{1}_{D_{v,n}})) = \tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D_{v,n}})$$

for all $g \in K(\mathfrak{h})$, and so $P_{\mathfrak{h}}(\tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D_{v,n}})) = \tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D_{v,n}})$. Thus

$$\begin{aligned} 0 &= \langle P_{\mathfrak{h}}\xi, \mathbf{1}_{D_{v,n}} \rangle' \\ &= \langle \xi, P_{\mathfrak{h}}(\tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D_{v,n}})) \rangle_{s_1, s_2} \\ &= \langle \xi, \tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D_{v,n}}) \rangle_{s_1, s_2} \\ &= (\nu_o \times \nu_o)(D_{v,n})b_{v,n}, \end{aligned}$$

so that $b_{v,n} = 0$. Similarly $P_{\mathfrak{h}}(\tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D'_{v,N}})) = \tilde{J}_{s_1, s_2}^{-1}(\mathbf{1}_{D'_{v,N}})$, and so $\delta_{v,N} = 0$. Hence F_N , defined in (2.10), equals

$$(2.12) \quad F = \sum_{\substack{v, w \in \partial \mathfrak{r} \\ v \neq w}} c_{v,w} \mathbf{1}_{\Omega_o(v) \times \Omega_o(w)} \in \mathcal{K}(\Omega \times \Omega).$$

But as $\langle F_N, f \rangle' \rightarrow \langle \xi, f \rangle'$, we see that $\langle F, f \rangle' = \langle \xi, f \rangle'$ for each $f \in \mathcal{K}(\Omega \times \Omega)$, and so $\xi = F \in \mathcal{K}(\Omega \times \Omega)$.

If $X \subset T$ is a finite set of vertices, let $\mathfrak{h}(X)$ denote the smallest complete subtree of T containing X . As we have already noted, each projection $P_{\mathfrak{h}}$ leaves $\mathcal{K}(\Omega \times \Omega)$ invariant, and when $o \in \mathfrak{h}$ and $f \in \mathcal{K}(\Omega \times \Omega)$ we have

$$(P_{\mathfrak{h}}f)(\omega_1, \omega_2) = \frac{1}{m(K(\mathfrak{h}))} \int_{K(\mathfrak{h})} f(g^{-1}\omega_1, g^{-1}\omega_2) dg.$$

We apply this formula to $f = F$, where $F \in \mathcal{H}'_{\mathfrak{r}}$ is given by (2.12). Observe that if $v, w \in \partial \mathfrak{r}$ are distinct, and if $\mathfrak{h} = \mathfrak{h}(\{o, v, w\})$ is a proper subtree of $\mathfrak{r}$, then $c_{v,w} = 0$. For we can choose $(\omega_1, \omega_2) \in \Omega_o(v) \times \Omega_o(w)$, and then $(g^{-1}\omega_1, g^{-1}\omega_2) \in \Omega_o(v) \times \Omega_o(w)$ for all $g \in K(\mathfrak{h})$. Hence $F(g^{-1}\omega_1, g^{-1}\omega_2) = c_{v,w}$ for all $g \in K(\mathfrak{h})$, so $0 = (P_{\mathfrak{h}}F)(\omega_1, \omega_2) = c_{v,w}$.

Assuming $\mathcal{H}'_{\mathfrak{r}} \neq \{0\}$, choose a nonzero $F \in \mathcal{H}'_{\mathfrak{r}}$. We know that F has the form (2.12), and as $F \neq 0$, we must have distinct $v, w \in \partial \mathfrak{r}$ such that $c_{v,w} \neq 0$. Hence $\mathfrak{h} = \mathfrak{h}(\{o, v, w\})$ must equal $\mathfrak{r}$. If $o \notin [v, w]$, then o is not an interior point of $\mathfrak{h}$, contradicting $o \in \text{Int}(\mathfrak{r})$ and $\mathfrak{h} = \mathfrak{r}$. Hence $o \in [v, w]$, and so $\mathfrak{h} = \mathfrak{h}(\{v, w\})$.

Suppose first that $\mathfrak{r}$ is the subtree of Figure 1(b). In this case we now know that $\mathcal{H}'_{\mathfrak{r}}$ consists of functions

$$F = \sum_{\substack{i, j=0 \\ i \neq j}}^q c_{i,j} \mathbf{1}_{\Omega_o(a_i)} \otimes \mathbf{1}_{\Omega_o(a_j)}.$$

The coefficients $c_{i,j}$ must satisfy the conditions of (2.2). For if we fix $i \in \{0, \dots, q\}$ and let $\mathfrak{h}$ consist of o , a_i and the edge joining them, then we see that $K(\mathfrak{h})$ acts transitively on $\{a_j : j \neq i\}$. Fix some $j \neq i$, and let $(\omega_1, \omega_2) \in \Omega_o(a_i) \times \Omega_o(a_j)$. For each $r \neq i$ choose $k_r \in K(\mathfrak{h})$ such that $a_j = k_r a_r$. Then $0 = (P_{\mathfrak{h}}F)(\omega_1, \omega_2)$ is the average of the integrals over the q cosets $K'k_r$ of K' in $K(\mathfrak{h})$. Here $K' = \{k \in G : ko = o, ka_i = a_i \text{ and } ka_j = a_j\}$. As $F(g^{-1}\omega_1, g^{-1}\omega_2) = c_{i,r}$ for all $g \in K'k_r$, we see that the first of the conditions (2.2) holds. Similarly, fixing $(\omega_1, \omega_2) \in \Omega_o(a_j) \times \Omega_o(a_i)$, where $i \neq j$, we find that the second of the conditions (2.2) holds. Using $\xi_{a_i} = (q+1)\mathbf{1}_{\Omega_o(a_i)} - \mathbf{1}$, and (2.2), we find that

$$\sum_{\substack{i, j=0 \\ i \neq j}}^q c_{i,j} \mathbf{1}_{\Omega_o(a_i)} \otimes \mathbf{1}_{\Omega_o(a_j)} = \frac{1}{(q+1)^2} \sum_{\substack{i, j=0 \\ i \neq j}}^q c_{i,j} \xi_{a_i} \otimes \xi_{a_j},$$

and so we have shown that F can be written in the form (2.1), with the new $c_{i,j}$'s, namely the $c_{i,j}/(q+1)^2$'s satisfying (2.2). This completes the proof of part (i) of the proposition.

Now suppose that $\mathfrak{x}$ is not the subtree of Figure 1(b), nor in its G -orbit. We know that $\mathfrak{x} = \eta(\{v, w\})$ for some $v, w \in \partial\mathfrak{x}$ such that $o \in [v, w]$. Replacing $\mathfrak{x}$ by $g\mathfrak{x}$ for some $g \in G$, we may suppose that $\mathfrak{x}$ is the subtree of the type described in Figure 1(c_r). Let $F \in \mathcal{H}_{\mathfrak{x}}$. We know that F is of the form (2.12). Now $c_{v,w} = 0$ for any distinct $v, w \in \partial\mathfrak{x}$ unless $v = a_i$ and $w = y_j$ or vice versa for some $i, j \in \{1, \dots, q\}$. For otherwise o, v and w are contained in a complete subtree $\eta \subsetneq \mathfrak{x}$, which implies that $c_{v,w} = 0$, as we saw above. So

$$(2.13) \quad F = \sum_{i,j=1}^q c_{i,j} \mathbf{1}_{\Omega_o(a_i)} \otimes \mathbf{1}_{\Omega_o(y_j)} + \sum_{i,j=1}^q d_{i,j} \mathbf{1}_{\Omega_o(y_j)} \otimes \mathbf{1}_{\Omega_o(a_i)}$$

for suitable constants $c_{i,j}$ and $d_{i,j}$. Fix i and j , let $(\omega_1, \omega_2) \in \Omega_o(a_i) \times \Omega_o(y_j)$, and let η be the complete subtree obtained by deleting the y_k 's and the edges $\{o, y_k\}$. Then $(P_{\eta}F)(\omega_1, \omega_2) = 0$ implies that $\sum_{r=1}^q c_{i,r} = 0$. Similarly, if we let η be the complete subtree obtained by deleting the a_k 's and the edges $\{o, a_k\}$, we obtain $\sum_{r=1}^q c_{r,j} = 0$. The same relations on the $d_{i,j}$'s are obtained by fixing $(\omega_1, \omega_2) \in \Omega_o(y_j) \times \Omega_o(a_i)$ and using the same η 's.

