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SPECTRAL LIFTING IN BANACH ALGEBRAS AND INTERPOLATION IN SEVERAL VARIABLES

GELU POPESCU

ABSTRACT. Let $\mathcal{A}$ be a unital Banach algebra and let J be a closed two-sided ideal of $\mathcal{A}$. We prove that if any invertible element of $\mathcal{A}/J$ has an invertible lifting in $\mathcal{A}$, then the quotient homomorphism $\Phi : \mathcal{A} \to \mathcal{A}/J$ is a spectral interpolant. This result is used to obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici, Foiaş, and Tannenbaum. This yields spectral versions of Sarason, Nevanlinna–Pick, and Carathéodory type interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra F_n^∞ and $B(\mathcal{K})$, the algebra of bounded operators on a finite dimensional Hilbert space $\mathcal{K}$. A spectral tangential commutant lifting theorem in several variables is considered and used to obtain a spectral tangential version of the Nevanlinna-Pick interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$.

In particular, we obtain interpolation theorems for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

1. Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disc in the complex plane, let $z_1, \ldots, z_k \in \mathbb{D}$ be given distinct points, and $F_1, \ldots, F_k$ be complex $m \times m$ matrices. The classical Nevanlinna–Pick problem [N], [P] consists in finding necessary and sufficient conditions for the existence of an analytic $m \times m$ matrix-valued function $F(z)$ with $F(z_j) = F_j$ $(1 \leq j \leq k)$ and such that $\|F\|_\infty \leq 1$.

Motivated by problems in control engineering, such as the design of feedback control systems in the presence of parameter uncertainty, Bercovici, Foiaş, and Tannenbaum proved in [BFT] a spectral generalization of the commutant lifting theorem [SzF1], and obtained a spectral version of the Nevanlinna–Pick problem, in which the infinity norm is replaced by

$$\rho(F) := \sup\{\|F(z)\|_{\rm sp} : \ z \in \mathbb{D}\}$$

($\|A\|_{\rm sp}$ denotes the spectral radius of an operator A).

The tangential Nevanlinna–Pick problem considered by Fedcina [F] is to find $F \in H^\infty(\mathbb{D}) \otimes \mathbb{C}^m$ with $F(z_j)u_j = v_j, \ j = 1, \ldots, k$, and $\|F\|_\infty \leq 1$, where $z_j \in \mathbb{D}$ and $u_j, v_j \in \mathbb{C}^m$ are prescribed. The spectral tangential Nevanlinna–Pick interpolation problem, considered by Bercovici and Foiaş [BF], is to find such an F for which $\rho(F) < 1$. This type of interpolation was also motivated by certain control engineering applications.

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In this paper we find noncommutative multivariable analogues of the abovementioned results obtained by Bercovici, Foiaş, and Tannenbaum (see [BFT] and [BF]) for the noncommutative analytic Toeplitz algebra F_n^∞. In particular, we obtain interpolation results (see Corollary 3.7 and Corollary 4.3) for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

We expect these results to play a role in multivariable control and systems theory, as it does in the case $n = 1$. We mention the papers [BV] and [B] for recent results in multivariable linear systems.

We need to recall some facts concerning the noncommutative analytic Toeplitz algebra F_n^∞ and its connection with the function theory on the open unit ball of $\mathbb{C}^n$. Let $F^2(H_n) = \mathbb{C}1 \oplus \bigoplus_{m\geq 1} H_n^{\otimes m}$ be the full Fock space on n generators, where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \dots, e_n\}$ if n is finite, and $\{e_1, e_2, \dots\}$ if $n = \infty$. For each $i = 1, 2, \dots$, define the left creation operator by $S_i\xi := e_i \otimes \xi,\ \xi \in F^2(H_n)$.

We shall denote by $\mathcal{P}$ the set of all $p \in F^2(H_n)$ which are finite sums of tensor monomials. Define F_n^∞ as the set of all $g \in F^2(H_n)$ such that

$$\|g\|_\infty := \sup\{\|g \otimes p\|_{F^2(H_n)} : p \in \mathcal{P}, \|p\|_{F^2(H_n)} \leq 1\} < \infty.$$

We denote by $\mathcal{A}_n$ the closure of $\mathcal{P}$ in $(F_n^\infty, \|\cdot\|_\infty)$. The Banach algebra F_n^∞ (resp. $\mathcal{A}_n$) can be viewed as a noncommutative analogue of the Hardy space $H^\infty(\mathbb{D})$ (resp. disc algebra $A(\mathbb{D})$); when $n = 1$ they coincide.

In [Po7, Theorem 3.1] we proved that $\mathcal{A}_n$ is completely isometrically isomorphic to the norm-closed algebra generated by any sequence $V_1, \dots, V_n$ of isometries with $V_1V_1^* + \dots + V_nV_n^* \leq I$, and the identity. It follows from [Po5, Theorem 4.3] that the noncommutative analytic Toeplitz algebra F_n^∞ can be identified with the WOT-closed algebra generated by the left creation operators $S_1, \dots S_n$, and the identity. The algebras F_n^∞ and $\mathcal{A}_n$ were introduced by the author in [Po3] in connection with a noncommutative von Neumann inequality, and have been studied in several papers [Po2], [Po5], [Po6], [Po7], [Po9], [ArPo1], and recently in [DP1], [DP2], [ArPo2], [DP3], and [Po8].

We established a strong connection between the algebra F_n^∞ and the function theory on the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ through the noncommutative von Neumann inequality [Po3] (see also [Po5], [Po7], and [Po9]). In particular, we proved that there is a completely contractive homomorphism

$$\Phi : F_n^\infty \to H^\infty(\mathbb{B}_n), \quad f(S_1, \dots, S_n) \mapsto f(\lambda_1, \dots, \lambda_n),$$

where $(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$. A characterization of the analytic functions in the range of the map Φ was obtained in [ArPo2] and [DP3]. W. Arveson proved that Φ is not surjective [Arv] and the functions in its range are the multipliers of a certain function Hilbert space. In [ArPo2], [DP3], it was proved that $F_n^\infty / \ker \Phi$ is an operator algebra which can be identified with $\mathcal{W}_n^\infty := P_{F_s^2} F_n^\infty|_{F_s^2}$, the compression to the symmetric Fock space $F_s^2 \subseteq F^2(H_n)$. In [Po8], [Po9], [Arv], [ArPo2], [DP3], [AMc], and [BTV], a good case is made that the appropriate commutative multivariable analogue of $H^\infty(\mathbb{D})$ is the algebra $\mathcal{W}_n^\infty$, which is the WOT-closed algebra generated by $B_i := P_{F_s^2} S_i|_{F_s^2}$, $i = 1, \dots, n$, and the identity. In this paper, we provide further evidence that F_n^∞ (resp. $\mathcal{W}_n^\infty$) is a noncommutative (resp. commutative) multivariate analogue of $H^\infty(\mathbb{D})$.

Let $\mathcal{A}$ be a unital Banach algebra and denote by $\mathrm{Inv}(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$. Given $a \in \mathcal{A}$, we define the $\mathcal{A}$-spectral radius of a by setting

$$\rho_{\mathcal{A}}(a) := \inf\{\|xax^{-1}\| : \ x \in \mathrm{Inv}(\mathcal{A})\}.$$

Since the spectral radius of $a \in \mathcal{A}$ is $\|a\|_{\mathrm{sp}} = \lim_{n\to\infty} \|a^n\|^{1/n}$, it is clear that $\|a\|_{\mathrm{sp}} = \|xax^{-1}\|_{\mathrm{sp}}$ for any $x \in \mathrm{Inv}(\mathcal{A})$. Now, it is easy to see that

$$\|a\|_{\mathrm{sp}} \le \rho_{\mathcal{A}}(a) \le \|a\|,$$

for any $a \in \mathcal{A}$. Note that if $\mathcal{A} = B(\mathcal{H})$ (or $\mathcal{A}$ is any C^*-subalgebra of $B(\mathcal{H})$) then $\|a\|_{\mathrm{sp}} = \rho_{\mathcal{A}}(a)$ (see [R]). There are some other examples of Banach algebras such that $\|a\|_{\mathrm{sp}} = \rho_{\mathcal{A}}(a)$ for any $a \in \mathcal{A}$. It was proved in [BFT] that this equality holds if $\mathcal{A}$ is the commutant of an isometry (resp. normal operator) on a Hilbert space.

Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras, and $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital contractive homomorphism. We say that Φ is a quotient interpolant if

$$\|b\| = \inf\{\|a\| : \ a \in \mathcal{A}, \Phi(a) = b\}$$

for any $b \in \mathcal{B}$. We say that $b \in \mathcal{B}$ with $\rho_{\mathcal{B}}(b) < 1$ has a spectral lifting if there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\rho_{\mathcal{A}}(a) < 1$. The homomorphism Φ is called a spectral interpolant if any $b \in \mathcal{B}$ has a spectral lifting.

Problem. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital contractive homomorphism which is also a quotient interpolant. When is Φ a spectral interpolant ?

We show, in Section 2, that this problem has a positive answer if $\mathrm{Inv}(\mathcal{B}) \subseteq \Phi(\mathrm{Inv}(\mathcal{A}))$. This relation holds, for example, if the group of invertible elements of $\mathcal{B}$ is connected (in particular, if $\mathcal{B}$ is finite dimensional or equal to $B(\mathcal{H})$).

The results of Section 2 are used in Section 3 to obtain a noncommutative multivariable analogue (see Theorem 3.1) of the spectral commutant lifting theorem of Bercovici-Foiaş-Tannenbaum. This yields spectral versions of Sarason ([S]), Nevanlinna–Pick , and Carathéodory type interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra F_n^∞ and $B(\mathcal{K})$, the algebra of bounded operators on a finite dimensional Hilbert space $\mathcal{K}$.

In Section 4, we obtain a spectral tangential commutant lifting theorem in several variables (see Theorem 4.1). This leads to a spectral tangential Nevanlinna-Pick interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$ (see Theorem 4.2).

Problems concerning the optimal solutions to these spectral interpolation problems in several variables, and explicit algorithm for finding the optimal interpolants will be considered in a future paper.

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2. Spectral lifting in Banach algebras

The notation and definitions from Section 1 are used throughout the paper. Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital contractive homomorphism. We call Φ a norm preserving interpolant if for any $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\|a\| = \|b\|$. Notice that any norm preserving interpolant is a quotient interpolant. Examples of norm preserving interpolants will be presented in Section 3.

Theorem 2.1. *Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital contractive homomorphism with the property that* $\operatorname{Inv}(\mathcal{B}) \subseteq \Phi(\operatorname{Inv}(\mathcal{A}))$ *and*

$$\|b\| = \inf\{\|a\| : \ a \in \mathcal{A}, \Phi(a) = b\}$$

for any $b \in \mathcal{B}$. Then

$$\rho_{\mathcal{B}}(b) = \inf\{\rho_{\mathcal{A}}(a) : \ a \in \mathcal{A}, \Phi(a) = b\} \tag{2.1}$$

for any $b \in \mathcal{B}$. In particular, Φ is a spectral interpolant.

Proof. Let $b \in \mathcal{B}$ and $a \in \mathcal{A}$ with $\Phi(a) = b$. Since Φ is a contractive homomorphism and $\Phi(\operatorname{Inv}(\mathcal{A})) \subseteq \operatorname{Inv}(\mathcal{B})$ we have

$$\begin{aligned}
\rho_{\mathcal{A}}(a) &= \inf\{\|waw^{-1}\| : \ w \in \operatorname{Inv}(\mathcal{A})\} \\
&\geq \inf\{\|\Phi(waw^{-1})\| : \ w \in \operatorname{Inv}(\mathcal{A})\} \\
&= \inf\{\|\Phi(w)b\Phi(w)^{-1}\| : \ w \in \operatorname{Inv}(\mathcal{A})\} \\
&\geq \inf\{\|zbz^{-1}\| : \ z \in \operatorname{Inv}(\mathcal{B})\} \\
&= \rho_{\mathcal{B}}(b).
\end{aligned}$$

Therefore,

$$\rho_{\mathcal{B}}(b) \leq \inf\{\rho_{\mathcal{A}}(a) : \ a \in \mathcal{A}, \Phi(a) = b\}.$$

Now, let $\epsilon > 0$ and choose $z \in \operatorname{Inv}(\mathcal{B})$ such that

$$\|zbz^{-1}\| \leq \rho_{\mathcal{B}}(b) + \frac{\epsilon}{2}. \tag{2.2}$$

Since $zbz^{-1} \in \mathcal{B}$, according to the hypothesis, for any $\epsilon > 0$, there exists $d \in \mathcal{A}$ such that

$$\Phi(d) = zbz^{-1} \quad \text{and} \quad \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2}. \tag{2.3}$$

Since $\Phi(\operatorname{Inv}(\mathcal{A})) \supseteq \operatorname{Inv}(\mathcal{B})$, we find $w \in \operatorname{Inv}(\mathcal{A})$ such that $\Phi(w) = z$. Notice that $y := w^{-1}dw \in \mathcal{A}$ and

$$\Phi(y) = \Phi(w)^{-1}\Phi(d)\Phi(w) = z^{-1}(zbz^{-1})z = b.$$

Now, using (2.2) and (2.3), we infer that

$$\rho_{\mathcal{A}}(y) \leq \|wyw^{-1}\| = \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2} \leq \rho_{\mathcal{A}}(b) + \epsilon.$$

Therefore,

$$\rho_{\mathcal{B}}(b) \geq \inf\{\rho_{\mathcal{A}}(a) : \ a \in \mathcal{A}, \phi(a) = b\}.$$

Using relation (2.1), it is easy to see that if $b \in \mathcal{B}$, then $\rho_{\mathcal{B}}(b) < 1$ if and only if there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\rho_{\mathcal{A}}(a) < 1$. This completes the proof. ∎

Corollary 2.2. *Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras such that the group* $\operatorname{Inv}(\mathcal{B})$ *is connected. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital contractive homomorphism which is also a quotient interpolant. Then Φ is a spectral interpolant.*

Proof. Let us prove that

$$\Phi(\operatorname{Inv}(\mathcal{A})) = \operatorname{Inv}(\mathcal{B}). \tag{2.4}$$

The inclusion $\Phi(\operatorname{Inv}(\mathcal{A})) \subseteq \operatorname{Inv}(\mathcal{B})$ is clear. Conversely, let $x \in \operatorname{Inv}(\mathcal{B})$. Since $\operatorname{Inv}(\mathcal{B})$ is connected, it is well known that

$$x = \exp(z_1) \cdots \exp(z_k)$$

for some $z_1, \dots, z_k \in \mathcal{B}$. Due to the hypothesis, there exist $w_1, \dots, w_k \in \mathcal{A}$ such that $\Phi(w_i) = z_i,\ i = 1, \dots, k$. Denote $y := \exp(w_1) \cdots \exp(w_k) \in \mathrm{Inv}(\mathcal{A})$ and notice that $\Phi(y) = \exp(\Phi(w_1)) \cdots \exp(\Phi(w_k)) = x$. Hence $\Phi(\mathrm{Inv}(\mathcal{A})) \supseteq \mathrm{Inv}(\mathcal{B})$ and (2.4) holds. ∎

Remark 2.3. If $\mathcal{B}$ is a finite dimensional algebra, then $\mathrm{Inv}(\mathcal{B}) = \exp(\mathcal{B})$, hence $\mathrm{Inv}(\mathcal{B})$ is connected.

Corollary 2.4. *Let $\mathcal{A}$ be a unital Banach algebra and let J be a closed two-sided ideal of $\mathcal{A}$. If any invertible element of $\mathcal{A}/J$ has an invertible lifting in $\mathcal{A}$, then the quotient homomorphism $\Phi : \mathcal{A} \to \mathcal{A}/J$ is a spectral interpolant, i.e., $\rho_{\mathcal{A}/J}(a+J) < 1$ if and only if there exists $b \in a + J$ such that $\rho_{\mathcal{A}}(b) < 1$.*

Proof. Apply Theorem 2.1 to the quotient homomorphism Φ. ∎

Let us remark that, in general, there are invertible elements in $\mathcal{A}/J$ which can not be lifted to invertible elements in $\mathcal{A}$. For example, if $\pi : B(H^2) \to B(H^2)/K(H^2)$ is the quotient homomorphism into the Calkin algebra, and S is the unilateral shift on the Hardy space H^2, then $\pi(S)$ is invertible and there is no invertible operator $T \in B(H^2)$ such that $\pi(T) = \pi(S)$.

