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SECOND CLASS PARTICLES AS MICROSCOPIC CHARACTERISTICS IN TOTALLY ASYMMETRIC NEAREST-NEIGHBOR K-EXCLUSION PROCESSES

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Abstract. We prove laws of large numbers for a second class particle in one-dimensional totally asymmetric K-exclusion processes, under hydrodynamic Euler scaling. The assumption required is that initially the ambient particle configuration converges to a limiting profile. The macroscopic trajectories of second class particles are characteristics and shocks of the conservation law of the particle density. The proof uses a variational representation of a second class particle, to overcome the problem of lack of information about invariant distributions. But we cannot rule out the possibility that the flux function of the conservation law may be neither differentiable nor strictly concave. To give a complete picture we discuss the construction, uniqueness, and other properties of the weak solution that the particle density obeys.

1. Introduction

The totally asymmetric nearest-neighbor K-exclusion process is an interacting system of indistinguishable particles on the one-dimensional integer lattice $\mathbf{Z}$. The admissible particle configurations are those that have at most K particles at each site. Independently at each site and at constant exponential rate one, one particle is moved to the next site on the right, provided the K-exclusion rule is not violated. The special case $K = 1$ is known as the totally asymmetric simple exclusion process, and the case $K = \infty$ is a member of the family of zero-range processes.

For general K, a hydrodynamic limit for totally asymmetric K-exclusion was proved in [24]. This is a law of large numbers according to which the empirical particle density converges to a weak solution of a nonlinear conservation law $\rho_t + f(\rho)_x = 0$, under the appropriate scaling of space and time. In the present paper we prove laws of large numbers for a second class particle added to the process, under the hydrodynamic scaling. A *second class particle* is an extra particle in the system that yields right of way to the regular, first class particles. The second class particle can jump only if there is no first class particle at its site who can jump, and a second class particle has to jump backward if first class particles fill its site.

Contrary to simple exclusion and the zero-range processes, the invariant distributions of K-exclusion for $1 < K < \infty$ are unknown, and even a complete existence proof is not yet available. The methods used to study hydrodynamics and second

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class particles for asymmetric simple exclusion and zero-range processes cannot be applied to general K-exclusion. For the hydrodynamic limit this problem was overcome in [24], and for the second class particle we do so in the present paper.

Due to the lack of information about invariant distributions, we do not presently know whether the flux function f of the conservation law is strictly concave or differentiable. Consequently we cannot use the Lax-Oleinik formula to define an entropy solution, or one-sided bounds on jumps as an entropy criterion. This shortcoming affects the macroscopic description of the trajectories of the second class particle. For this reason we address properties of the conservation law $\rho_t + f(\rho)_x = 0$ in the case where f is assumed only concave, but neither strictly concave nor differentiable. We prove that in the hydrodynamic limit the K-exclusion process chooses the unique weak solution $\rho(x,t)$ that maximizes current across any fixed position. In the cases $K = 1$ and $K = \infty$, where we know f is differentiable and strictly concave, this is the unique entropy solution defined by the Lax-Oleinik formula.

We turn to an overview of earlier results. Second class particles in asymmetric exclusion processes have been studied for several reasons. In a certain technically precise sense, the second class particle marks the location of a shock at the microscopic particle level. These ideas were developed by Ferrari, Fontes, Kipnis and Saada [5], [7], [10]. At this same time, Ferrari [5] discovered that in the hydrodynamic limit a second class particle follows the characteristics of the macroscopic equation. The connection between second class particles and characteristics was established in more general settings by Rezakhanlou [18] and Ferrari and Kipnis [9]. The second class particle also enters naturally in the study of the fluctuations of the current [7].

Another strain of work, technically related to the investigation of microscopic shocks, concerns the invariant distributions of the exclusion process as seen from a second class particle, and the invariant distributions of the two-species process of first and second class particles. This program culminated in the complete description of these measures by Derrida et al. [3] and Ferrari et al. [8]. These and other results, such as the central limit behavior of the second class particle, are discussed in Part III of [15] and in the notes of Chapter 9 in [12]. As mentioned, these earlier results are for systems whose invariant distributions are explicitly known. The proofs typically use precise and deep knowledge about those invariant measures.

Of the papers mentioned above, Ferrari's [5] and Rezakhanlou's [18] are the closest predecessors of our paper. Both derived laws of large numbers for the second class particle. The limiting trajectory is a characteristic or a shock of the conservation law of the particle density. Ferrari proved strong laws in an asymmetric simple exclusion process, mainly in these settings: the initial particle distribution is either an i.i.d. product measure, or a product measure that represents a shock profile that is constant on either side of the origin (this is known as the Riemann problem). Ferrari's proofs rely on couplings of processes of several species of particles, and knowledge of invariant distributions.

Rezakhanlou [18] proved L^1-laws of large numbers for the second class particle in asymmetric simple exclusion and in a totally asymmetric zero-range process. He assumed that initial distributions are product measures, and the initial macroscopic profiles are bounded integrable functions on $\mathbf{R}$, with some additional technical assumptions. His proof derives the limit of a single second class particle from the limit of a macroscopically visible interval of second class particles, and uses the hydrodynamic limits for exclusion and zero-range processes [17]. The invariant distributions of the processes enter the proofs through the macroscopic equations, which are known explicitly precisely because the invariant distributions are known.

