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S_∞ REPRESENTATIONS AND COMBINATORIAL IDENTITIES

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ABSTRACT. For various probability measures on the space of the infinite standard Young tableaux we study the probability that in a random tableau, the $(i, j)^{th}$ entry equals a given number n. Beside the combinatorics of finite standard tableaux, the main tools here are from the Vershik-Kerov character theory of S_∞. The analysis of these probabilities leads to many explicit combinatorial identities, some of which are related to hypergeometric series.

0. INTRODUCTION

Let S_n denote the symmetric group of the permutations on n letters, and S_∞ the group of all finitary permutations on a countable set. Let $Par(n)$ denote the partitions of n. The famous work of Frobenius and A. Young show how the character theory of S_n is determined by $Par(n)$.

The classical works of Thoma [T] and, more recently of Vershik and Kerov (the VK-theory, reviewed in Section 2) [VK1], [VK2], [VK3] show how the character theory of S_∞ is determined by the Young graph Y. The vertices of Y are $\bigcup_{n=0}^{\infty} Par(n)$; its edges are the pairs $(\lambda, \mu) \in Par(n) \times Par(n+1)$ such that the diagram of μ is obtained from λ by adding one box. Infinite paths in Y correspond to infinite standard tableaux.

In the VK-theory, the characters of S_∞ are described in terms of probability measures on the space of the infinite paths in Y (see Section 2). These measures form a convex set. The extreme points in this set are called "ergodic" measures and are related to the VK extended Schur functions (see (4.0.2)).

In addition, Kerov, Olshanski and Vershik recently discovered a two-parameter family of measures $M_{u,v}$ with some remarkable properties [K], [O], [KOV]. The measures $M_{u,v}$ are deformations of the so-called Plancherel measure.

The projective analogues are discussed in Part II.

In this work we indicate a certain method of applying the VK-theory, and its projective analogue, to deduce combinatorial identities. After proving the general theorems (3.1 and 10.1), we calculate explicitly several typical special cases.

The main problem studied in Part I is the following: Given a probability measure M on the space of the infinite standard tableaux, given a fixed box $(i, j) \in \mathbb{Z}_+^2$ and $n \in \mathbb{Z}_+$, what is then the probability $\mathcal{P}_{_M}\big(T(i, j) = n\big)$ that the (i, j) entry in a random infinite standard tableau is equal to n?

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The answer is given, in the form of a formula, in Theorem 3.1.a. That formula involves the measure M and the number d_μ of standard tableaux of shape μ. Explicit formulas are known for computing d_μ. Together with the formulas for the ergodic and the $M_{u,v}$ measures, they yield explicit formulas for the probabilities $\mathcal{P}_{_M}\big(T(i,j)=n\big)$. Each such measure M has a corresponding subset $R_{_M} \subseteq \mathbb{Z}_+^2$ of the M-reachable boxes (see (2.5)), and for $(i,j) \in R_{_M}$ we obtain the identity

$$(*) \qquad \sum_{n\geq 0} P_{_M}(T(i,j)=n) = 1\ .$$

Thus, to every such measure M and a box $(i,j) \in R_{_M}$ there corresponds the identity $(*)$. The left hand side of $(*)$ is an $i+j-2$ multisum. When M is ergodic or $M_{u,v}$, the left hand side of $(*)$ can be computed, yielding an explicit identity. For example, the measures $M_{u,v}$ and the box (2,1) yield the identity (9.1.1), which is closely related to the Gauss summation formula for ${}_2F_1$.

A large portion of the first part of this paper is devoted to the explicit computation of some cases of the identities $(*)$.

In addition, for the Plancherel measure, computer experiments lead to the discovery of the following remarkable phenomenon: for any $n \in \mathbb{Z}_+$,

$$\mathcal{P}\big(T(3,1)=n\big) = \mathcal{P}\big(T(2,2)=n+1\big) \qquad \text{(see Proposition 8.2).}$$

A computer proof was first given by D. Zeilberger [Z]. The proof given here is due to I.G. Macdonald [M1]. So far, no similar phenomenon has been observed between other boxes or for other measures.

Part II of this work is the "projective" analogue of Part I.

Projective representations were introduced and studied by I. Schur. The theory of the projective representations of S_n appeared in Schur's fundamental paper [S]. Now, an exact analogue of the VK-theory for the projective representations of S_∞ exists. It is mostly due to M. Nazarov [N] (see also [I]) and is reviewed in Section 10. The projective ergodic measures are given by the extended Schur P-functions. The projective analogues of the measures $M_{u,v}$ are the measures M_x which were discovered by Borodin [B].

In the projective theory the *strict* partitions $SPar(n)$ replace $Par(n)$. Accordingly, the Young graph Y is now replaced by the Schur-Young subgraph SY, spanned by $\bigcup_{n\geq 0} SPar(n)$. Diagrams λ are replaced by shifted diagrams $sh(\lambda)$ and d_λ by g^λ, the number of standard tableaux of shifted shape $sh(\lambda)$.

In Part II we analyze the probability $\mathcal{P}_{_M}\big(T(i,j)=n\big)$ on the space of the infinite *shifted* standard tableaux. There are corresponding ergodic measures, given by extended Schur P-functions (Section 11) and the family of measures M_x which replaces $M_{u,v}$ here (Section 12). These measures are again deformations of the corresponding projective Plancherel measure.

As in Part I, we calculate the "projective" probabilities $\mathcal{P}_{_M}\big(T(i,j)=n\big)$, then, for (i,j) reachable, deduce the identity

$$(p,*) \qquad \sum_{n} \mathcal{P}_{_M}\big(T(i,j)=n\big) = 1\ .$$

Again, a large portion of Part II is devoted to the computation of explicit $(p,*)$-identities. For example, in Section 12 we calculate the $(p,*)$-M_x-identity corresponding to the box (i,i) (the shifted analogue of $(i,1)$). For the box (2,2) this is, again, closely related to ${}_2F_1$ (see (12.2.1)), while for a general (i,i) that identity is closely related to higher (multivariate) hypergeometric series.

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1. The Main Results

Theorem 3.1 and its projective analogue 10.1 below are the main general theorems here (Section 5 indicates other such theorems). These theorems involve general probability measures on the space of infinite Young tableaux, and they are applied below with two specific families of such measures, yielding a variety of combinatorial identities. A large part of this work concerns the explicit computation of some of these identities.

Here is a description of these theorems and of the various identities — and their locations in this paper.

Theorem 3.1 is a source for infinitely many identities: for each reachable "box" $(i,j)\in\mathbb{Z}_+^2$ it gives the general identity

$$(*)\qquad \sum_{n=1}^{\infty}\ \sum_{\mu\in H'(i-1,j-1,n-1)} d_\mu \pi\big(s_{\mu^+(i,j)}\big)=1\ .$$

Here

$$H'(k,\ell,m)=\{\mu\vdash m\mid \mu_k\le \ell+1 \text{ and } \mu_{k+1}=\ell\}$$

(this is the set of partitions $\mu\vdash m$ to which the box $(k+1,\ell+1)$ can be added, yielding the partition $\mu^+(k+1,\ell+1)$). d_μ is the number of standard tableaux of shape μ; $\mu^+(i,j)$ is μ with the box (i,j) added; s_λ is the Schur function; Λ is the algebra of symmetric functions over $\mathbb{R}$ [M], and $\pi\colon\ \Lambda\to\mathbb{R}$ is any linear functional satisfying the three conditions in (2.7). By the Vershik-Kerov (VK-) theory about the characters of S_∞, such functionals π correspond to "central" measures on the space of the infinite standard tableaux (see Section 2).

Once d_μ and $\pi(s_\lambda)$ are calculated, and the reachable box (i,j) is fixed, $(*)$ yields an explicit identity. There are several well known formulas for d_μ — see (3.1.1), (3.1.2) and (3.1.3). For certain families of functionals π there are explicit formulas for calculating $\pi(s_\lambda)$. In Sections 4, 6, 7, 8 and 9 we study $(*)$ in detail for two specific families of functionals π: one corresponding to the "ergodic" measures $M_{(\alpha;\beta)}$ in the VK-theory, and the other to the measures $M_{u,v}$ below. In general these identities become more involved as $i+j$ increases.

The simplest such π's arise from the (so-called) Plancherel measures. Many explicit cases of $(*)$-identities are calculated from such π's: see (4.2.1), (4.2.1′), (7.3.2), (7.3.4), (7.3.5), (8.2.2), (8.2.4) and (8.2.5).

The $(*)$-identities of the ergodic and of the $M_{u,v}$ measures are deformations of the $(*)$-"Plancherel" identities. Special cases of the $(*)$-ergodic identities are q-analogues of the $(*)$-Plancherel identities: see Theorem 4.1.b, (6.1.2), (6.1.4), (6.1.6), (6.1.6′), (6.2.1)–(6.2.4), (7.2.1), (7.2.2), (7.2.2′), (7.3.1), (7.3.2), (7.3.3), (7.3.4) and (7.3.5).

Some special cases of the $(*)$-$M_{u,v}$ identities are calculated in Section 9, including those cases corresponding to the boxes $(i,j)=(k+1,1)$. Here, the relation to multivariate hypergeometric series is apparent. The $(i,j)=(2,1)$-identity (9.1.1) is closely related and can be deduced (also) from Gauss summation formula for ${}_2F_1$. The $(k+1,1)$-identities with $k \geq 2$, (9.2.1) and (9.3.1), seem more involved, having the square of the Vandermonde as a factor in them.

Part II (Sections 10–12) is the projective analogue of the first part. The projective analogue of the VK-theory, due mainly to Nazarov [N], is reviewed and applied. Theorem 10.1.c gives the projective analogue of $(*)$: for reachable (i,j),

$$(p,*) \qquad \sum_{n=1}^{\infty} \sum_{\mu \in SH'(i-1,j-1,n-1)} g^{\mu} \pi\big(P_{\mu^+(i,j)}\big) = 1 \ .$$

Here $SH'(k,\ell,m) = \{\mu \vdash m \text{ is strict} \mid \mu_k \geq \ell - k + 2 \geq 2 \text{ and } \mu_{k+1} \geq \ell - k\}$, g^{μ} is the number of standard tableaux of *shifted* shape μ, P_λ are the Schur P-functions, and π is a certain linear functional $\pi\colon\ \Gamma \to \mathbb{R}$, where $\Gamma \subseteq \Lambda$ is spanned by $\{P_\lambda \mid \lambda \text{ strict}\}$.

The structure of Part II is similar to that of the first part. Here g^{μ} is calculated by Schur's formula (10.1.1), and $\pi\big(P_{\mu^+(i,j)}\big)$ is calculated for two families of measures M, "ergodic" and M_x, with corresponding functional $M \leftrightarrow \pi$. Theorem 10.3.b is the ergodic case, while Theorem 10.5.b is the M_x case of $(p,*)$. Both of these cases are deformations of the Plancherel case, given by Theorem 10.4.b.

In the ergodic case, explicit formulas for $P_\lambda(x_1,\dots,x_n)$ in terms of Schur functions are known when $\ell(\lambda)$ equals n or $n-1$ [M, III, 8, Ex. 2]. Because of this restriction, the examples of the $(p,*)$-ergodic identities here are given only for $(i,j) \in \mathbb{Z}_+^2$ such that $i \leq j \leq i+2$. These are the identities calculated in Section 11.

Theorem 10.5.b allows, in principle, an explicit calculation of the $(p,*)$-M_x-identity for any box (k,ℓ) where $k \leq \ell$. Examples of such identities are given in Section 12. For brevity, only the boxes (k,k) are studied in general: see (12.1.1) and (12.1.2). Similar to (9.1.1), the box (2,2) (which is the projective — or shifted — analogue of the box (2,1)) yields the identity (12.2.1) — which also follows from the Gauss summation formula for ${}_2F_1$. Note that the same is also true for the box (2,3) (shifted analogue of (2,2)): see (12.3.1). For higher (k,k), (12.1.1) is related to higher hypergeometric functions.

The proofs of the key theorems 3.1 and 10.1 are obtained from the study of various probabilities on the infinite Young graph Y in the ordinary case, and on the subgraph of the strict partitions in the projective case.

Finally, notice the remarkable phenomenon exhibited in Proposition 8.2.

2. A Summary of the Vershik-Kerov Theory

Below is a brief summary of some of the main results of the Vershik-Kerov (VK-) theory which are needed here [VK1], [VK2], [VK3].

2.1. **Notation for partitions** [M]**.** $\ell(\mu)$ = the number of nonzero parts in μ.

$|\mu|$ = the sum of the parts of μ; we also write $\mu \vdash |\mu|$.

We identify partitions and Young diagrams.

For two Young diagrams μ and λ, we write $\mu \nearrow \lambda$ if μ is contained in λ and their difference λ/μ consists of a single box.

2.2. **Definition of the Young graph** Y**.** Its vertices are arbitrary partitions = Young diagrams, including the zero partition = the empty diagram. Its edges are couples $\mu \nearrow \lambda$. The graph is connected.

Let Y_n stand for the n^{th} floor of Y, i.e., the set of λ's with $|\lambda| = n$.

2.3. **Paths.** Assume that each edge $\mu \nearrow \lambda$ is oriented from μ to λ.

A *path* in Y is an oriented path, i.e., a sequence $\lambda^0 \nearrow \lambda^1 \nearrow \cdots$, finite or infinite.

A finite path $\mu = \lambda^0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^k = \lambda$ is the same as a standard Young tableau of the skew shape λ/μ. In particular, if $\mu = \emptyset$, then a finite path from $\emptyset$ to λ is the same as a standard tableau of the shape λ.

Given λ, let $\mathit{Tab}(\lambda)$ be the set of all standard tableaux of the shape λ, i.e., of the paths from $\emptyset$ to λ. We set $d_\lambda = \dim \lambda = |\mathit{Tab}(\lambda)|$. Let Tab_n be the union of all the sets $\mathit{Tab}(\lambda)$ with $|\lambda| = n$; this is a finite set. There is a natural projection $\mathit{Tab}_{n+1} \to \mathit{Tab}_n$ (in terms of tableaux, we delete the box containing $n+1$; in terms of paths, we delete the last edge).

Using these projections, we form the projective limit space

$$\mathit{Tab} = \lim_{\leftarrow} \mathit{Tab}_n \ , \qquad n \to \infty \ ,$$

whose elements are called *infinite tableaux*.

By the very definition, an element $T \in \mathit{Tab}$ is an infinite path in the graph Y, starting at $\emptyset$,

$$T = \emptyset \nearrow \lambda^1 \nearrow \lambda^2 \nearrow \cdots \ .$$

Clearly, $\emptyset \subset \lambda^1 \subset \lambda^2 \subset \cdots$, and we denote by $D(T)$ the union of the diagrams λ^k. This is an infinite Young diagram, i.e., a subset of $\{1, 2, \dots\} \times \{1, 2, \dots\} = \mathbb{Z}_+^2$ such that if a box (i, j) is contained in $D(T)$, then $D(T)$ also contains all the boxes (i', j') with $i' \le i$, $j' \le j$.

The space Tab is endowed with the projective limit topology. Since all the sets Tab_n are finite, Tab is a compact totally disconnected topological space.

2.4. **Cylindrical sets.** Given $\tau \in \mathit{Tab}_n$, we denote by $\mathit{Cyl}(\tau)$ the pull-back image of τ under the natural projection $\mathit{Tab} \to \mathit{Tab}_n$. This is an open and closed subset of Tab. It is called the *cylindrical set* with base τ. Clearly, $\mathit{Cyl}(\tau)$ consists of all the infinite paths in Y whose first n links are the edges of τ.

