

THE FINITE VERTEX-PRIMITIVE AND VERTEX-BIPRIMITIVE s -TRANSITIVE GRAPHS FOR $s \geq 4$

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ABSTRACT. A complete classification is given for finite vertex-primitive and vertex-biprimitive s -transitive graphs for $s \geq 4$. The classification involves the construction of new 4-transitive graphs, namely a graph of valency 14 admitting the Monster simple group M , and an infinite family of graphs of valency 5 admitting projective symplectic groups $\text{PSp}(4, p)$ with p prime and $p \equiv \pm 1 \pmod{8}$. As a corollary of this classification, a conjecture of Biggs and Hoare (1983) is proved.

1. INTRODUCTION

Denote by Γ a finite connected graph with vertex set $V\Gamma$ and edge set $E\Gamma$. For a positive integer s , an s -arc of Γ is an $(s+1)$ -tuple (v_0, v_1, \dots, v_s) of vertices such that $\{v_{i-1}, v_i\} \in E\Gamma$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. Let $\text{Aut } \Gamma$ denote the full automorphism group of Γ . If $G \leq \text{Aut } \Gamma$ and G is transitive on $V\Gamma$ and on the set of s -arcs of Γ , then Γ is called a (G, s) -arc transitive graph; while if in addition G is not transitive on the set of $(s+1)$ -arcs of Γ , then Γ is called a (G, s) -transitive graph. In particular, if $G = \text{Aut } \Gamma$, then a (G, s) -arc transitive or (G, s) -transitive graph is simply called an s -arc transitive graph or an s -transitive graph, respectively. (We remark that for any positive integer s a cycle is $(s+1)$ -arc transitive so that it is not s -transitive, so the valency of an s -transitive graph is greater than 2.) The main purpose of this paper is to classify finite vertex-primitive and vertex-biprimitive s -transitive graphs for $s \geq 4$.

Interest in s -transitive graphs stems from a beautiful result of Tutte (1947) who proved that there exist no finite s -transitive cubic graphs for $s \geq 6$. Tutte's Theorem was generalized by Weiss (1981) who proved that there exist no finite s -transitive graphs for $s = 6$ and $s \geq 8$. Since then, s -transitive graphs have received considerable attention in the literature (see, for example, [11, 13, 24, 34]), and investigating the following problem has become one of the central topics in algebraic graph theory:

Problem. Construct and characterize s -transitive graphs for $s \geq 2$, and in particular, for $s = 4, 5$ and 7.

It is known that for each value of $s \in \{1, 2, 3, 4, 5, 7\}$, there exist s -transitive graphs (see Section 2). However, it is widely believed that s -transitive graphs for

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large values of s , that is $s \in \{4, 5, 7\}$, are rare and hard to construct. An s -transitive graph is said to be *basic* if it is not a *normal cover* of a smaller s -transitive graph (see [26]). It was shown by Conway that any s -transitive graph has a nontrivial s -transitive normal cover (see [3, Section 19]). This suggests that the study of s -transitive graphs should be divided into two steps: studying basic s -transitive graphs and studying their normal covers (see [26]). Using such an approach, it is shown in [18, 19] that the order of a finite s -transitive graph for $s \geq 4$ is neither a prime-power nor an odd integer.

Let s be an integer in $\{4, 5, 7\}$. Up to now, the only known basic s -transitive graphs consist of the following graphs (see Sections 2 and 3): several infinite families of s -transitive graphs of valency 3 and valency 4, constructed in [4, 6, 7, 39]; the classical generalized $(s - 1)$ -gon graphs which are bipartite; and three graphs from sporadic simple groups: the Janko group J_3 , the Rudvalis group Ru and the Thompson group Th , which were constructed in [37] and [28]. The J_3 -graph has valency 5; the Ru -graph and the Th -graph have valency 6. These three graphs were the only known non-bipartite s -transitive graphs for $s \geq 4$ of valency greater than 4 until now. Here we construct a 4-transitive graph of valency 14 from the Monster sporadic simple group, and an infinite family of 4-transitive graphs of valency 5 from the projective symplectic groups $PSp(4, p)$ with p prime.

Theorem 1.1. *There exists exactly one 4-transitive graph Γ such that Γ is vertex-primitive and $\text{Aut } \Gamma = M$, the Monster simple group. Further, the graph Γ has valency 14.*

The Monster-graph is the only known non-bipartite basic s -transitive graph of valency greater than 6, for $s \geq 4$.

Theorem 1.2. *For each prime $p \equiv \pm 1 \pmod{8}$, there exists exactly one 4-transitive graph Γ such that Γ is vertex-primitive and $\text{Aut } \Gamma = PSp(4, p)$. Further, Γ has valency 5.*

The $PSp(4, p)$ -graphs form the only known infinite family of basic s -transitive graphs of valency greater than 4, for $s \geq 4$.

Moreover, it will be shown that the Monster-graph and the $PSp(4, p)$ -graphs, together with some other known graphs, form the class of vertex-primitive or vertex-biprimitive s -transitive graphs for $s \geq 4$. (A finite graph Γ is said to be *vertex-primitive* if $\text{Aut } \Gamma$ acts primitively on $V\Gamma$; while a bipartite graph Γ with bi-parts Δ_1 and Δ_2 is said to be *vertex-biprimitive* if the setwise stabilizer $(\text{Aut } \Gamma)_{\Delta_i}$ is primitive on Δ_i .)

For two groups G and F , let $G.F$ denote an extension of G by F , and $G \rtimes F$ denote a semidirect product of G by F . For a positive integer n , as in the Atlas [8], denote by $[n]$ a group of order n throughout the paper. Given a finite group G , a core-free subgroup H of G and a 2-element $g \in G \setminus H$ with $g^2 \in H$, the *coset graph* $\Gamma = \Gamma(G, H, HgH)$ is the graph with vertex set $[G : H]$, the set of right cosets of H in G , such that Hx, Hy are adjacent if and only if $yx^{-1} \in HgH$ (see Section 2 for more detailed discussions). The next theorem gives a complete classification of finite vertex-primitive s -transitive graphs for $s \in \{4, 5, 7\}$.

Theorem 1.3. *Let k, s be positive integers, and assume that $s \geq 4$. A finite graph Γ of valency k is vertex-primitive and s -transitive if and only if $\Gamma \cong \Gamma(G, H, HgH)$, where G, H, k, s are as listed in Table 1. Further, such a graph Γ is uniquely determined by its order $|V\Gamma|$.*

TABLE 1.

$\text{Aut } \Gamma = G$	H	k	s	reference
$\text{PSL}(2, p)$ $p \equiv \pm 1 \pmod{16}$ prime	S_4	3	4	Wong (1967)
$\text{PSL}(3, 3) \cdot \mathbb{Z}_2$	$S_4 \times \mathbb{Z}_2$	3	5	Wong (1967)
$\text{PSp}(4, p)$ $p \equiv \pm 1 \pmod{8}$ prime	$\mathbb{Z}_2^4 \rtimes \text{GL}(2, 4)$	5	4	Theorem 1.2
J_3	$\mathbb{Z}_2^4 \rtimes \text{GL}(2, 4)$	5	4	Weiss (1986)
Ru	$\mathbb{Z}_5^2 \rtimes \text{GL}(2, 5)$	6	4	Stroth & Weiss (1990)
Th	$\mathbb{Z}_5^2 \rtimes \text{GL}(2, 5)$	6	4	Stroth & Weiss (1990)
M	$\mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4 \cdot \text{PGL}(2, 13))$	14	4	Theorem 1.1

TABLE 2.

$T = \text{soc}(G)$	G	$H \cap T$	k	s	reference
$\text{PSL}(3, q)$	$G \leq \text{Aut}(T)$	$[q^2] \rtimes (\mathbb{Z}_l \cdot \text{PGL}(2, q))$ where $l = \frac{q-1}{(q-1, 3)}$	$q + 1$	4	Section 2
$\text{Sp}(4, 2^m)$	$G \leq \text{Aut}(T)$	$[2^{3m}] \rtimes \text{GL}(2, 2^m)$	$2^m + 1$	5	Section 2
$G_2(3^m)$	$G \leq \text{Aut}(T)$	$[3^{5m}] \rtimes \text{GL}(2, 3^m)$	$3^m + 1$	7	Section 2
$\text{PSL}(2, p)$ $p \equiv \pm 1 \pmod{8}$	$\text{PGL}(2, p)$	$H = S_4$	3	4	Biggs & Hoare [4]
A_6	$\text{P}\Gamma\text{L}(2, 9)$	$H = S_4 \times \mathbb{Z}_2$	3	5	Tutte's 8-Cage
M_{12}	$M_{12} \cdot \mathbb{Z}_2$	$H = \mathbb{Z}_3^2 \rtimes \text{GL}(2, 3)$	4	4	Weiss (1985)

Remark. The J_3 -graph was used to characterize the Janko simple group J_3 by Weiss [37]; the Ru-graph was used to characterize the Rudvalis simple group Ru by Stroth and Weiss [28].