An important particular case, when Corollary 2.4 can be applied, is when the quotient algebra A/J is finite dimensional. Applications of this result will be considered in the next section.

3. Noncommutative spectral commutant lifting and interpolation

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $s_1, \dots, s_n$, and let e be its neutral element. For any $\sigma := s_{i_1} \cdots s_{i_k} \in \mathbb{F}_n^+$ we define its length $|\sigma| := k$, and $|e| = 0$. On the other hand, if $T_i \in B(\mathcal{H}),\ i = 1, \dots, n$, we denote $T_\sigma := T_{i_1} \cdots T_{i_k}$ and $T_e := I_{\mathcal{H}}$.

Let us recall from [Po1], [Po2], and [Po4] some results concerning the noncommutative dilation theory for n-tuples of operators. A sequence of operators $\mathcal{T} := [T_1, \dots, T_n]$, $T_i \in B(\mathcal{H})$, $i = 1, \dots, n$, is called contractive (or row contraction) if $T_1T_1^* + \cdots + T_nT_n^* \le I_{\mathcal{H}}$. We say that a sequence of isometries $\mathcal{V} := [V_1, \dots, V_n]$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is a minimal isometric dilation of $\mathcal{T}$ if the following properties are satisfied:

(i) $V_1V_1^* + \cdots + V_nV_n^* \le I_{\mathcal{K}}$;
(ii) $V_i^*|_{\mathcal{H}} = T_i^*,\ i = 1, \dots, n$;
(iii) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

The minimal isometric dilation of $\mathcal{T}$ is uniquely determined up to an isomorphism. We need to recall the noncommutative commutant lifting theorem [Po4] (see [SzF1], [SzF2], [DMP] for the classical case).

Let $\mathcal{T} := [T_1, \dots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \dots, V_n]$ be its minimal isometric dilation on a Hilbert $\mathcal{K} \supseteq \mathcal{H}$. If $X \in B(\mathcal{H})$ and $XT_i = T_iX$ for any $i = 1, \dots, n$, then there exists $X_\infty \in B(\mathcal{K})$ satisfying the following properties:

(i) $X_\infty V_i = V_i X_\infty$, for any $i = 1, \dots, n$;
(ii) $X_\infty^*|_{\mathcal{H}} = X^*$;
(iii) $\|X_\infty\| = \|X\|$.

Let $\mathcal{T} := [T_1, \ldots, T_n]$ be a row contraction with $T_i \in B(\mathcal{H})$ and let $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $X \in \{T_1, \ldots, T_n\}'$, and denote

$$\mathrm{Dil}(X) := \{Y \in \{V_1, \ldots, V_n\}' : \ P_{\mathcal{H}} Y = X P_{\mathcal{H}}\},$$

where $P_{\mathcal{H}}$ is the orthogonal projection on $\mathcal{H}$. According to the noncommutative commutant lifting, we have $\mathrm{Dil}(X) \neq \emptyset$.

In what follows we obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici-Foiaş-Tannenbaum [BFT].

Theorem 3.1. *Let $\mathcal{T} := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $\mathcal{H}$ is finite dimensional and $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, then*

$$\rho_{\{T_1, \ldots, T_n\}'}(X) = \inf\{\rho_{\{V_1, \ldots, V_n\}'}(Y) : Y \in \mathrm{Dil}(X)\}$$

for any $X \in \{T_1, \ldots, T_n\}'$.

Proof. Let $\Phi : \{V_1, \ldots, V_n\}' \to \{T_1, \ldots, T_n\}'$ be defined by $\Phi(Y) := P_{\mathcal{H}} Y|_{\mathcal{H}}$. Since $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we have $Y^*(\mathcal{H}) \subseteq \mathcal{H}$ for any $Y \in \{V_1, \ldots, V_n\}'$. Since $\mathcal{V} := [V_1, \ldots, V_n]$ is the minimal isometric dilation of $\mathcal{T}$, we have $V_i^*|_{\mathcal{H}} = T_i^*, i = 1, \ldots, n$. Now, it is easy to see that

$$(P_{\mathcal{H}} Y|_{\mathcal{H}}) T_i = T_i (P_{\mathcal{H}} Y|_{\mathcal{H}}) \quad \text{for any } i = 1, 2, \ldots, n.$$

Therefore, the mapping Φ is well-defined. On the other hand, since $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we infer that Φ is a unital contractive homomorphism, and $\Phi(Y) = X$ is equivalent to $P_{\mathcal{H}} Y = X P_{\mathcal{H}}$. According to the noncommutative commutant lifting theorem, for any $X \in \{T_1, \ldots, T_n\}'$ there exists $Y \in \{V_1, \ldots, V_n\}'$ such that $P_{\mathcal{H}} Y = X P_{\mathcal{H}}$ and $\|Y\| = \|X\|$. Therefore, Φ is a norm preserving interpolant. Since $\mathcal{H}$ is finite dimensional, the algebra $\{T_1, \ldots, T_n\}'$ is finite dimensional. Applying Theorem 2.1 and Remark 2.3, in the particular case when $\mathcal{A} := \{V_1, \ldots, V_n\}'$ and $\mathcal{B} := \{T_1, \ldots, T_n\}'$, the result follows. ∎

Corollary 3.2. *Let $\mathcal{T} := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $\mathcal{H}$ is finite dimensional and $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, then, given $X \in \{T_1, \ldots, T_n\}'$, $\rho_{\{T_1, \ldots, T_n\}'}(X) < 1$ if and only if there exists $Y \in \mathrm{Dil}(X)$ such that $\rho_{\{V_1, \ldots, V_n\}'}(Y) < 1$.*

In what follows, we use the noncommutative spectral commutant lifting theorem to obtain spectral versions of Sarason, Nevanlinna–Pick, and Carathéodory type interpolation for $F_n^\infty \bar{\otimes} B(\mathcal{K})$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra F_n^∞ and $B(\mathcal{K})$. In particular, we obtain interpolation results for matrix-valued analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant.

According to Theorem 1.2 from [Po6], the commutant of F_n^∞, which we denote by R_n^∞, is equal to $U^* F_n^\infty U$, where U is the unitary operator on $F^2(H_n)$ defined by $U(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}$. Moreover, the commutant of R_n^∞ is equal to F_n^∞.

A complete description of the invariant subspace structure of F_n^∞ was obtained in [Po2, Theorem 2.2] (even in a more general setting). A subspace $\mathcal{N}$ of $F^2(H_n)$ is invariant under $S_1, \ldots, S_n$ if and only if $\mathcal{N} = \bigoplus_{\lambda \in \Lambda} U^* \varphi_\lambda U[F^2(H_n)]$, for some family $\{\varphi_\lambda \in F_n^\infty : \lambda \in \Lambda\}$ of isometries with orthogonal ranges (see also [Po6] and [DP1]). Let us remark that $\mathcal{M} \subseteq F^2(H_n)$ is hyperinvariant for $\{S_1, \dots, S_n\}$, i.e., invariant for $\{S_1, \dots, S_n\}'$, if and only if $U\mathcal{M}$ is invariant for $\{S_1, \dots, S_n\}$.