Now using these conditions on the $c_{i,j}$'s and $d_{i,j}$'s, together with $\xi_{a_i} = (q+1)\mathbf{1}_{\Omega_o(a_i)} - \mathbf{1}$ and $\xi_{y_j} = N_{r+1}\mathbf{1}_{\Omega_o(y_j)} - N_r\mathbf{1}_{\Omega_o(z)}$, we can write the sum in (2.13) in the form (2.3), where the (new) $c_{i,j}$'s and $d_{i,j}$'s satisfy (2.4). $\square$

Lemma 2.2. *For each $\mathfrak{x}$ such that $|\mathfrak{x}| > 1$, $\mathcal{H}'_{\mathfrak{x}}$ is contained in H_2 .*

Proof. Fix $\xi \in \mathcal{H}'_{\mathfrak{x}}$, and define $u : G \rightarrow \mathbb{C}$ by

$$u(g) = \langle \pi(g)\xi, \mathbf{1} \rangle_{s_1, s_2}.$$

Then $u(k_1 g k_2) = u(g)$ for all $g \in G$, $k_1 \in K$ and $k_2 \in K(\mathfrak{x})$. Also, if η denotes a finite complete subtree of T , then

$$(2.14) \quad \int_{K(\eta)} u(gk) dk = 0 \quad \text{for all } g \in G, \text{ if } \eta \subsetneq \mathfrak{x}.$$

Suppose first that we are not in the special case. Then because u is a bi- $K(\mathfrak{x})$ -invariant function satisfying (2.14), it must be supported in $\tilde{K}(\mathfrak{x})$ [2, Prop. III.3.2]. In Case (i) of Proposition 2.1 above, $\tilde{K}(\mathfrak{x}) = K$. As u is left K -invariant, it must be constant on its support. But $\eta = \{o\} \subsetneq \mathfrak{x}$, and so

$$u(1) = \langle \xi, \mathbf{1} \rangle_{s_1, s_2} = \langle \xi, P_{\eta}\mathbf{1} \rangle_{s_1, s_2} = \langle P_{\eta}\xi, \mathbf{1} \rangle_{s_1, s_2} = \langle 0, \mathbf{1} \rangle_{s_1, s_2} = 0,$$

and so $u = 0$.

If we are in Case (ii) of Proposition 2.1, then u is supported on $\tilde{K}(\mathfrak{x}) = (K_o \cap K_z) \cup (K_o \cap K_z)g_0$, where $g_0 o = z$ and $g_0 z = o$, and where $K_x = \{g \in G : gx = x\}$ for any $x \in T$. As u is left K -invariant, it must be constant on $K_o = K$. But $K \not\subset \tilde{K}(\mathfrak{x})$, and so u is zero on K , and hence on $K_o \cap K_z$. Note that $\pi(g_0)\xi \in \mathcal{H}'_{\mathfrak{x}}$. The same reasoning applied to $v(g) = \langle \pi(g)(\pi(g_0)\xi), \mathbf{1} \rangle_{s_1, s_2}$ shows that $u(g_0) = v(1) = 0$. It follows that $u = 0$ again.

In the special case, u is not supported on $\tilde{K}(\mathfrak{x})$, but is determined by $\alpha = u(1)$ and $\beta = u(g_0)$, where $g_0 o = a$ and $g_0 a = o$. To see this, (cf. [2, Proposition III.2.3]) let $\tilde{\mathcal{E}}$ denote the set of directed edges of T , i.e., the set of ordered pairs (x, y) , where

$x, y \in T$ and $d(x, y) = 1$. Defining $g.(x, y) = (gx, gy)$, we see that G acts transitively on $\tilde{\mathcal{E}}$. By right $K(\mathfrak{r})$ -invariance of u , we can define $\tilde{u} : \tilde{\mathcal{E}} \rightarrow \mathbb{C}$ by

$$\tilde{u}((go, ga)) = u(g).$$

We know that for $\mathfrak{y} = \{o\}$ and $\{a\}$,

$$\int_{K(\mathfrak{y})} u(gk) dk = 0 \quad \text{for all } g \in G.$$

Thus for each vertex v , the sum of $\tilde{u}((x, y))$ over the edges (x, y) for which $x = v$ is zero, as is the sum over the edges (x, y) for which $y = v$. Using this and the left $K(\mathfrak{r})$ -invariance of u , we see how u 's values are determined: each directed edge (x, y) lies on a doubly infinite geodesic of one of the following two types, and the values of $\tilde{u}$ are as indicated:

FIGURE 2(a).

FIGURE 2(b).

Arguing exactly as in Case (ii), we see that $u(1) = 0 = u(g_0)$. Hence $\alpha = \beta = 0$, and so $u = 0$. $\square$

Because $K(\mathfrak{r})$ is a normal subgroup of $\tilde{K}(\mathfrak{r})$, the group $\tilde{K}(\mathfrak{r})$ acts on the finite dimensional subspace $\mathcal{H}'_{\mathfrak{r}}$ of $\mathcal{H}_{s_1, s_2}$. Let τ denote the representation of $\tilde{K}(\mathfrak{r})$ thus obtained. By definition of $\mathcal{H}'_{\mathfrak{r}}$, τ is a sum of representations in $(\tilde{K}(\mathfrak{r}))_0^\wedge$. Let σ be the representation of G obtained by inducing τ from $\tilde{K}(\mathfrak{r})$ to G . Its representation space $\mathcal{H}_\sigma$ is the space of functions $f : G \rightarrow \mathcal{H}'_{\mathfrak{r}}$ such that $f(gk) = \tau(k^{-1})(f(g))$ for all $g \in G$ and $k \in \tilde{K}(\mathfrak{r})$ and such that $\int_G \|f(g)\|^2 dg < \infty$.

Lemma 2.3. *Assume that $|\mathfrak{r}| > 2$. Then $\sigma = \text{Ind}(\tau)$ is equivalent to the sum of the discrete series subrepresentations of π having minimal tree $\mathfrak{r}$.*

Proof. Let $g_0\tilde{K}(\mathfrak{r}), g_1\tilde{K}(\mathfrak{r}), \dots$ be the distinct left cosets of $\tilde{K}(\mathfrak{r})$ in G . Notice that, if $v, v' \in \mathcal{H}'_{\mathfrak{r}}$, then

$$(2.15) \quad \langle \pi(g_i)v, \pi(g_j)v' \rangle_{s_1, s_2} = \delta_{i,j} \langle v, v' \rangle_{s_1, s_2}$$

For if we set $u(g) = \langle \pi(g)v, v' \rangle_{s_1, s_2}$, then u is bi- $\tilde{K}(\mathfrak{r})$ -invariant, and satisfies (2.14) above. Hence u is supported on $\tilde{K}(\mathfrak{r})$ by [2, Prop. III.3.2], and so (2.15) holds. It follows that we may define $T : \mathcal{H}_\sigma \rightarrow \mathcal{H}_{s_1, s_2}$ by

$$Tf = (m(\tilde{K}(\mathfrak{r})))^{1/2} \sum_i \pi(g_i)(f(g_i)).$$

It is clear from (2.15) that T is an isometry. It intertwines σ and π , for if $g \in G$, we can write $g^{-1}g_i = g_jk$ for some $k \in \tilde{K}(\mathfrak{x})$ and index j depending on g and i . Now

$$(\sigma(g)f)(g_i) = f(g^{-1}g_i) = f(g_jk) = \pi(k^{-1})(f(g_j)),$$

so that

$$\pi(g_i)((\sigma(g)f)(g_i)) = \pi(g_i)(\pi(k^{-1})(f(g_j))) = \pi(g)(\pi(g_j)(f(g_j))).$$

As i varies over $0, 1, \dots$, so does j , and so $T(\sigma(g)f) = \pi(g)(Tf)$.