Some vertex-biprimitive s -transitive graphs are covers of vertex-primitive s -transitive graphs. For a (G, s) -transitive graph $\Sigma = \Gamma(G, H, HgH)$, let $F = G \times \langle z \rangle$ where z is an involution, and let $\Gamma = \Gamma(F, H, HgzH)$. Then Γ is an (F, s) -transitive bipartite graph, and is called the *standard double cover* of Σ . The following theorem gives a complete classification of finite vertex-biprimitive s -transitive graphs for $s \in \{4, 5, 7\}$.

Theorem 1.4. *Let k, s be positive integers, and assume that $s \geq 4$. A finite graph Γ of valency k is vertex-biprimitive and s -transitive if and only if either*

- (i) Γ is the standard double cover of a vertex-primitive s -transitive graph (classified in Theorem 1.3); or
- (ii) $\Gamma \cong \Gamma(G, H, HgH)$ such that $G, \text{soc}(G) \cap H, k, s$ are as listed in Table 2, where q is a prime-power, and p is a prime.

Further, such a graph Γ is uniquely determined by its order $|\text{V}\Gamma|$.

Remark. The M_{12} -graph was used to characterize the Mathieu group M_{12} by Weiss [36].

This result does not help to describe all s -transitive graphs for $s \geq 4$ as there is no reduction to the primitive and biprimitive case. The major problem remaining is to handle the quasiprimitive and bi-quasiprimitive cases (see [20]).

It follows from these theorems that there exist no finite vertex-primitive 7-transitive graphs, and that all finite vertex-biprimitive 7-transitive graphs of valency k admit $G_2(3^m)$ and have $k = 3^m + 1$, for some m .

Wong [39] constructed the $\text{PSL}(3, 3).\mathbb{Z}_2$ -graph (see Example 3.3) and proved that it is the unique vertex-primitive cubic 5-transitive graph. Biggs and Hoare [4] constructed an infinite family of bipartite 5-transitive cubic graphs from $\text{P}\Gamma\text{L}(2, p^2)$ (see Example 3.6). Motivated by Wong's classification, they further conjectured that each 5-transitive cubic biprimitive graph is either one of these graphs, or is a standard double cover of the $\text{PSL}(3, 3).\mathbb{Z}_2$ -graph given in Table 1. It is an immediate consequence of Theorems 1.3 and 1.4 that this conjecture is true.

Corollary 1.5. *The only vertex-biprimitive 5-transitive graphs of valency 3 are the $\text{P}\Gamma\text{L}(2, 9)$ -graph, which is Tutte's 8-Cage, and the standard double cover of the $\text{PSL}(3, 3).\mathbb{Z}_2$ -graph.*

In Section 2, we describe generalized polygon graphs in terms of coset graphs, and state the results of Weiss [33] and Gardiner [10], which provide a method for constructing s -transitive graphs for $s \geq 4$. Then in Section 3, we reproduce most of the known s -transitive graphs for $s \in \{4, 5, 7\}$ and construct some new 4-transitive graphs in a uniform manner. Finally, we complete the proofs of Theorems 1.1–1.4 in Section 4.

2. AMALGAMS OF s -TRANSITIVE GRAPHS FOR $s \in \{4, 5, 7\}$

In this section, we study a method of constructing s -transitive graphs in terms of coset graphs. Let G be a finite group, and let H be a subgroup of G . The subgroup H is said to be *core-free* if H contains no nontrivial normal subgroups of G . Denote by $[G : H]$ the set of right cosets of H in G , and for a subset S of G , denote by \underline{S} the set of right cosets of H contained in HS , that is, $\{Hs \mid s \in S\}$. Recall that for an element $g \in G \setminus H$ such that $g^2 \in H$, the coset graph $\Gamma(G, H, HgH)$ is the graph with vertex set $[G : H]$ and with Hx, Hy adjacent if and only if $yx^{-1} \in HgH$. By definition, the set of vertices Hx of $\Gamma(G, H, HgH)$ which are adjacent to the vertex H is equal to $\underline{HgH} = \{Hgh \mid h \in H\}$, denoted by $\Gamma(H)$ as usual. It follows that the group G in its coset action by right multiplication on $[G : H]$ is transitive on the vertex set of $\Gamma(G, H, HgH)$, and in this G -action the subgroup H is the stabilizer of the vertex H in G and acts transitively on $\Gamma(H)$. Thus G acts transitively on the vertex set and the arc set of $\Gamma(G, H, HgH)$, that is, $\Gamma(G, H, HgH)$ is a G -arc-transitive graph. The valency of $\Gamma(G, H, HgH)$ equals the cardinality of $\Gamma(H)$, that is, $|\underline{HgH} : H|$. Clearly, $|\underline{HgH} : H| = |[H : H \cap H^g]|$, and by definition, it is also clear that $\Gamma(G, H, HgH)$ is connected if and only if $\langle H, g \rangle = G$.

Let α and β denote the vertex H and the vertex Hg , respectively. Then, as observed before, $G_\alpha = H$, and further, $G_\beta = H^g$. Since $g^2 \in H$, g interchanges α and β , and so interchanges G_α and G_β . Now $G_{\alpha\beta}$, the subgroup of G which fixes both α and β , equals $H \cap H^g$; while $G_{\{\alpha, \beta\}}$, the subgroup of G which fixes the edge $\{\alpha, \beta\}$, equals $\langle H \cap H^g, g \rangle$.

As we have noticed, the subgroup H acts transitively on $\Gamma(H)$ and has point-stabilizer $H \cap H^g$. Thus the H -action on $\Gamma(H)$ is equivalent to the coset action of H on $[H : H \cap H^g]$. For convenience, we summarize the above observations in the following lemma (see [27]).

Lemma 2.1. *Let G be a finite group, and let $\Gamma = \Gamma(G, H, HgH)$ be a coset graph of G . Denote by α, β the vertices H, Hg , respectively, of Γ . Then the following are true:*

- (1) Γ is G -arc-transitive and has valency $|[H : H \cap H^g]|$;
- (2) Γ is connected if and only if $\langle H, g \rangle = G$;
- (3) $G_\alpha = H, G_\beta = H^g, G_{\alpha\beta} = H \cap H^g$ and $G_{\{\alpha, \beta\}} = \langle H \cap H^g, g \rangle$.

These simple properties suggest some method for constructing arc-transitive graphs with certain extra conditions.

Typical examples of s -transitive graphs for $s \geq 4$ come from the parabolic geometries of the rank 2 Lie type groups $A_2(q) \cong \text{PSL}(3, q)$, $B_2(2^m) \cong \text{Sp}(4, 2^m)$ and $G_2(3^m)$, corresponding to $s = 4, 5$ and 7 , respectively. These graphs are the so-called *classical generalized polygon graphs* (see [31] or [13, p. 423]). In this section, we first describe these graphs in terms of coset graphs, and then state a result of Weiss about automorphism groups of s -transitive graphs, that is the classification of the amalgams for s -transitive graphs for $s \in \{4, 5, 7\}$. Recall that for a positive integer n , $[n]$ denotes a group of order n .

A 4-transitive graph from $\text{PSL}(3, q)$. Let $T = \text{PSL}(3, q)$, where $q = p^m$, p is a prime and $m \geq 1$. Then T has two conjugacy classes of maximal parabolic subgroups, and has a graph automorphism g , which is of order 2; further, g fuses the two conjugacy classes of maximal parabolic subgroups. Let P be a maximal parabolic subgroup of T . Then $P = [q^2] \times (\mathbb{Z}_{\frac{q-1}{(q-1,3)}} \cdot \text{PGL}(2, q))$ and $P \cap P^g = [q^3] \times (\mathbb{Z}_{q-1} \times \mathbb{Z}_{\frac{q-1}{(q-1,3)}})$. Now $g^2 = 1 \in P, \langle P, g \rangle = \langle T, g \rangle$, and $|[P : P \cap P^g]| = q + 1$. Thus the coset graph

$$\Gamma = \Gamma(\langle T, g \rangle, P, PgP)$$

is a bipartite $(\langle T, g \rangle, 4)$ -transitive graph of valency $q + 1$ (see Theorem 2.2). Moreover, $\text{Aut } \Gamma = \text{Aut}(\text{PSL}(3, q))$, and thus Γ is $(G, 4)$ -transitive for each G with $\langle T, g \rangle \leq G \leq \text{Aut}(\text{PSL}(3, q))$; in particular, Γ is 4-transitive. Let d, f be a diagonal or a field automorphism, respectively, of T . Then $o(d) = (q - 1, 3), o(f) = m$, and d, f, P can be chosen so that d, f normalize P . Let R be a subgroup of $\langle d, f \rangle$, and let

$$G = T.R.\langle g \rangle.$$

Then R normalizes T, P and $P \cap P^g$. Thus $T.R$ has a subgroup $H := P.R$ such that $L := H \cap H^g = (P \cap P^g).R$. Then we have

$$H = ([q^2] \times (\mathbb{Z}_{\frac{q-1}{(q-1,3)}} \cdot \text{PGL}(2, q))).R, \quad L = ([q^3] \times (\mathbb{Z}_{q-1} \times \mathbb{Z}_{\frac{q-1}{(q-1,3)}})).R.$$

Finally, we note that $\mathbf{N}_G(L) = L.Z_2$ (see [16]). □

A 5-transitive graph from $\text{Sp}(4, 2^m)$. Let $T = \text{Sp}(4, q)$, where $q = 2^m$ for some $m \geq 1$. Then T has two conjugacy classes of maximal parabolic subgroups, which are fused in $\text{Aut}(T)$. Let f be a field automorphism, and let $g \in \text{Aut}(T) \setminus \text{Inn}(T).\langle f \rangle$. (Note that we are identifying T with $\text{Inn}(T)$.) Then $o(f) \mid m, \text{Aut}(T)/\langle T, f \rangle \cong \mathbb{Z}_2$ and $g^2 \in \langle f \rangle$. Let P be a maximal parabolic subgroup of T . Then g, f, P may be chosen so that f normalizes P but g does not, and $P = [q^3] \times \text{GL}(2, q)$. Let $R \leq \langle f \rangle$ contain g^2 , and set

$$G = T.\langle R, g \rangle.$$

TABLE 3.