Theorem 3.3. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and let $\mathcal{N} \subseteq F^2(H_n)$ be a finite dimensional subspace with the property that $\mathcal{N}$ and $U\mathcal{N}$ are invariant under $S_1^*, \dots, S_n^*$. Then $X \in B(\mathcal{N} \otimes \mathcal{K})$ commutes with each $P_\mathcal{N} S_i|_\mathcal{N} \otimes I_\mathcal{K}, i = 1, \dots, n$, and*

$$\rho_{P_\mathcal{N} R_n^\infty|_\mathcal{N} \bar{\otimes} B(\mathcal{K})}(X) < 1$$

if and only there exists $\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{K})$ such that

$$P_{\mathcal{N}\otimes\mathcal{K}}\Psi = X P_{\mathcal{N}\otimes\mathcal{K}} \quad \text{and} \quad \rho_{R_n^\infty \bar{\otimes} B(\mathcal{K})}(\Psi) < 1.$$

Proof. According to [Po8], we have

$$\mathcal{B} := \{P_\mathcal{N} S_i|_\mathcal{N} \otimes I_\mathcal{K},\ i = 1, \dots, n\}' = P_{\mathcal{N}\otimes\mathcal{K}}(R_n^\infty \bar{\otimes} B(\mathcal{K}))|_{\mathcal{N}\otimes\mathcal{K}}.$$

Notice that $\mathcal{B}$ is a finite dimensional algebra. Let $\mathcal{A} := R_n^\infty \bar{\otimes} B(\mathcal{K})$ and define $\Phi : \mathcal{A} \to \mathcal{B}$ by $\Phi(Y) = P_{\mathcal{N}\otimes\mathcal{K}} Y|_{\mathcal{N}\otimes\mathcal{K}}$. Since $S_i^*(U\mathcal{N}) \subseteq U\mathcal{N}$ for any $i = 1, \dots, n$, and $\{S_1 \otimes I_\mathcal{K}, \dots, S_n \otimes I_\mathcal{K}\}' = R_n^\infty \bar{\otimes} B(\mathcal{K})$, it is easy to see that $[F^2(H_n) \otimes \mathcal{K}] \ominus [\mathcal{N} \otimes \mathcal{K}]$ is hyperinvariant for $\{S_1 \otimes I_\mathcal{K}, \dots, S_n \otimes I_\mathcal{K}\}$ and the mapping Φ is a unital contractive homomorphism. Since $\mathcal{N}$ is invariant under $S_1^*, \dots, S_n^*$, it is clear that the operator matrix $[P_\mathcal{N} S_1|_\mathcal{N}, \dots, P_\mathcal{N} S_n|_\mathcal{N}]$ is a C_0-row contraction and its minimal isometric dilation is $[S_1, \dots, S_n]$ (see [Po1]). Therefore, the minimal isometric dilation of $[P_\mathcal{N} S_1|_\mathcal{N} \otimes I_\mathcal{K}, \dots, P_\mathcal{N} S_n|_\mathcal{N} \otimes I_\mathcal{K}]$ is $[S_1 \otimes I_\mathcal{K}, \dots, S_n \otimes I_\mathcal{K}]$. According to the noncommutative commutant lifting theorem, for any $X \in \mathcal{B}$ there exists $\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{K})$, such that $P_{\mathcal{N}\otimes\mathcal{K}}\Psi = X P_{\mathcal{N}\otimes\mathcal{K}}$ and $\|X\| = \|\Psi\|$. Therefore, $\Phi(\Psi) = X$ and Φ is a norm preserving interpolant. Applying Corollary 3.2, the result follows. ∎

Notice that the element Ψ in Theorem 3.3 satisfies $\|\Psi\|_{\rm sp} \le \rho_{R_n^\infty \bar{\otimes} B(\mathcal{K})}(\Psi) < 1$. It would be nice to know if $\rho_{R_n^\infty \bar{\otimes} B(\mathcal{K})}(\Psi) = \|\Psi\|_{\rm sp}$ for any $\Psi \in R_n^\infty \bar{\otimes} B(\mathcal{K})$. This equality holds if $n = 1$ (see [BFT]).

Let us remark that the finite dimensionality hypothesis can be dropped in Theorem 3.3 for those subspaces $\mathcal{N}$ and $\mathcal{K}$ for which one can prove that any invertible element $f \in P_\mathcal{N} R_n^\infty|_\mathcal{N} \bar{\otimes} B(\mathcal{K})$ can be lifted to an invertible element $g \in R_n^\infty \bar{\otimes} B(\mathcal{K})$, i.e., $P_{\mathcal{N}\otimes\mathcal{K}} g|_{\mathcal{N}\otimes\mathcal{K}} = f$. We do not have yet any nontrivial example when this lifting property holds and $\mathcal{N}$, $\mathcal{K}$ are infinite dimensional.

Let J be a WOT-closed, two-sided ideal of F_n^∞ and define $J(1) := \{\Psi(1) : \Psi \in J\}$ and $\mathcal{N}_J := F^2(H_n) \ominus J(1)\}$. Let us remark that $\mathcal{N}_J$ and $U\mathcal{N}_J$ are invariant subspaces under $S_i^*,\ i = 1, \dots, n$, therefore, Theorem 3.3 works in the case when $\dim \mathcal{N}_J < \infty$.

Corollary 3.4. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and let J be a WOT-closed two-sided ideal of F_n^∞ such that $\dim \mathcal{N}_J < \infty$. Then the quotient homomorphism*

$$\Phi : F_n^\infty \bar{\otimes} B(\mathcal{K}) \to F_n^\infty \bar{\otimes} B(\mathcal{K}) / (J \bar{\otimes} B(\mathcal{K}))$$

is a spectral interpolant.

Proof. According to [ArPo2], the quotient algebra $F_n^\infty \bar{\otimes} B(\mathcal{K}) / (J \bar{\otimes} B(\mathcal{K}))$ is completely isometrically isomorphic to $P_{\mathcal{N}_J} F_n^\infty|_{\mathcal{N}_J} \bar{\otimes} B(\mathcal{K})$, which is finite dimensional. Using Theorem 3.3, we infer that Φ is a spectral interpolant. The proof is complete. ∎

It will be interesting to see if this result remains true if $\mathcal{N}_J$ is infinite dimensional (at least for some particular cases, if not in general). The obstruction in the infinite dimensional case seems to be the lifting of the invertible elements of a quotient algebra $\mathcal{A}/J$ to invertible elements of $\mathcal{A}$ (see Section 2 for an example). In the finite dimensional case, Corollary 3.4 leads to our spectral interpolation results for F_n^∞ (see Theorem 3.6 and Theorem 3.8).

Let $F_s^2(H_n)$ be the symmetric Fock space and $\mathcal{W}_n^\infty$ be the WOT-closed algebra generated by $B_i := P_{F_s^2(H_n)}S_i|_{F_s^2(H_n)}$, $i=1,\dots,n$, and the identity. This algebra has been studied in [Po9], [Arv], [ArPo2], [DP3]. The following theorem can be seen as a spectral version of Sarason's interpolation theorem for $H^\infty(\mathbb{D})$ (see [S]), in a commutative and multivariable setting.

Theorem 3.5. *Let $\mathcal{E}\subseteq F_s^2(H_n)$ be a finite dimensional invariant subspace under $B_1^*,\dots,B_n^*$ and let $\mathcal{K}$ be a finite dimensional Hilbert space. Then $f\in B(\mathcal{E}\otimes\mathcal{K})$ commutes with each $P_\mathcal{E}B_i|_\mathcal{E}\otimes I_\mathcal{K}$, $i=1,\dots,n$, and*

$$\rho_{P_{\mathcal{E}\otimes\mathcal{K}}(\mathcal{W}_n^\infty\bar{\otimes}B(\mathcal{K}))|_{\mathcal{E}\otimes\mathcal{K}}}(f)<1$$

if and only if there exists $g\in\mathcal{W}_n^\infty\bar{\otimes}B(\mathcal{K})$ such that

$$P_{\mathcal{E}\otimes\mathcal{K}}g|_{\mathcal{E}\otimes\mathcal{K}}=f \quad \text{and} \quad \rho_{\mathcal{W}_n^\infty\bar{\otimes}B(\mathcal{K})}(g)<1.$$