The image of T is the sum of the discrete series subrepresentations of π having minimal tree $\mathfrak{x}$. For if σ_0 is such a subrepresentation, with representation space $\mathcal{H}_{\sigma_0} \subset \mathcal{H}_{s_1, s_2}$, then $\mathcal{H}_{\sigma_0}$ contains a nonzero $v_0 \in \mathcal{H}'_{\mathfrak{x}}$. Define $f \in \mathcal{H}_{\sigma}$ by $f(k) = \pi(k^{-1})v_0$ if $k \in \tilde{K}(\mathfrak{x})$ and $f(k) = 0$ otherwise. Then $T(f) = v_0$. As the image of T is closed and π invariant, it contains $\mathcal{H}_{\sigma_0}$. $\square$

The last lemma reduces the problem of determining the decomposition (1.2) to (A): for $\mathfrak{x}$ as in cases (i) and (ii) of Proposition 2.1, determining the decomposition into irreducibles of the representation τ of $\tilde{K}(\mathfrak{x})$ defined before Lemma 2.3; and (B): determining which of the special representations of G occur in (1.2), and their multiplicities. These steps are carried out in the next three sections.

3. CASE (i) OF PROPOSITION 2.1

In this case, $\mathcal{H}'_{\mathfrak{x}}$ is $q(q-1)-1$ dimensional, and consists of elements $\sum_{a,b \in \mathcal{C}_1} t_{a,b} \xi_a \otimes \xi_b$, where $\mathcal{C}_1 = \{x \in T : d(o, x) = 1\}$, and where the numbers $t_{a,b} \in \mathbb{C}$, $a, b \in \mathcal{C}_1$, satisfy the conditions

$$(3.1) \quad \begin{aligned} & t_{a,a} = 0 \text{ for each } a \in \mathcal{C}_1, \\ & \sum_{a \in \mathcal{C}_1} t_{a,b} = 0 \text{ for each } b \in \mathcal{C}_1, \quad \text{and} \quad \sum_{b \in \mathcal{C}_1} t_{a,b} = 0 \text{ for each } a \in \mathcal{C}_1. \end{aligned}$$

Proposition 3.1. *The representation $g \mapsto \pi(g)|_{\mathcal{H}'_{\mathfrak{x}}}$ of $\tilde{K}(\mathfrak{x})$ on $\mathcal{H}'_{\mathfrak{x}}$ is (a) one dimensional if $q = 2$, and (b) the sum of an irreducible subrepresentation of dimension $(q+1)(q-2)/2$ and one of dimension $q(q-1)/2$ when $q > 2$. These subrepresentations are described below.*

Proof. In this case $\tilde{K}(\mathfrak{x}) = K = \{g \in G : go = o\}$. Suppose that $g \in \tilde{K}(\mathfrak{x})$. Then

$$\begin{aligned} \pi(g) \left(\sum_{a,b \in \mathcal{C}_1} t_{a,b} \xi_a \otimes \xi_b \right) &= \sum_{a,b \in \mathcal{C}_1} t_{a,b} \xi_{ga} \otimes \xi_{gb} \\ &= \sum_{a,b \in \mathcal{C}_1} t_{g^{-1}a, g^{-1}b} \xi_a \otimes \xi_b. \end{aligned}$$

Now $\mathcal{H}'_{\mathfrak{x}}$ may be decomposed into a sum of two subspaces invariant under $\pi(g)$ for each $g \in \tilde{K}(\mathfrak{x})$. In fact, $\mathcal{H}'_{\mathfrak{x}} = V_+ \oplus V_-$, where V_+ (resp. V_-) is the subspace of $\mathcal{H}'_{\mathfrak{x}}$ consisting of elements $\sum_{a,b \in \mathcal{C}_1} t_{a,b} \xi_a \otimes \xi_b$ for which $t_{a,b} = t_{b,a}$ (resp. $t_{a,b} = -t_{b,a}$) for all $a, b \in \mathcal{C}_1$. It is easy to see that $\dim(V_+) = (q+1)(q-2)/2$ and that $\dim(V_-) = q(q-1)/2$ ($= \dim(V_+) + 1$). Notice that $V_+ = \{0\}$ if $q = 2$.

Restriction to V_+ and V_- gives representations π_+ and π_- of $\tilde{K}(\mathfrak{x})$. It is well known that they are irreducible. For if we write $\mathcal{C}_1 = \{a_0, \dots, a_q\}$, then for $g \in \tilde{K}(\mathfrak{x})$, we have $ga_i = a_{f(g)i}$ for some $f(g) \in S_{q+1}$. This defines a surjective homomorphism $f : \tilde{K}(\mathfrak{x}) \rightarrow S_{q+1}$ (with kernel $K(\mathfrak{x})$). Let $U = \{(x_j) \in \mathbb{C}^{q+1} : \sum_{j=0}^q x_j =$

$0\}$, and let λ be the natural representation of S_{q+1} on U : $\lambda(\mu)(x_0, \dots, x_q) = (x_{\mu^{-1}0}, \dots, x_{\mu^{-1}q})$. This is the irreducible representation of S_{q+1} corresponding to the partition $(q, 1)$ of $q+1$ (see [3, Lemma 2.2.19(iii)]), i.e. to the Young diagram

The representation π of $\tilde{K}(\mathfrak{x})$ on $\mathcal{H}'_{\mathfrak{x}}$ is equivalent to a subrepresentation of $(\lambda \otimes \lambda) \circ f$, an intertwining operator being

$$\sum_{i,j=0}^q t_{i,j} \xi_{a_i} \otimes \xi_{a_j} \mapsto \sum_{i,j=0}^q t_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j,$$

where $\mathbf{e}_0, \dots, \mathbf{e}_q$ is the usual basis of $\mathbb{C}^{q+1}$. But the decomposition of $\lambda \otimes \lambda$ into irreducible subrepresentations is given on page 97 in [3] (where $n = q+1$). Two of these irreducible components are the representations $\pi_{q-1,2}$ and $\pi_{q-1,1,1}$ of S_{q+1} corresponding to the partitions $(q-1, 2)$ and $(q-1, 1, 1)$ of $q+1$, i.e. to the Young diagrams

Comparing the dimensions of these two representations (see [3, Theorem 2.3.21]), we see that these match those of π_+ and π_- , respectively. The above intertwining operator therefore gives equivalences between π_+ and $(\pi_{q-1,2}) \circ f$ and between π_- and $(\pi_{q-1,1,1}) \circ f$. When $q = 2$, it is easy to verify that the representation $g \mapsto \pi(g)|_{\mathcal{H}'_{\mathfrak{x}}}$ of $\tilde{K}(\mathfrak{x})$ on $\mathcal{H}'_{\mathfrak{x}}$ is (equivalent to) $\epsilon \circ f$, where ϵ is the sign character of S_3 . $\square$