$k = q + 1$	s	$G_\alpha = H$	$G_{\alpha\beta} = L$
$q + 1$	4	$([q^2] \rtimes (\mathbb{Z}_{\frac{q-1}{(q-1,3)}} \cdot \text{PGL}(2, q))).R$	$([q^3] \rtimes (\mathbb{Z}_{q-1} \times \mathbb{Z}_{\frac{q-1}{(q-1,3)}})).R$
$2^m + 1$	5	$([q^3] \rtimes \text{GL}(2, q)).R$	$([q^4] \rtimes \mathbb{Z}_{q-1}^2).R$
$3^m + 1$	7	$([q^5] \rtimes \text{GL}(2, q)).R$	$([q^6] \rtimes \mathbb{Z}_{q-1}^2).R$

Then $T.R$ has a subgroup $H := P.R$ such that $g^2 \in H$ and $\langle H, g \rangle = G$. Thus the coset graph $\Gamma = \Gamma(G, H, HgH)$ is a $(G, 5)$ -transitive graph of valency $q + 1$ (see Theorem 2.2). Moreover, $\text{Aut } \Gamma = \text{Aut}(T)$, and Γ is 5-transitive. The subgroups $H, L < G$ have the form

$$H = ([q^3] \rtimes \text{GL}(2, q)).R, \quad L = ([q^4] \rtimes \mathbb{Z}_{q-1}^2).R.$$

We notice that $\mathbf{N}_G(L) = L.\mathbb{Z}_2$ (see [16]). □

A 7-transitive graph from $G_2(3^m)$. Let $T = G_2(q)$, where $q = 3^m$ for some $m \geq 1$. Then T has two conjugacy classes of maximal parabolic subgroups, and has a graph automorphism g , which is of order 2; further, g fuses the two conjugacy classes of parabolic subgroups. Let P be a parabolic subgroup of T . Then $P = [q^5] \rtimes \text{GL}(2, q)$, and P, g may be chosen so that $P \cap P^g = [q^6] \rtimes \mathbb{Z}_{q-1}^2$. Now $g^2 = 1 \in P$, $\langle P, g \rangle = \langle T, g \rangle$, and $|[P : P \cap P^g]| = q + 1$. Thus the coset graph

$$\Gamma = \Gamma(\langle T, g \rangle, P, P^gP)$$

is a bipartite $(\langle T, g \rangle, 7)$ -transitive graph of valency $q + 1$ (see Theorem 2.2). Moreover, $\text{Aut } \Gamma = \text{Aut}(G_2(q))$, and thus Γ is $(G, 7)$ -transitive for each G with $\langle T, g \rangle \leq G \leq \text{Aut}(G_2(q))$; in particular, Γ is 7-transitive. Choose a field automorphism f of T such that f normalizes P . Let R be a subgroup of $\langle f \rangle$, and let

$$G = T.R.\langle g \rangle.$$

Then R normalizes T, P and $P \cap P^g$. Thus $T.R$ has a subgroup $H := P.R$ such that $L := H \cap H^g = (P \cap P^g).R$. Then we have

$$H = ([q^5] \rtimes \text{GL}(2, q)).R, \quad L = ([q^6] \rtimes \mathbb{Z}_{q-1}^2).R.$$

Finally, we note that, by [15, Theorem B], $\mathbf{N}_G(L) = L.\mathbb{Z}_2$. □

These three classes of graphs are important because of the following result (see [10] and [33, 35], also refer to [13, Theorem 2.1.5]).

Theorem 2.2. *Let Γ be a connected graph of valency k , and let $G \leq \text{Aut } \Gamma$ be such that Γ is transitive on $V\Gamma$. Let $\alpha, \beta \in V\Gamma$ be a pair of adjacent vertices.*

- (1) (Weiss [33, 35]) *Assume that $s \geq 4$ and that Γ is (G, s) -transitive. Then $s \in \{4, 5, 7\}$, $k = q + 1$ where q is a prime power, and the stabilizers G_α and $G_{\alpha\beta}$ are completely known, as in Table 3. Further, either Γ is s -transitive, or $k = 3$, $s = 4$, and Γ is 5-transitive.*
- (2) (Gardiner [10, Theorem 3.1]) *Assume that H is a subgroup of G which satisfies line 1, 2 or 3 of Table 3. If there exists an arc-transitive graph $\Gamma(G, H, HgH)$ of valency k , then $\Gamma(G, H, HgH)$ is s -transitive for $s = 4, 5$ or 7 , respectively.*

Remark on Theorem 2.2. The valency of the graph Γ and the arc-stabilizer L are uniquely determined by the point-stabilizer H . Actually, L is the normalizer of a Sylow p -subgroup of H , where q is a p -power, and the valency is equal to $|H|/|L|$.

Hence, in particular, all subgroups of H isomorphic to L are conjugate (in H) to L .

To end this section, we describe a possible (and often efficient) method for constructing s -arc transitive graphs for $s \geq 4$. Let $\Gamma(G, H, HgH)$ be a coset graph, and let $L = H \cap H^g$ and $K = \langle L, g \rangle$. Then we have three subgroups H, L, K of G , which have the properties:

$$H \cap K = L, |[K : L]| = 2, \text{ and } |[H : L]| \text{ equals the valency of } \Gamma(G, H, HgH).$$

A triple (H, L, K) with these properties is called an *amalgam* (see [11, 13]). Thus, constructing arc-transitive graphs with certain extra property is equivalent to seeking suitable groups G and suitable pairs $\{H, g\}$; the latter is in turn equivalent to seeking suitable amalgams (H, L, K) . A procedure for constructing s -transitive graphs may be stated as follows:

- (A) find a suitable group G which has a subgroup H as listed in Table 3;
- (B) find a subgroup L of H such that H acting on $[H : L]$ is of degree $q + 1$;
- (C) analyze $\mathbf{N}_G(L)$ to decide whether there exists a 2-element $g \in \mathbf{N}_G(L) \setminus L$ such that $g^2 \in H$ and $\langle H, g \rangle = G$.

We note that there may be different 2-elements $g \in \mathbf{N}_G(L) \setminus L$ such that $g^2 \in H$ and $\langle H, g \rangle = G$. Thus for a pair (G, H) , there may be different arc-transitive graphs $\Gamma(G, H, HgH)$, which in general are not isomorphic. However, for certain s -transitive graphs for $s \geq 4$, we have the uniqueness of such graphs as in Lemma 2.3.

Lemma 2.3. *Let $\Gamma = \Gamma(G, H, HgH)$ be a connected (G, s) -transitive graph, where $s \in \{4, 5, 7\}$, and let $L = H \cap H^g$. Assume that $\mathbf{N}_G(L) = K = L\mathbb{Z}_2$. For any 2-element $f \in G$ with $f^2 \in H$, if $\Gamma(G, H, HfH)$ is connected and s -transitive, then*

$$\Gamma(G, H, HfH) = \Gamma(G, H, HgH).$$

Proof. By Theorem 2.2, $H \cap H^f \cong L$. It then follows from the remark on Theorem 2.2 that $H \cap H^f$ is conjugate in H to L . Hence

$$H \cap H^{h^{-1}fh} = (H \cap H^f)^h = L$$

for some $h \in H$. Since $(h^{-1}fh)^2 \in H$, we have that

$$L^{h^{-1}fh} = (H \cap H^{h^{-1}fh})^{h^{-1}fh} = H^{h^{-1}fh} \cap H = L,$$

that is, $h^{-1}fh \in \mathbf{N}_G(L)$. Since $|\mathbf{N}_G(L)/L| = 2$ and $g \in \mathbf{N}_G(L) \setminus L$, we have $\mathbf{N}_G(L) = L \cup Lg$. Hence $h^{-1}fh \in \mathbf{N}_G(L) \setminus L = Lg$, and so $h^{-1}fh = h'g$ for some $h' \in L$. Therefore, $HfH = Hh^{-1}fhH = Hh'gH = HgH$, and thus we have that $\Gamma(G, H, HfH) = \Gamma(G, H, HgH)$. □

An immediate consequence of this lemma is

Corollary 2.4. *The classical generalized polygon graphs (constructed above) are uniquely determined by their automorphism groups.*

3. CONSTRUCTING s -ARC TRANSITIVE GRAPHS

In this section, we will first describe the constructions of most known examples of s -arc transitive graphs for $s \in \{4, 5, 7\}$ in a uniform way, apart from the polygonal graphs described in Section 2, and then construct some new 4-transitive graphs: one from the Monster simple group M ; an infinite family of graphs from $\text{PSp}(4, p)$ with p a prime. (As usual, for positive integers m and n , by the notation $m \mid n$ we

mean m divides n , and by $m \parallel n$ we mean m strictly divides n , that is, $m \mid n$ and m is coprime to n/m .)