Proof. Since $F_s^2(H_n)$ is invariant under each S_i^*, $i=1,\dots,n$, it is easy to see that $\mathcal{E}$ has the same property. Taking into account that $\mathcal{W}_n^\infty$ is the compression of F_n^∞ to the symmetric Fock space, one can see that f commutes with $P_{\mathcal{E}\otimes\mathcal{K}}(S_i\otimes I_\mathcal{K})|_{\mathcal{E}\otimes\mathcal{K}}$. As in the proof of Theorem 3.3, using the noncommutative commutant lifting theorem, we find $\phi\in F_n^\infty\bar{\otimes}B(\mathcal{K})$ such that $P_{\mathcal{E}\otimes\mathcal{K}}(U^*\otimes I_\mathcal{K})\phi(U\otimes I)|_{\mathcal{E}\otimes\mathcal{K}}=f$ and $\|f\|=\|\phi\|$. Hence, $P_{\mathcal{E}\otimes\mathcal{K}}\phi|_{\mathcal{E}\otimes\mathcal{K}}=f$. Setting $g:=P_{F_s^2(H_n)\otimes\mathcal{K}}\phi|_{F_s^2(H_n)\otimes\mathcal{K}}\in\mathcal{W}_n^\infty\bar{\otimes}B(\mathcal{K})$, we have $P_{\mathcal{E}\otimes\mathcal{K}}g|_{\mathcal{E}\otimes\mathcal{K}}=f$ and $\|f\|\le\|g\|\le\|\phi\|=\|f\|$. This shows that $\|f\|=\|g\|$. Define $\mathcal{A}:=\mathcal{W}_n^\infty\bar{\otimes}B(\mathcal{K})$, $\mathcal{B}:=P_{\mathcal{E}\otimes\mathcal{K}}(\mathcal{W}_n^\infty\bar{\otimes}B(\mathcal{K}))|_{\mathcal{E}\otimes\mathcal{K}}$ and let $\Phi:\mathcal{A}\to\mathcal{B}$ be defined by $\Phi(g):=P_{\mathcal{E}\otimes\mathcal{K}}(g)_{\mathcal{E}\otimes\mathcal{K}}$. We just proved that Φ is a unital contractive homomorphism and also a norm preserving interpolant. Now, the result follows by applying the results of Section 2 in our setting. ∎

Let us remark that a result similar to Corollary 3.4 holds for the algebra $\mathcal{W}_n^\infty\bar{\otimes}B(\mathcal{K})$.

In what follows we obtain a spectral version of Nevanlinna-Pick interpolation for the noncommutative analytic Toeplitz algebra F_n^∞(see [ArPo2], [DP3], and [Po8]). As mentioned in the first section, there exists a unital contractive homomorphism

$$\Psi:F_n^\infty\bar{\otimes}B(\mathcal{K})\to H^\infty(\mathbb{B}_n)\bar{\otimes}B(\mathcal{K})$$

defined by $[\Psi(f)](\lambda):=f(\lambda)$, $\lambda\in\mathbb{B}_n$.

Theorem 3.6. *Let $\mathcal{K}$ be a finite dimensional Hilbert space, $W_j\in B(\mathcal{K})$, and λ_j, $j=1,\dots,k$, be distinct elements in $\mathbb{B}_n$. Then there exists $\Phi\in F_n^\infty\bar{\otimes}B(\mathcal{K})$ such that*

$$\rho_{F_n^\infty\bar{\otimes}B(\mathcal{K})}(\Phi)<1 \quad \text{and} \quad \Phi(\lambda_j)=W_j,\ j=1,\dots,k,$$

if and only if there exist invertible operators $M_j\in B(\mathcal{K})$, $j=1,\dots,k$, such that

$$\left[\frac{I_\mathcal{K}-(M_iW_iM_i^{-1})(M_jW_jM_j^{-1})^*}{1-\langle\lambda_i,\lambda_j\rangle}\right]_{1\le i,j\le k}>0. \tag{3.1}$$

Proof. Let $\lambda_j := (\lambda_{j1}, \dots, \lambda_{jn}) \in \mathbb{B}_n$, $j = 1, \dots, k$. For any $\alpha := s_{j_1} s_{j_2} \dots s_{j_m}$ in $\mathbb{F}_n^+$, let $\lambda_{j\alpha} := \lambda_{jj_1}\lambda_{jj_2}\dots\lambda_{jj_m}$ and $\lambda_e := 1$. Define $z_{\lambda_j} \in F^2(H_n)$ by setting

$$z_{\lambda_j} := \sum_{\alpha \in \mathbb{F}_n^+} \overline{\lambda}_{j\alpha} e_\alpha, \quad j = 1, 2, \dots, k.$$

Let $\mathcal{N} := \operatorname{span}\{z_{\lambda_j} : j = 1, \dots, k\}$ and $X \in B(\mathcal{N} \otimes \mathcal{K})$ be defined by

$$X^*(z_{\lambda_j} \otimes h) := z_{\lambda_j} \otimes W_j^* h, \qquad h \in \mathcal{K}. \tag{3.2}$$

Notice that $S_i^* z_{\lambda_j} = \overline{\lambda}_{ji} z_{\lambda_j}$ for any $i = 1, \dots, n$; $j = 1, \dots, k$. Hence, the subspaces $\mathcal{N}$ and $U\mathcal{N}$ are invariant under each S_i^*, $i = 1, \dots, n$. Define $T_i \in B(\mathcal{N} \otimes \mathcal{K})$ by $T_i := P_{\mathcal{N}} S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}$. Since $z_{\lambda_1}, \dots, z_{\lambda_k}$ are linearly independent, the operator $X \in B(\mathcal{N} \otimes \mathcal{K})$ given by (3.2) is well defined.

Notice that $XT_i = T_i X$ for any $i = 1, \dots, k$. Indeed,

$$\begin{aligned} T_i^* X^*(z_{\lambda_j} \otimes h) &= T_i^*(z_{\lambda_j} \otimes W_j^* h) = S_i^* z_{\lambda_j} \otimes W_j^* h \\ &= \overline{\lambda}_{ji} z_{\lambda_j} \otimes W_j^* h \end{aligned}$$

and

$$X^* T_i^*(z_{\lambda_j} \otimes h) = X^*(\overline{\lambda}_{ji} z_{\lambda_j} \otimes h) = \overline{\lambda}_{ji} z_{\lambda_j} \otimes W_j^* h.$$

Applying Theorem 3.3, we infer that

$$\rho_{\{T_1, \dots, T_n\}'}(X) < 1 \tag{3.3}$$

if and only there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that

$$P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I) = X P_{\mathcal{N} \otimes \mathcal{K}} \quad \text{and} \quad \rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < 1. \tag{3.4}$$

Since $[F^2(H_n) \otimes \mathcal{K}] \ominus [\mathcal{N} \otimes \mathcal{K}]$ is hyperinvariant for $\{S_1 \otimes I_{\mathcal{K}}, \dots, S_n \otimes I_{\mathcal{K}}\}$, the first relation in (3.4) is equivalent to

$$P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I)|_{\mathcal{N} \otimes \mathcal{K}} = X. \tag{3.5}$$

Since $U(z_{\lambda_j}) = z_{\lambda_j}$, $j = 1, \dots, k$, and $\langle \phi, z_{\lambda_i} \rangle = \phi(\lambda_i)$ for any $\phi := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha$ in $F^2(H_n)$, it is easy to see that

$$\begin{aligned} &\langle (U^* \otimes I)\Phi(U \otimes I)(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle \\ &\qquad = \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle \Phi(\lambda_j) h, h' \rangle = \langle X(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle \\ &\qquad = \langle \Phi(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle = \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle W_j h, h' \rangle. \end{aligned}$$

for any $j = 1, \dots, k$, and $h, h' \in \mathcal{K}$. This shows that (3.5) holds if and only if $\Phi(\lambda_j) = W_j$ for any $j = 1, \dots, k$. Notice that relation (3.3) holds if and only if there exists $M \in \operatorname{Inv}(\{T_1, \dots, T_n\}')$ such that $\|MXM^{-1}\| < 1$. It is easy to see that $M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes M_j^* h$, $h \in \mathcal{K}$, for some invertible operators $M_j \in B(\mathcal{K})$, $j = 1, \dots, k$. On the other hand, notice that

$$M^{*-1} X^* M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes (M_j W_j M_j^{-1})^* h$$

and $\|MXM^{-1}\| < 1$ is equivalent to $I_{\mathcal{N} \otimes \mathcal{K}} - (MXM^{-1})(MXM^{-1})^* > 0$, which is equivalent to (3.1). This completes the proof. ■

Let us remark that the inequality (3.1) can be replaced with

$$\rho_{P_{\mathcal{N}} F_n^\infty|_{\mathcal{N}} \bar{\otimes} B(\mathcal{K})}(X) < 1. \tag{3.6}$$

In the particular case when $n = 1$, we find again Theorem 4 from [BFT]. As mentioned in [BFT], since $P_\mathcal{N} F_n^\infty|_\mathcal{N} \bar{\otimes} B(\mathcal{K})$ is finite dimensional, conditions of type (3.6) can be checked using computer algorithms.