2.5. **Measures on paths and reachable boxes.** We shall deal with probability measures on the space Tab. Given such a measure M, we shall look at (Tab, M) as a probability space and examine certain random variables defined on it.

A box (i, j) is called *reachable* (with respect to M) if it is contained in $D(T)$ for almost all T. This means that a random path passes through (i, j) with probability 1. The set of all M-reachable boxes is an infinite Young diagram. Let $D(M)$ denote the set of the M-reachable points: $D(M) \subseteq \mathbb{Z}_+^2$.

Each $T \in \mathit{Tab}$ can be viewed as a function $T(i, j)$ from the boxes $(i, j) \in D(T)$ to the numbers $1, 2, \dots$. If (i, j) is a reachable box, then the function $T \mapsto T(i, j)$ is defined almost everywhere.

2.6. **Central measures.** A probability measure M on Tab is called *central* if for any diagram λ, all the cylindrical sets $\mathit{Cyl}(\tau)$ with $\tau \in \mathit{Tab}(\lambda)$ have the same mass.

2.7. **Positive functionals.** Let Λ stand for the algebra of symmetric functions over the base field $\mathbb{R}$, and let $\{s_\lambda\}$ be the basis of Λ formed by the Schur functions [M].

Proposition. *There is a bijective correspondence $M \leftrightarrow \pi$ between the central measures M on* Tab *and the linear functionals $\pi\colon\ \Lambda \to \mathbb{R}$ satisfying the following three conditions:*

- $\pi(1) = 1$;
- *π factors through the algebra $\Lambda/(s_{(1)} - 1)\Lambda$;*
- *$\pi(s_\lambda) \geq 0$ for any λ.*

Under this correspondence, for any λ and any $\tau \in \mathrm{Tab}(\lambda)$,

$$M\big(\mathrm{Cyl}(\tau)\big) = \pi(s_\lambda) \ .$$

2.8. **Ergodic measures.** The central measures form a convex set. Its extreme points are called *ergodic* (central) measures.

2.9. **Theorem** [VK3]. *In terms of the correspondence $M \leftrightarrow \pi$, M is ergodic if and only if π is multiplicative, i.e., $\pi(fg) = \pi(f)\pi(g)$ for any $f, g \in \Lambda$.*

2.10. **Thoma's simplex.** Let Ω be the set of pairs $\omega = (\alpha, \beta)$ of weakly decreasing sequences of nonnegative real numbers, $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0)$, such that

$$\sum_{i=1}^{\infty}(\alpha_i + \beta_i) \leq 1 \ .$$

The set Ω is equipped with the topology of coordinatewise convergence. With respect to this topology, it is a compact topological space.

We set

$$\gamma = 1 - \sum_{i=1}^{\infty}(\alpha_i + \beta_i) \ .$$

Note that γ is *not* a continuous function of ω.

The set Ω is an infinite-dimensional simplex, called *Thoma's simplex*.

2.11. **Extended symmetric functions.** Let $C(\Omega)$ be the Banach algebra of continuous real-valued functions on Ω. There exists an algebra morphism $\Lambda \to C(\Omega)$ which factors through $\Lambda/(s_{(1)} - 1)\Lambda$ and determines an embedding of the latter algebra into $C(\Omega)$ with closed image.

This morphism $f \mapsto \widetilde{f}$ is defined as follows: Recall [M], [I] that Λ coincides with the polynomial algebra $\mathbb{R}[p_1, p_2, \dots]$, where $p_1, p_2, \dots$ are Newton power sums. Given $\omega = (\alpha, \beta) \in \Omega$, let us specialize:

$$p_1 \mapsto \widetilde{p}_1(\omega) = 1 \ ,$$

$$p_k \mapsto \widetilde{p}_k(\omega) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum \beta_i^k \ , \qquad k \geq 2 \ .$$

Then we obtain an algebra morphism $f \mapsto \widetilde{f}(\omega)$ from Λ to $\mathbb{R}$. Since $\widetilde{p}_1(\omega)$, $\widetilde{p}_2(\omega), \widetilde{p}_3(\omega), \dots$ are continuous in ω, we get that $\widetilde{f}(\omega)$ is continuous in ω for any $f \in \Lambda$.

The continuous function $\widetilde{f} \in C(\Omega)$ obtained in this way is called the *extended version* of $f \in \Lambda$. (Note that the function $\omega \mapsto \sum \alpha_i + \sum \beta_i$ is not continuous on Ω.)

In particular, the functions $\widetilde{s}_\lambda(\omega)$ are called *extended Schur functions*.

In Sec. 4 we indicate how to compute $\widetilde{s}_\lambda(\omega)$ explicitly.

2.12. **Description of central measures.**

Theorem. *(1) There exists a bijective correspondence $M \leftrightarrow P$ between the central measures M on Tab and the probability measures P on Ω, characterized by the following property: For any λ, and each $\tau \in Tab(\lambda)$,*

$$M\big(Cyl(\tau)\big) = \int_\Omega \widetilde{s}_\lambda(\omega) P(d\omega) \ .$$

(2) Under the bijection $M \leftrightarrow P$, the ergodic central measures M correspond to the Dirac delta measures on Ω, i.e., to points $\omega \in \Omega$:

$$M\big(Cyl(\tau)\big) = \widetilde{s}_\lambda(\omega) \ .$$

2.13. **The diagram attached to an ergodic measure.** Let M be the ergodic measure corresponding to a point $\omega = (\alpha, \beta) \in \Omega$. Then there exists an infinite diagram $D = D(\omega)$ of the M reachable points such that $D(T) = D$ for almost all (with respect to M) paths $T \in Tab$. This diagram looks as follows.

Proposition. *(i) If there exist $k, \ell \in \{0, 1, 2, \dots\}$, $k + \ell \geq 1$, such that $\alpha_{k+1} = \alpha_{k+2} = \cdots = \beta_{\ell+1} = \beta_{\ell+2} = \cdots = \gamma = 0$,*

$$\alpha_k > 0 \ , \qquad \beta_\ell > 0 \ ,$$

then $D(\omega)$ is a hook shape:

$$D(\omega) = \big\{(i,j) \mid i \leq k \quad or \quad j \leq \ell\big\} \ .$$

(ii) Otherwise (i.e., if all α_i's are strictly positive or all β_i's are strictly positive or γ is strictly positive) $D(\omega)$ coincides with the whole set $\{1, 2, \dots\} \times \{1, 2, \dots\}$.

3. Applications of the Ordinary S_∞-Representations

The main result here is Theorem 3.1. To formulate it, we introduce additional notations.

Let M be a probability measure on *Tab*. Let $\mathbb{Z}_+ = \{1, 2, \dots\}$. Given $i, j, n \in \mathbb{Z}_+$, denote

$$Tab\big(T(i,j) = n\big) = \big\{T \in Tab \mid T(i,j) = n\big\}$$

and define

$$\mathcal{P}_{_M}\big(T(i,j) = n\big) = M\big(Tab\big(T(i,j) = n\big)\big) \ ,$$

the M probability that a random $T \in Tab$ satisfies $T(i,j) = n$. Denote $Tab(i,j) = \bigcup_{n \geq 1} Tab\big(T(i,j) = n\big)$, the set of tableaux such that the corresponding diagrams contain the box $(i.j)$. This is a disjoint union; hence

$$M\big(Tab(i,j)\big) = \sum_{n \geq 1} \mathcal{P}_{_M}\big(T(i,j) = n\big).$$

By definition, $(i,j) \in \mathbb{Z}_+^2$ is reachable if and only if $M\big(Tab(i,j)\big) = 1$.

Let $\mu \vdash n-1$, $\nu \vdash n$ be such that ν/μ is the box (i,j), then write $\nu = \mu^+(i,j)$. Denote

$$H'(i-1, j-1, n-1) = \big\{(\mu_1, \mu_2, \dots) \vdash n-1 \mid \mu_{i-1} \geq j \text{ and } \mu_i = j-1\big\} \ .$$

If $\mu \in H'(i-1, j-1, n-1)$, then $\nu = \mu^+(i,j) = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots)$ is a partition. Conversely, let T_ν be a finite standard tableau of shape $\nu \vdash n$, with

n in the (i,j) box. Deleting n from T_ν gives a standard tableau T_μ of shape μ, $\mu \in H'(i-1, j-1, n-1)$, and $\nu = \mu^+(i,j)$.

We can now formulate

Theorem 3.1. *Fix $i, j, n \in \mathbb{Z}_+$. Let M be a central measure on* Tab *with corresponding $M \leftrightarrow \pi$, $\pi\colon\ \Lambda \to \mathbb{R}$ (see 2.7). Then*

(a)

$$\mathcal{P}_{M}\big(T(i,j) = n\big) = \sum_{\mu \in H'(i-1,j-1,n-1)} d_\mu \pi(s_{\mu^+(i,j)}) \ .$$

By summing over all $n \ge 1$, we obtain

(b)

$$\sum_{n=1}^{\infty} \sum_{\mu \in H'(i-1,j-1,n-1)} d_\mu \pi(s_{\mu^+(i,j)}) = M\big(\mathrm{Tab}(i,j)\big) \ .$$

In particular, if (i,j) is M-reachable, the right-hand side equals 1 *and we obtain*

(c) If (i,j) is M-reachable, then

$$\sum_{n=1}^{\infty} \sum_{\mu \in H'(i-1,j-1,n-1)} d_\mu \pi(s_{\mu^+(i,j)}) = 1 \ .$$

To state Theorem 3.1 more explicitly requires some additional notations and explicit formulas for d_λ. We list three such formulas:

The Young-Frobenius formula. Let $\mu = (\mu_1, \dots, \mu_k)$, $p_r = \mu_r + k - r$, $1 \le r \le k$. Then

$$d_\mu = \frac{|\mu|!V(p)}{p!} \ , \tag{3.1.1}$$

where $V(p) = \prod_{r<r'}(p_r - p_{r'})$ and $p! = \prod_r p_r!$.

The hook formula. For $x = (r,s) \in \mu$, $h(x) = \mu_r + \mu'_s - r - s + 1$ is the corresponding hook number. Denote $H(\mu) = \prod_{x\in\mu} h(x)$; then

$$d_\mu = \frac{|\mu|!}{H(\mu)} \ . \tag{3.1.2}$$

d_μ in (k,ℓ)-Frobenius-type coordinates. Let $H_0(k,\ell) \subseteq H(k,\ell)$ denote the partitions containing the box (k,ℓ) but not $(k+1,\ell+1)$: $H_0(k,\ell) = \{\mu \mid \mu_k \ge \ell,\ \mu_{k+1} \le \ell\}$. Given $\mu \in H_0(k,\ell)$, let

$$p_r = \mu_r + k - \ell - r \ , \quad 1 \le r \le k \ , \qquad q_s = \mu'_s + \ell - k - s \ , \quad 1 \le s \le \ell \ .$$

Thus $(p \mid q) = (p_1, \dots, p_k \mid q_1, \dots, q_\ell)$ are the (k,ℓ)-(Frobenius-)type coordinates of μ, where $p = (p_1 > \cdots > p_k \ge 0)$, $q = (q_1 > \cdots > q_\ell \ge 0)$. Trivially, $|\mu| = \sum p_r + \sum q_s + \frac{1}{2}\big[k + \ell - (k-\ell)^2\big]$. Note that $(p \mid q) = \mu \in H'(k,\ell,|\mu|)$ if and only if $p_k, q_\ell \ge 1$. In that case $\mu^+(k+1,\ell+1)$ is also a partition, denoted by $\mu^+(k+1,\ell+1) = (p \mid q)^+$.

For $\mu = (p \mid q)$ in $H_0(k,\ell)$ the previous two formulas for d_μ imply that

$$d_\mu = d_{(p|q)} = \frac{|\mu|!V(p)V(q)}{p!q!\prod_{r,s}(p_r + q_s + 1)} \ , \tag{3.1.3}$$

where $V(p) = \prod_{r<r'}(p_r - p_{r'})$, $V(q) = \prod_{s<s'}(q_s - q_{s'})$, $p! = \prod p_r!$ and $q! = \prod q_s!$.

We can now restate

Theorem 3.1.c. *Denote $k = i-1$ and $\ell = j-1$. Then*

$$\sum_{\substack{p_1>\cdots>p_k\geq 1\\ q_1>\cdots>q_\ell\geq 1}} \frac{|(p\mid q)|!V(p)V(q)}{p!q!\prod_{r,s}(p_r+q_s+1)}\pi(s_{(p|q)^+}) = 1 \ ,$$

where, again,

$$|(p\mid q)| = \sum_{r=1}^{k} p_r + \sum_{s=1}^{\ell} q_s + \frac{1}{2}\big[k+\ell-(k-\ell)^2\big],$$

$V(p) = \prod_{r<r'}(p_r - p_{r'})$, $V(q) = \prod_{s<s'}(q_s - q_{s'})$, $p! = \prod_r p_r!$ *and* $q! = \prod_s q_s!$.

The proof of Theorem 3.1 follows from Lemmas 3.3–3.5 below.

Let $\mu \vdash n$. Define $A(\mu) = \bigcup_{\tau\in Tab(\mu)} Cyl(\tau)$ (a disjoint union), i.e., $A(\mu)$ is the subset of *Tab* of the paths that go through μ: $A(\mu) = \{\lambda^0 \nearrow \lambda^1 \nearrow \cdots \mid \lambda^n = \mu\}$.

Lemma 3.3. *Let $\mu \vdash n$. Let M be a central measure on Tab, with $M \leftrightarrow \pi$, $\pi\colon \Lambda \to \mathbb{R}$ (*see *(2.7)). Then $M\big(A(\mu)\big) = d_\mu \pi(s_\mu)$.*

Proof. Indeed,

$$M\big(A(\mu)\big) = \sum_{\tau\in Tab(\mu)} M\big(Cyl(\tau)\big) = \sum_{\tau\in Tab(\mu)} \pi(s_\mu) = d_\mu \pi(s_\mu) \ .$$

□

Given $\mu \nearrow \nu$, $\mu \vdash n-1$, $\nu \vdash n$, let

$$A(\mu\nearrow\nu) = \{\lambda^0 \nearrow \lambda^1 \nearrow \cdots \mid \lambda^{n-1} = \mu \text{ and } \lambda^n = \nu\} \ ,$$

the (infinite) paths passing through both μ and ν.

Lemma 3.4. *Let $\mu \nearrow \nu$ and let M be a central measure on Tab, with corresponding $M \leftrightarrow \pi$, $\pi\colon \Lambda \to \mathbb{R}$. Then*

$$M\big(A(\mu\nearrow\nu)\big) = d_\mu \pi(s_\nu) \ .$$

Proof. Let $\tau \in Tab(\mu)$: $\tau = \lambda^0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^{n-1} = \mu$. Then τ uniquely defines $\tau^+ \in Tab(\nu)$ via $\tau^+ = \lambda^0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^{n-1} \nearrow \lambda^n = \nu$. Thus $A(\mu\nearrow\nu) = \bigcup_{\tau\in Tab(\mu)} Cyl(\tau^+)$, a disjoint union. Since $\tau^+ \in Tab(\nu)$, therefore $M\big(Cyl(\tau^+)\big) = \pi(s_\nu)$. Thus

$$M\big(A(\mu\nearrow\nu)\big) = \sum_{\tau\in Tab(\mu)} M\big(Cyl(\tau^+)\big) = \sum_{\tau\in Tab(\mu)} \pi(s_\nu) = d_\mu \pi(s_\nu) \ .$$

□

Given $\mu \nearrow \nu$, let $Tab(\mu\nearrow\nu) = \{\lambda^0 \nearrow \lambda^1 \nearrow \cdots \nearrow \mu \nearrow \nu\} \subseteq Tab(\nu)$. Note that $A(\mu\nearrow\nu) = \bigcup_{\tau\in Tab(\mu\nearrow\nu)} Cyl(\tau)$, a disjoint union.