In the following, for a finite group G and a prime $p \mid |G|$, denote by G_p a Sylow p -subgroup of G . The first lemma provides a method to construct certain arc-transitive graphs.

Lemma 3.1. *Let G be a finite group. Assume that G has a maximal subgroup H which is core-free, and assume further that H contains a subgroup P such that $|\mathbf{N}_G(P)|/|\mathbf{N}_H(P)|$ is even. Then there exists $g \in G \setminus H$ such that $\Gamma := \Gamma(G, H, HgH)$ is a connected G -arc transitive graph of valency $|[H : H \cap H^g]|$.*

Proof. By the assumption, there exists a 2-element $g \in \mathbf{N}_G(P) \setminus \mathbf{N}_H(P)$ such that $g^2 \in \mathbf{N}_H(P)$. Thus we have $g^2 \in H$, and $g \notin H$. Since H is a maximal subgroup of G , $G = \langle H, g \rangle$. By Lemma 2.1(1), the coset graph $\Gamma(G, H, HgH)$ is a connected G -arc transitive graph of valency $|[H : H \cap H^g]|$. \square

Lemma 3.2. *Let G be a finite group, and let K be a 2-subgroup of G . Assume that $|G|/|K|$ is even. Then $\mathbf{N}_G(K)/K$ is of even order.*

Proof. Let G_2 be a Sylow 2-subgroup of G which contains K . Then $G_2 > K$, and by [30, p. 88], $\mathbf{N}_{G_2}(K) > K$, so the conclusion of the lemma follows. \square

In the rest of this section, we describe some constructions of s -transitive graphs for $s \in \{4, 5, 7\}$. All of them involve nonabelian simple groups.

3.1. Graphs of valency 3 and 4. Note that $\mathrm{GL}(2, 2) \cong \mathrm{S}_3$, $\mathbb{Z}_2^2 \rtimes \mathrm{GL}(2, 2) \cong \mathrm{S}_4$, and $\mathbb{Z}_2^3 \rtimes \mathrm{GL}(2, 2) \cong \mathrm{S}_4 \times \mathbb{Z}_2$. Thus, for cubic s -transitive graphs, the amalgams have been often written as special forms, that is, either

$$\begin{aligned} s = 4 \text{ and } (H, L) &= (\mathrm{S}_4, \mathrm{D}_8), \text{ or} \\ s = 5 \text{ and } (H, L) &= (\mathrm{S}_4 \times \mathbb{Z}_2, \mathrm{D}_8 \times \mathbb{Z}_2). \end{aligned}$$

In particular, in either case L is a 2-group. Assume that G is a finite group which has subgroups H and L satisfying these conditions. By Lemma 3.2, if $|G|/|L|$ is even, then $\mathbf{N}_G(L)/L$ is of even order, and thus there exists a 2-element $g \in \mathbf{N}_G(L) \setminus H$ such that $g^2 \in H$. Assume further that H is a maximal subgroup. Then $\langle H, g \rangle = G$. Therefore, we have the following examples of s -transitive graphs for $s = 4$ or 5. These were first constructed by Wong (1967).

Example 3.3. Let

$$G = \mathrm{Aut}(\mathrm{PSL}(3, 3)) = \mathrm{PSL}(3, 3).\mathbb{Z}_2.$$

By the Atlas [8], G has a maximal subgroup

$$H \cong \mathrm{S}_4 \times \mathbb{Z}_2,$$

and $|G|/|H|$ is even. Thus there exists a vertex-primitive $(G, 5)$ -transitive cubic graph $\Gamma = \Gamma(G, H, HgH)$ such that $\mathrm{Aut} \Gamma = G$ and

$$L = H \cap H^g \cong \mathrm{D}_8 \times \mathbb{Z}_2.$$

Noting that $2^5 \parallel |G|$ (see the Atlas [8]), it follows from Lemma 3.2 that $\mathbf{N}_G(L) = [2^5]$, and so $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 5)$ -transitive cubic graph.

Example 3.4. Let

$$G = \text{PSL}(2, p) \text{ where } p \equiv \pm 1 \pmod{16} \text{ is a prime.}$$

Then by [30, p. 417], G has a maximal subgroup

$$H \cong S_4,$$

and thus $|G|/|H|$ is even. So there exists a vertex-primitive $(G, 4)$ -transitive cubic graph $\Gamma = \Gamma(G, H, HgH)$ such that $\text{Aut } \Gamma = G$ and

$$L = H \cap H^g \cong D_8.$$

Checking subgroups of G which contain L (see [30, p.417]), we conclude that $\mathbf{N}_G(L) \cong D_{16}$ and so $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 4)$ -transitive cubic graph.

Assume now that $|G|/|L|$ is odd but $|\text{Aut}(G)|/|L|$ is even. Then in some circumstances there exists a 2-element $g \in \mathbf{N}_{\langle G, g \rangle}(L) \setminus H$ such that $g^2 \in H$. If, in addition, H is a maximal subgroup of G , then $\langle H, g \rangle = \langle G, g \rangle$. Thus we have the next examples of bipartite s -transitive graphs for $s = 4$ or 5 . These were first constructed by Biggs and Hoare [4] (also see [3, 18b]).

Example 3.5. Let $T = \text{PSL}(2, p)$ where $p \equiv \pm 1 \pmod{8}$. Then T has two conjugacy classes of maximal subgroups S_4 , which are fused in

$$G = \text{PGL}(2, p),$$

see [30, p. 417]. Let H be such that

$$S_4 \cong H < T.$$

Then $|T|/|H|$ is odd but $|G|/|H|$ is even. Thus there exists a 2-element $g \in G \setminus H$ such that $g^2 \in H$, $\langle H, g \rangle = G$ and $H \cap H^g = L$. Thus there exists a bipartite $(G, 4)$ -transitive cubic graph $\Gamma = \Gamma(G, H, HgH)$ such that $\text{Aut } \Gamma = G.\mathbb{Z}_2 = \text{PGL}(2, p)$ and

$$L = H \cap H^g \cong D_8.$$

Checking subgroups of G which contain L (see [30, p. 417]), we may conclude that $\mathbf{N}_G(L)$ is a Sylow 2-subgroup of G , which is of order 16. Thus $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 4)$ -transitive cubic graph.

The 5-transitive cubic graphs given in the next example were also constructed by Biggs and Hoare [4].

Example 3.6. Let $T = \text{PSL}(2, p^2)$ where p is a prime and $p^2 \equiv \pm 1 \pmod{8}$. Then $\text{Out}(T) \cong \mathbb{Z}_2^2$ and

$$G := \text{Aut}(T) = \text{PTL}(2, p^2).$$

By [30, p. 417], T has two conjugacy classes of subgroups which are isomorphic to S_4 , which are fused in G . It follows that the group $\text{P}\Sigma\text{L}(2, p^2)$ has two conjugacy classes of subgroups that are isomorphic to $S_4 \times \mathbb{Z}_2$, which are fused in G . Let H be a subgroup of $\text{P}\Sigma\text{L}(2, p^2)$ such that

$$H \cong S_4 \times \mathbb{Z}_2.$$

Then a Sylow 2-subgroup L of H is isomorphic to $D_8 \times \mathbb{Z}_2$, and is a proper subgroup of a Sylow 2-subgroup S of G . Therefore, there exists a 2-element $g \in S \setminus L$ normalizing L , and so

$$H \cap H^g = L \cong D_8 \times \mathbb{Z}_2.$$

Now it is easily shown that $\langle H, g \rangle$ is contained in no proper subgroup of G , so $\langle H, g \rangle = G$. Therefore, by Lemma 3.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(G, H, HgH)$ is a 5-transitive cubic graph, which is bipartite. Analyzing subgroups of G which contain L (see [30, p. 417]), we may obtain that $\mathbf{N}_G(L)$ is a Sylow 2-subgroup of G , which has order 32. Hence $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 5)$ -transitive cubic graph. Since H is maximal in $\text{PSL}(2, p^2)$ if and only if $p = 3$, Γ is biprimitive if and only if $p = 3$. We note that $\text{PSL}(2, 9) \cong A_6$, and the corresponding graph is Tutte’s 8-Cage.