Corollary 3.7. *Let $\mathcal{K}$ be a finite dimensional Hilbert space, $W_j \in B(\mathcal{K})$, and λ_j, $j = 1, \dots, k$, be distinct elements in $\mathbb{B}_n$. If there exist invertible operators $M_j \in B(\mathcal{K})$, $j = 1, \dots, k$, such that*

$$\left[\frac{I_\mathcal{K} - (M_i W_i M_i^{-1})(M_j W_j M_j^{-1})^*}{1 - \langle \lambda_i, \lambda_j \rangle}\right]_{1 \le i,j \le k} > 0,$$

then there exists $f \in H^\infty(\mathbb{B}_n) \bar{\otimes} B(\mathcal{K})$ such that

$$f(\lambda_j) = W_j,\ j = 1, \dots, k, \quad \text{and} \quad \sup_{\lambda \in \mathbb{B}_n} \|f(\lambda)\|_{\text{sp}} < 1.$$

Proof. Using Theorem 3.6, we find $f \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that $f(\lambda_j) = W_j$, $i = 1, \dots, k$, and $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(f) < 1$. As in the proof of Theorem 2.1, we infer that

$$\|\Psi(f)\|_{\text{sp}} \le \rho_{H^\infty(\mathbb{B}_n) \bar{\otimes} B(\mathcal{K})}(\Psi(f)) \le \rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(f) < 1.$$

On the other hand, similarly to [BFT, Proposition 3], one can prove that

$$\|\Psi(f)\|_{\text{sp}} = \sup_{\lambda \in \mathbb{B}_n} \|f(\lambda)\|_{\text{sp}}.$$

This completes the proof. ∎

Let $\mathcal{P}_m$ be the set of all polynomials in $F^2(H_n)$ of degree $\le m$, and let $\mathcal{P}_m^\infty := \{p(S_1, \dots, S_n) : \ p \in \mathcal{P}_m\}$. Let $J_{>m}^\infty$ be the WOT-closed two-sided ideal of F_n^∞ generated by $\{S_\alpha : \alpha \in \mathbb{F}_n^+, |\alpha| = m + 1\}$. The following result is a spectral version of the noncommutative Carathéodory interpolation problem for F_n^∞ (see [Po6] and [Po8]).

Theorem 3.8. *Let $\mathcal{K}$ be a finite dimensional Hilbert space and let $p \in \mathcal{P}_m^\infty \bar{\otimes} B(\mathcal{K})$. Then there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ with*

$$\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < 1$$

such that $\Phi = p + g$ for some $g \in J_{>m}^\infty \bar{\otimes} B(\mathcal{K})$ if and only if

$$\rho_\mathcal{C}[P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I) p (U \otimes I)|_{\mathcal{P}_m \otimes \mathcal{K}}] < 1 \tag{3.7}$$

where $\mathcal{C} := P_{\mathcal{P}_m \otimes \mathcal{K}}(R_n^\infty \bar{\otimes} B(\mathcal{K}))|_{\mathcal{P}_m \otimes \mathcal{K}}$.

Proof. Let $\mathcal{N} := \mathcal{P}_m$ and $X := P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I) p (U \otimes I)|_{\mathcal{P}_m \otimes \mathcal{K}}$. Notice that X commutes with each $P_{\mathcal{P}_m} S_i|_{\mathcal{P}_m} \otimes I_\mathcal{K}$, $i = 1, \dots, n$, and $\mathcal{P}_m = U \mathcal{P}_m$ is invariant under each $S_1^*, \dots, S_n^*$. According to Theorem 3.3, relation (3.7) holds if and only if there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ with $P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I) = X P_{\mathcal{P}_m \otimes \mathcal{K}}$ and $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < 1$. Hence, we infer that

$$P_{\mathcal{P}_m \otimes \mathcal{K}}(U^* \otimes I)(\Phi - p)(U \otimes I)|_{\mathcal{P}_m \otimes \mathcal{K}} = 0. \tag{3.8}$$

On the other hand, every element $f \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ has a unique Fourier expansion $f \sim \sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes W_{(\alpha)}$ determined by

$$f(1 \otimes h) = \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes W_{(\alpha)} h \in F^2(H_n) \otimes \mathcal{K},$$

where $W_{(\alpha)} \in B(\mathcal{K})$ are given by $\langle W_{(\alpha)} h, k\rangle = \langle f(1\otimes h), e_\alpha \otimes k\rangle$ for any $h, k \in \mathcal{K}$, and $\alpha \in \mathbb{F}_n^+$ (see [Po8]). Using now relation (3.8), one can easily see that $g := \Phi - p \in J^\infty_{>m} \bar{\otimes} B(\mathcal{K})$. This completes the proof. ∎

Using Theorem 3.5, one can obtain a version of Theorem 3.8 for the algebra $\mathcal{W}_n^\infty \bar{\otimes} B(\mathcal{K})$, in a similar manner. We leave this task to the reader.

4. Spectral tangential commutant lifting in several variables

Let $\mathcal{T} := [T_1, \ldots, T_n]$ be a row contraction with $T_i \in B(\mathcal{H})$, and $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Let $\mathcal{M} \subseteq \mathcal{H}$ be an invariant subspace under each T_i^*, $i = 1, \ldots, n$, and $X \in B(\mathcal{H})$ be such that $X\mathcal{H} \subseteq \mathcal{M}$ and

$$(P_\mathcal{M} T_i|_\mathcal{M})X = XT_i, \quad \text{for any } i = 1, \ldots, n. \tag{4.1}$$

According to the noncommutative commutant lifting theorem, there exists $Y \in \{V_1, \ldots, V_n\}'$ with $P_\mathcal{M} Y = XP_\mathcal{H}$. Define

$$\mathrm{Dil}_\mathcal{M}(X) := \{Y \in \{V_1, \ldots, V_n\}' : \ P_\mathcal{M} Y = XP_\mathcal{H}\}$$

and

$$\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}'}(X) := \inf\{\|P_{Z^*\mathcal{M}} Z^{-1} X Z\| : \ Z \in \mathrm{Inv}(\{T_1, \ldots, T_n\}')\}.$$

Notice that if $\mathcal{M} = \mathcal{H}$, then $\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}'}(X) = \rho_{\{T_1, \ldots, T_n\}'}(X)$.

In what follows we extend the spectral tangential commutant lifting theorem of Bercovici and Foiaş [BF] to our noncommutative multivariable setting.

Theorem 4.1. *Let $\mathcal{T} := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $\mathcal{H}$ is finite dimensional, $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, and $\mathcal{M} \subseteq \mathcal{H}$ is an invariant subspace under each T_i^*, $i = 1, \ldots, n$, then, for every $X \in B(\mathcal{H})$ such that $X\mathcal{H} \subseteq \mathcal{M}$ and $(P_\mathcal{M} T_i|_\mathcal{M})X = XT_i$, $i = 1, \ldots, n$, we have*

$$\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}'}(X) = \inf\{\rho_{\{V_1, \ldots, V_n\}'}(Y) : Y \in \mathrm{Dil}_\mathcal{M}(X)\}. \tag{4.2}$$

Proof. Denote the right hand side of (4.2) by t. Let $\epsilon > 0$ and choose $Y \in \mathrm{Dil}_\mathcal{M}(X)$ such that $\rho_{\{V_1, \ldots, V_n\}'}(Y) < t + \epsilon$. Hence, there is $W \in \mathrm{Inv}(\{V_1, \ldots, V_n\}')$ such that $\|W^{-1} Y W\| < t + \epsilon$. Since $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we infer that $P_\mathcal{H} W P_\mathcal{H} = P_\mathcal{H} W$. Let $Z := P_\mathcal{H} W|_\mathcal{H}$ and notice that $Z \in \mathrm{Inv}(\{T_1, \ldots, T_n\}')$ and

$$Z^{-1} = P_\mathcal{H} W^{-1}|_\mathcal{H}. \tag{4.3}$$

The subspace $\mathcal{M}_* := Z^*\mathcal{M}$ is invariant under each T_i^*, $i = 1, \ldots, n$, and satisfies $\mathcal{M}_* = \mathcal{H} \ominus Z^{-1}(\mathcal{H} \ominus \mathcal{M})$. Hence, we deduce the relations

$$P_{\mathcal{M}_*} Z^{-1} = P_{\mathcal{M}_*} Z^{-1} P_\mathcal{M} \quad \text{and} \quad P_\mathcal{M} Z = P_\mathcal{M} Z P_{\mathcal{M}_*}. \tag{4.4}$$