Our approach is different from the earlier ones. We use a representation of the second class particle in the variational coupling method. To preview this, let us switch from the exclusion process to the associated marching soldiers process (in other words, consider the interface process instead of the increments process). In this approach a process $z(i,t)$ from a general initial interface is coupled with auxiliary processes $w^k(i,t)$ that evolve from translated wedge-shaped initial profiles. The coupling is arranged so that $z(i,t)$ is the envelope of the auxiliary processes:

$$z(i,t) = \sup_{k \in \mathbf{Z}} w^k(i-k,t). \tag{1.1}$$

Let $X(t)$ denote the position of a second class particle added to the exclusion process $\eta(i,t) = z(i,t) - z(i-1,t)$. The new tool is this variational expression:

$$X(t) = \inf\{i : z(i,t) = w^k(i-k,t) \text{ for some } k \geq X(0)\}. \tag{1.2}$$

Variational formula (1.1) is a microscopic analogue of the Hopf-Lax formula of viscosity solutions of Hamilton-Jacobi equations. In this same vein, (1.2) corresponds to the definition of forward characteristics through the Hopf-Lax formula. This justifies regarding the second class particle as a microscopic analogue of the characteristic, as expressed in the title of the paper.

Variational formulas (1.1)–(1.2) permit us to work directly on the paths of the process, and we do not need a particular form for the initial particle distributions. We need to assume that the initial particle profile and the initial position of the second class particle both converge under the appropriate space scaling. The result is a law of large numbers for the second class particle. The theorems give either convergence in probability or almost surely, depending on which kind is assumed initially. The limit is a characteristic or a shock of the macrosopic partial differential equation.

Then we run into the difficulty mentioned above. For $2 \leq K < \infty$ (cases other than exclusion or zero-range) we do not have sufficient control of the regularity of the flux f of the conservation law. For this reason we cannot assert that macroscopically the second class particle follows the solution of $\dot{x} = f'(\rho(x,t))$, where $\rho(x,t)$ is the macroscopic particle density. Following [18], we use the Hopf-Lax formula to define the macroscopic trajectory of the second class particle.

If the macroscopic characteristics are unique (there are no rarefaction fans) we do not need the assumption of an initial law of large numbers for the second class particle. In this case the second class particle follows, in the hydrodynamic scale, the macroscopically defined characteristic that emanates from its random initial position.

As a by-product of our proof technique, the laws of large numbers of [5] and [18] are improved to permit general initial distributions, instead of only product distributions. In another sense the results of [5] and [18] remain more general than ours: their results cover an asymmetric simple exclusion process where particles can jump both right and left, while our results are restricted to *totally* asymmetric processes where particles jump only to the right. This is because a way to apply the variational coupling approach without total asymmetry has yet to be discovered.

While the problem of invariant distributions of K-exclusion complicates the study of second class particles, this problem also partly motivates the present work on the second class particle. The reason is that sufficiently sharp bounds on the speed of a second class particle would enable us to draw conclusions about the regularity of the flux f. This in turn would be helpful for the existence proof of invariant distributions.

Overview of the paper. In Section 2 we discuss the results. Section 2.1 reviews the hydrodynamics of totally asymmetric K-exclusion and zero-range processes (Theorem 1). We state a uniqueness criterion for the weak solution chosen by the particle system, valid for a concave flux that is not necessarily differentiable or strictly concave (Theorem 2). Section 2.2 states the laws of large numbers for the second class particle (Theorems 3 and 4). Section 2.3 comments on the various assumptions and conclusions.

In Section 2.4 we describe a generalization of the K-exclusion and zero-range processes to which our results apply. In this model particles jump in batches of varying sizes from one site to the next. Such a generalization is natural from a queueing perspective, where the particles are customers moving through a sequence of service stations.

In Section 3 we discuss the role of the regularity of f. If the graph of f has a line segment, the Lax-Oleinik formula cannot be used, and the entropy criterion is necessarily weakened to permit both upward and downward jumps across the densities on the line segment (Section 3.1). Without differentiability of f we loose some good properties of characteristics (Section 3.2). In Section 3.3 we prove Theorem 2, the uniqueness result.

Section 4 reviews the essentials of the variational coupling approach to exclusion and zero-range processes from [22], [23], and [24], and develops the formula for the second class particle. Section 5 proves the laws of large numbers for the second class particle. The proofs are an interplay of the particle level properties from Section 4 and the properties of the macroscopic characteristics.

Notation. The set of natural numbers is $\mathbf{N} = \{1, 2, 3, \dots\}$, and the set of nonnegative integers $\mathbf{Z}_+ = \{0, 1, 2, 3, \dots\}$.

2. Results

2.1. Hydrodynamic limits and entropy solutions. We start by reviewing the hydrodynamic limit of totally asymmetric K-exclusion. Throughout the paper, the parameter K is a fixed positive finite or infinite integer $K \in \mathbf{N} \cup \{\infty\}$. The process consists of indistinguishable particles that move on the one-dimensional integer lattice $\mathbf{Z}$, subject to the restriction that a site $i \in \mathbf{Z}$ can be occupied by at most K particles. If $K = \infty$ this constraint is absent. The state space of the process is $\mathcal{S} = \{0, 1, 2, \dots, K\}^{\mathbf{Z}}$ for finite K, and $\mathcal{S} = \mathbf{Z}_+^{\mathbf{Z}}$ if $K = \infty$. A generic element of $\mathcal{S}$ is $\eta = (\eta(i) : i \in \mathbf{Z})$, where $\eta(i)$ denotes the number of particles present at site i.

The dynamics is such that at exponential rate 1, one particle moves from site i to $i+1$, provided there is at least one particle at i and at most $K-1$ particles at $i+1$. The jump attempts happen independently at all sites i.

To realize this dynamic, let $\{D_i : i \in \mathbf{Z}\}$ denote a collection of mutually independent rate 1 Poisson point processes on the time axis $(0, \infty)$. We usually think of D_i as the random set of jump times (or *epochs*, to borrow Feller's term) rather than as the corresponding random step function. At each epoch of D_i a jump from site i to site $i+1$ is attempted, and successfully executed if the rules permit. The state of the process at time t is the configuration $\eta(t) = (\eta(i,t) : i \in \mathbf{Z})$ of occupation numbers.