4. CASE (ii) OF PROPOSITION 2.1

Referring to Figure 1(c_r), let A denote the set of neighbors of o other than z_1 , and let Y denote the neighbors of z other than z_{r-1} . Then $\mathcal{H}'_{\mathfrak{x}}$ consists of the elements

$$\sum_{a \in A, y \in Y} s_{a,y} \xi_a \otimes \xi_y + \sum_{a \in A, y \in Y} t_{a,y} \xi_y \otimes \xi_a,$$

where

$$(4.1) \quad \begin{aligned} \sum_{a \in A} s_{a,y} &= \sum_{a \in A} t_{a,y} = 0 \quad \text{for each } y \in Y, \text{ and} \\ \sum_{y \in Y} s_{a,y} &= \sum_{y \in Y} t_{a,y} = 0 \quad \text{for each } a \in A. \end{aligned}$$

Proposition 4.1. *The representation $g \mapsto \pi(g)|_{\mathcal{H}'_{\mathfrak{x}}}$ of $\tilde{K}(\mathfrak{x})$ on $\mathcal{H}'_{\mathfrak{x}}$ is the sum of two inequivalent irreducible subrepresentations, each of dimension $(q-1)^2$. The two subrepresentations are described below.*

Proof. Suppose that $g \in \tilde{K}(\mathfrak{x})$. Then either (i): $go = o$ and $gz = z$, in which case g permutes A and Y , and

$$\begin{aligned} \pi(g) & \left(\sum_{a \in A, y \in Y} c_{a,y} \xi_a \otimes \xi_y + \sum_{a \in A, y \in Y} d_{a,y} \xi_y \otimes \xi_a \right) \\ &= \sum_{a \in A, y \in Y} c_{a,y} \xi_{ga} \otimes \xi_{gy} + \sum_{a \in A, y \in Y} d_{a,y} \xi_{gy} \otimes \xi_{ga} \\ &= \sum_{a \in A, y \in Y} c_{g^{-1}a, g^{-1}y} \xi_a \otimes \xi_y + \sum_{a \in A, y \in Y} d_{g^{-1}a, g^{-1}y} \xi_y \otimes \xi_a, \end{aligned}$$

or (ii): $go = z$ and $gz = o$, in which case g interchanges A and Y , and we find using (4.1) that

$$\begin{aligned} \pi(g) & \left(\sum_{a \in A, y \in Y} c_{a,y} \xi_a \otimes \xi_y + \sum_{a \in A, y \in Y} d_{a,y} \xi_y \otimes \xi_a \right) \\ &= \left(\frac{s_2}{s_1} \right)^r \sum_{a \in A, y \in Y} c_{a,y} \xi_{ga} \otimes \xi_{gy} + \left(\frac{s_1}{s_2} \right)^r \sum_{a \in A, y \in Y} d_{a,y} \xi_{gy} \otimes \xi_{ga} \\ &= \left(\frac{s_1}{s_2} \right)^r \sum_{a \in A, y \in Y} d_{g^{-1}y, g^{-1}a} \xi_a \otimes \xi_y + \left(\frac{s_2}{s_1} \right)^r \sum_{a \in A, y \in Y} c_{g^{-1}y, g^{-1}a} \xi_y \otimes \xi_a. \end{aligned}$$

Let $\epsilon : \tilde{K}(\mathfrak{x}) \rightarrow \{-1, 1\}$ be the character defined by setting $\epsilon(g) = +1$ in case (i) and -1 in case (ii). Now $\mathcal{H}'_{\mathfrak{x}}$ may be decomposed into a sum of two subspaces invariant under $\pi(g)$ for each $g \in \tilde{K}(\mathfrak{x})$. In fact, $\mathcal{H}'_{\mathfrak{x}} = V_+ \oplus V_-$, where for $\delta = \pm 1$, V_{δ} is the subspace of $\mathcal{H}'_{\mathfrak{x}}$ consisting of elements of the form

$$\sum_{a \in A, y \in Y} c_{a,y} \xi_a \otimes \xi_y + \delta \left(\frac{s_2}{s_1} \right)^r \sum_{a \in A, y \in Y} c_{a,y} \xi_y \otimes \xi_a.$$

Restriction to V_+ and V_- gives representations π_+ and π_- of $\tilde{K}(\mathfrak{x})$. Clearly π_- is equivalent to the product of π_+ by ϵ .

The group S_2 of permutations of $\{1, 2\}$ acts on the product $S_q \times S_q$ of 2 copies of S_q : for $p \in S_2$ and $\mu_1, \mu_2 \in S_q$ we set

$$p \cdot (\mu_1, \mu_2) = (\mu_{p^{-1}1}, \mu_{p^{-1}2}).$$

We can therefore form the semidirect product $(S_q \times S_q) \rtimes S_2$ (the *wreath product* S_q wr S_2 of S_q by S_2 — see [3, p. 132]), whose elements are triples (μ_1, μ_2, p) , where $p \in S_2$ and $\mu_1, \mu_2 \in S_q$, and where multiplication is defined by

$$(\mu_1, \mu_2, p)(\mu'_1, \mu'_2, p') = (\mu_1 \mu'_{p^{-1}1}, \mu_2 \mu'_{p^{-1}2}, pp').$$

There is a group homomorphism $f : \tilde{K}(\mathfrak{x}) \rightarrow (S_q \times S_q) \rtimes S_2$, defined as follows. Write $A = \{a_1, \dots, a_q\}$ and $Y = \{y_1, \dots, y_q\}$. For g as in case (i), we have $ga_i = a_{\varphi(g)i}$ and $gy_i = y_{\vartheta(g)i}$ for some $\varphi(g), \vartheta(g) \in S_q$, and we define $f(g)$ to be $(\varphi(g), \vartheta(g), 1)$; for g as in case (ii), $ga_i = y_{\varphi'(g)i}$ and $gy_i = a_{\vartheta'(g)i}$ for some $\varphi'(g), \vartheta'(g) \in S_q$, and we set $f(g) = (\vartheta'(g), \varphi'(g), (1\ 2))$. It is easy to check that f is a homomorphism, which is clearly surjective.

Let $U = \{(x_j) \in \mathbb{C}^q : \sum_{j=1}^q x_j = 0\}$, and let λ be the natural representation of S_q on U : $\lambda(\mu)(x_1, \dots, x_q) = (x_{\mu^{-1}1}, \dots, x_{\mu^{-1}q})$. This is the irreducible representation of S_q corresponding to the partition $(q-1, 1)$ of q (cf. Section 3 above). There is

a representation $\tilde{\lambda}$ of $(S_q \times S_q) \rtimes S_2$ on $U \otimes U$ determined by

$$\tilde{\lambda}(\mu_1, \mu_2, p)(u_1 \otimes u_2) = (\lambda(\mu_1)(u_{p^{-1}1})) \otimes (\lambda(\mu_2)(u_{p^{-1}2}))$$

(see [3, p. 147]). This is clearly irreducible, because its restriction to the index 2 subgroup $\{(\mu_1, \mu_2, 1) : \mu_1, \mu_2 \in S_q\} \cong S_q \times S_q$ of $(S_q \times S_q) \rtimes S_2$ is already irreducible, being equivalent to the outer tensor product of two copies of λ on $S_q \times S_q$.