Lemma 3.5. *With the above notations,*

$$Tab\big(T(i,j)=n\big) = \bigcup_{\mu\in H'(i-1,j-1,n-1)} A\big(\mu\nearrow\mu^+(i,j)\big) \ ,$$

a disjoint union.

Proof. The inclusion $\supseteq$ is obvious. Conversely, let $T \in \mathit{Tab}$ with $T(i,j) = n$. By standardness, the integers $1, 2, \ldots, n-1$ in the (infinite) tableau T form a standard subtableau T_{n-1} of shape μ, where $\mu \in H'(i-1, j-1, n-1)$. Adding the cell (i,j) to T_{n-1} — with n in it — gives T_n, the standard tableau of shape $\mu^+(i,j)$, and obviously

$$T \in A\big(\mu \nearrow \mu^+(i,j)\big) \ .$$

$\square$

The proof of Theorem 3.1 now easily follows from Lemma 3.5 by applying the central measure M. Note that by definition,

$$M\big(\mathit{Tab}(T(i,j) = n)\big) = P_{_M}\big(T(i,j) = n\big),$$

and by Lemma 3.4

$$M\big(A(\mu \nearrow \mu^+(i,j))\big) = d_\mu \pi(s_{\mu^+(i,j)}) \ . \qquad \square$$

4. Two Families of Measures

We begin with a more explicit description of the functions $\widetilde{s}_\lambda(\alpha;\beta)$ (see (2.11)), $(\alpha;\beta) = \omega \in \Omega$, hence, by Theorem 2.12.2, of the ergodic measures M_ω on *Tab*. Denote

$$\widetilde{s}_\lambda(\alpha;\beta) = \widetilde{s}_\lambda(\alpha;\beta;\gamma) \quad \text{and} \quad M_{(\alpha;\beta)} = M_{(\alpha;\beta;\gamma)} \ ,$$

where $\gamma = 1 - \sum_i (\alpha_i + \beta_j)$; then

$$M_{(\alpha;\beta;\gamma)}\big(\mathit{Cyl}(\tau)\big) = \widetilde{s}_\lambda(\alpha;\beta;\gamma)$$

for any $\tau \in \mathit{Tab}(\lambda)$. Also

$$M_{(\alpha;\beta;\gamma)} \leftrightarrow \pi_{(\alpha;\beta;\gamma)} \colon \Lambda \to \mathbb{R} \colon \ \pi_{(\alpha;\beta;\gamma)}(s_\lambda) = \widetilde{s}_\lambda(\alpha;\beta;\gamma) \ .$$

The extended "complete" symmetric functions $\widetilde{h}_n(\alpha;\beta;\gamma)$ are introduced via the generating function

$$\sum_{n=0}^{\infty} \widetilde{h}_n(\alpha;\beta;\gamma) z^n = e^{\gamma z} \prod_{i=1}^{\infty} \frac{1+\beta_i z}{1-\alpha_i z} \ . \tag{4.0.1}$$

The extended Schur functions are given by a Jacobi-Trudi type formula:

$$\widetilde{s}_\mu(\alpha;\beta;\gamma) = \det\big(\widetilde{h}_{\mu_i - i + j}(\alpha;\beta;\gamma)\big) \ . \tag{4.0.2}$$

As a corollary of Theorem 3.1.c we obtain

Theorem 4.1. *Let $\omega = (\alpha;\beta) \in \Omega$ (2.10). Then*

(a)

$$\mathcal{P}_{_{M_{(\alpha;\beta;\gamma)}}}\big(T(i,j) = n\big) = \sum_{\mu \in H'(i-1,j-1,n-1)} d_\mu \widetilde{s}_{\mu^+(i,j)}(\alpha;\beta;\gamma) \ .$$

If (i,j) is $M_{(\alpha;\beta;\gamma)}$-reachable (see (2.13)), then

(b)

$$\sum_{n=1}^{\infty} \sum_{\mu \in H'(i-1,j-1,n-1)} d_\mu \widetilde{s}_{\mu^+(i,j)}(\alpha;\beta;\gamma) = 1 \ ,$$

or, more explicitly, with $(k,\ell)=(i-1,j-1)$,

$$\sum_{\substack{p_1>\cdots>p_k\geq 1\\ q_1>\cdots>q_\ell\geq 1}} \frac{|(p\mid q)|!V(p)V(q)}{p!q!\prod_{r,s}(p_r+q_s+1)}\ \widetilde{s}_{(p|q)^+}(\alpha;\beta;\gamma)=1\ .$$

The Plancherel measure $M_{(0;0;1)}$**.** Let $\alpha_1=\beta_1=0$; then by (4.0.1) we have $\widetilde{h}_n(0;0;1)=\frac{1}{n!}$, and hence, by (4.0.2) and by [J, 19.5],

$$\widetilde{s}_\mu(0;0;1)=\frac{1}{H(\mu)}=\frac{d_\mu}{|\mu|!}\ , \tag{4.1.1}$$

where $H(\mu)$ is the product of the hook numbers of μ.

In the VK-terminology, $M_{(0;0;1)}$ is called the Plancherel measure. It is the limit of $M_{(\alpha;\beta;\gamma)}$, where all $\alpha_i,\beta_i\to 0$.

Theorem 4.1 clearly implies

Corollary 4.2. *(a)*

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(i,j)=n\big)=\sum_{\mu\in H'(i-1,j-1,n-1)}\frac{d_\mu}{H\big(\mu^+(i,j)\big)}\ .$$

(b) By (2.13) $D\big(M_{(0;0;1)}\big)=\mathbb{Z}_+^2$, *and hence for any* $(i,j)\in\mathbb{Z}_+^2$

$$\sum_{n=1}^{\infty}\ \sum_{\mu\in H'(i-1,j-1,n-1)}\frac{d_\mu}{H\big(\mu^+(i,j)\big)}=1\ . \tag{4.2.1}$$

Denote $(i-1,j-1)=(k,\ell)$ *and let* $\mu=(p\mid q)$ *in the* (k,ℓ) *coordinates (see (3.1.3)). By an easy calculation,* $H(\mu^+)=H(\mu)\left(\prod_{r=1}^k\frac{p_r+1}{p_r}\right)\left(\prod_{s=1}^\ell\frac{q_s+1}{q_s}\right)$, *and since* $d_\mu=\frac{|(p|q)|!}{H(\mu)}$ *, clearly*

$$\frac{d_\mu}{H(\mu^+)}=\frac{|(p\mid q)|!V^2(p)V^2(q)}{(p!)^2(q!)^2\prod_{r,s}(p_r+q_s+1)^2}\left(\prod_{r=1}^k\frac{p_r}{p_r+1}\right)\left(\prod_{s=1}^\ell\frac{q_s}{q_s+1}\right)\ . \tag{4.2.2}$$

Both parts of Corollary 4.2 are given explicitly by (4.2.2). For example, Corollary 4.2.b becomes:

For any $k,\ell\geq 0$,

$$\sum_{\substack{p_1>\cdots>p_k\geq 1\\ q_1>\cdots>q_\ell\geq 1}}\frac{|(p\mid q)|!V^2(p)V^2(q)}{(p!)^2(q!)^2\prod_{r=1}^k\prod_{s=1}^\ell(p_r+q_s+1)^2}\times\left(\prod_{r=1}^k\frac{p_r}{p_r+1}\right)\left(\prod_{s=1}^\ell\frac{q_s}{q_s+1}\right)=1 \tag{4.2.1$'$}$$

where $(p\mid q)=\sum_r p_r+\sum_s q_s+\frac{1}{2}\big[(k+\ell-(k-\ell)^2\big]$.

The Plancherel measure $M_{(0;0;1)}$ is revisited in Section 8. See also (6.1.8).

The non-ergodic central measures $M_{u,v}$ ([K], [O], [KOV]). Given $u,v\in\mathbb{C}$, $M_{u,v}$ is given by $M_{u,v}\leftrightarrow\pi_{u,v}:\ \Lambda\to\mathbb{R}$ (see (2.7)), where

$$\pi_{u,v}(s_\mu)=\prod_{(i,j)\in\mu}(u+j-i)(v+j-i)\frac{d_\mu}{uv(uv+1)\cdots(uv+n-1)n!}\ , \tag{4.3.1}$$

where $|\mu|=n$.

The parameters u, v should satisfy one of the following two conditions:

1. $u = \overline{v} \in \mathbb{C}$ is a non-integer.
2. $u, v \in (m, m+1)$ for some $m \in \mathbb{Z}$ (hence again $u, v \notin \mathbb{Z}$).

The reachable boxes are $D(M_{u,v}) = \mathbb{Z}_+^2$. In the limit, as $u = \overline{v}$ tends to infinity, $M_{u,v}(\mu) \to M_{(0;0;1)}(\mu)$. Thus, both $M_{(\alpha;\beta;\gamma)}$ and $M_{u,v}$ are deformations of the Plancherel measure $M_{(0;0;1)}$.

Similarly to Theorem 4.1, we have

Theorem 4.3. *Let $u = \overline{v} \in \mathbb{C} - \mathbb{Z}$ or $u, v \in (m, m+1)$ for some $m \in \mathbb{Z}$. Then:*
(a)

$$\mathcal{P}_{M_{u,v}}\big(T(i,j) = n\big) = \sum_{\mu \in H'(i-1,j-1,n-1)} \frac{d_\mu}{H\big(\mu^+(i,j)\big)} \times \frac{\prod_{x \in \mu^+(i,j)} \big[(u + c(x))(v + c(x))\big]}{uv(uv+1)\cdots(uv+n-1)} .$$

(b) Since $D(M_{u,v}) = \mathbb{Z}_+^2$, for any $(i,j) \in \mathbb{Z}_+^2$

$$\sum_{n=1}^{\infty} \sum_{\mu \in H'(i-1,j-1,n-1)} \frac{d_\mu}{H\big(\mu^+(i,j)\big)} \times \frac{\prod_{x \in \mu^+(i,j)} \big[(u + c(x))(v + c(x))\big]}{uv(uv+1)\cdots(uv+n-1)} = 1$$

(here $c(x)$ is the content of x [M], [I]*: $c(k,\ell) = \ell - k$).*

Denote $(z)_n = z(z+1)\cdots(z+n-1)$, the Pochhammer symbol. Let $(k,\ell) = (i-1, j-1)$ and $\mu = (p \mid q)$ in the (k,ℓ) coordinates. A simple calculation shows that

$$\prod_{x \in \mu^+(i,j)} \big(u + c(x)\big) = \frac{(u + \ell - k) \prod_{r=1}^{k} (u - r + 1)_{p_r - k + \ell + r} \prod_{s=1}^{\ell} (u - q_s - k + \ell)_{q_s + k - \ell + s}}{\prod_{r=1}^{k} (u - r + 1)_\ell} .$$

If we combine this with (4.2.2), Theorem 4.3.b becomes: *Let $k, \ell \geq 0$, $u, v \in \mathbb{C}$, $u = \overline{v}$ or $u, v \in (m, m+1)$, $m \in \mathbb{Z}$. Then*

$$\begin{aligned} &\sum_{\substack{p_1 > \cdots > p_k \geq 1 \\ q_1 > \cdots > q_\ell \geq 1}} |(p \mid q)|! \frac{V^2(p) V^2(q)}{(p!)^2 (q!)^2 \prod_{r=1}^{k} \prod_{s=1}^{\ell} (p_r + q_s + 1)^2} \left(\prod_{r=1}^{k} \frac{p_r}{p_r + 1} \right) \\ &\qquad \times \left(\prod_{s=1}^{\ell} \frac{q_s}{q_s + 1} \right) \frac{f(u) f(v)}{\prod_{r=1}^{k} \big[(u - r + 1)_\ell (v - r + 1)_\ell\big] (uv)_{|(p|q)|+1}} = 1 , \end{aligned} \tag{4.3.2}$$

where

$$f(w) = (w + \ell - k) \prod_{r=1}^{k} \big[(w - r + 1)_{p_r - k + \ell + r}\big] \prod_{s=1}^{\ell} \big[(w - q_s - k + \ell)_{q_s + k - \ell + s}\big]$$

and where $|(p \mid q)| = \sum_r p_r + \sum_s q_s + \frac{1}{2}\big[k + \ell - (k - \ell)^2\big]$.

In Sections 6 and 7 it is shown that certain specializations of Theorem 4.1 produce q-analogues of the "Plancherel" identities, given by Corollary 4.2. Examples of explicit identities produced by Theorem 4.3 are given in Section 8, identities which are different kind of deformations of the "Plancherel" identities.

5. Extensions to Probability of Joint Events

Similar to $\mathcal{P}_{M}\big(T(i,j)=n\big)$, it is possible to define the probability of two events:

$$\mathcal{P}_{M}\big(T(i,j)=m \text{ and } T(a,b)=n\big)\ , \qquad m<n\ ,$$

and similarly for more events.

By arguments similar to those in the proof of Theorem 3.1 one proves

Theorem 5.1. *Fix $i,j,a,b,m,n\in\mathbb{Z}$, $m<n$. Let M be a central measure on Tab with corresponding $M\leftrightarrow\pi$, $\pi\colon\Lambda\to\mathbb{R}$. Then*

(5.1.1)

$$\mathcal{P}_{M}\big(T(i,j)=m \text{ and } T(a,b)=n\big) = \sum_{\mu\in H'(i-1,j-1,m-1)}\ \sum_{\mu^+(i,j)\subseteq\lambda\in H'(a-1,b-1,n-1)} d_\mu d_{\lambda/\mu^+(i,j)}\pi(s_{\lambda^+(a,b)})\ .$$

Analogous theorems can be proved for more events.

Remark 5.2. A conditional reachability for the box (a,b) can be defined by the equation

$$\mathcal{P}_{M}\big(T(i,j)=m\big)=\sum_{n=1}^{\infty}\mathcal{P}_{M}\big(T(i,j)=m \text{ and } T(a,b)=n\big)\ . \tag{5.2.1}$$

In some cases of such reachability, and when $d_{\lambda/\mu^+(i,j)}$ can be calculated effectively, Theorem 5.1 yields explicit identities. For brevity, we consider only the following example:

$$\mathcal{P}_{M}\big(T(i,j)=ij \text{ and } T(i,j+1)=n\big)\ \big(\text{or } \mathcal{P}_{M}\big(T(i,j)=ij \text{ and } T(i+1,j)=n\big)\big)\ ,$$

where $M=M_{(0;0;1)}$ and $\mathcal{P}=\mathcal{P}_{M_{(0;0;1)}}$.

First, $H'(i-1,j-1,ij-1)=\{\mu\}$, where $\mu^+(i,j)=(j^i)$, and $\mathcal{P}\big(T(i,j)=ij\big)$ can easily be calculated by Corollary 4.2.b. Next, apply Theorem 5.1. Notice that here $\lambda/\big(\mu^+(i,j)\big)$ is a disjoint union of two ordinary diagrams, say $\eta\cup\theta$, and

$$d_{\lambda/\mu^+(i,j)}=d_{\eta\cup\theta}=\binom{|\eta|+|\theta|}{|\eta|}d_\eta d_\theta\ .$$

Finally, apply (5.2.1) to deduce the identity. The essence of this calculation is given in Figure 1.