The dynamics can be represented by the generator L that acts on bounded cylinder functions f on $\mathcal{S}$:

$$Lf(\eta) = \sum_{i\in\mathbf{Z}} \mathbf{1}\{\eta(i) \geq 1, \eta(i+1) \leq K-1\}[f(\eta^{i,i+1}) - f(\eta)]. \tag{2.1}$$

Here $\eta^{i,i+1}$ is the configuration that results from the jump of a single particle from site i to site $i+1$:

$$\eta^{i,i+1}(j) = \begin{cases} \eta(i)-1, & j=i, \\ \eta(i+1)+1, & j=i+1, \\ \eta(j), & j \neq i, i+1. \end{cases}$$

Special cases. (a) When $K=1$ this process is known as the totally asymmetric simple exclusion process (TASEP). The Bernoulli probability measures $\{\nu^\rho\}$ on $\mathcal{S}$, indexed by density $\rho \in [0,1]$, are invariant for the process. These are defined by the condition

$$\nu^\rho\{\eta(i_1) = \eta(i_2) = \cdots = \eta(i_m) = 1\} = \rho^m$$

for any finite set $\{i_1, \dots, i_m\}$ of sites.

(b) If $K=\infty$ this process is a member of the family of totally asymmetric zero-range processes (TAZRP). The i.i.d. geometric probability measures $\{\nu^\rho\}$, $\rho \in [0,\infty)$, are invariant. These are given by

$$\nu^\rho\{\eta(i_1) = k_1, \eta(i_2) = k_2, \dots, \eta(i_m) = k_m\} = \prod_{s=1}^{m} \frac{1}{1+\rho}\left(\frac{\rho}{1+\rho}\right)^{k_s} \tag{2.2}$$

for distinct sites $i_1, \dots, i_m$ and arbitrary integers $k_1, \dots, k_m \in \mathbf{Z}_+$.

(c) For $2 \leq K < \infty$ we call this the totally asymmetric K-exclusion process. For this case the existence of spatially ergodic invariant measures for all densities has not been proved. Note, however, that for the *symmetric* K-exclusion there are reversible product measures. This can be easily checked through detailed balance. But this does not help in our present study of the asymmetric process.

The basic result about the large-scale behavior of these processes is the following hydrodynamic scaling limit. Let ρ_0 be a bounded measurable function on $\mathbf{R}$ such that $0 \leq \rho_0(x) \leq K$. Assume given a probability space on which are defined a sequence of initial configurations $(\eta_n(i,0) : i \in \mathbf{Z})$, $n \in \mathbf{N}$, and the Poisson jump time processes $\{D_i\}$. The processes $\{\eta_n(\cdot)\}$ are constructed on this probability space by the standard graphical method. Assume that at time zero the following strong law of large numbers is valid:

$$\text{For all } -\infty < a < b < \infty, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=[na]+1}^{[nb]} \eta_n(i,0) = \int_a^b \rho_0(x)dx \quad \text{a.s.} \tag{2.3}$$

Theorem 1. *Under assumption* (2.3) *the following strong law of large numbers holds. There exists a deterministic measurable function $\rho(x,t)$ such that, for all $t>0$ and all $-\infty < a < b < \infty$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=[na]+1}^{[nb]} \eta_n(i,nt) = \int_a^b \rho(x,t)dx \quad \text{a.s.} \tag{2.4}$$

The limiting density $\rho(x,t)$ is a weak solution of the Cauchy problem

$$\rho_t + f(\rho)_x = 0\,, \quad \rho(x,0) = \rho_0(x). \tag{2.5}$$

The flux function f of the conservation law (2.5) *is a fixed, deterministic concave function on $[0,K]$ that depends on K.*

Recall this standard definition: a locally bounded measurable function $\lambda(x,t)$ is a *weak solution* of (2.5) if for all compactly supported and continuously differentiable test functions ϕ on $\mathbf{R}\times[0,\infty)$,

$$\int_0^\infty \int_{\mathbf{R}} [\lambda(x,t)\phi_t(x,t) + f(\lambda(x,t))\phi_x(x,t)]dxdt + \int_{\mathbf{R}} \rho_0(x)\phi(x,0)dx = 0. \tag{2.6}$$

The limit (2.3) is the only assumption needed for Theorem 1 in the following strong sense: The joint distribution of the initially defined random variables $\{(\eta_n(i,0) : i \in \mathbf{Z}) : n \in \mathbf{N}\}$ and $\{D_i : i \in \mathbf{Z}\}$ may be arbitrary. As long as the marginal distribution of $\eta_n(i,0)$ satisfies (2.3), and marginally $\{D_i\}$ are i.i.d. rate 1 Poisson point processes, Theorem 1 is valid. A second point is that if (2.3) holds in probability, then the conclusion (2.5) also holds in probability.

For exclusion ($K=1$) and zero-range ($K=\infty$) Theorem 1 has gone through many stages of generalization, beginning with the seminal work of Rost [20]. The entropy technique of Rezakhanlou [17] proves it for multidimensional exclusion and zero-range processes with initial product distributions, without requiring total asymmetry. The variational coupling method of [22] and [23] permits general initial distributions for the one-dimensional, totally asymmetric case. For $2 \le K < \infty$ Theorem 1 was first proved in [24].

At the particle level the flux $f(\rho)$ is the average current at density ρ. If a family $\{\nu^\rho\}_{\rho\in[0,K]}$ of spatially ergodic invariant distributions is known, then $f(\rho)$ is the ν^ρ-expectation of the rate factor in the generator (2.1):

$$f(\rho) = \nu^\rho\{\eta(i) > 0, \eta(i+1) < K\}.$$

So for the cases where invariant distributions are known, we get $f(\rho) = \rho(1-\rho)$ for $0 \le \rho \le 1$ for exclusion ($K=1$), and $f(\rho) = \rho/(1+\rho)$ for $0 \le \rho < \infty$ for zero-range ($K=\infty$).