Next we consider 4-valent graphs. Note that $\text{GL}(2, 3) \cong Q_8 \rtimes S_3$. Thus, for s -transitive graphs of valency 4 with $s \in \{4, 5, 7\}$, the amalgams also have a special forms, that is, either

$$s = 4 \text{ and } (H, L) = (\mathbb{Z}_3^2 \rtimes (Q_8 \rtimes S_3), [3^3] \rtimes \mathbb{Z}_2^2),$$

or

$$s = 7 \text{ and } (H, L) = ([3^5] \rtimes (Q_8 \rtimes S_3), [3^6] \rtimes \mathbb{Z}_2^2).$$

We note that in this case $L = \mathbf{N}_H(L_3)$, where L_3 is a Sylow 3-subgroup of H , and thus $\mathbf{N}_G(L) \leq \mathbf{N}_G(L_3)$. The 4-transitive graph given in the next lemma was constructed and used to characterize the group M_{12} by Weiss [36].

Lemma 3.7. *Let $T = M_{12}$, the Mathieu simple group of degree 12, and let $G = \text{Aut}(T) = T.\mathbb{Z}_2$. Then there exists a unique connected 4-transitive graph Γ of valency 4 such that Γ is biprimitive and $\text{Aut } \Gamma = G$.*

Proof. By the Atlas [8], T has two conjugacy classes of maximal subgroups isomorphic to $\mathbb{Z}_3^2 \rtimes (\mathbb{Z}_2.S_4)$, which are fused in G into maximal subgroups of G isomorphic to $3_+^{1+2} \rtimes D_8$. Let H be a subgroup of T such that

$$H \cong \mathbb{Z}_3^2 \rtimes (\mathbb{Z}_2.S_4).$$

(We note that in this case, H is isomorphic to $\mathbb{Z}_3^2 \rtimes \text{GL}(2, 3)$.) Let H_3 be a Sylow 3-subgroup of H and let $L = \mathbf{N}_H(H_3)$. Then $H_3 \cong 3_+^{1+4}$, and

$$L \cong 3_+^{1+4} \rtimes [2^2].$$

It follows from the Atlas [8] that $\mathbf{N}_G(L) = 3_+^{1+4} \rtimes D_8$. Thus there exists a 2-element $g \in \mathbf{N}_G(L) \setminus H$ such that $L = H \cap H^g$. Since H is a maximal subgroup of T , it follows that $\langle H, g \rangle = G$. Thus, by Lemma 2.1, the coset graph $\Gamma = \Gamma(G, H, HgH)$ is a connected arc-transitive graph of valency 4, and further by Lemma 3.1 and Theorem 2.2, Γ is 4-transitive. Since $g \notin T$, T is not transitive on $V\Gamma$, and thus Γ is bipartite, say with parts Δ_1 and Δ_2 . As H is a maximal subgroup of T , Γ is biprimitive. It follows from Theorem 2.2 that $\text{Aut } \Gamma = G$. Since $\mathbf{N}_G(L)/L \cong \mathbb{Z}_2$, by Lemma 2.3, Γ is the unique $(G, 4)$ -transitive graph of valency 4. \square

An infinite family of non-bipartite 5-transitive cubic graphs was constructed by Conder (1988) from the alternating groups A_n and the symmetric groups S_n . Very recently, an infinite family of non-bipartite 7-transitive graphs of valency 4 was constructed by Conder and Walker (1998), also from the alternating groups A_n and the symmetric groups S_n . These graphs are neither vertex-primitive nor vertex-biprimitive.

3.2. Graphs of valency 5. Weiss (1986) constructed a 4-transitive graph of valency 5 from the Janko simple group J_3 , and used the graph to characterize the simple group. Here we first give a description for the Weiss graph, and then construct an infinite family of 4-transitive graphs of valency 5 from a class of symplectic simple groups. We notice that for a $(G, 4)$ -transitive graph of valency 5,

$$\mathbb{Z}_2^4 \rtimes \text{GL}(2, 4) \leq H \leq (\mathbb{Z}_2^4 \rtimes \text{GL}(2, 4)).\mathbb{Z}_2,$$

and

$$[2^6].\mathbb{Z}_3^2 \leq L \leq [2^6].\mathbb{Z}_3^2.\mathbb{Z}_2;$$

in particular, L is a Hall $\{2, 3\}$ -subgroup of H .

Lemma 3.8. *Let T be the Janko simple group J_3 . Then there exists a unique 4-transitive graph Γ of valency 5 such that $\text{Aut } \Gamma = \text{Aut}(T) = T.\mathbb{Z}_2$ and Γ is vertex-primitive.*

Proof. Let $G = \text{Aut}(T)$. By the Atlas [8], G has one conjugacy class of maximal subgroups isomorphic to $\mathbb{Z}_2^4 \rtimes (\mathbb{Z}_3 \times A_5).\mathbb{Z}_2 \cong \mathbb{Z}_2^4 \rtimes \text{GL}(2, 4).\mathbb{Z}_2$. Let $H < G$ be such that

$$H \cong \mathbb{Z}_2^4 \rtimes \text{GL}(2, 4).\mathbb{Z}_2,$$

and let L be a Hall $\{2, 3\}$ -subgroup of H . Then

$$L \cong [2^6].\mathbb{Z}_3^2.\mathbb{Z}_2,$$

and by the Atlas [8], $\mathbf{N}_G(L) = 2^{2+4} \rtimes (S_3 \times S_3) = L.\mathbb{Z}_2$. Thus there exists a 2-element g in $\mathbf{N}_G(L) \setminus H$ such that $\langle H, g \rangle = G$, $g^2 \in H$, $L = H \cap H^g$ and $|H : L| = 5$. Therefore, by Lemma 2.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(G, H, HgH)$ is a 4-transitive graph of valency 5, which is the Weiss graph given in [37]. It follows from Theorem 2.2 that $\text{Aut } \Gamma = G$. Finally, by Lemma 2.3, Γ is a unique $(G, 4)$ -transitive graph of valency 5. \square

Next we construct an infinite family of 4-transitive graphs of valency 5.

Lemma 3.9. *Let $G = \text{PSp}(4, p)$, where p is a prime and $2^4.3 \parallel (p^2 - 1)$. Then there exists a unique 4-transitive graph Γ of valency 5 such that $G_\alpha \cong \mathbb{Z}_2^4 \rtimes S_5$, $\text{Aut } \Gamma = G$ and Γ is vertex-primitive.*

Proof. By [16, Proposition 4.6.9], G has one conjugacy class of maximal subgroups isomorphic to $\mathbb{Z}_2^4.(\Omega_4^-(2).\mathbb{Z}_2) \cong \mathbb{Z}_2^4.\text{PGL}(2, 4) \cong \mathbb{Z}_2^4.S_5$. Let H be a maximal subgroup of G such that

$$H \cong \mathbb{Z}_2^4.S_5,$$

and let L be a subgroup of H such that

$$L \cong \mathbb{Z}_2^4.S_4.$$

Write $H = N_1.S_5$, and $L = N_1.(N_2.S_3)$, where $N_1 \cong \mathbb{Z}_2^4$ and $N_2 \cong \mathbb{Z}_2^2$. Let L_2 be a Sylow 2-subgroup of L , and let G_2 be a Sylow 2-subgroup of G containing L_2 . Then $|G_2| = 2^8$, $|L_2| = 2^7$, and so $G_2 > L_2$. Now $M := \mathbf{O}_2(L) = N_1.N_2$. Since $\mathbf{C}_H(N_1) = N_1$, we have that M is not abelian, and hence M has exponent 4. Let $b \in M$ be of order 4. Then $\bar{b} = bN_1 \in (N_1.A_4)/N_1$ is an involution. It is clear that each involution of A_4 is a square of an element of S_4 of order 4, that is, there exists $\bar{c} \in L/N_1 = S_4$ such that $\bar{c}^2 = \bar{b}$. Let c be a preimage of \bar{c} in L . Then the order of c is divisible by 8, and so L_2 is of exponent 8.