Here are a few low cases:

Example 5.3. $\underline{(i,j)=(1,2)}$. Here $\mathcal{P}\big(T(1,2)=2\big)=\frac{1}{2}$ and we deduce the two identities

$$\sum_{n=1}^{\infty}\sum_{a=0}^{n-1}\frac{\binom{n}{a}}{(n-a-1)!a!(n-a+1)(a+2)(n+2)}=\frac{1}{2} \tag{5.3.1}$$

and

$$\sum_{v=3}^{\infty}\sum_{n=0}^{\infty}\sum_{a=0}^{\infty}\frac{\binom{n}{a}}{(n+v-a-3)!a!(n+v-a-1)(a+2)(n+v)}=\frac{1}{2}\ . \tag{5.3.2}$$

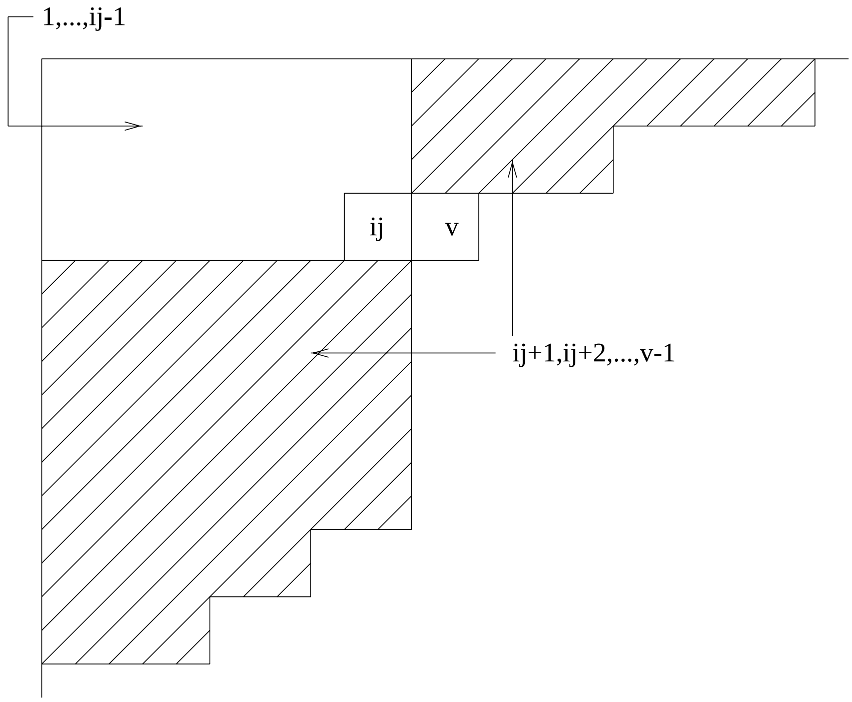

FIGURE 1.

$\underline{(i,j)=(1,3)}$. Now $\mathcal{P}\big(T(1,3)=3\big)=\frac{1}{6}$ and, for example, one identity deduced here is

$$\sum_{n=2}^{\infty}\sum_{c=0}^{n}\sum_{b=0}^{\left[\frac{n-c}{2}\right]}\frac{\binom{n}{c}(n-c)!(n-c-2b+1)^2}{g(n,b,c)}=\frac{1}{6}\ , \tag{5.3.3}$$

$g(n,b,c)=(b+1)!(n-b-c+2)!c!(c+2)(b-1)!(n-b-c)!(b+c+2)(n-b+3)$.

6. APPLICATION I: $M=M_{(\alpha;0;0)}$

Here, and in the next three sections, Theorem 3.1 is applied in some special cases, and we shall see that it yields certain explicit summation-identities. We study first a special case of the ergodic measure $M_{(\alpha;\beta;\gamma)}$.

Fix m and let $\omega_m=(\alpha;0;0)$ be the following points in the Thoma simplex: $\gamma=0$, $\beta_1=\beta_2=\cdots=0$, $\alpha_m>0$ and $\alpha_{m+1}=\alpha_{m+2}=\cdots=0$. Then the reachable points of $M=M_{(\alpha;0;0)}$ are $D(M)=\big\{(i,j)\in\mathbb{Z}_+^2\mid i\le m\big\}$, and $\widetilde{s}_\lambda(\alpha;\beta;\gamma)=s_\lambda(\alpha_1,\dots,\alpha_m)$ is the ordinary Schur function. Specialize $\alpha_1,\dots,\alpha_m$ as follows: Let $0<q<1$, $r=\big[\sum_{i=1}^m q^{i-1}\big]^{-1}=\frac{1-q}{1-q^m}$, and let $\alpha_i=rq^{i-1}$, $1\le i\le m$, and $\alpha_{m+1}=0$. Note that

$$\lim_{m\to\infty}\omega_m=(\overline{\alpha};0;0)\ ,$$

where $\overline{\alpha}_i = (1-q)q^{i-1}$, $i = 1, 2, \dots$, and every $(i,j) \in \mathbb{Z}_+^2$ is reachable. Recall the definitions of $h(x)$, of $c(x)$ and also of $n(\lambda) = \sum_{i\geq 1} \binom{\lambda_i'}{2}$ [M], [I]. Applying Theorem 3.1 and [M, Ex. I.3.1], we deduce

Proposition 6.1. *Let $0 < q < 1$, $r = \frac{1-q}{1-q^m}$, $\alpha_i = rq^{i-1}$, $1 \leq i \leq m$, $\alpha_{m+1} = \beta_1 = \gamma = 0$. Then, for any j,*

(a)

$$\begin{aligned} \mathcal{P}_{M_{(\alpha;0;0)}}(T(i,j)=u) = &\sum_{\mu \in H'(i-1,j-1,u-1)} d_\mu \left(\frac{1-q}{1-q^m}\right)^u q^{n(\mu^+(i,j))} \\ &\times \prod_{x \in \mu^+(i,j)} \frac{1-q^{m+c(x)}}{1-q^{h(x)}} \end{aligned} \tag{6.1.1}$$

(note that the last factor equals zero unless, also, $\mu^+(i,j) \in H(m,0,u)$). Since (i,j) is $M_{(\alpha;0;0)}$ reachable, hence

$$\begin{aligned} &\sum_{u=1}^{\infty} \sum_{\mu \in H'(i-1,j-1,u-1)} d_\mu \left(\frac{1-q}{1-q^m}\right)^u q^{n(\mu^+(i,j))} \\ &\times \prod_{x \in \mu^+(i,j)} \frac{1-q^{m+c(x)}}{1-q^{h(x)}} = 1 \ . \end{aligned} \tag{6.1.2}$$

(b) Letting $m \to \infty$ in (6.1.1), we obtain

$$\begin{aligned} &\mathcal{P}_{M_{(\alpha;0;0)}}\big(T(i,j)=u\big) \\ &\quad = \sum_{\mu \in H'(i-1,j-1,u-1)} d_\mu (1-q)^u q^{n(\mu^+(i,j))} \prod_{x \in \mu^+(i,j)} \frac{1}{1-q^{h(x)}} \ . \end{aligned} \tag{6.1.3}$$

Also

$$\sum_{u=1}^{\infty} \sum_{\mu \in H'(i-1,j-1,u-1)} d_\mu (1-q)^u q^{n(\mu^+(i,j))} \prod_{x \in \mu^+(i,j)} \frac{1}{1-q^{h(x)}} = 1 \tag{6.1.4}$$

(see the Remark, below).

(c) Since

$$s_\lambda \underbrace{\left(\frac{1}{m}, \cdots, \frac{1}{m}\right)}_{m} = \left(\frac{1}{m}\right)^{|\lambda|} \prod_{x \in \lambda} \frac{m + c(x)}{h(x)}$$

[M], [I], *by substituting $\alpha_1 = \cdots = \alpha_m = \frac{1}{m}$ ($\alpha_{m+1} = \beta_1 = \gamma = 0$) we obtain from Theorem 4.1*

$$\begin{aligned} &\mathcal{P}_{M_{(\alpha;0;0)}}\big(T(i,j)=u\big) \\ &\quad = \sum_{\mu \in H'(i-1,j-1,u-1)} d_\mu \left(\frac{1}{m}\right)^u \prod_{x \in \mu^+(i,j)} \frac{m+c(x)}{h(x)} \end{aligned} \tag{6.1.5}$$

and

$$\sum_{u=1}^{\infty} \sum_{\mu \in H'(i-1,j-1,u-1)} \frac{d_\mu}{H\big(\mu^+(i,j)\big)} \left(\frac{1}{m}\right)^u \prod_{x \in \mu^+(i,j)} \big(m + c(x)\big) = 1 \ . \tag{6.1.6}$$

Note that (6.1.5) is obtained from (6.1.1) by formally letting $q \to 1$*. If in (6.1.1)* $\mathcal{P}_{M_{(\alpha;0;0)}} \xrightarrow{u\to\infty} 0$ *uniformly, independent of* q*, then the limit* $q \to 1$ *exists and gives — again — (6.1.6).*

(d) Letting $q \to 1$ *in (b) or* $m \to \infty$ *in (c) or* $q \to 1$ *and* $m \to \infty$ *in (a), we obtain*

$$\mathcal{P}_{M_{(\alpha;0;0)}}\big(T(i,j)=u\big) = \sum_{\mu\in H'(i-1,j-1,u-1)} \frac{d_\mu}{H\big(\mu^+(i,j)\big)}\ . \tag{6.1.7}$$

Hence

$$\sum_{u=1}^{\infty} \sum_{\mu\in H'(i-1,j-1,u-1)} \frac{d_\mu}{H\big(\mu^+(i,j)\big)} = 1\ , \tag{6.1.8}$$

just like (4.2.1), (4.2.1′).

Remark. Any extended Schur function is continuous on the Thoma simplex. So, if a sequence of points, say, ω_m tends to a limit point ω in the topology of the Thoma simplex, then the corresponding values converge too. Identity (6.1.4) is obtained by applying Theorem 3.1.c to the ergodic measure $M_{(\overline{\alpha};0;0)}$, $\overline{\alpha}$ as above. One could also deduce (6.1.4) directly from (6.1.2) by letting $m \to \infty$, provided one can prove directly that such a limit exists.

By applying the $(k,\ell) = (i-1, j-1)$ *type coordinates, it is possible to rewrite the above formulas, and, in particular, part (c), more explicitly. Here, for example, is the case* $i = k+1$ *and* $j = 1$ $(\ell = 0)$*. Now* $\mu = (p \mid \phi) = (p)$, $p_1 > \cdots > p_k \geq 1$, $|\mu| = \sum_r p_r - \binom{k}{2}$, $\prod_{x\in\mu^+(k+1,1)}\big(m + c(x)\big) = \prod_{r=1}^{k}(m-r+1)_{p_r-k+r}$*, and* $\frac{d_\mu}{H(\mu^+)}$ *is obtained from (4.2.2) with* $\ell = 0$*. Thus*

$$\sum_{p_1>\cdots>p_k\geq 1} \frac{|(p)|!V^2(p)}{p!} \left[\prod_{r=1}^{k} \frac{p_r}{p_r+1}\right] \left(\frac{1}{m}\right) \prod_{r=1}^{k} (m-r+1)_{p_r=k+r} = 1\ , \tag{6.1.6′}$$

where $|(p)| = \sum_r p_r - \binom{k}{2}$.

Clearly, (a) is a q-analogue of (c), while (b) is a q-analogue of (d). Note also that as $m \to \infty$ and $q \to 1$, all $\alpha_i \to 0$, while all $\beta_i = 0$ by definition. We shall see in Proposition 8.1 that $\mathcal{P}_{M_{(0;0;1)}}\big(T(i,j)=u\big)$ is the right hand side of (6.1.7); hence we shall deduce (6.1.8) again from the case $(0;0;1)$. Thus, in a sense, the case $(\alpha;0;0)$ (or $(0;\beta;0)$) here is a q-analogue of the case $(0;0;1)$.

Example 6.2. When $i = 2$ and $j = 1$, (6.1.2) becomes

$$\sum_{u=2}^{\infty} \left(\frac{1-q}{1-q^m}\right)^u q \frac{\prod_{k=-1}^{u-2}(1-q^{m+k})}{(1-q)(1-q^u)\prod_{\ell=1}^{u-2}(1-q^\ell)} = 1\ , \tag{6.2.1}$$

(6.1.4) yields

$$\sum_{u=2}^{\infty} (1-q)^u q \frac{1}{(1-q)(1-q^u)\prod_{\ell=1}^{u-2}(1-q^\ell)} = 1\ , \tag{6.2.2}$$

while (6.1.6) (or formally $q \to 1$ in (6.2.2)) implies that

$$\sum_{u=2}^{\infty} \left(\frac{1}{m}\right)^u \frac{(m-1)m\cdots(m+u-2)}{(u-2)!u} = 1\ . \tag{6.2.3}$$

Finally, (6.1.8) (i.e., $m \to \infty$, $q \to 1$ in (6.2.1)) yields

$$\sum_{u=2}^{\infty} \frac{1}{(u-2)!u} = 1 \tag{6.2.4}$$

(which is easy to verify, since $\frac{1}{(u-2)!u} = \frac{1}{(u-1)!} - \frac{1}{u!}$). Thus, (6.2.1)–(6.2.3) are generalizations, and q-analogue generalizations of (6.2.4).

Similar examples can be calculated for any $(i,j) \in \mathbb{N}^2$. However, the identities corresponding to $(i,j) \in \mathbb{N}^2$ become much more complicated as $i+j$ increases.

7. Application II: $M = M_{(\alpha;\beta;0)}$

Here we let $\gamma = 0$ and $(\alpha;\beta;\gamma) = (\alpha;\beta;0) \equiv (\alpha;\beta) = (\alpha_1,\dots,\alpha_i;\beta_1,\dots,\beta_j)$ (i.e., $\alpha_{i+1} = \beta_{j+1} = 0$). The $M_{(\alpha;\beta;0)}$ reachable points are $\{(a,b) \mid a \le i,\ b \le j\}$ (2.13). Again, for each $(i,j) \in \mathbb{N}^2$ we deduce identities with ("double") q-analogues.