For $1 < K < \infty$ we know the concavity and continuity of f, the symmetry $f(\rho) = f(K-\rho)$, and these bounds:

$$\min\{\rho(1-\rho)\,,\, 1/4\} \le f(\rho) \le \frac{\rho}{1+\rho} \text{ for } 0 \le \rho \le K/2. \tag{2.7}$$

As suggested by the cases $K=1$ and ∞, we would expect f to be C^1 and strictly concave for all K, in other words, for its graph to have no corners or line segments. We do not know this at present.

A basic fact about nonlinear conservation laws such as (2.5) is that they have multiple weak solutions. To pick the unique physically relevant solution, the initial value problem (2.5) is supplemented with additional conditions that determine the so-called *entropy solution.* The well-known uniqueness theorems based on the work of Kruzkov, Lax, Oleinik and others (see [1], [4], [13], [14], [16] and their references) require either strict concavity (or convexity), or continuous differentiability of f. Except for $K=1$ and ∞, we do not have such information about f. So for $1 < K < \infty$ we complement Theorem 1 with a uniqueness criterion that does not assume any regularity on f beyond continuity and concavity. First we explain how the solution $\rho(x,t)$ that the particle system chooses in Theorem 1 is constructed.

Define an antiderivative u_0 of ρ_0 by

$$u_0(0) = 0 \text{ and } u_0(b) - u_0(a) = \int_a^b \rho_0(x)dx \quad \text{for all } a < b.$$

Let g be the nonincreasing, nonnegative convex function on $\mathbf{R}$ defined by

$$g(x) = \sup_{0 \le \rho \le K} \{f(\rho) - x\rho\}. \tag{2.8}$$

Symmetry of f and bounds (2.7) imply that $f'(0) = 1$ and $f'(K) = -1$, the existence of the derivatives being part of the conclusion. It follows that g satisfies

$$g(x) = 0 \text{ for } x \ge 1 \text{ and } g(x) = -Kx \text{ for } x \le -1. \tag{2.9}$$

Then for $x \in \mathbf{R}$ set $u(x, 0) = u_0(x)$, and for $t > 0$

$$u(x,t) = \sup_{y \in \mathbf{R}} \left\{ u_0(y) - tg\left(\frac{x-y}{t}\right)\right\}. \tag{2.10}$$

The supremum is attained at some $y \in [x - t, x + t]$. The function u is uniformly Lipschitz on $\mathbf{R} \times [0, \infty)$, nonincreasing in t and nondecreasing in x. (2.10) is known as the *Hopf-Lax formula*. It defines $u(x, t)$ as a *viscosity solution* of the Hamilton-Jacobi equation (Ch. 10 in [4])

$$u_t + f(u_x) = 0\,, \qquad u(x, 0) = u_0(x). \tag{2.11}$$

The viscosity solution is unique by Theorem 2.1 of [11] that applies to (2.11) because f is continuous and u is uniformly continuous. Let

$$\rho(x,t) = \frac{\partial}{\partial x} u(x,t) \tag{2.12}$$

be the a.e. defined x-derivative. By the proof of Thm. 3.4.2 in [4], $\rho(x, t)$ is a weak solution of (2.5) in the integral sense (2.6). The solution $\rho(x, t)$ defined by (2.12) is the one chosen by the particle system in Theorem 1.

To complement the characterization of $\rho(x, t)$ by (2.12) we give a uniqueness criterion that does not appeal to the Hamilton-Jacobi equation (2.11). As already emphasized, the point is that this criterion does not require strict concavity or differentiability of f, only concavity.

Theorem 2. *Among the weak solutions of* (2.5)*, $\rho(x, t)$ defined by* (2.12) *is characterized by maximal current over time. More precisely, suppose $\lambda(x, t)$ is a nonnegative, locally bounded measurable function on $\mathbf{R} \times (0, \infty)$ that satisfies the integral criterion* (2.6)*. Fix $t > 0$. Then for almost all $x \in \mathbf{R}$,*

$$\int_0^t f(\lambda(x,s))ds \le \int_0^t f(\rho(x,s))ds. \tag{2.13}$$

Conversely, if equality holds in (2.13) *for almost all (x, t), then $\lambda(x, t) = \rho(x, t)$ for almost all (x, t).*

Thus the particle system chooses the weak solution that transports the most material. Theorem 2 is proved in Section 3.3. The system maximizes the current because f is concave. With a convex flux the current would be minimized, as in the stick model version of Hammersley's process [21].

In cases $K = 1$ and ∞ where we know f is differentiable and strictly concave, $\rho(x, t)$ can be equivalently defined by the Lax-Oleinik formula, or it can be characterized as the unique weak solution that satisfies $\rho(x-, t) \le \rho(x+, t)$ so that all the discontinuities are upward jumps. For $2 \le K < \infty$, $\rho(x,t)$ cannot be defined by the Lax-Oleinik formula, and we cannot rule out discontinuities of the wrong kind. More on this in Sections 3.1–3.2.

For $K = 1$ the function g in (2.8)–(2.9) is given by $g(x) = (1/4)(1-x)^2$ for $-1 \le x \le 1$. For $K = \infty$, $g(x) = (1-\sqrt{x}\,)^2$ for $0 \le x \le 1$, $g \equiv 0$ on $(1,\infty)$ and we can take $g \equiv \infty$ on $(-\infty, 0)$. The supremum in (2.10) is attained at some $y \in [x-t, x]$. Now there is no finite uniform bound K on $\rho(x,t)$, but the boundedness assumption on ρ_0 guarantees that u is again Lipschitz on $\mathbf{R} \times [0,\infty)$.

2.2. Second class particles and characteristics. In this section we state the laws of large numbers for the second class particle. The representation of the second class particle in the variational coupling is in Section 4.2.