Since $G_2 > L_2$, there exists $g \in \mathbf{N}_G(L_2) \setminus L_2$ such that $g^2 \in L_2$. Suppose that $M^g \neq M$. Then, since M, M^g are subgroups of L_2 of index 2, $\langle M, M^g \rangle = M \circ M^g = L_2$. However, as M is of exponent 4, every element of $M \circ M^g$ is of order at most 4, which is a contradiction since $L_2 = M \circ M^g$ is of exponent 8. Hence $M^g = M$, that is g normalizes M . In particular, M is a normal subgroup of $\langle L, g \rangle$. It is now easy to obtain that $\langle L, g \rangle/M$ is of order 12 and thus g normalizes L , so that $L = H \cap H^g$. Hence the coset graph $\Gamma = \Gamma(G, H, HgH)$ is arc-transitive and of valency 5. It then follows from Theorem 2.2 that Γ is 4-transitive and $\text{Aut } \Gamma = G$. Analyzing maximal subgroups of G which contain L (see [16]), we may conclude that $\mathbf{N}_G(L) = \langle L, g \rangle \cong L.\mathbb{Z}_2$. By Lemma 2.3, Γ is a unique $(G, 4)$ -transitive graph of valency 5. □

3.3. Graphs of valency greater than 5. Here we will construct two 4-transitive graphs of valency 6, and a 4-transitive graph of valency 14. We observe that if $k = p + 1 \geq 6$ for some prime p , then a $(G, 4)$ -arc transitive graph of valency k is 4-transitive;

$$H = \mathbb{Z}_p^2 \rtimes \mathbb{Z}_{\frac{p-1}{(p-1,3)}}. \text{PGL}(2, p),$$

and

$$L = [p^3] \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{\frac{p-1}{(p-1,3)}}).$$

In particular, $L = \mathbf{N}_H(L_p)$ for a Sylow p -subgroup L_p . Thus $\mathbf{N}_G(L) \leq \mathbf{N}_G(L_p)$. The construction of the graphs of valency 6 in the next lemma was first obtained by Stroth and Weiss [28].

Lemma 3.10. *Let T be one of the sporadic simple groups Ru, Th or M. Then there exists a unique 4-transitive graph Γ such that $\text{Aut } \Gamma = T$, and further, Γ has valency 6 in the cases where $T = \text{Ru}$ or Th, while Γ has valency 14 when $T = \text{M}$.*

Proof. Since the construction for the Ru-graph is similar to the construction for the Th-graph, we only describe the latter.

Let T be the Thompson group Th. By the Atlas [8], T has one conjugacy class of maximal subgroups that are isomorphic to $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4.S_5 \cong \mathbb{Z}_5^2 \rtimes \text{GL}(2, 5)$. Let H be a subgroup of T such that

$$H \cong \mathbb{Z}_5^2 \rtimes \text{GL}(2, 5),$$

and let H_5 be a Sylow 5-subgroup of H . Then

$$L := \mathbf{N}_H(H_5) \cong 5_+^{1+2}.(\mathbb{Z}_4.\mathbb{Z}_4),$$

and however, by the Atlas [8], $\mathbf{N}_T(H_5) \cong 5_+^{1+2}.(\mathbb{Z}_4.S_4)$. Let M be a Hall $\{2, 5\}$ -subgroup of $\mathbf{N}_T(H_5)$ which contains L . Then L has index 2 in M , and thus there exists a 2-element $g \in M \setminus L$ such that $g^2 \in L < H$ and g normalizes L . It follows that $L = H \cap H^g$. Now $|H : L| = 6$, and as H is a maximal subgroup of T , $\langle H, g \rangle =$

T . Therefore, by Lemma 3.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(T, H, HgH)$ is a connected 4-transitive graph of valency 6. It also follows from Theorem 2.2 that $\text{Aut } \Gamma = T$. Now H has a normal subgroup N such that $5_+^{1+2}.\mathbb{Z}_4 \cong N \triangleleft L$. Thus $N \triangleleft L \triangleleft \mathbf{N}_T(L) \leq \mathbf{N}_T(H_5) = N.S_4$. Thus $\mathbb{Z}_4 \cong L/N \triangleleft \mathbf{N}_T(L)/N \leq S_4$, and so $\mathbf{N}_T(L)/N \cong D_8$. In particular, $\mathbf{N}_T(L)/L \cong \mathbb{Z}_2$, and so by Lemma 2.3, Γ is a unique $(T, 4)$ -transitive graph of valency 6.

Now let T be the Monster simple group M . By the Atlas [8], T has one conjugacy class of maximal subgroups which are isomorphic to $\mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4.\text{PSL}(2, 13).\mathbb{Z}_2)$. Let H be a subgroup of T such that

$$H \cong \mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4.\text{PSL}(2, 13).\mathbb{Z}_2),$$

and let H_{13} be a Sylow 13-subgroup of H . Then

$$L := \mathbf{N}_H(H_{13}) \cong 13_+^{1+2}.\langle \mathbb{Z}_3 \times (\mathbb{Z}_4.\mathbb{Z}_4) \rangle,$$

and however, by the Atlas [8], $\mathbf{N}_T(H_{13}) \cong 13_+^{1+2} \rtimes (\mathbb{Z}_3 \times \mathbb{Z}_4.S_4)$. Let $L_{\{2,3\}}$ and P be Hall $\{2, 3\}$ -subgroups of L and $\mathbf{N}_T(H_{13})$, respectively, such that $L_{\{2,3\}} < P$. It follows that either $L_{\{2,3\}} \triangleleft P$, or $\mathbf{N}_P(L_{\{2,3\}}) \cong \mathbb{Z}_3 \times [2^5]$. In either case, $|P|/|L_{\{2,3\}}|$ is even, and thus there exists a 2-element $g \in \mathbf{N}_T(H_{13}) \setminus L$ such that g normalizes L and $g^2 \in L < H$. Now $|H : L| = 14$, and as H is a maximal subgroup of T , $\langle H, g \rangle = T$. Therefore, by Lemma 3.1 and Theorem 2.2, the coset graph $\Gamma = \Gamma(T, H, HgH)$ is a connected 4-transitive graph of valency 14. Note that H has a normal subgroup N such that $13_+^{1+2}.\mathbb{Z}_{12} \cong N \triangleleft L \triangleleft \mathbf{N}_T(L) \leq \mathbf{N}_T(H_{13}) = N.S_4$. Hence $\mathbb{Z}_4 \cong L/N \triangleleft \mathbf{N}_T(L)/N \leq S_4$, and so $\mathbf{N}_T(L)/N \cong D_8$. In particular, $\mathbf{N}_T(L)/L \cong \mathbb{Z}_2$, and so by Lemma 2.3, Γ is a unique $(T, 4)$ -transitive graph of valency 14.

To complete the proof of this lemma, we only need to prove that $\text{Aut } \Gamma = T$. Suppose that $\text{Aut } \Gamma \neq T$. Since H is a maximal subgroup of T , T is primitive on $V\Gamma$. Thus any group X with $T < X \leq \text{Aut } \Gamma$ is primitive on $V\Gamma$. Choose X such that T is maximal in X . Since $\text{Aut}(T) = T$ and T is primitive on $V\Gamma$, it follows that $\mathbf{N}_{\text{Aut } \Gamma}(T) = T$. Since Γ is of valency 14, all prime divisors of $|X_\alpha|$ are less than 14. As 17 exactly divides $|T|$, we have that 17 exactly divides $|X|$ and note that 17 divides $|V\Gamma|$. It then follows from the O’Nan-Scott theorem (see [24]) that X is an almost simple group. Now both T and X_α are maximal subgroups of X , and $X = TX_\alpha$, which is a maximal factorization of X . By Corollary 4 and Theorems B and C of [21], there exists no such triple (T, X_α, X) , which is a contradiction. Therefore, $\text{Aut } \Gamma = T$, and Lemma 3.10 is proved. \square

4. PROOFS OF THE THEOREMS

This section is devoted to proving Theorems 1.1–1.4. Let Γ be a (G, s) -transitive graph, where $G \leq \text{Aut } \Gamma$ and $s \in \{4, 5, 7\}$, such that G is primitive or biprimitive on $V\Gamma$. If Γ is bipartite with parts Δ_1 and Δ_2 , then $G_{\Delta_1} = G_{\Delta_2}$ is primitive on Δ_1 and Δ_2 . Let

$$\begin{cases} \Omega = V\Gamma, & G^+ = G, & \text{if } \Gamma \text{ is non-bipartite,} \\ \Omega = \Delta_1, & G^+ = G_{\Delta_1}, & \text{if } \Gamma \text{ is bipartite with parts } \Delta_i. \end{cases}$$

Then G^+ is a primitive permutation group on Ω (see [25, Theorem 2.1]). Thus for $\alpha \in \Omega$, $G_\alpha = G_\alpha^+$ is a maximal subgroup of G^+ . The first lemma gives a reduction to almost simple groups.

Lemma 4.1. *Let Γ be a graph such that $G \leq \text{Aut } \Gamma$ is primitive or biprimitive on $V\Gamma$ and Γ is (G, s) -transitive for some $s \in \{4, 5, 7\}$. Then G^+ is an almost simple group, and further, either*

- (i) G is an almost simple group, or
- (ii) Γ is the double cover of a vertex-primitive (G^+, s) -transitive graph.

Proof. By Theorem 2.2, for $\alpha \in \Omega$, $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive permutation group on $\Gamma(\alpha)$ such that $\text{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \text{PSL}(2, q)$, where q is a power of a prime p with $|\Gamma(\alpha)| = q + 1$. Since G^+ is primitive, by Praeger [24, 25] and in terms of [26], G^+ is of type HA, AS, PA or TW.