Note that if $\mu \in H'(i-1,j-1,u-1)$, then $\mu^+ = \mu^+(i,j) \supseteq R_{i,j}$, the $i \times j$ rectangle. Thus $\mu^+(i,j)$ defines the two partitions,

$$\nu(\mu^+) = (\mu_1 - j, \dots, \mu_{i-1} - j, 0) \quad \text{and} \quad \eta(\mu^+) = (\mu'_1 - i, \dots, \mu'_{j-1} - i, 0)\ .$$

Since here $\widetilde{s}_\lambda(\alpha;\beta;0) = s_\lambda(\alpha_1,\dots,\alpha_i;\beta_1,\dots,\beta_j)$ are the super (or hook) Schur functions, by [BR, Theorem 6.20] (or by a special case of the Sergeev-Pragacz formula [M, I.3], and since $\mu^+ \supseteq R_{i,j}$,

$$\begin{aligned} &s_{\mu^+(i,j)}(\alpha_1,\dots,\alpha_i;\beta_1,\dots,\beta_j) \\ &\quad = \left[\prod_{k=1}^{i}\prod_{\ell=1}^{j}(\alpha_k+\beta_\ell)\right] s_{\nu(\mu^+)}(\alpha_1,\dots,\alpha_i) s_{\eta(\mu^+)}(\beta_1,\dots,\beta_j)\ . \end{aligned}$$

Now let $0 < p, q < 1$, $r = \left(\sum_{k=0}^{i-1} p^k + \sum_{\ell=0}^{j-1} q^\ell\right)^{-1}$, and substitute $\alpha_k = rp^{k-1}$, $1 \le k \le i$, and $\beta_\ell = rq^{\ell-1}$, $1 \le \ell \le j$. Thus $\sum_{k\ge1}(\alpha_k+\beta_k) = 1$; hence $\gamma = 0$. Clearly, $s_\nu(r, rp, \dots, rp^{i-1}) = r^{|\nu|} s_\nu(1, p, \dots, p^{i-1})$, and similarly for $s_\eta(r, rq, \dots, rq^{j-1})$. Applying [M, Ex. I.3.1], we obtain the value of $s_{\mu^+}(\alpha;\beta)$. Combined with Theorems 3.1.a, this implies

Proposition 7.1. *Let $0 < p, q < 1$, $r = \left[\sum_{k=0}^{i-1} p^k + \sum_{\ell=0}^{j-1} q^\ell\right]^{-1}$, $\gamma = 0$, and let $(\alpha;\beta) = (r, rp, \dots, rp^{i-1}; r, rq, \dots, rq^{j-1})$ as above. Then*
(a)

$$\begin{aligned} \mathcal{P}_{M_{(\alpha;\beta;0)}}\big(T(i,j) = u\big) &= \sum_{\mu \in H'(i-1,j-1,u-1)} d_\mu r^u \left[\prod_{k=0}^{i-1}\prod_{\ell=0}^{j-1}(p^k+q^\ell)\right] \\ &\quad \times p^{n(\nu)} q^{n(\eta)} \left[\prod_{x\in\nu(\mu^+)} \frac{1-p^{i+c(x)}}{1-p^{h(x)}}\right] \left[\prod_{x\in\eta(\mu^+)} \frac{1-q^{j+c(x)}}{1-q^{h(x)}}\right]\ . \end{aligned} \tag{7.1.1}$$

(b) As in Proposition 6.1.c, let $(\alpha;\beta) = \left(\frac{1}{i+j},\dots,\frac{1}{i+j};\frac{1}{i+j},\dots,\frac{1}{i+j}\right)$*, or formally let* $p,q\to 1$ *in (7.1.1). Then*

$$\mathcal{P}_{M_{(\alpha;\beta;0)}}\big(T(i,j)=u\big) = \sum_{\mu\in H'(i-1,j-1,u-1)} d_\mu \left(\frac{1}{i+j}\right)^u 2^{ij} \times \left[\prod_{x\in\nu(\mu^+)} \frac{i+c(x)}{h(x)}\right]\left[\prod_{x\in\eta(\mu^+)} \frac{j+c(x)}{h(x)}\right] . \tag{7.1.2}$$

Again, $n(\lambda)$, $h(x)$ and $c(x)$ are given in [M, I.1].

Clearly, (7.1.1) is a ("double") q-analogue of (7.1.2). By (2.13) the box (i,j) is $M_{(\alpha;\beta;0)}$-reachable. Hence, by Theorem 3.1.c (i.e., by summing over all u's in (7.1.1) and in (7.1.2)) we obtain

Proposition 7.2. *In the notations of Proposition 7.1,*

(a)

$$\sum_{u=1}^{\infty}\sum_{\mu\in H'(i-1,j-1,u-1)} d_\mu r^u \left[\prod_{k=0}^{i-1}\prod_{\ell=0}^{j-1}(p^k+q^\ell)\right] p^{n(\nu(\mu^+))} \times q^{n(\eta(\mu^+))}\left[\prod_{x\in\nu(\mu^+)}\frac{1-p^{i+c(x)}}{1-p^{h(x)}}\right]\left[\prod_{x\in\eta(\mu^+)}\frac{1-q^{j+c(x)}}{1-q^{h(x)}}\right] = 1 . \tag{7.2.1}$$

(b) From (7.1.2) (or formally letting $p,q\to 1$ *in (a)), we obtain*

$$\sum_{u=1}^{\infty}\sum_{\mu\in H'(i-1,j-1,u-1)} d_\mu \left(\frac{1}{i+j}\right)^u 2^{ij} \times \left[\prod_{x\in\nu(\mu^+)} \frac{i+c(x)}{h(x)}\right]\left[\prod_{x\in\eta(\mu^+)} \frac{j+c(x)}{h(x)}\right] = 1 . \tag{7.2.2}$$

Note that (7.2.2) is obtained from (7.2.1) by formally letting $p,q\to 1$. Thus, again, (7.2.1) is a double (or p-) q analogue of (7.2.2).

Let $(i,j)=(k+1,\ell+1)$ and rewrite (7.2.2) with the (k,ℓ) coordinates as follows (compare with (4.3.2) and (6.1.6′)). First, the summation $\sum_u\sum_{\mu\cdots}$ becomes $\sum_{\substack{p_1>\cdots>p_k\ge 1\\ q_1>\cdots>q_\ell\ge 1}}$. Now

$$d_\mu = d_{(p|q)} = \frac{|(p\mid q)|!V(p)V(q)}{p!q!\prod_{r,s}(p_r+q_s+1)} ,$$

where $|(p\mid q)| = \sum_{r=1}^k p_r + \sum_{s=1}^\ell q_s + \frac12\big[k+\ell-(k-\ell)^2\big]$,

$$\frac{1}{H\big(\nu(\mu^+)\big)} = \frac{V(p)}{p!}\left(\frac{p_r}{p_r+1}\right) , \qquad \frac{1}{H\big(\nu(\mu^+)\big)} = \frac{V(q)}{q!}\prod_s\left(\frac{q_s}{q_s+1}\right) ,$$

$$\prod_{x\in\nu(\mu^+)}\big(i+c(x)\big) = \prod_{x\in\nu(\mu^+)}\big(k+1+c(x)\big) = \prod_{r=1}^{k}(k+2-r)_{p_r-k+r}$$

and

$$\prod_{x\in\eta(\mu^+)} \big(j + c(x)\big) = \prod_{s=1}^{\ell} (\ell + 2 - s)_{q_s-\ell+s} \ .$$

Thus, (7.2.2) now reads as follows:

Proposition 7.2.b′.

(7.2.2′)

$$\sum_{\substack{p_1>\cdots>p_k\geq 1\\ q_1>\cdots>q_\ell\geq 1}} \frac{|(p\mid q)|!V^2(p)V^2(q)}{(p!)^2(q!)^2\prod_{r=1}^{k}\prod_{s=1}^{\ell}(p_r+q_s+1)} \left(\prod_{r=1}^{k}\frac{p_r}{p_r+1}\right)\left(\prod_{s=1}^{\ell}\frac{q_s}{q_s+1}\right) \times \left(\frac{1}{k+\ell+2}\right)^{|(p|q)|+1} 2^{(k+1)(\ell+1)} \left(\prod_{r=1}^{k}(k+2-r)_{p_r-k+r}\right) \times \left(\prod_{s=1}^{\ell}(\ell+2-s)_{q_s-\ell+s}\right) = 1 \ .$$

Some special cases. By symmetry, the cases (i, j) and (j, i) are the same. The case $i = j = 1$ being trivial, we first consider

$\underline{i = 2,\ j = 1}$: Here (7.2.1) specializes to

$$\sum_{u=2}^{\infty}\left(\frac{1}{p+2}\right)^{u} 2(1+p)\frac{1-p^{u-1}}{1-p} = 1 \ , \tag{7.3.1}$$

while (7.2.2) (or $p \to 1$ in (7.3.1)) becomes

$$\sum_{u=2}^{\infty}\left(\frac{1}{3}\right)^{u}(u-1) = \frac{1}{4} \ , \tag{7.3.2}$$

which can easily be verified directly.

$\underline{i = j = 2}$: By (7.2.1),

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\binom{k+\ell+2}{k+1}\left(\frac{1}{2+p+q}\right)^{k+\ell+4}(1+1)(1+q)(p+1)(p+q) \times \left(\frac{1-p^{k+1}}{1-p}\right)\left(\frac{1-q^{\ell+1}}{1-q}\right) = 1 \ , \tag{7.3.3}$$

and by (7.2.2) (or by $p, q \to 1$ in (7.3.3))

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\binom{k+\ell+2}{k+1}\left(\frac{1}{4}\right)^{k+\ell}(k+1)(\ell+1) = 16 \ . \tag{7.3.4}$$

$\underline{i = 3,\ j = 2}$: Here (7.2.1) implies a p, q identity of a type similar to, but more involved than (7.3.3), while (7.2.2) implies the identity

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(k+2\ell+m+5)!(k+1)^2(k+\ell+2)}{\ell!m!(k+\ell+2)!(\ell+m+3)(k+\ell+m+4)}\left(\frac{1}{5}\right)^{k+2\ell+m} = \frac{5^6}{2^5} = 488.28125 \ . \tag{7.3.5}$$

8. Application III: $M = M_{(0;0;1)}$

We now examine more closely some special cases, arising from the Plancherel measure $M_{(0;0;1)}$. It is shown below that the probabilities in (8.2.1) and (8.2.3) below possess a remarkable property.

Example 8.1.

<u>(3,1)</u>: $i = 3$, $j = 1$ (or $i = 1$, $j = 3$). By Corollary 4.2,

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(3,1) = u\big) = \sum_{k=0}^{\left[\frac{u-3}{2}\right]} \frac{(u-1)!(u-2k-2)^2}{k!(k+2)!(u-k-2)!(u-k)!} \ ; \tag{8.2.1}$$

hence (also by (6.1.8))

$$\sum_{u=3}^{\infty} \sum_{k=0}^{\left[\frac{u-3}{2}\right]} \frac{(u-1)!(u-2k-2)^2}{k!(k+2)!(u-k-2)!(u-k)!} = 1 \ . \tag{8.2.2}$$

<u>(2,2)</u>: $i = j = 2$. By Corollary 4.2,

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(2,2) = u\big) = \sum_{k=0}^{u-4} \frac{(u-2)!}{k!(k+2)!(u-k-4)!(u-k-2)!(u-1)} \ ; \tag{8.2.3}$$

hence (or by (6.1.8))

$$\sum_{u=4}^{\infty} \sum_{k=0}^{u-4} \frac{(u-2)!}{k!(k+2)!(u-k-4)!(u-k-2)!(u-1)} = 1 \ . \tag{8.2.4}$$

<u>(4,1)</u>: $i = 4$, $j = 1$ or $i = 1$, $j = 4$. Here we deduce from Corollary 4.2 (or from (6.1.8)) that

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} f(a,b,c) = 1 \ , \tag{8.2.5}$$

where

$$f(a,b,c) = \frac{A}{B \times C} \ ,$$

$$\begin{aligned} A &= (a+2b+3c+3)!(b+1)^2(a+1)^2(a+b+2)^2 \ , \\ B &= c!(c+1)!(b+c+1)!(b+c+2)!(a+b+c+2)!(a+b+c+3)! \ , \\ C &= (c+2)(b+c+3)(a+b+c+4) \ . \end{aligned}$$

Computer experiments lead us to observe the following rather remarkable phenomenon:

Proposition 8.2. *For all $u \ge 1$ (i.e., $u \ge 3$),*

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(3,1) = u\big) = \mathcal{P}_{M_{(0;0;1)}}\big(T(2,2) = u+1\big) \ .$$

(So far, no similar phenomenon has been observed between other entries or for other measures.)

Comparing (8.2.1) with (8.2.3) for u even and for u odd, we see that Proposition 8.2 is equivalent to the following two "binomial" identities:

Lemma 8.3. *(1)*

$$\sum_{k=0}^{n-2} \binom{2n}{k} \binom{2n}{k+2} (n-k-1)^2 = n \sum_{k=0}^{n-2} \binom{2n-1}{k} \binom{2n-1}{k+2} \ . \tag{8.3.1}$$

(2)

$$\begin{aligned} &\sum_{k=0}^{n-1} \binom{2n+1}{k} \binom{2n+1}{k+2} (2n-2k-1)^2 \\ &= (2n+1) \sum_{k=0}^{2n-2} \binom{2n}{k} \binom{2n}{k+2} \ . \end{aligned} \tag{8.3.2}$$

Proof (I.G. Macdonald). The first proof of 8.3 was a computer proof, given by D. Zeilberger (and Shalosh B. Ekhad) [Z]. Here are the main steps of a "human" proof, due to I.G. Macdonald (private letter). We verify, for example, (8.3.1). Note first that, on both sides, $\sum_{k=0}^{n-2}$ can be replaced by $\frac{1}{2}\sum_{k=0}^{2n-2}$. Evaluate the coefficient of t^{2n-3} in $(1+t)^{4n-2} = (1+t)^{2n-1}(1+t)^{2n-1}$ to conclude that

$$\sum_{k=0}^{2n-3} \binom{2n-1}{k} \binom{2n-1}{k+2} = \binom{4n-2}{2n-3}.$$

This simplifies the right hand side of (8.3.1). Now simplify the left hand side: expand $(n-k-1)^2 = (n-1)^2 - 2nk + k(k+2)$, so that the left hand side is

$$\frac{1}{2}\big[(n-1)^2 A - 2nB + C\big]$$

with corresponding A, B and C, which simplify by similar arguments. For example,

$$\begin{aligned} B &= \sum_{k=0}^{2n-2} k \binom{2n}{k} \binom{2n}{k+2} = 2n \sum_{k=0}^{2n-2} \binom{2n-1}{k-1} \binom{2n}{2n-k-2} \\ &= 2n\big[\text{the coefficient of } t^{2n-3} \text{ in } (1+t)^{4n-1} = (1+t)^{2n-1}(1+t)^{2n}\big] \\ &= 2n \binom{4n-1}{2n-3} \ . \end{aligned}$$

After simplifying A, B and C, it is easy to verify that the left hand side equals the right hand side. ☐

8.1. **Probability of congruences.** Given $q, r \in \mathbb{N}$ and M, we consider

$$\mathcal{P}_M\big(T(i,j) \equiv r \pmod q\big),$$

the M-probability that $T(i,j) \equiv r \pmod q$.

Trivially, $\mathcal{P}_M\big(T(i,j) = u\big) = 0$ if $u < ij$. Therefore we can replace $0 \le r < q$ by $ij \le r < ij + q$; then clearly,

$$\mathcal{P}_M\big(T(i,j) \equiv r \pmod q\big) = \sum_{k=0}^{\infty} \mathcal{P}_M\big(T(i,j) = kq + r\big) \ . \tag{8.4.1}$$

Below we consider two examples, both in the case $M = M_{(0;0;1)}$ and with $(i,j) = (1,2)$.

8.1.1. *Example:* $(i,j)=(1,2)$ *(or* $(i,j)=(2,1)$*).* Let $2\le r<2+q$; then

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(1,2)\equiv r\ (\mathrm{mod}\, q)\big)=\sum_{k=0}^{\infty}\frac{1}{(qk+r-2)!(qk+r)}\ . \tag{8.4.2}$$

This follows since, by Proposition 8.1, $\mathcal{P}_{M_{(0;0;1)}}\big(T(1,2)=u\big)=\frac{1}{(u-2)!u}$. Since $\frac{1}{(u-2)!u}=\frac{1}{(u-1)!}-\frac{1}{u!}$, this implies that for $q=2$

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(1,2)\equiv 1\ (\mathrm{mod}\, 2)\big)=\frac{1}{e}\ , \tag{8.4.3}$$

a phenomenon which was observed in [R].