Let $X(t)$ denote the position of a single second class particle added to the process $\eta(t)$. $X(t)$ is rigorously defined as follows. Assume given a probability space on which are defined these random variables: the Poisson jump time processes $\{D_i\}$, an initial particle configuration $(\eta(i,0) : i \in \mathbf{Z})$, and an initial location $X(0)$. Define a second initial configuration $(\widetilde{\eta}(i,0) : i \in \mathbf{Z})$ that differs from η only at site $X(0)$: $\widetilde{\eta}(X(0),0) = \eta(X(0),0) + 1$ and $\widetilde{\eta}(i,0) = \eta(i,0)$ for $i \ne X(0)$. [Of course we must have $\eta(X(0),0) \le K-1$ a.s. for this to be possible.]

Run the processes $\eta(t)$ and $\widetilde{\eta}(t)$ so that they read the same Poisson jump time processes $\{D_i\}$. This is known as the *basic coupling* of $\eta(t)$ and $\widetilde{\eta}(t)$. There is always a unique site $X(t)$ at which the two processes differ: $\widetilde{\eta}(X(t),t) = \eta(X(t),t)+1$ and $\widetilde{\eta}(i,t) = \eta(i,t)$ for $i \ne X(t)$. This defines the position $X(t)$ of the second class particle.

The way in which the second class particle yields to first class particles becomes evident when we consider the two types of transitions that can happen to it. Suppose $X(\tau-) = i$.

(i) Suppose τ is a jump epoch for Poisson process D_i. The second class particle can jump from i to $i+1$ if there is space at site $i+1$, but only if there is no first class particle to jump. In other words, if $\eta(i,\tau-) = 0$ and $\eta(i+1,\tau-) < K$, then $X(\tau) = i+1$. Otherwise the second class particle does not move and $X(\tau) = X(\tau-) = i$.

(ii) Suppose τ is a jump epoch for D_{i-1}. The second class particle jumps from i to $i-1$ if first class particles fill site i. In other words, if $\eta(i-1,\tau-) \ge 1$ and $\eta(i,\tau-) = K-1$, then $X(\tau) = i-1$.

If $K = \infty$ the second class particle never jumps left and only the first kind of transition happens to it.

The definition in terms of η and $\widetilde{\eta}$ generalizes naturally to a two-species process of first and second class particles. But in this paper we consider only a single second class particle added to the process.

Next we discuss the characteristics of the conservation law (2.5). Since existence of f' is a problem, we cannot use the well-known description of characteristics as generalized solutions of the ordinary differential equation $dx/dt = f'(\rho(x,t))$ [2]. As in [18], we use the Hopf-Lax formula to define characteristics.

Once $u(x,t)$ is defined by (2.10), a semigroup property is in force: for all $0 \le s < t$ and $x \in \mathbf{R}$,

$$u(x,t) = \sup_{y\in\mathbf{R}} \left\{ u(y,s) - (t-s)\, g\!\left(\frac{x-y}{t-s}\right) \right\}. \tag{2.15}$$

The supremum is assumed at some point $y \in [x-(t-s), x+(t-s)]$. (If $K=\infty$ this range would actually be $[x-(t-s), x]$.) By continuity the supremum in (2.15) is attained at the least and largest maximizers, defined by

$$y^-(x;s,t) = \inf\left\{ y \ge x-(t-s) : u(x,t) = u(y,s) - (t-s)g\left(\frac{x-y}{t-s}\right)\right\} \tag{2.16}$$

and

$$y^+(x;s,t) = \sup\left\{ y \le x+(t-s) : u(x,t) = u(y,s) - (t-s)g\left(\frac{x-y}{t-s}\right)\right\}. \tag{2.17}$$

Abbreviate $y^\pm(x,t) = y^\pm(x;0,t)$. As functions of x for fixed $0 \le s < t$, $y^+(x;s,t)$ is right-continuous, $y^-(x;s,t)$ is left-continuous, both are nondecreasing, and trivially from the definition $\lim_{x\to-\infty} y^\pm(x;s,t) = -\infty$ and $\lim_{x\to\infty} y^\pm(x;s,t) = \infty$. See comment 2.3.1 below for justification of the restriction $x-(t-s) \le y \le x+(t-s)$ in the definitions of $y^\pm(x;s,t)$.

The minimal and maximal forward characteristics are defined for $b \in \mathbf{R}$, $0 \le s < t$, as

$$x^-(b;s,t) = \inf\{x : y^+(x;s,t) \ge b\} \tag{2.18}$$

and

$$x^+(b;s,t) = \sup\{x : y^-(x;s,t) \le b\}. \tag{2.19}$$

Immediate properties are these:

$$b-(t-s) \le x^-(b;s,t) \le x^+(b;s,t) \le b+(t-s).$$

As functions of b, $x^+(b;s,t)$ is right-continuous, $x^-(b;s,t)$ is left-continuous, and both are nondecreasing. For the $s=0$ case abbreviate $x^\pm(b,t) = x^\pm(b;0,t)$. As functions of t, $x^\pm(b;s,t)$ are Lipschitz and satisfy the initial condition $x^\pm(b;s,s) = b$. When characteristics are unique we write $x(b;s,t) = x^\pm(b;s,t)$.