Finally, the representations π_+ and $\tilde{\lambda} \circ f$ of $\tilde{K}(\mathfrak{x})$ are equivalent, an intertwining operator being

$$\sum_{i,j=1}^q c_{i,j} \left(\xi_{a_i} \otimes \xi_{y_j} + \left(\frac{s_2}{s_1} \right)^r \xi_{y_j} \otimes \xi_{a_i} \right) \mapsto \sum_{i,j=1}^q c_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j,$$

where $\mathbf{e}_1, \dots, \mathbf{e}_q$ is the usual basis of $\mathbb{C}^q$. □

5. THE SPECIAL CASE

Let $\mathfrak{x}$ be as in Figure 1(a). We must first describe $\mathcal{H}'_{\mathfrak{x}}$. For $n \geq 1$, let

$$X_n = \{(\omega_1, \omega_2) \in \Omega_o(a) \times \Omega_o(a) : \omega_1 \neq \omega_2 \text{ and } d(o, \omega_1 \wedge \omega_2) = n\}$$

($= D_{a,n}$, see (2.5)). Let $\Omega'_o(x)$ denote $\Omega \setminus \Omega_o(x)$, and for $n \geq 0$ let

$$Y_n = \{(\omega_1, \omega_2) \in \Omega'_o(a) \times \Omega'_o(a) : \omega_1 \neq \omega_2 \text{ and } d(o, \omega_1 \wedge \omega_2) = n\}.$$

Also, let

$$\Delta_a = \{(\omega_1, \omega_2) \in \Omega_o(a) \times \Omega_o(a) : \omega_1 = \omega_2\}$$

and

$$\Delta'_a = \{(\omega_1, \omega_2) \in \Omega'_o(a) \times \Omega'_o(a) : \omega_1 = \omega_2\}.$$

Let $Z = \Omega_o(a) \times \Omega'_o(a)$ and $Z' = \Omega'_o(a) \times \Omega_o(a)$. Then

$$\Omega \times \Omega = \Delta_a \cup \Delta'_a \cup \bigcup_{n \geq 1} X_n \cup \bigcup_{n \geq 0} Y_n \cup Z \cup Z',$$

a disjoint union. We also need to define

$$X'_M = \{(\omega_1, \omega_2) \in \Omega_o(a) \times \Omega_o(a) : \omega_1 = \omega_2, \text{ or } \omega_1 \neq \omega_2 \text{ and } d(o, \omega_1 \wedge \omega_2) > M\}.$$

for $M \geq 1$, and

$$Y'_M = \{(\omega_1, \omega_2) \in \Omega'_o(a) \times \Omega'_o(a) : \omega_1 = \omega_2, \text{ or } \omega_1 \neq \omega_2 \text{ and } d(o, \omega_1 \wedge \omega_2) > M\}.$$

If $F \in \mathcal{K}(\Omega \times \Omega)$ satisfies $P_{\mathfrak{x}} F = F$, then as in the case $|\mathfrak{x}| > 2$, we can write

$$F = \sum_{n=1}^M x_n \mathbf{1}_{X_n} + x \mathbf{1}_{X'_M} + \sum_{n=0}^M y_n \mathbf{1}_{Y_n} + y \mathbf{1}_{Y'_M} + z \mathbf{1}_Z + z' \mathbf{1}_{Z'},$$

for some integer $M \geq 1$ and constants x_n, x, y_n, y, z, z' .

Lemma 5.1. *For f equal to any of $\mathbf{1}_{X_n}$, $\mathbf{1}_{Y_n}$, $\mathbf{1}_{X'_n}$ or $\mathbf{1}_{Y'_n}$, the following estimates hold for the norm $\|f\|_{s_1, s_2}$ of f in $\mathcal{H}_{s_1, s_2}$.*

$$\|f\|_{s_1, s_2} = \begin{cases} O\left(n^{\frac{1}{2}} \max\left\{\frac{1}{q^n}, \left(\frac{|s_1 s_2|}{\sqrt{q}}\right)^n\right\}\right) & \text{if } s_1^2, s_2^2 \in (1/q, q), \\ O\left(\left(\frac{|s_2|}{\sqrt{q}}\right)^n\right) & \text{if } |s_1| = 1 \text{ and } s_2^2 \in (1/q, q), \\ O\left(\left(\frac{|s_1|}{\sqrt{q}}\right)^n\right) & \text{if } s_1^2 \in (1/q, q) \text{ and } |s_2| = 1, \\ O\left(\frac{1}{q^{n/2}}\right) & \text{if } |s_1|, |s_2| = 1. \end{cases}$$

Proof. Let us write $\tilde{J}$ in place of $\tilde{J}_{s_1, s_2}$. Recall that if $v \in T$ and $n, M \geq |v| = d(o, v)$, then $D_{v, n}$ and $D'_{v, M}$ are defined as in (2.5) and (2.8). Let $b \in T$ satisfy $|b| = 1$. Then X_n is $D_{a, n}$, and Y_n is the union of the sets $D_{b, n}$, where $|b| = 1$ and $b \neq a$. Similarly, X'_n is $D'_{a, n}$, and Y'_n is the union of the sets $D'_{b, n}$, where $|b| = 1$ and $b \neq a$. So it is enough to fix b with $|b| = 1$, and estimate $\langle f_n, f_n \rangle_{s_1, s_2} = \langle f_n, \tilde{J}f_n \rangle'$ and $\langle f'_n, f'_n \rangle_{s_1, s_2} = \langle f'_n, \tilde{J}f'_n \rangle'$, where

$$f_n = \mathbf{1}_{D_{b, n}} \quad \text{and} \quad f'_n = \mathbf{1}_{D'_{b, n}}.$$

Moreover, since $f_n = f'_{n-1} - f'_n$, it is sufficient to estimate $\langle f'_n, \tilde{J}f'_n \rangle'$

To do this, let $n \geq 1$, let $|x| = n$, and consider the following element of $\mathcal{K}(\Omega \times \Omega)$:

$$(5.1) \quad \Sigma_x = \sum_{\substack{y: |y|=n+1, \\ y'=x}} \xi_y \otimes \xi_y.$$

Clearly Σ_x is supported on $\Omega_o(x) \times \Omega_o(x)$. Let us fix $\omega_1, \omega_2 \in \Omega_o(x)$, and evaluate both sides of (5.1) at (ω_1, ω_2) . Let z_1 and z_2 denote the $(n+1)$ th vertices on $[o, \omega_1)$ and $[o, \omega_2)$, respectively. If $z_1 \neq z_2$ (which just means that $(\omega_1, \omega_2) \in D_{x, n}$), then $\xi_y(\omega_1)\xi_y(\omega_2) = -N_n(N_{n+1} - N_n)$ if $y = z_1$ or z_2 , and $\xi_y(\omega_1)\xi_y(\omega_2) = (-N_n)^2$ for the remaining $(q-2)$ y 's satisfying $|y| = n+1$ and $y' = x$. Hence

$$\Sigma_x(\omega_1, \omega_2) = (q-2)N_n^2 - 2N_n(N_{n+1} - N_n) = Aq^{2n}$$

for $A = -(q+1)^2/q$. If $z_1 = z_2$ (which just means that $(\omega_1, \omega_2) \in D'_{x, n}$), then $\xi_y(\omega_1)\xi_y(\omega_2) = (N_{n+1} - N_n)^2$ if $y = z_1 = z_2$, and $\xi_y(\omega_1)\xi_y(\omega_2) = (-N_n)^2$ for the remaining $(q-1)$ y 's satisfying $|y| = n+1$ and $y' = x$. Hence

$$\Sigma_x(\omega_1, \omega_2) = (q-1)N_n^2 + (N_{n+1} - N_n)^2 = Bq^{2n}$$

for $B = (q-1)(q+1)^2/q = -(q-1)A$.