For $q=3$ the corresponding probabilities are

(8.4.4)

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(1,2)\equiv 2\ (\mathrm{mod}\, 3)\big)=\frac{2}{3\sqrt{e}}\left[\cos\left(\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right)-\cos\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)\right]\ ,$$

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(1,2)\equiv 3\ (\mathrm{mod}\, 3)\big)=1-\frac{2}{3\sqrt{e}}\left[\cos\left(\frac{\sqrt{3}}{2}\right)+\cos\left(\frac{\sqrt{3}}{2}-\frac{\pi}{2}\right)\right]\ ,$$

and

$$\mathcal{P}_{M_{(0;0;1)}}\big(T(1,2)\equiv 4\ (\mathrm{mod}\, 3)\big)=\frac{2}{3\sqrt{e}}\left[\cos\left(\frac{\sqrt{3}}{2}\right)-\cos\left(\frac{\sqrt{3}}{2}+\frac{\pi}{2}\right)\right]\ .$$

9. Examples of Some $M_{u,v}$-Identities

Theorem 4.3.b is applied for an explicit calculation of the $M_{u,v}$-identity corresponding to the box $(k+1,1)$, k arbitrary, and to (2,2).

$\underline{k=1}$ (i.e., the box $(2,1)$): Theorem 4.3.b implies that

$$\sum_{n\ge 2}\frac{1}{(n-2)!n}\frac{(u-1)_n(v-1)_n}{(uv)_n}=1\ . \tag{9.1.1}$$

A priori, (9.1.1) holds if $u=\overline{v}\in\mathbb{C}$ or $u,v\in(m,m+1)$ for some $m\in\mathbb{Z}$. However, using a summation formula of Gauss, we show that (9.1.1) holds for (u,v) in a much larger domain in $\mathbb{C}^2$: $0\ge v\in\mathbb{Z}$ or $\mathrm{Re}(uv)>\mathrm{Re}(u)+\mathrm{Re}(v)$ (G. Olshanski):

Since $\frac{1}{(n-2)!n}=\frac{1}{(n-1)!}-\frac{1}{n!}$, the left hand side of (9.1.1) can be rewritten as

$$\frac{(u-1)(v-1)}{uv}\sum_{n\ge 2}\frac{(u)_{n-1}(v)_{n-1}}{(uv+1)_{n-1}(n-1)!}-\sum_{n\ge 2}\frac{(u-1)_n(v-1)_n}{(uv)_n n!}\ .$$

Now apply the Gauss formula

$$\sum_{n\ge 0}\frac{(a)_n(b)_n}{(c)_n n!}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ ,$$

where $0\ge b\in\mathbb{Z}$ or $\mathrm{Re}(c)>\mathrm{Re}(a)+\mathrm{Re}(b)$, apply $\Gamma(z+1)=z\Gamma(z)$, and notice that the summations above start with $n\ge 2$. The proof of (9.1.1) follows.

Recall that, given $m=(m_1,\dots,m_k)$,

$$V(m)=V(m_1,\dots,m_k)=\prod_{1\le i<j\le k}(m_i-m_j)\ .$$

<u>$k=2$</u>: Here Theorem 4.3.b implies that

$$\sum_{m_1,m_2\geq 0}\frac{(2)_{m_1+m_2}V^2(m)(u)_{m_1}(u)_{m_2}(v)_{m_1}(v)_{m_2}}{m_1!m_2!(3)_{m_1}(3)_{m_2}(uv+2)_{m_1+m_2}} = \frac{8uv(uv+1)}{(u-1)(u-2)(v-1)(v-2)}\ , \tag{9.2.1}$$

at least when $u=\overline{v}\in\mathbb{C}$ or $u,v\in(M,M+1)$, $M\in\mathbb{Z}$.

<u>$k\geq 3$</u>: The $M_{u,v}$-identity corresponding to the box $(k+1,1)$ is

$$\sum_{m_1,\dots,m_k\geq 0}\frac{\big(|m|-\frac{1}{2}k(k-3)\big)!V^2(m)\prod_{i=1}^{k}\big[(u+2-k)_{m_i}(v+2-k)_{m_i}\big]}{\prod_{i=1}^{k}\big[m_i!(3)_{m_i}\big](uv)_{|m|-\frac{1}{2}k(k-3)+1}} = \frac{2^k k!\prod_{i=1}^{k-2}\big[(u+2-k)_i(v+2-k)_i\big]}{(u-k)(u-k+1)(v-k)(v-k+1)}\ , \tag{9.3.1}$$

at least when $u=\overline{v}$ or $u,v\in(M,M+1)$, $M\in\mathbb{Z}$. Here $|m|=m_1+\cdots+m_k$.

Notice that without the factor $V^2(m)$, the left hand side (of (9.2.1) and) of (9.3.1) is one of the multivariate hypergeometric series, evaluated at $(1,\dots,1)$.

As an additional example, we also give the case of

<u>The box (2,2)</u>: Let $u=\overline{v}\in\mathbb{C}$ or $u,v\in(M,M+1)$, $M\in\mathbb{Z}$. Let

$$p(k,n)=\frac{(n-2)!}{k!(k+2)!(n-k-4)!(n-k-2)!(n-1)}$$

(compare with (8.2.3)) and let

$$g(u,k,n)=(u)_{k+2}(u-n+k+3)_{n-k-2}\ ;$$

then

$$\mathcal{P}_{M_{u,v}}\big(T(2,2)=n\big)=\sum_{k=0}^{n-4}\frac{p(k,n)g(u,k,n)g(v,k,n)}{(uv)_n}$$

and

$$\sum_{n=4}^{\infty}\sum_{k=0}^{n-4}\frac{p(k,n)g(u,k,n)g(v,k,n)}{(uv)_n}=1\ . \tag{9.4.1}$$

10. The Analogous Projective Theory

A partition is strict if all its parts are distinct. These partitions span the Schur-Young subgraph SY of Y. In the theory of the projective representations of S_n and of S_∞, SY replaces Y. An exact analogue of the VK-theory exists for SY, and is mostly due to Nazarov [N] (see also Ivanov [I]).

Here *Tab* is replaced by $STab\subset Tab$, the infinite paths T in SY. A strict partition λ now corresponds both to its ordinary and to its shifted Young diagram [M, III], [HH]. We continue to identify λ with its ordinary diagram, and will denote by $sh(\lambda)$ its shifted diagram. For example,

$$(5,2,1)\equiv\begin{matrix}x & x & x & x & x\\ x & x & & & \\ x & & & & \end{matrix}$$

and

$$sh(5,2,1) \equiv \begin{array}{ccccc} x & x & x & x & x \\ & x & x & & \\ & & x & & \end{array}$$

Each infinite path $T \in STab$ now corresponds, in an obvious way, to an infinite shifted standard tableau.

The algebra Λ of the symmetric functions is replaced here by its subalgebra Γ, spanned, for example, by the Schur P functions $\{P_\lambda \mid \lambda \text{ strict}\}$ [M, III, 8]. Here d_λ is replaced by g^λ, the number of standard tableaux of (shifted) shape $sh(\lambda)$.

Theorem (Schur. See [M, III, 8]). *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$. Then*

$$g^\lambda = \frac{n!}{\lambda_1! \cdots \lambda_k!} \prod_{1 \leq i < j \leq k} \frac{(\lambda_i - \lambda_j)}{(\lambda_i + \lambda_j)} \ . \tag{10.1.1}$$

Let $\lambda \vdash n$ be strict. There is an obvious bijection between paths

$$\phi = \lambda^0 \nearrow \lambda^1 \nearrow \cdots \nearrow \lambda^n = \lambda$$

and standard tableaux of shape $sh(\lambda)$. Denote by $STab(\lambda)$ all such paths, i.e., all such standard tableaux of shape $sh(\lambda)$: $|STab(\lambda)| = g^\lambda$.

S-cylindrical sets are defined in analogy to the Y case: given $\lambda \vdash n$ strict and $\tau \in STab(\lambda)$, $SCyl(\tau)$ are the infinite paths in $STab$ with first n links equal to τ. The definition of central measures on $STab$ is analogous (is central if for any strict λ, all the S-cylindrical sets $SCyl(\tau)$, $\tau \in STab(\lambda)$, have the same mass).

Proposition (see [N], [I]). *There is a bijective correspondence $M \leftrightarrow \pi$ between the central measures M on STab and the linear functionals $\pi\colon\ \Gamma \to \mathbb{R}$ satisfying the following conditions:*

- $\pi(1) = 1$.
- *π factors through the algebra $\Gamma/(s_{(1)} - 1)\Gamma$. Note that $s_{(1)} = P_{(1)}$.*
- *$\pi(P_\lambda) \geq 0$ for any strict λ.*

Under this correspondence, for any strict λ and any $\tau \in STab(P_\lambda)$

$$M\big(SCyl(\tau)\big) = \pi(P_\lambda) \ .$$

Shifted analogues (of the Y case) are now introduced in an obvious way:

$S(\mathbb{Z}_+^2) = \big\{(i,j) \in \mathbb{Z}_+^2 \mid i \leq j\big\}$ (analogue of $\mathbb{Z}_+^2$).

The shifted diagram $SD(T) \subseteq S(\mathbb{Z}_+^2)$ of $T \in STab$ (analogue of $D(T)$).

$\mathcal{SP}_{M}\big(T(i,j) = n\big)$, where M is a central measure on $STab$, $(i,j) \in S(\mathbb{Z}_+^2)$, $n \in \mathbb{Z}_+$ (analogue of $\mathcal{P}_{M}\big(T(i,j) = n\big)$).

$$STab(i,j) = \big\{T \in STab \mid (i,j) \in SD(T)\big\} \ ,$$

$$M\big(STab(i,j)\big) = \sum_{n \geq 1} SP_{M}\big(T(i,j) = n\big) \ .$$

Call $(i,j) \in S(\mathbb{Z}_+^2)$ M-reachable if $M\big(STab(i,j)\big) = 1$.

Given $\ell \geq k$, $H'(k,\ell,n)$ is now replaced by $SH'(k,\ell,n) = \big\{(\lambda_1, \lambda_2, \ldots) \vdash n \mid \lambda_1 > \lambda_2 > \cdots,\ \lambda_k \geq \ell - k + 2 \text{ and } \lambda_{k+1} = \ell - k\big\}$. Note that if $\mu \in SH'(k,\ell,n)$, then $\mu^+(k,\ell) = (\mu_1, \ldots, \mu_k, \mu_{k+1} + 1, \mu_{k+2}, \ldots)$ is also a strict partition.

In exact analogy with Theorem 3.1 we have here

Theorem 10.1. *Let $(i,j) \in S(\mathbb{Z}_+^2)$, $n \in \mathbb{Z}_+$. Let M be a central measure on $STab$ with corresponding $M \leftrightarrow \pi$, $\pi\colon\ \Gamma \to \mathbb{R}$. Then*

(a)

$$\mathcal{SP}_{M}\big(T(i,j)=n\big) = \sum_{\mu \in SH'(i-1,j-1,n-1)} g^{\mu}\pi\big(P_{\mu^+(i,j)}\big)\ .$$

By summing over all $n \geq 1$ we obtain

(b)

$$\sum_{n=1}^{\infty} \sum_{\mu \in SH'(i-1,j-1,n-1)} g^{\mu}\pi\big(P_{\mu^+(i,j)}\big) = M\big(STab(i,j)\big)\ .$$

(c) In particular, if (i,j) is M reachable, then

$$\sum_{n=1}^{\infty} \sum_{\mu \in SH'(i-1,j-1,n-1)} g^{\mu}\pi\big(P_{\mu^+(i,j)}\big) = 1\ .$$

Similarly to the ordinary theory, we now study two special families of central measures M on $STab$. One family is the ergodic measures $M_{(\alpha;\gamma)}$, and the second is a remarkable one parameter family of measures $\big\{M_x \mid x \in \mathbb{R}^+ \cup \{\infty\}\big\}$.

The measure $M_{(\alpha;\gamma)}$. Here α is $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ with $\sum_{i=1}^{\infty}\alpha_i \leq 1$, and $\gamma = 1 - \sum_{i=1}^{\infty}\alpha_i$. To define $M_{(\alpha;\gamma)}$ we need the functions $\widetilde{P}_\lambda(\alpha;\gamma)$ which generalize the Schur P functions $P_\lambda(x)$: $\widetilde{P}_\lambda(\alpha;0) = P_\lambda(\alpha)$. The $\widetilde{P}_\lambda(\alpha;\gamma)$'s are implicitly given as combinations of the functions $\widetilde{s}_\lambda(\alpha;0;\gamma)$.

To compute $\widetilde{P}_\lambda(\alpha;\gamma)$, start with $\widetilde{q}_k = q_k(\alpha;\gamma)$: these are given by

$$\widetilde{Q}(t) = 1 + \sum_{k=1}^{\infty}\widetilde{q}_k t^k = e^{2\gamma t}\prod_i \frac{1+\alpha_i t}{1-\alpha_i t} \tag{10.1.2}$$

(compare with [M, III, (8.1)]). Since $\widetilde{Q}(t)\widetilde{Q}(-t)=1$, follow [M, III, 8], replacing $q_r = q_r(x)$ by $\widetilde{q}_r = \widetilde{q}_r(\alpha;\gamma)$. Then $\widetilde{Q}_\lambda = Pf(\widetilde{M}_\lambda)$, the Pfaffian of the matrix $\widetilde{M}_\lambda = (\widetilde{Q}_{(\lambda_i,\lambda_j)})$ [M, III, (8.11)]. Here $\widetilde{Q}_{(r,s)} = \widetilde{q}_r\widetilde{q}_s + 2\sum_{i=1}^{s}(-1)^i\widetilde{q}_{r+i}\widetilde{q}_{s-i}$ [M, III, (8.10)].

Finally, $\widetilde{P}_\lambda = 2^{-\ell(\lambda)}\widetilde{Q}_\lambda$ [M, III, (8.7)].

The ergodic measure. $M_{(\alpha;\gamma)}$ is given by its corresponding functional

$$M_{(\alpha;\gamma)} \leftrightarrow \pi_{(\alpha;\gamma)}\colon\ \Gamma \to \mathbb{R}\ ,$$

$$\pi_{(\alpha;\gamma)}(P_\lambda) = \widetilde{P}_\lambda(\alpha;\gamma)\ .$$

Reachable boxes. (i) If there exist $k \in \mathbb{Z}_+$ such that $\alpha_{k+1} = \cdots = \gamma = 0$, $\alpha_k > 0$, then the reachable boxes are the shifted k-strip $\big\{(i,j) \mid i \leq k,\ i \leq j\big\}$.

(ii) Otherwise (i.e. if all α_i are strictly positive or if $\gamma > 0$), the reachable boxes are $S(\mathbb{Z}_+^2)$.

Let $\lambda \vdash n$ be strict. Note that

$$\frac{g^\lambda}{n!} = \frac{1}{\prod_{x \in sh(\lambda)} h_{D(\lambda)}(x)}$$

in the sense of [M, III, 8, Ex. 12] (with the correction $D(\lambda) = (\lambda_1, \lambda_2, \dots \mid \lambda_1 - 1, \lambda_2 - 1, \dots)$. In fact, if $D(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \dots \mid \lambda_1, \lambda_2, \dots)$, then

$$\frac{2^{-\ell(\lambda)} g^\lambda}{n!} = \frac{1}{\prod_{x \in sh(\lambda)} h_{D(\lambda)}(x)}) \ .$$

Thus, the projective analogue of (4.1.1), and of [J, 19.5], is

Theorem 10.2. *Let $\alpha = 0$. Then $\gamma = 1$, and*

$$\widetilde{P}_\lambda(0;1) = \frac{2^{n-\ell(\lambda)} g^\lambda}{n!} \ ,$$

where $|\lambda| = n$. This is the Plancherel measure on STab.