Now consider the setting of the hydrodynamic limit Theorem 1, with a sequence of processes η_n that satisfy the initial limit (2.3). Let $X_n(t)$ denote the position of a second class particle in the process η_n. The initial position $X_n(0)$ may be deterministic or random, and may depend in an arbitrary fashion on the initial particle configurations $(\eta_n(i,0))$ and the Poisson processes $\{D_i\}$. For the first result assume a law of large numbers at time zero: for a deterministic point $b \in \mathbf{R}$

$$\lim_{n\to\infty} \frac{X_n(0)}{n} = b \quad \text{a.s.} \tag{2.20}$$

Theorem 3. *Under assumptions* (2.3) *and* (2.20), *for any fixed* $t > 0$,

$$x^-(b,t) \le \liminf_{n\to\infty} \frac{X_n(nt)}{n} \le \limsup_{n\to\infty} \frac{X_n(nt)}{n} \le x^+(b,t) \tag{2.21}$$

almost surely. In particular, if $x(b,t) = x^\pm(b,t)$, *then we have a strong law of large numbers:*

$$\lim_{n\to\infty} \frac{X_n(nt)}{n} = x(b,t) \quad \textit{a.s.} \tag{2.22}$$

In the special case where characteristics are unique we can dispense with assumption (2.20). The second class particle tracks the macroscopically defined characteristic that emanates from its random initial position, as stated in part (a) of the next theorem. We only need to assume that the fluctuations of the initial location are macroscopically bounded:

$$-\infty < \liminf_{n\to\infty} \frac{X_n(0)}{n} \le \limsup_{n\to\infty} \frac{X_n(0)}{n} < \infty \qquad \text{almost surely.} \tag{2.23}$$

The uniqueness of characteristics is true if we look at characteristics started at a time $s > 0$. But only if the flux function is differentiable, so this assumption needs to be made explicitly for $2 \le K < \infty$ in part (b).

Theorem 4. *Assume* (2.3) *and* (2.23).

(a) *If $x(y,t) = x^{\pm}(y,t)$ for all $y \in \mathbf{R}$, then almost surely*

$$\lim_{n\to\infty} \left| \frac{X_n(nt)}{n} - x\left(\frac{X_n(0)}{n}\,,\, t\right)\right| = 0. \tag{2.24}$$

(b) *Let $0 < s < t$. In the case $2 \le K < \infty$ suppose f is differentiable on $[0,K]$. For $K = 1$ and $K = \infty$ this is known and does not need to be assumed. Then $x(y;s,t) = x^{\pm}(y;s,t)$ for all $y \in \mathbf{R}$, and almost surely*

$$\lim_{n\to\infty} \left| \frac{X_n(nt)}{n} - x\left(\frac{X_n(ns)}{n}\,; s,\, t\right)\right| = 0. \tag{2.25}$$

The meaning of $x\left(n^{-1}X_n(0)\,,\, t\right)$ is that the deterministic function $x(\cdot,t)$ defined by (2.18)–(2.19) is evaluated at the random point $n^{-1}X_n(0)$. If assumption (2.23) is strengthened to $na \le X_n(0) \le nb$ for all large enough n, then for Theorem 4(a) it is sufficient to assume $x(y,t) = x^{\pm}(y,t)$ for $a \le y \le b$. As for Theorem 1, if the assumptions are valid in probability, then so are the conclusions.

2.3. Comments on the assumptions and results.

2.3.1. Differentiability and strict convexity. Since g is convex, left and right derivatives

$$g'_-(x) = \lim_{\varepsilon\searrow 0} \frac{g(x-\varepsilon)-g(x)}{-\varepsilon} \qquad \text{and} \qquad g'_+(x) = \lim_{\varepsilon\searrow 0} \frac{g(x+\varepsilon)-g(x)}{\varepsilon}$$

exist at all points x. Same is true for f, and for both f and g differentiability on an open interval is equivalent to continuous differentiability. *Strict convexity* means that there are no $x_1 < x_2$ such that $g'_+(x_1) = g'_-(x_2)$, which is the same as not having a line segment in the graph of g. From the duality (2.8) it follows that f is differentiable on $[0,K]$ iff g is strictly convex on $[-1,1]$, and f is strictly concave on $[0,K]$ iff g is differentiable on $\mathbf{R}$. Consult [19] for a general treatment of convex analysis.

Theorem 4(b) is true because strict convexity of g guarantees that $x(y;s,t) = x^{\pm}(y;s,t)$ for all $y \in \mathbf{R}$, if $s > 0$. This is proved in Proposition 3.1 in Section 3.2.

The restriction $x-(t-s) \le y \le x+(t-s)$ in definition (2.16)–(2.17) of $y^{\pm}(x;s,t)$ is necessary because g is linear outside $[-1,1]$. For example, take as initial profile $\rho_0(x) = K$ for $x \le 0$ and $\rho_0(x) = 0$ for $x > 0$. Then for $x > t$, the entire interval $[0, x-t]$ consists of maximizers in the Hopf-Lax formula. Without the restriction in (2.16) we would have $y^-(x,t) = 0$ for all $x > t$, and then $x^+(0,t) = \infty$ which is not the appropriate characteristic.

2.3.2. Shocks. Consider the simplest shock case (Riemann problem) with initial profile $\rho_0(x) = \alpha\mathbf{1}\{x < 0\} + \beta\mathbf{1}\{x > 0\}$ where $\alpha < \beta$. The solution (2.12) is $\rho(x,t) = \alpha\mathbf{1}\{x < \xi t\} + \beta\mathbf{1}\{x > \xi t\}$ with shock speed $\xi = (f(\beta) - f(\alpha))/(\beta - \alpha)$. The unique characteristic from the origin is $x(0,t) = \xi t$. Theorem 3 says that the second class particle converges to the shock. This generalizes results of Ferrari [5] from exclusion to K-exclusion, and from initial product distributions to arbitrary initial distributions that have the shock profile.

2.3.3. Rarefaction fan. Ferrari and Kipnis [9] proved convergence in probability in (2.25) for the following case of TASEP: initial distribution is product measure with density ρ to the left and density λ to the right of the origin, with $\rho > \lambda$, and the second class particle initially at the origin. This initial profile $\rho_0(x) = \rho\mathbf{1}\{x < 0\} + \lambda\mathbf{1}\{x > 0\}$ has an inadmissible shock at the origin, which produces a "rarefaction fan" of characteristics with $x^-(0,t) = (1-2\rho)t$ and $x^+(0,t) = (1-2\lambda)t$. Ferrari and Kipnis also proved that weakly $n^{-1}X_n(nt)$ converges to the uniform distribution on $[x^-(0,t), x^+(0,t)]$. Consequently one cannot hope for a law of large numbers when $x^-(0,t) < x^+(0,t)$.