Hence

$$\Sigma_x = Aq^{2n}\mathbf{1}_{D_{x, n}} + Bq^{2n}\mathbf{1}_{D'_{x, n}}.$$

If we now fix $b \in T$ satisfying $|b| = 1$, and sum this last identity over all x such that $b \in [o, x]$ and $|x| = n$, we obtain

$$\sum_{\substack{x: b \in [o, x], \\ |x|=n}} \Sigma_x = Aq^{2n}f_n + Bq^{2n}f'_n.$$

Now apply $\tilde{J}$ to both sides. Write $\tilde{j}_n(s)$ to denote either 1 if $|s| = 1$, or $j_n(s)$ if $s^2 \in (1/q, q)$. Using (1.3), we get

$$\begin{aligned} \tilde{j}_{n+1}(s_1)\tilde{j}_{n+1}(s_2)(Aq^{2n}f_n + Bq^{2n}f'_n) &= \tilde{j}_{n+1}(s_1)\tilde{j}_{n+1}(s_2)\left(\sum_{\substack{x: b \in [o, x], \\ |x|=n}} \Sigma_x\right) \\ &= \tilde{J}\left(\sum_{\substack{x: b \in [o, x], \\ |x|=n}} \Sigma_x\right) \\ &= Aq^{2n}\tilde{J}f_n + Bq^{2n}\tilde{J}f'_n. \end{aligned}$$

Cancelling Aq^{2n} , using $B = -(q-1)A$, and writing j_{n+1}^* in place of $\tilde{j}_{n+1}(s_1)\tilde{j}_{n+1}(s_2)$, we get

$$(5.2) \quad \tilde{J}f_n - (q-1)\tilde{J}f'_n = j_{n+1}^*(f_n - (q-1)f'_n) \quad \text{if } n \geq 1.$$

If $n > 1$, we also have $f_n + f'_n = f'_{n-1}$, and hence

$$(5.3) \quad \tilde{J}f_n + \tilde{J}f'_n = \tilde{J}f'_{n-1} \quad \text{if } n > 1.$$

It is clear from (5.2) and (5.3) that for all $n \geq 1$,

$$(5.4) \quad \tilde{J}f'_n = \frac{1}{q^{n-1}} \tilde{J}f'_1 + \sum_{k=1}^n \beta'_{n,k} f_k + C'_n f'_n$$

for certain numbers $\beta'_{n,k}, C'_n$. Indeed, if we set $\beta'_{1,1} = C'_1 = 0$, then (5.4) holds for $n = 1$. Assume that $n > 1$, and that (5.4) holds for $n - 1$ in place of n . If we subtract (5.2) from (5.3), we obtain

$$q\tilde{J}f'_n = \tilde{J}f'_{n-1} - j_{n+1}^* f_n + (q-1)j_{n+1}^* f'_n,$$

and so

$$\tilde{J}f'_n = \frac{1}{q} \left(\frac{1}{q^{n-2}} \tilde{J}f'_1 + \sum_{k=1}^{n-1} \beta'_{n-1,k} f_k + C'_{n-1} f'_{n-1} \right) - \frac{j_{n+1}^*}{q} f_n + \frac{(q-1)j_{n+1}^*}{q} f'_n,$$

which gives formula (5.4) if we set

$$(5.5) \quad \begin{aligned} \beta'_{n,k} &= \frac{1}{q} \beta'_{n-1,k} \quad \text{if } 1 \leq k \leq n-1, \\ \beta'_{n,n} &= \frac{1}{q} (C'_{n-1} - j_{n+1}^*), \\ C'_n &= \frac{1}{q} (C'_{n-1} + (q-1)j_{n+1}^*). \end{aligned}$$

We have used $f'_{n-1} = f_n + f'_n$ again here.

Now

$$\begin{aligned} \langle f'_n, f'_n \rangle_{s_1, s_2} &= \langle f'_n, \tilde{J}f'_n \rangle' \\ &= \langle f'_n, \frac{1}{q^{n-1}} \tilde{J}f'_1 + \sum_{k=1}^n \beta'_{n,k} f_k + C'_n f'_n \rangle' \\ &= \frac{1}{q^{n-1}} \langle f'_n, \tilde{J}f'_1 \rangle' + C'_n \langle f'_n, f'_n \rangle'. \end{aligned}$$

Now

$$\langle f'_n, f'_n \rangle' = (\nu_o \times \nu_o)(D'_{b,n}) = \frac{1}{(q+1)^2 q^n} = O\left(\frac{1}{q^n}\right).$$

Also,

$$\langle f'_n, \tilde{J}f'_1 \rangle' = O\left((\nu_o \times \nu_o)(D'_{b,n})\right) = O\left(\frac{1}{q^n}\right)$$

because $\tilde{J}f'_1$ is a bounded function. Hence

$$(5.6) \quad \langle f'_n, f'_n \rangle_{s_1, s_2} = \langle f'_n, \tilde{J}f'_n \rangle' = O\left(\frac{1}{q^{2n}}\right) + O\left(\frac{C'_n}{q^n}\right).$$

So we need only estimate C'_n . Using the third equation in (5.5), we see by induction that if $n \geq 2$, then

$$(5.7) \quad C'_n = \frac{q-1}{q^{n+2}} (j_3^* q^3 + \cdots + j_{n+1}^* q^{n+1}).$$

To sum the finite series (5.7), we must now consider the various cases:

When $|s_1| = |s_2| = 1$, Then $j_k^* = 1$ for each k , and we get $C'_n = 1 - 1/q^{n-1}$.

When $|s_1| = 1$ and $s_2^2 \in (1/q, q)$, (5.7) gives

$$C'_n = \frac{D_1}{q^n} + D_2 s_2^{2n} = O(|s_2|^{2n}),$$

for certain constants D_1 and D_2 . When $s_1^2 \in (1/q, q)$ and $|s_2| = 1$, (5.7) similarly gives $C'_n = D_1/q^n + D_2 s_1^{2n}$ for certain constants D_1 and D_2 .

Finally, when $s_1^2, s_2^2 \in (1/q, q)$, we get

$$C'_n = \frac{D_1}{q^n} + D_2 (s_1 s_2)^{2n} \quad \text{if } (s_1 s_2)^2 q \neq 1,$$

for certain constants D_1 and D_2 . Also,

$$C'_n = \frac{D_1^*(n-1)}{q^n} \quad \text{if } (s_1 s_2)^2 q = 1,$$

for some constant D_1^* .

With these formulas for C'_n , Lemma 5.1 is proved. $\square$

Lemma 5.2. Suppose that $\xi \in \mathcal{H}'_{\mathbf{x}}$. Form

$$(5.8) \quad \begin{aligned} x_n &= \langle \xi, \mathbf{1}_{X_n} \rangle' / (\nu_o \times \nu_o)(X_n) & \text{if } n \geq 1, & \quad z = \langle \xi, \mathbf{1}_Z \rangle' / (\nu_o \times \nu_o)(Z), \\ y_n &= \langle \xi, \mathbf{1}_{Y_n} \rangle' / (\nu_o \times \nu_o)(Y_n) & \text{if } n \geq 0, & \quad z' = \langle \xi, \mathbf{1}_{Z'} \rangle' / (\nu_o \times \nu_o)(Z'). \end{aligned}$$

Then the series

$$(5.9) \quad \xi = \sum_{k=1}^{\infty} x_k \mathbf{1}_{X_k} + \sum_{k=0}^{\infty} y_k \mathbf{1}_{Y_k} + z \mathbf{1}_Z + z' \mathbf{1}_{Z'}.$$

converges in $\mathcal{H}_{s_1, s_2}$ to ξ . Moreover,

$$(5.10) \quad \begin{aligned} y_{2n} &= \frac{-1}{(q-1)(s_1 s_2)^{2n}} (z + z'), & \text{if } n \geq 0, \\ y_{2n+1} &= \frac{1}{(q-1)(s_1 s_2)^{2n}} \left(\frac{z}{s_2^2} + \frac{z'}{s_1^2} \right), & \text{if } n \geq 0, \\ x_{2n} &= -q y_{2n} & \text{if } n \geq 1, \\ x_{2n+1} &= -q y_{2n+1} & \text{if } n \geq 0, \end{aligned}$$

and so ξ is determined by z and z' . Thus $\mathcal{H}'_{\mathbf{x}}$ is two dimensional.