Proof. Show that $\widetilde{Q}_\lambda(0;1) = \frac{2^n g^\lambda}{n!}$ by following the steps mentioned above.

First, by (10.1.2) with $\alpha = 0$, we have $\widetilde{q}_r = \frac{2^r}{r!}$; consequently

$$\widetilde{Q}_{(r,s)} = \frac{2^r}{r!}\frac{2^s}{s!} + 2\sum_{i=1}^{s}(-1)^i \frac{2^{r+i}}{(r+i)!}\frac{s^{s-i}}{(s-i)!} \ .$$

Standard "binomial" calculations give

$$\widetilde{Q}_{(r,s)} = \frac{2^{r+s}}{r!s!} \times \frac{(r-s)}{(r+s)} \ .$$

Now write $\lambda = (\lambda_1, \dots, \lambda_{2m})$, $\lambda_1 > \cdots > \lambda_{2m} \geq 0$; then

$$\widetilde{M}_\lambda = \left(\frac{2^{\lambda_i + \lambda_j}}{\lambda_i! \lambda_j!} \times \frac{(\lambda_i - \lambda_j)}{(\lambda_i + \lambda_j)} \right)_{1 \leq i,j \leq 2m} \ .$$

Note that the i^{th} row of $\widetilde{M}_\lambda$ is divisible by $\frac{2^{\lambda_i}}{\lambda_i!}$, and the j^{th} column by $\frac{2^{\lambda_j}}{\lambda_j!}$. Hence

$$\det(\widetilde{M}_\lambda) = \left(\prod_{i,j} \frac{2^{\lambda_i}}{\lambda_i!}\frac{2^{\lambda_j}}{\lambda_j!} \right) \det\left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) = \left(\frac{2^n}{\lambda_1! \cdots \lambda_{2m}!} \right)^2 \det\left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right) \ .$$

Thus

$$Pf(\widetilde{M}_\lambda) = \frac{2^n}{\lambda_1! \cdots \lambda_{2m}!} Pf(\frac{\lambda_i - \lambda_j}{\lambda_i + j}) \ .$$

It is well known that

$$Pf(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}) = \prod_{i<j}(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}) \ ,$$

and the proof now follows from a formula for g^λ [M, III, 8, Ex. 12, (2)]. □

From Theorem 10.1 we now deduce

Theorem 10.3. *For the ergodic measure $M_{(\alpha;\gamma)}$*

(a)

$$\mathcal{SP}_{M_{(\alpha;\gamma)}}\big(T(i,j) = n\big) \sum_{\mu \in SH'(i-1,j-1,n-1)} g^\mu \widetilde{P}_{\mu^+(i,j)}(\alpha;\gamma) \ .$$

(b) If $(i,j) \in S(\mathbb{Z}_+^2)$ is $M_{(\alpha;\gamma)}$ reachable, then

$$\sum_{n=1}^{\infty} \sum_{\mu \in SH'(i-1,j-1,n-1)} g^\mu \widetilde{P}_{\mu^+(i,j)}(\alpha;\gamma) = 1 \ .$$

In the special case $(\alpha;\gamma) = (0;1)$ Theorems 10.1 and 10.2 imply

Theorem 10.4. *Let* $(i,j) \in S(\mathbb{Z}_+^2)$. *Then*

(a)

$$\mathcal{SP}_{M_{(0;1)}}\big(T(i,j)=n\big) = \sum_{\mu\in SH'(i-1,j-1,n-1)} \frac{g^\mu 2^{n-\ell(\mu^+(i,j))} g^{\mu^+(i,j)}}{n!} \ .$$

Hence, since any (i,j) *is reachable,*

(b)

$$\sum_{n=1}^{\infty} \sum_{\mu\in SH'(i-1,j-1,n-1)} \frac{g^\mu 2^{n-\ell(\mu^+(i,j))} g^{\mu^+(i,j)}}{n!} = 1 \ .$$

Clearly, Theorem 10.3 is a certain deformation of Theorem 10.4.

Similarly to the measures $M_{u,v} \leftrightarrow \pi_{u,v}$ (4.3.1), there exists here a remarkable one parameter family of (non-ergodic) central measures M_x. These were discovered in [B], and are given, on strict partitions $\lambda \vdash n$, by $M_x \leftrightarrow \pi_x$,

$$\pi_x(P_\lambda) = \frac{2^{n-\ell(\lambda)} g^\lambda}{n!} \times \frac{\prod_{(i,j)\in sh\lambda}\big[(j-i)(j-i+1)+x\big]}{x(x+2)(x+4)\cdots\big(x+2(n-1)\big)} \ . \tag{10.1.3}$$

Here $x \in \mathbb{R}^+ \cup \{\infty\}$ and λ is interpreted as a shifted diagram. When λ is interpreted as an ordinary diagram, M_x is given by

$$\pi_x(P_\lambda) = \frac{2^{n-\ell(\lambda)} g^\lambda}{n!} \times \frac{\prod_{(i,j)\in \lambda}\big[(j-1)j+x\big]}{x(x+2)(x+4)\cdots\big(x+2(n-1)\big)} \ . \tag{10.1.4}$$

The reachable boxes of M_x comprise the whole of $S(\mathbb{Z}_+^2)$.

Note that when $x=\infty$, the "x factor" equals 1; hence $M_\infty = M_{(0;1)}$ is the Plancherel measure. Thus M_x are (again) deformations of the Plancherel measure.

From Theorem 10.1 we deduce

Theorem 10.5. *Interpret* μ *as an ordinary diagram. For any* $(k,\ell) \in S(\mathbb{Z}_+^2)$

(a)

$$\mathcal{SP}_{M_x}\big(T(k,\ell)=n\big) = \sum_{\mu\in SH'(k-1,\ell-1,n-1)} \frac{2^{n-\ell(\mu^+)} g^\mu g^{\mu^+}}{n!} \times \frac{\prod_{(i,j)\in\mu^+}\big[(j-1)j+x\big]}{x(x+2)\cdots\big(x+2(n-1)\big)} \ .$$

Hence, since (k,ℓ) *is reachable,*

(b)

$$\sum_{n=1}^{\infty} \sum_{\mu\in SH'(k-1,\ell-1,n-1)} \frac{2^{n-\ell(\mu^+)} g^\mu g^{\mu^+}}{n!} \times \frac{\prod_{(i,j)\in\mu^+}\big[(j-1)j+x\big]}{x(x+2)\cdots\big(x+2(n-1)\big)} = 1 \ . \tag{10.2.1}$$

Here $x \in \mathbb{R}^+ \cup \{\infty\}$, $\mu^+ = \mu^+(k,\ell)$, *and in "*$(i,j)\in\mu^+$*",* μ^+ *is interpreted as an ordinary (not shifted) diagram. Clearly, Theorem 10.4 is the case* $x=\infty$ *here.*

Theorems 10.3, 10.4 and 10.5 can be made more explicit by applying formula (10.1.1) for g^λ. This is done in Sections 11 and 12, where several special cases are treated.

Extensions to the probability of joint events. Let $j \geq i$, $s \geq r$ and $n > m$, and consider the probability $\mathcal{SP}_{_M}\big(T(i,j) = m \text{ and } T(r,s) = n\big)$, and similarly for more events. In complete analogy with Theorem 5.1, we now have

Theorem 10.6. *Let M be a central measure on $STab, (i,j),(r,s) \in S(\mathbb{Z}_+^2)$, $m < n \in \mathbb{Z}_+$. Then*

$$\mathcal{SP}_{_M}\big(T(i,j) = m \textit{ and } T(r,s) = n\big)$$
$$= \sum_{\mu \in SH'(i-1,j-1,m-1)} \ \sum_{\mu^+(i,j) \subseteq \lambda \in SH'(r-1,s-1,n-1)} g^\mu g^{\lambda/\mu^+(i,j)} \pi\big(\lambda^+(r,s)\big) \ .$$

Here $g^{\lambda/\mu^+(k,\ell)}$ is the number of standard tableaux of shifted skew shape $\lambda/\mu^+(k,\ell)$.

The "shifted" analogue of Remark 5.2 would lead to many new identities. For brevity, no examples of such identities are given here.

11. Applications IV: $M = M_{(\alpha;0)}$

To apply Theorem 10.3.b when $\gamma = 0$, we shall need an explicit formula for $P_\lambda(\alpha_1, \ldots, \alpha_m)$.

Fix $i \in \mathbb{Z}_+$ and let $(i,j) \in S(\mathbb{Z}_+^2)$. Let $m = i$ or $m = i+1$, with $\alpha_1 \geq \cdots \geq \alpha_m \geq 0$, $\alpha_1 + \cdots + \alpha_m = 1$, $\alpha_{m+1} = \gamma = 0$: m is the number of variables in $P_\lambda(\alpha)$. Let $M_{(\alpha;0)}$ be the corresponding ergodic measure. Then (i,j) is $M_{(\alpha;0)}$ reachable and Theorem 10.3.b applies.

Let $u \in \mathbb{Z}_+$, $\mu \in SH'(i-1, j-1, u-1)$ and $\mu^+ = \mu^+(i,j)$. Then the decomposition

$$P_{\mu^+}(\alpha) = s_{\mu^+ - \delta}(\alpha)\alpha_\delta(\alpha) \tag{11.1.1}$$

holds, provided $\ell(\mu^+)$ equals m or $m-1$ [M, III, 8, Ex. 2]. Here $\delta = (m-1, m-2, \ldots, 1, 0)$.

Claim. *Let $i \leq j \leq i+1$, $m = i$ or $m = i+1$. Then (11.1.1) holds. Indeed, if $\mu \in SH'(i-1, j-1, n-1)$, then $\ell(\mu^+) = i$ (equals m or $m-1$).*

Claim. *Let $j = i+2$, $m = i+1$. Then again (11.1.1) holds, since in that case, $\ell(\mu^+) = i$ or $\ell(\mu^+) = i+1$.*

However, if $j \geq i+3$, then $\ell(\mu^+) = i$ for some μ's and $\ell(\mu^+) = j-1 \geq i+2$ for other μ's in $SH'(i-1, j-1, n-1)$. Hence (11.1.1) might be false.

Therefore, explicit identities are deduced here from Theorem 10.3.b only for $(i,j) \in S(\mathbb{Z}_+^2)$, $i \leq j \leq i+2$.

Obviously, Theorem 10.3.b now implies

Theorem 11.1. *Let $(i,j) \in S(\mathbb{Z}_+^2)$.*

(a) Let $i \leq j \leq i+2$. Then

$$\sum_{u=1}^{\infty} \ \sum_{\mu \in SH'(i-1,j-1,u-1)} g^\mu s_{\mu^+-\delta}(\alpha_1, \ldots, \alpha_{i+1}) s_\delta(\alpha_1, \ldots, \alpha_{i+1}) = 1 \ .$$

Here $\alpha_1 \geq \cdots \geq \alpha_{i+1} \geq 0$, $\alpha_1 + \cdots + \alpha_{i+1} = 1$ and $\delta = (i, i-1, \ldots, 1, 0)$.

(b) Let $i \le j \le i+1$. Then, in addition,

$$\sum_{u=1}^{\infty} \sum_{\mu \in SH'(i-1,j-1,u-1)} g^\mu s_{\mu^+-\delta}(\alpha_1,\dots,\alpha_i) s_\delta(\alpha_1,\dots,\alpha_i) = 1 \ .$$

Here $\alpha_1 \ge \cdots \ge \alpha_i$, $\alpha_1 + \cdots + \alpha_i = 1$ and $\delta = (i-1,\dots,1,0)$.

Similarly to Section 6, specialize $\alpha_t = rq^{t-1}$, $1 \le t \le i+1$, where $0 < q < 1$ and $r = 1 + q + \cdots + q^i = \frac{1-q}{1-q^{i+1}}$. Thus, $r \to \frac{1}{i+1}$ when $q \to 1$. For $x \in \lambda$, denote $R_\lambda(x) = \frac{1-q^{i+1+c(x)}}{1-q^{h(x)}}$ (again, $c(x)$ is the "content", $h(x)$ the "hook number" of $x \in \lambda$). By Theorem 11.1a and [M, I.3, Ex. 1], for $1 \le i \le j \le i+2$ and $\delta = (i,\dots,1,0)$,

$$\begin{aligned} &\sum_{u=1}^{\infty} \left(\frac{1-q}{1-q^{i+1}}\right)^u \sum_{\mu \in SH'(i-1,j-1,u-1)} g^\mu q^{n(\mu^+)} \\ &\quad \times \left[\prod_{x \in \mu^+-\delta} R_{\mu^+-\delta}(x)\right] \left[\prod_{x\in\delta} R_\delta(x)\right] = 1 \ . \end{aligned} \tag{11.2.1}$$

Substitute $\alpha_1 = \cdots = \alpha_{i+1} = \frac{1}{i+1}$ in Theorem 11.1a (or, formally, let $q \to 1$ in (11.2.1)), then deduce that

$$\begin{aligned} &\sum_{u=1}^{\infty} \left(\frac{1}{i+1}\right)^u \sum_{\mu \in SH'(i-1,j-1,u-1)} g^\mu \\ &\quad \times \left[\prod_{x \in \mu^+-\delta} \frac{i+1+c(x)}{h(x)}\right] \left[\prod_{x\in\delta} \frac{i+1+c(x)}{h(x)}\right] = 1 \ , \end{aligned}$$

and since $\prod_{x\in\delta} \frac{i+1+c(x)}{h(x)} = 2^{\binom{i+1}{2}}$, deduce that:

If $1 \le i \le j \le i+1$ and $\delta = (i,\dots,1,0)$, then

$$\sum_{u=1}^{\infty} \left(\frac{1}{i+1}\right)^u \sum_{\mu \in SH'(i-1,j-1,u-1)} g^\mu \prod_{x \in \mu^+-\delta} \frac{i+1+c(x)}{h(x)} = 2^{-\binom{i+1}{2}} \ . \tag{11.2.2}$$

Clearly, (11.2.1) is a q analogue of (11.2.2).

Similar identities can be derived for $1 \le i \le j \le i+1$, with $m = i$ replacing $m = i+1$ in the above summands.

As before, (11.2.2) can be made more explicit by analyzing g^μ and $\frac{i+1+c(x)}{h(x)}$. Here is

<u>The case $i = j$, $m = i+1$</u>: $\mu \in SH'(i-1, j-1, u-1)$ implies

$$\mu = (\mu_1,\dots,\mu_{i-1}) \ , \qquad \mu_1 > \cdots > \mu_{i-1} \ge 2 \ , \qquad \mu^+(i,i) = (\mu_1,\dots,\mu_{i-1},1) \ .$$

Let $p_r = \mu_r - 2$, $1 \le r \le i-1$. Then (11.2.2) becomes

(11.2.3)
$$\sum_{p_1 > \cdots > p_{i-1} \ge 0} \left(\frac{1}{i+1}\right)^{|p|+2i-1} \frac{(|p|+2i-2)! V^2(p)}{\prod_{r=1}^{i-1} [p_r!(p_r+2)!] \prod_{1 \le r < s \le i-1} (p_r + p_s + 2)} \times \prod_{r=1}^{i-1} (i+2-r)_{p_r+r-i+1} = 2^{-\binom{i+1}{2}}$$

(because of the factor $V^2(p) = \prod_{r<s}(p_r - p_s)^2$, the condition "$p_1 > \cdots > p_{i-1}$" can be replaced by "$p_1 \ge \cdots \ge p_{i-1}$".