Since $Y \in \mathrm{Dil}_\mathcal{M}(X)$ and $\mathcal{K} \ominus \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we can use (4.4) and (4.3) to infer that

$$\begin{aligned}
\|P_{\mathcal{M}_*} Z^{-1} X Z\| &= \|P_{\mathcal{M}_*} Z^{-1}(P_\mathcal{M} Y|_\mathcal{H}) Z\| = \|P_{\mathcal{M}_*} Z^{-1}(P_\mathcal{H} Y|_\mathcal{H}) Z\| \\
&= \|P_{\mathcal{M}_*}(P_\mathcal{H} W^{-1}|_\mathcal{H})(P_\mathcal{H} Y|_\mathcal{H})(P_\mathcal{H} W|_\mathcal{H})\| \le \|P_\mathcal{H}(W^{-1} Y W)|_\mathcal{H}\| \\
&\le \|W^{-1} Y W\| < t + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$, we deduce that $\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}'}(X) \le t$.

Now, let us prove the converse. Let $\epsilon > 0$ and choose $Z \in \operatorname{Inv}(\{T_1, \dots, T_n\}')$ such that

$$\|P_{\mathcal{M}_*} Z^{-1} X Z\| \le \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon. \tag{4.5}$$

Since $\{T_1, \dots, T_n\}'$ is finite dimensional, we use Theorem 2.1 and Remark 2.3 when $\Phi : \{V_1, \dots, V_n\}' \to \{T_1, \dots, T_n\}'$ and $\Phi(W) = P_{\mathcal{H}} W|_{\mathcal{H}}$, to find $W \in \operatorname{Inv}(\{V_1, \dots, V_n\}')$ such that $Z = P_{\mathcal{H}} W|_{\mathcal{H}}$. Denote $X_* := P_{\mathcal{M}_*} Z^{-1} X Z$ and notice that

$$(P_{\mathcal{M}_*} T_i|_{\mathcal{M}_*}) X_* = X_* T_i, \qquad i = 1, \dots, n. \tag{4.6}$$

Indeed, since $\mathcal{M}_*$ is invariant under each T_i^*, $i = 1, \dots, n$, we have $P_{\mathcal{M}_*} T_i P_{\mathcal{M}_*} = P_{\mathcal{M}_*} T_i$, $i = 1, \dots, n$. Using this relation together with (4.1) and (4.4), we infer that, for any $i = 1, \dots, n$,

$$\begin{aligned}
X_* T_i &= P_{\mathcal{M}_*} Z^{-1} X Z T_i = P_{\mathcal{M}_*} Z^{-1} X T_i Z \\
&= P_{\mathcal{M}_*} Z^{-1} (P_{\mathcal{M}} T_i|_{\mathcal{M}}) X Z = P_{\mathcal{M}_*} Z^{-1} T_i X Z \\
&= P_{\mathcal{M}_*} T_i Z^{-1} X Z = P_{\mathcal{M}_*} T_i P_{\mathcal{M}_*} Z^{-1} X Z \\
&= P_{\mathcal{M}_*} T_i X_*.
\end{aligned}$$

According to (4.6), the noncommutative commutant lifting theorem, and relation (4.5), we find $Y_* \in \operatorname{Dil}_{\mathcal{M}_*}(X_*)$ satisfying

$$\|Y_*\| = \|X_*\| \le \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon. \tag{4.7}$$

Set $Y := W Y_* W^{-1}$ and let us show that $Y \in \operatorname{Dil}_{\mathcal{M}}(X)$. Notice that

$$X = P_{\mathcal{M}} Z X_* Z^{-1}. \tag{4.8}$$

Indeed, using (4.4), we have

$$\begin{aligned}
P_{\mathcal{M}} Z X_* Z^{-1} &= P_{\mathcal{M}} Z (P_{\mathcal{M}_*} Z^{-1} X Z) Z^{-1} = P_{\mathcal{M}} Z P_{\mathcal{M}_*} Z^{-1} X \\
&= P_{\mathcal{M}} Z Z^{-1} X = P_{\mathcal{M}} X = X.
\end{aligned}$$

Since $P_{\mathcal{M}_*} Y_* = X_* P_{\mathcal{H}}$, $Z^{-1} = P_{\mathcal{H}} W^{-1}|_{\mathcal{H}}$, and $Y(\mathcal{K} \ominus \mathcal{H}) \subseteq \mathcal{K} \ominus \mathcal{H}$, we can use relation (4.8) to obtain

$$\begin{aligned}
X P_{\mathcal{H}} &= P_{\mathcal{M}} Z X_* Z^{-1} P_{\mathcal{H}} = P_{\mathcal{M}} Z P_{\mathcal{M}_*} Y_* Z^{-1} P_{\mathcal{H}} \\
&= P_{\mathcal{M}} Z P_{\mathcal{H}} Y_* Z^{-1} P_{\mathcal{H}} = P_{\mathcal{M}} (P_{\mathcal{H}} Z|_{\mathcal{H}})(P_{\mathcal{H}} Y_*|_{\mathcal{H}})(P_{\mathcal{H}} W^{-1}|_{\mathcal{H}}) P_{\mathcal{H}} \\
&= P_{\mathcal{M}} (P_{\mathcal{H}} W Y_* W^{-1}|_{\mathcal{H}}) P_{\mathcal{H}} = P_{\mathcal{M}} Y P_{\mathcal{H}} = P_{\mathcal{M}} Y.
\end{aligned}$$

According to (4.7), we have $\|W^{-1} Y W\| = \|Y_*\| \le \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'} + \epsilon$. Hence $\rho_{\{V_1, \dots, V_n\}'}(Y) \le \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon$ and $t \le \rho_{\mathcal{M}, \{T_1, \dots, T_n\}'}(X) + \epsilon$. This completes the proof. ∎

The following result is a spectral version of the tangential Nevanlinna-Pick interpolation problem for F_n^∞ (see [Po8]).

Theorem 4.2. *Let λ_j, $j = 1, \dots, k$, be distinct elements in $\mathbb{B}_n$ and let $\mathcal{K}$ be a finite dimensional Hilbert space. If $u_1, \dots, u_k, v_1, \dots, v_k \in \mathcal{K}$ with $u_i \ne 0, j = 1, \dots, k$, and $\delta > 0$, then there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that*

$$\Phi(\lambda_j)^* u_j = v_j, \ j = 1, \dots, k, \quad \textit{and} \quad \rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < \delta$$

if and only if there exist invertible operators $Z_j \in B(\mathcal{K})$, $j = 1, \ldots, k$, such that

$$\left[\frac{\langle \delta Z_j u_j, \delta Z_i u_i\rangle - \langle Z_j v_j, Z_i v_i\rangle}{1 - \langle \lambda_j, \lambda_i\rangle}\right]_{1\leq i,j\leq k} > 0. \tag{4.9}$$

Proof. Let $\mathcal{N} := \text{span}\{z_{\lambda_j} : j = 1, \ldots, k\}$ and $\mathcal{M} := \mathbb{C}z_{\lambda_1} \otimes u_1 + \cdots + \mathbb{C}z_{\lambda_k} \otimes u_k$ be a subspace of $\mathcal{N} \otimes \mathcal{K}$. Define $X(\{\lambda_j\}, \{u_j\}, \{v_j\}) \in B(\mathcal{N} \otimes \mathcal{K}, \mathcal{M})$ by setting $X(\{\lambda_j\}, \{u_j\}, \{v_j\})^*(z_{\lambda_j} \otimes u_j) := z_{\lambda_j} \otimes v_j$, $j = 1, \ldots, k$. For each $i = 1, \ldots, n$, define $T_i := P_{\mathcal{N}} S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}$ and notice that $T_i^* X^* = X^* T_i^*|_{\mathcal{M}}$, where $X := X(\{\lambda_j\}, \{u_j\}, \{v_j\})$. Hence, $XT_i = P_{\mathcal{M}} T_i X$ for any $i = 1, \ldots, n$.