Suppose first that G^+ is of type HA or TW. Then $\text{soc}(G^+)$ is regular on Ω . Let P be the normal subgroup of G_α of order q^{s-2} , and let $X = \text{soc}(G^+) \rtimes P$. Then $X \triangleleft G^+$, and so $1 \neq X_\alpha^{\Gamma(\alpha)} \triangleleft G_\alpha^{\Gamma(\alpha)}$. As $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive, $X_\alpha^{\Gamma(\alpha)}$ is transitive on $\Gamma(\alpha)$, which is a contradiction since $|\Gamma(\alpha)| = q + 1$ and $X_\alpha^{\Gamma(\alpha)}$ is a p -group.

Suppose now that G^+ is of type PA, and let $N = \text{soc}(G^+)$. Then $1 \neq N_\alpha^{\Gamma(\alpha)} \triangleleft G_\alpha^{\Gamma(\alpha)}$. As $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive, $N_\alpha^{\Gamma(\alpha)}$ is transitive on $\Gamma(\alpha)$, and hence $N_\alpha^{\Gamma(\alpha)} \geq \text{PSL}(2, q)$. It follows that the unique insoluble composition factor of G_α (isomorphic to $\text{PSL}(2, q)$) is a composition factor of N_α . However, as G^+ is of type PA, $N_\alpha = R_1 \times \cdots \times R_k$, where $k \geq 2$ and $R_1 \cong \cdots \cong R_k$. This is not possible. Therefore, G^+ is almost simple, that is, $\text{soc}(G^+) = T$ is a nonabelian simple group.

Assume that G is not an almost simple group. It follows that $\mathbf{C}_G(T) \neq 1$. Since $|G|/|G^+| = 2$ and $\mathbf{C}_{G^+}(T) = 1$, we have that $C := \mathbf{C}_G(T) \cong \mathbb{Z}_2$. Now $C \triangleleft G$, and it then follows that $G = G^+ \times C$. Let $\Sigma = \Gamma_C$, the quotient of Γ induced by C . Then Γ is the standard double cover of Σ . Further, $G^+ \cong G/C \leq \text{Aut } \Sigma$, G/C is primitive on $V\Sigma$, and by [24, Theorem 4.1], Σ is $(G/C, s)$ -transitive. \square

Thus, to complete the proofs of Theorems 1.3 and 1.4, we only need to consider almost simple groups. So we assume that G is an almost simple group with socle $T = \text{soc}(G) = \text{soc}(G^+)$, and that Γ is a (G, s) -transitive graph with $s \in \{4, 5, 7\}$. We analyse the finite nonabelian simple groups class by class.

Lemma 4.2. *Assume that $T = \text{soc}(G) \cong A_n$ for some $n \geq 5$. Then $n = 6$, and Γ is a bipartite 5-transitive cubic graph, which is isomorphic to Tutte’s 8-Cage (see Example 3.6).*

Proof. Assume that $n = 6$. Then $T \cong \text{PSL}(2, 9)$, and $\text{PSL}(2, 9) \leq G \leq \text{P}\Gamma\text{L}(2, 9)$. By information given in the Atlas [8], $\text{P}\Gamma\text{L}(2, 9)$ has a maximal subgroup $S_4 \times \mathbb{Z}_2$, and further we may conclude that the pair $(\text{P}\Gamma\text{L}(2, 9), S_4 \times \mathbb{Z}_2)$ gives rise to the smallest graph Γ constructed in Example 3.6, which is bipartite, cubic and 5-transitive.

Suppose that $n \neq 6$. Then either $G = G^+$, or $G^+ = T$ and $G \cong S_n$. In particular, $G^+ \cong A_n$ or S_n . Now G^+ naturally acts on $\Omega = \{1, 2, \dots, n\}$, and G_α^+ is a maximal subgroup of G^+ . Consider the G_α^+ -action on Ω . Note that G_α^+ has the form of H given in Table 3 of Theorem 2.2, and in particular, the unique insoluble composition factor of G_α^+ is $\text{PSL}(2, q)$. By the O’Nan-Scott theorem (see [21, p. 130]), we may conclude that $n = p^2$ and $G_\alpha^+ = \text{AGL}(2, p) \cap G^+ = (\mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1} \cdot \text{PSL}(2, p) \cdot \mathbb{Z}_2)) \cap G^+$. Suppose that $T_\alpha = \mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1} \cdot \text{PSL}(2, p) \cdot \mathbb{Z}_2) = \text{AGL}(2, p)$. Then there exists an involution σ of $S_n \setminus A_n$ which normalizes T_α . However, by the O’Nan-Scott theorem, $\langle T_\alpha, \sigma \rangle$ is not a subgroup of S_n , which is a contradiction. Thus $T_\alpha = \mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1} \cdot \text{PSL}(2, p))$. If Γ is non-bipartite, then Γ is (T, s) -transitive, which is a

contradiction since T_α does not satisfy Theorem 2.2. If Γ is bipartite, then $G^+ = T$ and so $G_\alpha^+ = T_\alpha = \mathbb{Z}_p^2 \rtimes (\mathbb{Z}_{p-1} \cdot \text{PSL}(2, p))$, which is again a contradiction since G^+ does not satisfy Theorem 2.2. \square

Lemma 4.3. *Assume that $T = \text{soc}(G)$ is a sporadic simple group. Then $T = M_{12}, J_3, \text{Ru}, \text{Th}$ or M , and Theorems 1.3 and 1.4 hold.*

Proof. For a vertex α of Γ , T_α satisfies Theorem 2.2. Thus, by information given in the Atlas [8] and the recent result in [23, 38], the pair (T, T_α) is one of the following:

- $(M_{12}, \mathbb{Z}_3^2 \rtimes \text{GL}(2, 3)); (M_{22}, \mathbb{Z}_2^4 \rtimes \text{PSL}(2, 4).\mathbb{Z}_2); (M_{23}, \mathbb{Z}_2^4 \rtimes \text{GL}(2, 4).\mathbb{Z}_2);$
- $(J_3, \mathbb{Z}_2^4 \rtimes \text{GL}(2, 4)); (\text{McL}, \mathbb{Z}_3^4 \rtimes M_{10}); (\text{He}, \mathbb{Z}_7^2 \rtimes \text{SL}(2, 7));$
- $(\text{Ru}, \mathbb{Z}_5^2 \rtimes \text{GL}(2, 5)); (\text{Th}, \mathbb{Z}_5^2 \rtimes \text{GL}(2, 5)); (M, \mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4 \cdot \text{PSL}(2, 13).\mathbb{Z}_2)).$

Suppose that $T = M_{22}$ or M_{23} . Then the graph Γ has valency 5, and $|T : T_\alpha|$ is odd, which is a contradiction. Next suppose $(T, T_\alpha) = (\text{McL}, \mathbb{Z}_3^4 \rtimes M_{10})$. If there exists such a graph Γ , then $T_\alpha \cap T_\alpha^g = T_3 \rtimes Q_8$ where $T_3 \cong \mathbb{Z}_3^4$ is a Sylow 3-subgroup of T and g normalizes $T_3 \rtimes Q_8$. Thus we have $\mathbf{N}_T(T_3) = T_3 \rtimes [16]$. However, by the Atlas [8], this is not possible. An argument for $T = \text{He}$ using Sylow 7-subgroups, which is analogous to the argument for McL using Sylow 3-subgroups, proves that $T \neq \text{He}$.

The pair $(T, T_\alpha) = (M_{12}, \mathbb{Z}_3^2 \rtimes \text{GL}(2, 3))$ gives rise to the bipartite graph Γ described in Lemma 3.7 such that $\text{Aut } \Gamma = \text{Aut}(M_{12}) = M_{12}.\mathbb{Z}_2$. The pair $(T, T_\alpha) = (J_3, \mathbb{Z}_2^4 \rtimes \text{GL}(2, 4))$ gives rise to the Weiss graph (see Lemma 3.8). The pair $(\text{Ru}, \mathbb{Z}_5^2 \rtimes \text{GL}(2, 5))$ gives rise to the Stroth-Weiss graph, one of the graphs constructed in Lemma 3.10; while the pairs $(\text{Th}, \mathbb{Z}_5^2 \rtimes \text{GL}(2, 5))$ and $(M, \mathbb{Z}_{13}^2 \rtimes (\mathbb{Z}_4 \cdot \text{PSL}(2, 13).\mathbb{Z}_2))$ give rise to the other two graphs constructed in Lemma 3.10. \square

We note that, by Lemma 3.10 and Lemma 4.3, we have completed the proof of Theorem 1.1.

Assume next that $\text{soc}(G)$ is a classical simple group of Lie type. Note that H is a maximal local subgroup of G^+ . All possibilities for H are described in [16], and we will analyse the classification in [16] to complete this case. For convenience, we first deal with some small groups, using information given in the Atlas [8].