As before, $|p| = p_1 + \cdots + p_{i-1}$ and $(a)_n = a(a+1)\cdots(a+n-1)$.

The case $i = j = m$ is omitted for brevity.

Below we list a few (low) case examples of (11.2.1), (11.2.2) and (11.2.3).

$\underline{i = j = 2,\ m = 3}$: By (11.2.3)

$$\sum_{u=3}^{\infty} \left(\frac{1}{3}\right)^u \frac{(u-1)(u-2)}{2} = 2^{-3} ,$$

which can be verified directly. A q-analogue can be obtained from (11.2.1).

$\underline{i = j = 3,\ m = 4}$: By (11.2.3)

$$\sum_{u=6}^{\infty} \left(\frac{1}{4}\right)^u \sum_{b=0}^{\left[\frac{u-6}{2}\right]} \binom{u-2}{b+1} (b+1)(u-b-4)(u-b-3)(u-2b-5)^2 = \frac{3}{16} .$$

When $j = i+1 = 3$ and $j = i+1 = 4$, both cases $m = i$ and $m = i+1$ of Theorem 11.1 are calculated below.

$\underline{j = i+1 = 3}$: By the $m = i$-analogue of (11.2.1) we deduce the identity

$$\sum_{u=5}^{\infty} (u-3) \left(\frac{1}{1+q}\right)^{u-1} q^2 \frac{1-q^{u-4}}{1-q} = 1 \tag{11.3.1}$$

and by the $m = i$-analogue of (11.2.2) (or, formally, $q \to 1$ in (11.3.1)),

$$\sum_{u=5}^{\infty} (u-3)(u-4) \left(\frac{1}{2}\right)^{u-4} = 1$$

(which can be verified directly).

$\underline{j = i+1 = m = 3}$: Deduce that

$$\sum_{u=5}^{\infty} (u-3) \left(\frac{1-q}{1-q^3}\right)^u q \frac{(1-q^{u-2})(1-q^{u-4})}{(1-q)^2} \times \frac{(1-q^2)(1-q^4)}{(1-q)^2} = 1 , \tag{11.3.2}$$

and ($q \to 1$)

$$\sum_{u=5}^{\infty} (u-2)(u-3)(u-4) \left(\frac{1}{3}\right)^u = \frac{1}{8}$$

(can be verified directly).

$\underline{j = i+1 = 4,\ m = 3}$: Get

$$\sum_{u=9}^{\infty}\left(\frac{1-q}{1-q^3}\right)^u \sum_{b=1}^{\left[\frac{u-3}{2}\right]}\binom{u-1}{b}\frac{(u-2b-2)(b-1)(u-b-3)}{(u-2)(b+1)} \times q^{b+4}\frac{(1-q^{b-2})(1-q^{u-2b-2})(1-q^{u-b-4})}{(1-q)^2(1-q^2)} \times \frac{(1-q^2)(1-q^4)}{(1-q)^2} = 1 \tag{11.4.1}$$

and $(q \to 1)$

$$\sum_{u=9}^{\infty}\left(\frac{1}{3}\right)^u \sum_{b=1}^{\left[\frac{u-3}{2}\right]}\binom{u-1}{b} \times \frac{(u-2b-2)^2(b-1)(u-b-3)(b-2)(u-b-4)}{(u-2)(b+1)} = \frac{1}{4}\ . \tag{11.4.2}$$

$\underline{j = i+1 = m = 4}$: The q analogue is left for the reader. By (11.2.2) (which is "$q \to 1$" of that q analogue),

$$\sum_{u=9}^{\infty}\left(\frac{1}{4}\right)^u \sum_{b=3}^{\left[\frac{u-2}{2}\right]}\binom{u-1}{b+1} \times \frac{(u-2b-2)^2 b(b-1)(b-2)(u-b-2)(u-b-3)(u-b-4)}{(u-2)(u-b-1)} = \frac{3}{32}\ . \tag{11.4.3}$$

$\underline{j = i+2 = 4,\ m = i+1 = 3}$: Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. From (11.2.1) we deduce that

$$\sum_{u=7}^{\infty}\left(\frac{1-q}{1-q^3}\right)^u \binom{u-1}{2}(u-5)f(q,u) = 1\ , \tag{11.5.1}$$

where

$$f(q,u) = \frac{q^3(1-q^3)(1-q^2)(1-q^4)(1-q^{u-3})(1-q^{u-6})}{2(u-1)(1-q)^5} + \frac{q^5(u-6)(1-q^2)(1-q^4)(1-q^{u-5})(1-q^{u-7})}{3(u-2)(1-q)^4}$$

and by (11.2.2) (i.e., $q \to 1$)

$$\sum_{u=7}^{\infty}\left(\frac{1}{3}\right)^u \binom{u-1}{2}(u-5) \times \left[\frac{12(u-3)(u-6)}{(u-1)} + \frac{8(u-5)(u-6)(u-7)}{3(u-2)}\right] = 1\ . \tag{11.5.2}$$

12. Applications V: $M = M_x$

Applying Theorem 10.5.b, we calculate the general case $(k, \ell) = (m+1, m+1)$, m arbitrary, as well as a few other low cases.

Proposition 12.1. *Let $u, v \in \mathbb{C}$ satisfy $u + v = 1$ and $uv \in \mathbb{R}_+$. Let $m \in \mathbb{Z}_+$. Then*

(a)

$$\frac{1}{m!}\sum_{\mu_1,\dots,\mu_m=1}^{\infty}\frac{|\mu|!}{(\mu!)^2}\left[\prod_{r=1}^{m}\frac{\mu_r-1}{\mu_r+1}\right]\left[\prod_{1\le r<s\le m}\frac{\mu_r-\mu_s}{\mu_r+\mu_s}\right]^2 \times \frac{uv\left(\prod_{r=1}^{m}(u)_{\mu_r}\right)\left(\prod_{s=1}^{m}(v)_{\mu_s}\right)}{\left(\frac{uv}{2}\right)_{|\mu|+1}} = 2^{m+1}\ , \tag{12.1.1}$$

(b)

$$\frac{1}{m!}\sum_{\mu_1,\dots,\mu_m=1}^{\infty}\frac{2^{|\mu|+1}|\mu|!}{(\mu!)^2}\left[\prod_{r=1}^{m}\frac{\mu_r-1}{\mu_r+1}\right]\left[\prod_{1\le r<s\le m}\frac{\mu_r-\mu_s}{\mu_r+\mu_s}\right]^2 = 2^{m+1}\ . \tag{12.1.2}$$

Here $|\mu| = \mu_1+\cdots+\mu_m$ *and* $\mu! = \mu_1!\cdots\mu_m!$.

Proof. When $x=\infty$ the x-factor in (10.2.1) equals 1, and part (b) will follow from the values of $\frac{2^{n-\ell(\mu^+)}g^\mu g^{\mu^+}}{n!}$ given below ($n=|\mu|+1$).

To prove (a), note first that in (10.2.1) the summation $\sum_n \sum_{\mu\in SH'(m,m,n-1)}$ can be replaced by $\sum_{\mu_1>\cdots>\mu_m\ge 2}$, $\mu=(\mu_1,\dots,\mu_m)$, $\mu^+=(\mu_1,\dots,\mu_m,1)$, $\ell(\mu^+)=m+1$ and $|\mu|=\mu_1+\cdots+\mu_m=n-1$.

Denote $uv=2x$ and let $2x$ replace x in (10.2.1); then deduce that

$$\sum_{\mu_1>\cdots>\mu_m\ge 2}\frac{2^{-(m+1)}g^\mu g^{\mu^+}}{(|\mu|+1)!}\times\frac{\left(\prod_{r=1}^{m}\ \prod_{s=1}^{\mu_r}\left[(s-1)s+2x\right]\right)2x}{(x)_{|\mu|+1}} = 1\ .$$

By (10.1.1), for such $\mu=(\mu_1,\dots,\mu_m)$

$$\frac{g^\mu g^{\mu^+}}{(|\mu|+1)!} = \frac{|\mu|!}{(\mu!)^2}\left[\prod_{r=1}^{m}\frac{\mu_r-1}{\mu_r+1}\right]\left[\prod_{1\le r<s\le m}\frac{\mu_r-\mu_s}{\mu_r+\mu_s}\right]^2\ .$$

By symmetry and because of the factors μ_r-1 and $\mu_r-\mu_s$, the summation $\sum_{\mu_1>\cdots>\mu_m\ge 2}$, in both (a) and (b), can now be replaced by

$$\frac{1}{m!}\sum_{\mu_1,\dots,\mu_m=1}^{\infty}\ .$$

This proves (b).

Let $c=s-1$. Since $u+v=1$ and $uv=2x$, $(c+u)(c+v)=(s-1)s+2x$. Thus

$$\prod_{s=1}^{\mu_r}\left[(s-1)s+2x\right] = \prod_{c=0}^{\mu_r-1}\left[(c+u)(c+v)\right] = (u)_{\mu_r}(v)_{\mu_r}\ .$$

This completes the proof of (a). □

Example 12.1. Let $m=1$ in the above. Then (12.1.2) yields the identity

$$\sum_{n=3}^{\infty}\frac{n-2}{n!}2^n = 4\ ,$$

which is obvious. Here (12.1.1) gives the identity

$$\sum_{n=3}^{\infty}\frac{(n-2)}{n!}\frac{uv(u)_{n-1}(v)_{n-1}}{\left(\frac{uv}{2}\right)_n} = 4\ , \tag{12.2.1}$$

$u+v=1$, $uv\in\mathbb{R}_+$, which is not that obvious.

The following argument, due to G. Olshanski, shows that (12.2.1) follows from the Gauss summation formula for ${}_2F_1$:

Let q be the left hand side of (12.2.1); then

$$\begin{aligned} q &= \sum_{n=3}^{\infty}\frac{1}{(n-1)!}\frac{(u)_{n-1}(v)_{n-1}uv}{\left(\frac{uv}{2}\right)_n} - 2\sum_{n=3}^{\infty}\frac{1}{n!}\frac{(u)_{n-1}(v)_{n-1}uv}{\left(\frac{uv}{2}\right)_n} \\ &= \frac{uv}{\left(\frac{uv}{2}\right)}\sum_{k=0}^{\infty}\frac{(u)_k(v)_k}{k!\left(\frac{uv}{2}+1\right)_i} - \frac{2uv}{(u-1)(v-1)}\sum_{n=3}^{\infty}\frac{(u-1)_n(v-1)_n}{n!\left(\frac{uv}{2}\right)_n}\ . \end{aligned}$$

Note that $(u-1)(v-1)=uv-(u+v)+1=uv$. The proof now follows from the Gauss summation formula for ${}_2F_1$ (compare with (9.1.1)).

Example 12.3. The box $(k,\ell)=(2,3)$. By Theorem 10.4.b or 10.5.b with $x=\infty$,

$$\sum_{n=5}^{\infty}\frac{(n-1)(n-3)(n-4)2^n}{n!}=8$$

(easy to verify directly). Similarly to Example 12.2, deduce here the identity

$$\sum_{n=5}^{\infty}\frac{(n-1)(n-3)(n-4)}{n!}\frac{(u)_{n-2}(v)_{n-2}}{\left(\frac{uv}{2}+2\right)_{n-2}}=2 \tag{12.3.1}$$

for $u,v\in\mathbb{C}$ satisfying $u+v=1$ and $uv\in\mathbb{R}_+$.

Similarly to Example 12.2, it is now possible to derive (12.3.1) from the Gauss summation formula for ${}_2F_1$ (expand $\frac{(n-1)(n-3)(n-4)}{n!}=\frac{1}{(n-3)!}-\frac{5}{(n-2)!}+\frac{12}{(n-1)!}-\frac{12}{n!}$, then proceed as in Example 12.2).

Remark. The box $(1,\ell)$: Since $SH'(0,\ell-1,n-1)$ has $\le\binom{\ell}{2}$ elements, (10.2.1) implies that the sum of a certain number $\le\binom{\ell}{2}$ of rational functions (in x) equals 1. Each such identity can easily be verified algebraically. For example,
The box $(1,3)$ yields the trivial identity $\frac{2(x+6)}{3(x+4)}+\frac{x}{3(x+4)}=1$.

References

[BR] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. Math. **64** (1987), 118–175. MR **88i:**20006

[B] A.M. Borodin, *Multiplicative central measure on Schur graph*, Zap. Nauc. Sem. POMI **240** (199), 44–52 (Russian). English translation in *J. Math. Sci.* **96** (1999), 3472–3477. MR **2000g:**43001

[HH] P. Hoffman and J.F. Humphreys, *Projective Representation of the Symmetric Groups*, Oxford Mathematical Monographs, 1992. MR **94f:**20047

[I] V.N Ivanov, *Dimensions of skew shifted Young diagrams and projective characters of the infinite symmetric group*, Zap. Nauc. Sem. POMI **240** (1997), 116–136 (Russian). English translation in *J. Math. Sci.* **96** (1999), 3517–3530. MR **2000j:**05124

[J] G.D. James, *The Representation Theory of the Symmetric Groups*, Springer Lecture Notes in Mathematics **682** (1978). MR **80g:**20019

[K] S. Kerov, *Anisotropic Young diagrams and Jack symmetric functions*, Funct. Anal. Appl. **34** (2000), 41–51. CMP 2000:12

[KOV] S. Kerov, G. Olshanski and A. Vershik, *Harmonic analysis on the infinite symmetric group. A deformation of the regular representation*, C.R. Acad. Sci. Paris, Ser. I, Math. **316** (8) (1993), 773–778. MR **94e:**20019

[M] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2^{nd} Edition, Oxford, 1995. MR **96h:**05207

[M1] I.G. Macdonald, Private letter.

[N] M. Nazarov, *Projective representation of the infinite symmetric group*, Representation Theory and Dynamical Systems (A. M. Vershik, editor), Advances in Soviet Math., vol. 9, Amer. Math. Soc., Providence, RI, 1992, pp. 115–130. MR **93j:**20032

[O] G. Olshanski, *Point processes and the infinite symmetric group. Part I: The general formalism and the density function*, Preprint, math.RT/9804086, 1998. To appear.

[R] A. Regev, *Asymptotics of degrees of some S_n-subregular representations*, Isr. J. Math. **113** (1999), 15-28. MR **2001b:**05226

[S] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare substitutionen*, J. Reine Angew. Math. **139** (1911), 155–250.

[T] E. Thoma, *Die unzerlegbaren, positive-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe*, Math. Zeitschr. **85** (1964), 40–61. MR **30:**3382

[VK1] A.M. Vershik and S.V. Kerov, *Asymptotics of the Plancherel measure of the symmetric group and the limit form of Young tableaux*, Dokl. Akad. Nauk SSSR **233** (1977), 1024–1027; English transl., Soviet Math. Dokl. **18** (1977), 527–531. MR **58:**562

[VK2] A.M. Vershik and S. Kerov, *Asymptotics of maximal and typical dimensions of irreducible representations of the symmetric group*, Funkts. Anal. Prilozhen **19** (1985), 25–36; English transl., Funct. Anal. Appl. **19** (1985), 21–31. MR **86k:**11051

[VK3] A.M Vershik and S. Kerov, *Asymptotic theory of characters of the symmetric group*, Funct. Anal. Appl. **15** (1981), 246–255. MR **84a:**22016

[Z] D. Zeilberger, Shalosh, Private communication.

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