As in the proof of Theorem 3.3, the minimal isometric dilation of the sequence $[T_1, \ldots, T_n]$ is $[S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}}]$ and $[F^2(H_n) \otimes \mathcal{K}] \ominus [\mathcal{N} \otimes \mathcal{K}]$ is hyperinvariant for $\{S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}}\}$. Since $\mathcal{M} \subseteq \mathcal{N} \otimes \mathcal{K}$ is invariant under each T_i^*, $i = 1, \ldots, n$, we can apply Theorem 4.1 and infer that

$$\rho_{\mathcal{M},\{T_1,\ldots,T_n\}'}(X) = \inf\{\rho_{\{S_1\otimes I_{\mathcal{K}},\ldots,S_n\otimes I_{\mathcal{K}}\}'}(Y) : Y \in \text{Dil}_{\mathcal{M}}(X)\}.$$

Since $\{S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}}\}' = U^* F_n^\infty U \bar{\otimes} B(\mathcal{K})$, we can see that

$$\rho_{\mathcal{M},\{T_1,\ldots,T_n\}'}(X) < \delta \tag{4.10}$$

if and only if there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K})$ such that $\rho_{F_n^\infty \bar{\otimes} B(\mathcal{K})}(\Phi) < \delta$ and

$$P_{\mathcal{M}}(U^* \otimes I)\Phi(U \otimes I) = XP_{\mathcal{N}\otimes\mathcal{K}}. \tag{4.11}$$

Notice that

$$\begin{aligned}\langle P_{\mathcal{M}}(U^* \otimes I)\Phi(U \otimes I)(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j)\rangle &= \langle \Phi(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j)\rangle \\ &= \langle z_{\lambda_i}, z_{\lambda_j}\rangle\langle \Phi(\lambda_j)k, u_j\rangle \\ &= \langle z_{\lambda_i}, z_{\lambda_j}\rangle\langle k, \Phi(\lambda_j)^* u_j\rangle\end{aligned}$$

and $\langle X(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j\rangle = \langle z_{\lambda_i}, z_{\lambda_j}\rangle\langle k, v_j\rangle$ for any $k \in \mathcal{K}$ and $i, j = 1, \ldots, k$. Therefore, the relation (4.11) holds if and only if $\Phi(\lambda_j)^* u_j = v_j$, $j = 1, \ldots, k$. On the other hand, if $Z \in \{T_1, \ldots, T_n\}'$ then

$$Z^*(z_{\lambda_j} \otimes k) = z_{\lambda_j} \otimes Z_j k, \quad k \in \mathcal{K}, \tag{4.12}$$

for some $Z_j \in B(\mathcal{K}), j = 1, \ldots, k$. Notice that Z is invertible if and only if Z_j is invertible for any $j = 1, \ldots, k$. Moreover, using the definition of $X = X(\{\lambda_j\}, \{u_j\}, \{v_j\})$ and (4.12), we have

$$Z^* X^*(\{\lambda_j\}, \{u_j\}, \{v_j\}) Z^{*-1}|_{Z^*\mathcal{M}} = X^*(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}).$$

Therefore,

$$\rho_{\mathcal{M},\{T_1,\ldots,T_n\}'}(X) = \inf\{\|X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\})\| : \ Z_j \in B(\mathcal{K}) \text{ are invertible}\}$$

and relation (4.10) holds if and only if there exist invertible operators $Z_j \in B(\mathcal{K})$ such that $\|X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\})\| < \delta$. This inequality is equivalent to

$$\delta^2 I - X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}) X^*(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}) > 0,$$

which is equivalent to (4.9). This completes the proof. ∎

We remark that (4.9) can be replaced by relation (4.10). As a consequence of Theorem 4.2, when the distinct elements in $\mathbb{B}_n$ are $\overline{\lambda}_j$, $j = 1, \ldots, k$, we infer the following spectral tangential interpolation result for matrix-valued bounded analytic functions in the unit ball of $\mathbb{C}^n$.

Corollary 4.3. *Let λ_j, $j = 1, \ldots, k$, be distinct elements in $\mathbb{B}_n$ and let $\mathcal{K}$ be a finite dimensional Hilbert space. If $u_1, \ldots, u_k, v_1, \ldots, v_k \in \mathcal{K}$ with $u_i \neq 0, j = 1, \ldots, k$, $\delta > 0$, and there exist invertible operators $Z_j \in B(\mathcal{K})$, $j = 1, \ldots, k$, such that*

$$\left[\frac{\langle \delta Z_j u_j, \delta Z_i u_i\rangle - \langle Z_j v_j, Z_i v_i\rangle}{1 - \langle \lambda_i, \lambda_j\rangle}\right]_{1\leq i,j\leq k} > 0,$$

then there exists $F \in H^\infty(\mathbb{B}_n)\bar{\otimes}B(\mathcal{K})$ such that

$$\sup_{\lambda\in\mathbb{B}_n} \|F(\lambda)\|_{sp} < \delta \quad \text{ and } \quad F(\lambda_j)u_j = v_j,\ j = 1, \ldots, k.$$

Let us make some remarks on the dependence of $\rho_{\mathcal{M},\{T_1,\ldots,T_n\}'}(X)$ on the given interpolation data. For each $m = 1, \ldots, k$, we define

$$\rho_m := \inf\{\|X(\{\lambda_j\}_{j=1}^m, \{Z_j u_j\}_{j=1}^m, \{Z_j v_j\}_{j=1}^m)\| : \ Z_j \in B(\mathcal{K}) \text{ are invertible}\}.$$

A multivariable analogue of [BF, Proposition 4] holds. More precisely, one can prove that if u_k and v_k are linearly independent, then $\rho_{k-1} = \rho_k$. Indeed, suppose that $\rho_{k-1} < \rho_k$. Using Theorem 4.2, we find $\Phi \in F_n^\infty\bar{\otimes}B(\mathcal{K})$ such that $\rho_{F_n^\infty\bar{\otimes}B(\mathcal{K})}(\Phi) < \rho_k$ and $\Phi(\lambda_j)^* u_j = v_j$, $j = 1, \ldots, k-1$. We may suppose that $\Phi(\lambda_k)^* \notin \mathbb{C}I_\mathcal{K}$ because, otherwise, we can replace Φ by $\Phi + \Psi$ for some $\Psi \in F_n^\infty\bar{\otimes}B(\mathcal{K})$ satisfying $\Phi(\lambda_j) = 0$, $j = 1, \ldots, k-1$, and $\Psi(\lambda_k) \notin \mathbb{C}I_\mathcal{K}$. Since we can choose Ψ with very small norm we have $\rho_{F_n^\infty\bar{\otimes}B(\mathcal{K})}(\Phi + \Psi) < \rho_k$.

Therefore, since $\Phi(\lambda_k)^* \notin \mathbb{C}I_\mathcal{K}$, there exist linearly independent vectors u and v such that $\Phi(\lambda_k)^* u = v$. Since u_k, v_k are linearly independent, we can find $Z_k \in B(\mathcal{K})$ invertible with $Z_k u_k = u$ and $Z_k v_k = v$. Hence, we infer that $\rho_k \leq \rho_{F_n^\infty\bar{\otimes}B(\mathcal{K})}(\Phi) < \rho_k$, which is a contradiction. Since $\rho_{k-1} \leq \rho_k$, we must have $\rho_{k-1} = \rho_k$. This shows that in Theorem 4.2 we can assume, without loss of generality, that $v_j = \mu_j u_j$, for some $\mu_j \in \mathbb{C}$, $\mu_j \neq 0$, $j = 1, \ldots, k$. Similarly to [BF, Proposition 5], one can show that if $k \leq \dim \mathcal{K}$, then

$$\rho_k = \max\{|\mu_1|, \ldots, |\mu_k|\}.$$

The case when the number of dependent vector pairs (u_j, v_j) exceeds the dimension of $\mathcal{K}$, and the problem of optimal solutions will be considered in a future paper.

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Division of Mathematics and Statistics, The University of Texas at San Antonio, San Antonio, Texas 78249

E-mail address: `gpopescu@math.utsa.edu`