If f is not differentiable at ρ, then the situation $x^-(b,t) < x^+(b,t)$ happens even for the constant profile $\rho(x,t) \equiv \rho$. See Example 3.1 in Section 3.2. In this case we cannot deduce convergence of $n^{-1}X_n(nt)$ from Theorem 3, only the bounds (2.24), unless we can prove the differentiability of f.

2.3.4. The characteristic ordinary differential equation. When f is strictly concave and differentiable, Theorem 3 can be strengthened with this statement: The characteristics $x(t) = x^\pm(b,t)$ are *Filippov solutions* of the o.d.e.

$$\frac{dx}{dt} = f'(\rho(x,t)) \ , \ x(0) = b \,, \tag{2.26}$$

where $\rho(x,t)$ is the Lax-Oleinik solution of (2.5). Since $\rho(x,t)$ does not exist at all x, (2.26) is interpreted as

$$\operatorname*{ess\,inf}_x f'(\rho(x,t)) \le \frac{dx}{dt} \le \operatorname*{ess\,sup}_x f'(\rho(x,t)) \qquad \text{for a.e. } t. \tag{2.27}$$

Here the essential infimum relative to x, at the point (x,t), is defined by

$$\operatorname*{ess\,inf}_x f'(\rho(x,t)) = \lim_{\delta\searrow 0} \operatorname*{ess\,inf}_{x-\delta<y<x+\delta} f'(\rho(y,t)),$$

and similarly for the essential supremum. The Lax-Oleinik formula (we review it in Section 3.1 below) shows that (2.27) is equivalent to

$$f'\left(-g'\left(\frac{x-y^+(x,t)}{t}\right)\right) \le \frac{dx}{dt} \le f'\left(-g'\left(\frac{x-y^-(x,t)}{t}\right)\right) \tag{2.28}$$

for almost every t. These results are developed in [18] for the exclusion and zero-range processes.

Drop the assumption of strict concavity of f, but retain the differentiability assumption. Then (2.26) remains valid in the sense (2.27), but $\rho(x,t)$ must be defined by (2.12) instead of by the Lax-Oleinik formula. (We show in Section 3.1 that without strict concavity the Lax-Oleinik formula cannot be used.) If we drop the differentiability assumption on f the interpretation of (2.26) becomes a problem. For example, if f' does not exist at ρ and we consider the constant solution $\rho(x,t) \equiv \rho$, then $f'(\rho(x,t))$ does not exist at any (x,t).

Without any assumptions on f beyond concavity, bounds (2.28) are valid for $x(t) = x^\pm(b,t)$ in the form

$$f'_+\left(-g'_+\left(\frac{x-y^+(x,t)}{t}\right)\right) \le \frac{dx}{dt} \le f'_-\left(-g'_-\left(\frac{x-y^-(x,t)}{t}\right)\right) \tag{2.29}$$

for almost every t. We omit the proof of (2.29) and the statement in the previous paragraph.

2.4. Generalization: particles jump in batches. We discuss here a generalization of the K-exclusion and zero-range processes for which the variational coupling technique works and Theorems 1–4 are true.

From a queueing perspective, the particles are customers moving through an infinite sequence of servers. Each server has space for K customers in its queue, and $\eta(i,t)$ is the number of customers waiting at server i at time t. In the formulation (2.1), each server serves at exponential rate 1, and only one customer is served at a time and moves on to the next server. We generalize this by permitting customers to move in batches of varying sizes, with exponential rates and independently for each batch size. Let β_h be the rate at which a batch of h customers can jump from i to $i+1$, $h = 1,2,3,\dots$. If the state just before the jump does not permit h customers to jump [so either $\eta(i) < h$ or $\eta(i+1) > K-h$], we do not suppress the entire jump but instead move as many customers as possible without violating the constraints.

The family of processes is parametrized by K and a sequence of nonnegative numbers $(\beta_h : h \in \mathbf{N})$. If $K < \infty$ we may assume $\beta_h = 0$ for $h > K$. To realize this dynamics take a collection of mutually independent Poisson point processes $\{D_i^h : i \in \mathbf{Z}, h \in \mathbf{N}\}$ on the time axis $(0,\infty)$, so that the rate of D_i^h is β_h. At each epoch of D_i^h, a batch of h particles (customers) attempts to jump from site i to site $i+1$. The generator becomes

$$Lf(\eta) = \sum_{i\in\mathbf{Z}} \sum_{h\in\mathbf{N}} \beta_h [f(\eta^{h,i,i+1}) - f(\eta)],$$

where $\eta^{h,i,i+1}$ is the configuration that results from an attempt to move h particles from site i to $i+1$. Denote the actual number of particles that can move by

$$b = b(h, \eta(i), \eta(i+1)) = \min\{h,\, \eta(i),\, K - \eta(i+1)\}.$$

Then the new configuration is

$$\eta^{h,i,i+1}(j) = \begin{cases} \eta(i) - b, & j = i, \\ \eta(i+1) + b, & j = i+1, \\ \eta(j), & j \neq i, i+1. \end{cases}$$

The definition of a second class particle is the same as before. From the queueing perspective it is a second class customer. He can move forward only if first class customers do not fill the batch that is served, and he has to move back to the previous server if arriving first class customers fill the capacity of the queue.

The limit theorems 1, 3 and 4 are valid as stated for the batch process, provided the jump rates are restricted so that we can expect finite limits. It is enough to require $\sum_{h\in\mathbf{N}} \beta_h h^2 < \infty$. Under this assumption a weaker version of Theorem 1 with convergence in probability was proved in [25]. That paper also shows how to adapt the variational coupling approach to batch jumps.