Proof. If $\mathfrak{y} = \{o\}$, then it is easy to see that $P_{\mathfrak{y}}(\mathbf{1}_{X_n}) = (\mathbf{1}_{X_n} + \mathbf{1}_{Y_n})/(q+1)$ and $P_{\mathfrak{y}}(\mathbf{1}_{Y_n}) = q(\mathbf{1}_{X_n} + \mathbf{1}_{Y_n})/(q+1)$ for $n \geq 1$. If $\xi \in \mathcal{H}'_{\mathbf{x}}$, then $P_{\mathfrak{y}}\xi = 0$ therefore implies that $x_n + qy_n = 0$ for $n = 1, 2, \dots$, because $(\nu_o \times \nu_o)(Y_n) = q(\nu_o \times \nu_o)(X_n)$. Also, using $P_{\mathfrak{y}}\mathbf{1}_{Y_0} = ((q-1)/(q+1))(\mathbf{1}_{Y_0} + \mathbf{1}_Z + \mathbf{1}_{Z'})$, plus $(\nu_o \times \nu_o)(Y_0) = q(q-1)/(q+1)^2 = (q-1)((\nu_o \times \nu_o)(Z)) = (q-1)((\nu_o \times \nu_o)(Z'))$, we find that $(q-1)y_0 + z + z' = 0$.

If $\mathfrak{y} = \{a\}$, then we can calculate

$$(5.11) \quad \begin{aligned} P_{\mathfrak{y}}\mathbf{1}_{X_n} &= \frac{q}{q+1} \left\{ \mathbf{1}_{X_n} + \left(\frac{s_1 s_2}{q} \right)^2 \mathbf{1}_{Y_{n-2}} \right\} & \text{if } n \geq 2, \\ P_{\mathfrak{y}}\mathbf{1}_{X_1} &= \frac{q-1}{q+1} \left\{ \mathbf{1}_{X_1} + \left(\frac{s_2^2}{q} \right) \mathbf{1}_Z + \left(\frac{s_1^2}{q} \right) \mathbf{1}_{Z'} \right\}, \\ P_{\mathfrak{y}}\mathbf{1}_{Y_n} &= \frac{1}{q+1} \left\{ \mathbf{1}_{Y_n} + \left(\frac{q}{s_1 s_2} \right)^2 \mathbf{1}_{X_{n+2}} \right\} & \text{if } n \geq 0, \\ P_{\mathfrak{y}}\mathbf{1}_Z &= \frac{1}{q+1} \left\{ \left(\frac{q}{s_2^2} \right) \mathbf{1}_{X_1} + \mathbf{1}_Z + \left(\frac{s_1}{s_2} \right)^2 \mathbf{1}_{Z'} \right\}, & \text{and} \\ P_{\mathfrak{y}}\mathbf{1}_{Z'} &= \frac{1}{q+1} \left\{ \left(\frac{q}{s_1^2} \right) \mathbf{1}_{X_1} + \mathbf{1}_{Z'} + \left(\frac{s_2}{s_1} \right)^2 \mathbf{1}_Z \right\}. \end{aligned}$$

For example, to see the first of these, consider $(\omega_1, \omega_2) \in \Omega \times \Omega$, $g \in K(\mathfrak{y})$, and ask when $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$. If $(\omega_1, \omega_2) \in X_n$, there is a neighbor b of a other than o such that $\omega_1, \omega_2 \in \Omega_o(b)$. We find that $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$ and $\delta(o, go, \omega_1) = \delta(o, go, \omega_2) = 0$ unless $go = b$. If $(\omega_1, \omega_2) \in Y_{n-2}$, then $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$ and $\delta(o, go, \omega_1) = \delta(o, go, \omega_2) = -2$ unless $go = o$. Also, $(g^{-1}\omega_1, g^{-1}\omega_2) \in X_n$ for no other $(\omega_1, \omega_2) \in \Omega \times \Omega$ and $g \in K(\mathfrak{y})$.

As $\xi \in \mathcal{H}'_{\mathfrak{x}}$, $P_{\mathfrak{y}}\xi = 0$ implies that

$$(5.12) \quad 0 = \langle P_{\mathfrak{y}}\xi, \mathbf{1}_{X_n} \rangle' = \langle \xi, P_{\mathfrak{y}}\tilde{J}_{s_1, s_2}^{-1} \mathbf{1}_{X_n} \rangle_{s_1, s_2}.$$

Because $o \notin \mathfrak{y}$, $P_{\mathfrak{y}}$ and $\tilde{J}_{s_1, s_2}^{-1}$ no longer commute, but the first equation in (2.11) is still valid, and therefore $P_{\mathfrak{y}}\tilde{J}_{s_1, s_2}^{-1} = \tilde{J}_{s_1, s_2}^{-1}\tilde{P}_{\mathfrak{y}}$, where $\tilde{P}_{\mathfrak{y}}$ is defined as was $P_{\mathfrak{y}}$, but with (s_1, s_2) replaced by $(\tilde{s}_1, \tilde{s}_2)$. Using the first of the equations (5.11), with (s_1, s_2) replaced by $(\tilde{s}_1, \tilde{s}_2)$, together with $(\nu_o \times \nu_o)(Y_{n-2}) = q^3(\nu_o \times \nu_o)(X_n)$, we obtain from (5.12)

$$0 = x_n + \frac{q}{(s_1 s_2)^2} y_{n-2} \quad \text{if } n \geq 2.$$

The same equations are obtained using the third equation in (5.11). Using any of the other three equations in (5.11), we also obtain

$$0 = (q-1)x_1 + \frac{q}{s_2^2} z + \frac{q}{s_1^2} z'.$$

It is now elementary to obtain the formulas (5.10) for the x_n 's and y_n 's.

Using the estimates of Lemma 5.1 and the estimates $x_k, y_k = O(1/|s_1 s_2|^k)$ which follow from (5.10), we see that the series converges in $\mathcal{H}_{s_1, s_2}$. Let $\xi^* \in \mathcal{H}_{s_1, s_2}$ denote the sum. To see that $\xi^* = \xi$, it is enough to check that $\langle \xi^*, f \rangle' = \langle \xi, f \rangle'$ for each $f = \mathbf{1}_{X_n}, \mathbf{1}_{X'_n}$, $n \geq 1$, for $f = \mathbf{1}_{Y_n}, \mathbf{1}_{Y'_n}$, $n \geq 0$, and for $f = \mathbf{1}_Z$ and $\mathbf{1}_{Z'}$. By (5.8), we need only check the cases $f = \mathbf{1}_{X'_n}$, $n \geq 1$, and $f = \mathbf{1}_{Y'_n}$, $n \geq 0$. Now

$$\mathbf{1}_{\Omega_o(a) \times \Omega_o(a)} = \mathbf{1}_{X_1} + \cdots + \mathbf{1}_{X_n} + \mathbf{1}_{X'_n},$$