Lemma 4.4. *The socle $T = \text{soc}(G)$ of G is not one of the following groups:*

- (1) $\text{PSL}(3, 2), \text{PSL}(4, 2), \text{PSL}(5, 2), \text{PSL}(4, 3);$
- (2) $\text{PSU}(4, 2), \text{PSU}(5, 2), \text{PSU}(6, 2), \text{PSU}(3, 3), \text{PSU}(4, 3);$
- (3) $\text{PSp}(6, 2), \text{PSp}(8, 2), \text{PSp}(4, 3) (\cong \text{PSL}(4, 2)), \text{PSp}(6, 3);$
- (4) $\text{O}_8^+(2), \text{O}_8^-(2), \text{O}_{10}^+(2), \text{O}_{10}^-(2), \text{O}_7(3), \text{O}_8^+(3), \text{O}_8^-(3).$

Proof. For the groups listed in (1), from information given in the Atlas [8], if $T = \text{PSL}(3, 2)$, then $H = S_4$, but T is 2-transitive on $[T : H]$, a contradiction; while if T is one of the other groups in (1), there is no a subgroup H of T satisfying Theorem 2.2.

Now consider the groups listed in (2). By the Atlas [8], only $\text{PSU}(4, 2)$ and $\text{PSU}(4, 3)$ have subgroups satisfying Theorem 2.2. If $T = \text{PSU}(4, 2)$, then H is isomorphic to $[2^4] \rtimes A_5$ or $[3^3] \rtimes 2A_4$. Thus the valency $k = 5$ or 4 , and $K \cong ([2^6].3).2$ or $[3^4].[4]$, but $\text{PSU}(4, 2)$ has no such subgroups. If $T = \text{PSU}(4, 3)$, then H is isomorphic to $3^4 \rtimes A_6$ or $[3^3].2S_4$. Thus K should be isomorphic to $[3^6].[8]$, but $\text{PSU}(4, 3)$ has no such subgroups.

Similarly, from the information given in the Atlas [8], we may easily conclude that T is not one of the groups in (3) or (4). \square

Next we check subgroups in \mathcal{C}_i for $1 \leq i \leq 8$ described in [16]. Observing the form of the point-stabilizer in G , by Sections 4.2, 4.4, 4.5, 4.7 and 4.8 and by Lemma 4.4, we may conclude that H is not in $\mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_7 \cup \mathcal{C}_8$. Thus we only need to consider the remaining three cases.

Lemma 4.5. *Assume that $T = \text{soc}(G)$ is a classical simple group of Lie type. Then $T = A_6, \text{PSL}(3, 3), \text{PSL}(3, q), \text{PSp}(4, 2^m), \text{PSL}(2, p)$, or $\text{PSp}(4, p)$ with $p \equiv \pm 1 \pmod{8}$, and Γ is one of the graphs given in Tables 1 and 2.*

Proof. First assume that H is a subgroup of G in \mathcal{C}_1 , namely a reducible subgroup. Checking Section 4.1 of [16] and using Lemma 4.4, we find that one of the following holds:

$$T = \text{PSL}(3, q) \text{ and } T_\alpha = [q^2].\mathbb{Z}_{\frac{q-1}{(q-1,3)}} \text{PGL}(2, q) \text{ (see [16, Proposition 4.1.17]);}$$

$$T = \text{PSp}(4, q) \text{ and } T_\alpha = [q^3].\text{GL}(2, q) \text{ where } q \text{ is a 2-power (see [16, Proposition 4.1.19]).}$$

We note that although for $T = \text{PSU}(4, q)$ where q is a 3-power, T has a subgroup M which has a normal subgroup N of order q^5 and has a unique insoluble composition factor isomorphic to $\text{PSL}(2, q)$, $M/N \cong \text{GL}(2, q)$. A similar situation happens for the group $\text{PSL}(4, q)$. Therefore, T is isomorphic to $\text{PSL}(3, q)$ or $\text{PSp}(4, q)$. The former gives rise to the generalized 3-gon graphs, and the latter gives rise to the generalized 4-gon graphs described in Section 2.

Now assume that H is a subgroup of type \mathcal{C}_3 , that is a field extension subgroup. Then by Section 4.3 of [16], we may conclude that $T = \text{PSL}(2, 3^2) \cong A_6$, and $H \cap T \cong \text{PGL}(2, 3) \cong S_4$. This case gives rise to one of the graphs Γ constructed in Example 3.6 such that $\text{Aut } \Gamma = \text{PTL}(2, 3^2)$.

Finally, we assume that H is a subgroup of G in \mathcal{C}_6 , namely a symplectic-type normalizer. Checking Section 4.6 of [16] and by Lemma 4.4, we may conclude that one of the following cases holds:

- (1) $T = \text{PSL}(3, 3)$, and $H \cap T \cong \text{PSL}(2, 3).\mathbb{Z}_2 \cong S_4$ (see [16, Proposition 4.6.5]);
- (2) $T = \text{PSL}(2, p)$ with $p \equiv \pm 1 \pmod{8}$ and $H \cong S_4$ (see [16, Proposition 4.6.7]);
- (3) $T = \text{PSp}(4, p)$ with $p \equiv \pm 1 \pmod{8}$, and $H \cong \mathbb{Z}_2^4.(\Omega_4^-(2).\mathbb{Z}_2) \cong \mathbb{Z}_2^4.\text{GL}(2, 4)$ (see [16, Proposition 4.6.9]).

Case (1) gives rise to the Wong-graph described in Example 3.3. Case (2) gives rise to the Biggs-Hoare-graphs Γ described in Example 3.5 such that $\text{Aut } \Gamma = \text{PGL}(2, p)$; if further $p \equiv \pm 1 \pmod{16}$, then Case (2) gives rise to the Wong-graphs described in Example 3.4. Case (3) gives rise to the graphs constructed in Lemma 3.9. This completes the proof of the lemma. \square

We remark that, by Lemma 3.9 and Lemma 4.5, we have completed the proof of Theorem 1.2.

Finally, we deal with exceptional simple groups of Lie type.

Lemma 4.6. *If T is an exceptional simple group of Lie type, then $T = G_2(q)$ with $q = 3^m$ for some $m \geq 1$, and $T_\alpha \cong [q^5] \rtimes \text{GL}(2, q)$, as in Theorem 1.4.*

Proof. Let $T = \text{soc}(G)$ be an exceptional simple group of Lie type over $GF(t^f)$ where t is a prime. Let H be a maximal subgroup of T which is isomorphic to T_α .

Assume first that $T = F_4(q), E_i(q)$ for $i \in \{6, 7, 8\}$, or ${}^2E_6(q)$.

Let E be a minimal normal subgroup of H , and let $\text{InnDiag}(T)$ be the group generated by all inner and diagonal automorphisms of T . Then E is elementary abelian and $H = \mathbf{N}_T(E)$. By [5], there are four cases to be considered:

- (i) $E < \text{InnDiag}(T)$ and H is of maximal rank;
- (ii) $E < \text{InnDiag}(T)$ and H is not of maximal rank;
- (iii) $E \not< \text{InnDiag}(T)$;
- (iv) $t \mid |E|$, so that H is a maximal parabolic subgroup of G .

Suppose that case (i) holds. Then all possibilities for H are listed in [5, Tables 5.1 and 5.2]. We may easily conclude that no subgroups H satisfy Theorem 2.2, which is a contradiction.

Suppose that case (ii) holds. Then all possibilities are listed in [5, Table 1]. Checking the list, we find that H has a nonabelian simple composition factor not isomorphic to $\text{PSL}(2, q)$, which is a contradiction.

Suppose that case (iii) holds. Then $E \cap \text{InnDiag}(T) = 1$. It follows from [12, Sections 7 and 9] that $H \cap T = \mathbf{C}_T(\alpha) \neq 1$ for some outer automorphism α of T of prime order. Therefore, by [2, Section 19] and [5, Proposition 2.7], either $\mathbf{C}_T(\alpha)$ is a group of Lie type over a maximal subfield of $GF(t^f)$, or $\mathbf{C}_T(\alpha)$ is ${}^2\text{F}_4(q)$, ${}^2\text{E}_6(q^{1/2})$, $\text{C}_4(q)$ with q odd, or $\text{F}_4(q)$. Each of them contains an insoluble composition factor not isomorphic to $\text{PSL}(2, q)$, which is a contradiction.

Suppose that case (iv) holds. Then H is an extension of a t -group by the Chevalley group determined by a maximal subdiagram of the Dynkin diagram of T . It follows that either H contains a nonabelian composition factor not isomorphic to $\text{PSL}(2, q)$, or $T = \text{F}_4(2)$. The former is not the case. By the Atlas [8], $\text{F}_4(2)$ has no maximal subgroup satisfying Theorem 2.2, which is a contradiction.

Now consider the remaining exceptional simple groups T of Lie type. All maximal subgroups of such groups T are completely known: [29] for $T = \text{Sz}(q)$, [22] for $T = {}^2\text{F}_4(q^2)$, [15] for $T = {}^2\text{G}_2(q)$, and [14] for $T = {}^3\text{D}_4(q)$. Checking these lists of maximal subgroups of T , we may conclude that $T = \text{G}_2(q)$ with $q = 3^m$ and $T_\alpha = [q^5] \rtimes \text{GL}(2, q)$. This gives rise to the generalized 6-gon graphs described in Section 3. The proof of the lemma is completed. \square

Combining Lemmas 4.1–4.6, we have completed the proofs of Theorem 1.3 and Theorem 1.4.

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