and so to check that $\langle \xi^*, \mathbf{1}_{X'_n} \rangle' = \langle \xi, \mathbf{1}_{X'_n} \rangle'$, we need only show that $\langle \xi^*, \mathbf{1}_{X'_n} \rangle', \langle \xi, \mathbf{1}_{X'_n} \rangle' \rightarrow 0$. To do this, it is enough to show that $\|\tilde{J}_{s_1, s_2}^{-1} \mathbf{1}_{X'_n}\|_{s_1, s_2} \rightarrow 0$ as $n \rightarrow \infty$. But

$$\begin{aligned} \|\tilde{J}_{s_1, s_2}^{-1} \mathbf{1}_{X'_n}\|_{s_1, s_2}^2 &= \langle \tilde{J}_{s_1, s_2}^{-1} \mathbf{1}_{X'_n}, \tilde{J}_{s_1, s_2}^{-1} \mathbf{1}_{X'_n} \rangle_{s_1, s_2} \\ &= \langle \tilde{J}_{s_1, s_2}^{-1} \mathbf{1}_{X'_n}, \mathbf{1}_{X'_n} \rangle' \\ &= \|\mathbf{1}_{X'_n}\|_{s_1^{-1}, s_2^{-1}}^2 \\ &\rightarrow 0 \quad \text{by Lemma 5.1, with } (s_1, s_2) \text{ replaced by } (s_1^{-1}, s_2^{-1}). \end{aligned}$$

The corresponding facts for Y'_n in place of X'_n are proved in the same way. $\square$

Proposition 5.3. *Let $\mathcal{H}_{\text{sp}}$ denote the closed linear span of $\{\pi(g)\xi : \xi \in \mathcal{H}'_{\mathfrak{x}}\}$. Then $\mathcal{H}_{\text{sp}}$ is the sum $\mathcal{H}_+ \oplus \mathcal{H}_-$ of two invariant subspaces. The restrictions of π to $\mathcal{H}_+$ and $\mathcal{H}_-$ are equivalent to the two distinct special representations of G .*

Proof. Let $g_0 \in G$ satisfy $g_0 o = a$, $g_0 a = o$ and $g_0^2 = 1$. Let us find $z, z' \in \mathbb{C}$ such that the corresponding element ξ^+ of $\mathcal{H}'_{\mathfrak{x}}$ satisfies $\pi(g_0)\xi^+ = \xi^+$. It is easy to

calculate that

$$\begin{aligned}\pi(g_0)\mathbf{1}_{X_k} &= \frac{s_1 s_2}{q} \mathbf{1}_{Y_{k-1}} \quad \text{for } k \geq 1, \\ \pi(g_0)\mathbf{1}_{Y_k} &= \frac{q}{s_1 s_2} \mathbf{1}_{X_{k+1}} \quad \text{for } k \geq 0, \\ \pi(g_0)\mathbf{1}_Z &= \frac{s_1}{s_2} \mathbf{1}_{Z'}, \\ \pi(g_0)\mathbf{1}_{Z'} &= \frac{s_2}{s_1} \mathbf{1}_Z.\end{aligned}$$

Using these, we see that the condition $\pi(g_0)\xi^+ = \xi^+$ amounts to the conditions

$$y_k = \frac{s_1 s_2}{q} x_{k+1} \quad \text{for } k \geq 0, \quad \text{and} \quad z' = \frac{s_1}{s_2} z,$$

and using (5.10), we see that the second of these conditions implies the first. So if we set $z = s_2$, $z' = s_1$, and substitute these values of z, z' into the formulas (5.10) for the x_k 's and y_k 's, we get an element $\xi^+ \in \mathcal{H}'_{\mathfrak{r}}$ such that $\pi(g_0)\xi^+ = \xi^+$. In fact, $\pi(g)\xi^+ = \xi^+$ for all $g \in \tilde{K}(\mathfrak{r})$. Notice that

$$y_n = \frac{(-1)^{n-1}(s_1 + s_2)}{(q-1)(s_1 s_2)^n} \quad \text{for } n \geq 0,$$

and so in the case $s_2 = -s_1$, $\xi^+ = s_1(\mathbf{1}_{Z'} - \mathbf{1}_Z)$ is in $\mathcal{K}(\Omega \times \Omega)$.

Similarly, if we seek $\xi^- \in \mathcal{H}'_{\mathfrak{r}}$ satisfying $\pi(g_0)\xi^- = -\xi^-$, we are led to the conditions

$$y_k = -\frac{s_1 s_2}{q} x_{k+1} \quad \text{for } k \geq 0, \quad \text{and} \quad z' = -\frac{s_1}{s_2} z.$$

Again, the second of these conditions implies the first. So if we set $z = s_2$, $z' = -s_1$, and substitute these values of z, z' into the formulas (5.10) for the x_k 's and y_k 's, we get an element $\xi^- \in \mathcal{H}'_{\mathfrak{r}}$ such that $\pi(g_0)\xi^- = -\xi^-$. Notice that

$$y_n = \frac{(s_1 - s_2)}{(q-1)(s_1 s_2)^n} \quad \text{for } n \geq 0,$$

and so in the case $s_1 = s_2$, $\xi^- = s_1(\mathbf{1}_Z - \mathbf{1}_{Z'})$ is in $\mathcal{K}(\Omega \times \Omega)$.

If we let $u(g) = \langle \pi(g)\xi^+, \xi^- \rangle_{s_1, s_2}$, then u is bi- $K(\mathfrak{r})$ -invariant and satisfies (2.14), and so determined by $u(1)$ and $u(g_0)$ (see the proof of Lemma 2.2). Now

$$u(g_0) = \langle \pi(g_0)\xi^+, \xi^- \rangle_{s_1, s_2} = \langle \xi^+, \xi^- \rangle_{s_1, s_2} = u(1),$$

and also

$$u(g_0) = \langle \pi(g_0)\xi^+, \xi^- \rangle_{s_1, s_2} = \langle \xi^+, \pi(g_0^{-1})\xi^- \rangle_{s_1, s_2} = -\langle \xi^+, \xi^- \rangle_{s_1, s_2} = -u(1).$$

It follows that $u = 0$. Hence the linear span $\mathcal{H}_+$ of $\{\pi(g)\xi^+ : g \in G\}$ is orthogonal to the linear span $\mathcal{H}_-$ of $\{\pi(g)\xi^- : g \in G\}$. Clearly $\mathcal{H}_{\text{sp}} = \mathcal{H}_+ \oplus \mathcal{H}_-$. We also know that $\mathcal{H}_{\text{sp}} \subset H_2$, by Lemma 2.2. Moreover, $\mathcal{H}_{\text{sp}}$ can contain no nonzero $\xi \in \mathcal{H}'_{\mathfrak{r}}$ for any $\mathfrak{r}'$ as in Figure 1(b) or Figure 1(c_r). For if $\eta \in \mathcal{H}_{\text{sp}}$, then $u(g) = \langle \pi(g)\xi, \eta \rangle_{s_1, s_2}$ would be right $K(\mathfrak{r}')$ -invariant, left $K(\mathfrak{r})$ -invariant, and satisfy (2.14) (with $\mathfrak{r}$ there replaced by $\mathfrak{r}'$), and by [2, Proposition III.3.2] would therefore be supported in the empty set $\{g \in G : g\mathfrak{r}' \subset \mathfrak{r}\}$.

Hence the restrictions of π to $\mathcal{H}_+$ and $\mathcal{H}_-$ are equivalent to special representations of G . They are inequivalent, for if $T : \mathcal{H}_+ \rightarrow \mathcal{H}_-$ intertwines these two restrictions, then $\xi = T\xi^+$ satisfies $\pi(g_0)\xi = \xi$, and so ξ is a multiple of ξ^+ . But $\mathcal{H}_+$ and $\mathcal{H}_-$ are orthogonal, and so $\xi = 0$, and therefore $T = 0$. $\square$

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