If $K = \infty$, the i.i.d. geometric probability measures (2.2) are again invariant, for any choice of the rates (β_h). In this case the flux function is

$$f(\rho) = \sum_{h\in\mathbf{N}} \beta_h \left[\frac{\rho}{1+\rho} + \left(\frac{\rho}{1+\rho}\right) + \cdots + \left(\frac{\rho}{1+\rho}\right)^h \right].$$

3. Scalar conservation laws with concave flux functions

In this section we investigate the consequences of strict concavity (Section 3.1) and differentiability (Section 3.2) of the flux function. In Section 3.3 we prove the uniqueness criterion of Theorem 2 which assumes neither strict concavity nor differentiability of the flux.

3.1. Strict concavity of the flux and entropy shocks. We begin by recalling the Lax-Oleinik formula from Section 3.4 in [4], and then show how it depends on the strict concavity of the flux function.

Lax-Oleinik formula. Suppose f is strictly concave and differentiable on its domain $[0, K]$. Define two functions

$$\rho^{\pm}(x,t) = -g'\left(\frac{x - y^{\pm}(x,t)}{t}\right). \tag{3.1}$$

For fixed $t > 0$, under these assumptions $y^+(x,t) = y^-(x,t)$ for all but countably many x, and consequently the Lax-Oleinik solution

$$\rho_{\mathrm{LO}}(x,t) = \rho^{\pm}(x,t) \tag{3.2}$$

is well-defined for all but countably many x. It satisfies $\rho_{\mathrm{LO}}(x,t) = u_x(x,t)$ a.e., and is a weak solution of (2.5) in the integral sense (2.6). $\square$

In particular, when it exists ρ_{LO} is the solution defined in (2.12) for the hydrodynamic limit. One can check that $\rho_{\mathrm{LO}}(\cdot\,,t)$ is continuous at all but the countably many x at which $\rho^-(x,t) \neq \rho^+(x,t)$. If $\rho_{\mathrm{LO}}(\cdot,t)$ is discontinuous at x, there must be a jump upward: $\rho_{\mathrm{LO}}(x-,t) < \rho_{\mathrm{LO}}(x+,t)$. This is because g' is nondecreasing and $y^-(x,t) \leq y^+(x,t)$. Let us call such a jump an *entropy shock* for a strictly concave flux function. [For a strictly convex flux entropy shocks are downward jumps.]

There are two ways definition (3.1)–(3.2) can run into a problem:

(i) If f is not strictly concave, then g' fails to exist at some points and the right-hand side of (3.1) may not be defined.

(ii) If f' does not exist everywhere, then $y^-(x,t) < y^+(x,t)$ may happen for a nontrivial interval of points x, and formula (3.2) is not true a.e. (See Example 3.1 in Section 3.2.)

It turns out that if f is strictly concave definition (3.1)–(3.2) works and all discontinuities are entropy shocks. Even though $y^-(x,t) = y^+(x,t)$ may fail on a whole interval if f fails to be differentiable at some point.

Theorem 5. *Suppose the flux f is strictly concave. Then for a fixed $t > 0$ the solution $\rho(x,t) = u_x(x,t)$ defined by* (2.12) *exists and is continuous on a set $H \subseteq \mathbf{R}$ that contains all but countably many x. On H $\rho(x,t) = \rho^{\pm}(x,t)$ so the Lax-Oleinik formula is valid. For all x $\rho(x-,t) \leq \rho(x+,t)$, so the discontinuities of $\rho(\cdot,t)$ are entropy shocks.*

Conversely, suppose f has a linear segment. Then there is a range of densities ρ such that the constant solution $\rho(x,t) \equiv \rho$ cannot be represented by any left or right derivative of g, so the Lax-Oleinik formula (3.1)–(3.2) *fails. Furthermore, the relevant solution defined by* (2.12) *can have shocks of both types, namely $\rho(x-,t) > \rho(x+,t)$ and $\rho(x-,t) < \rho(x+,t)$.*

Proof. Strict concavity of f implies that g is differentiable. Inequalities (3.3)–(3.4) below then imply that the right and left partial x-derivatives $u_{x\pm}$ of u exist and are given by

$$u_{x\pm}(x,t) = -g'\left(\frac{x - y^{\pm}(x,t)}{t}\right) = \rho^{\pm}(x,t).$$

Since g' is continuous and $y^+(\cdot\,,t)$ nondecreasing and right-continuous, we see that $\rho^+(\cdot\,,t)$ is right-continuous, has at most countably many discontinuities, and has left limits at all points. And correspondingly for $\rho^-(x,t)$.

By the Lipschitz property $u(x,t)$ is differentiable at a.e. x and is the integral of its derivative. Thus

$$u(x,t) = \int_0^x u_x(\xi,t)d\xi = \int_0^x \rho^{\pm}(\xi,t)d\xi.$$

Consequently $\rho(x,t) \equiv u_x(x,t)$ exists and equals $\rho^{\pm}(x,t)$ at any x where one of $\rho^{\pm}(x,t)$ is continuous. This defines the set H. From this follows $\rho(x-,t) = \rho^-(x,t) \le \rho^+(x,t) = \rho(x+,t)$.

For the converse, suppose $f'(\rho) = \xi$ for $\rho_0 < \rho < \rho_1$. Then g has a corner at ξ, with $g'_-(\xi) = -\rho_1$ and $g'_+(\xi) = -\rho_0$. The values $\rho \in (\rho_0, \rho_1)$ are not taken by $g'_\pm$ at any point, so a constant solution $\rho(x,t) \equiv \rho \in (\rho_0,\rho_1)$ cannot be represented by a derivative of g.

The initial profile $\rho_0(x) = K$ for $x \le 0$ and $\rho_0(x) = 0$ for $x > 0$ gives the solution $u(x,t) = -tg(x/t)$. Then $\rho((t\xi)-,t) = \rho_1 > \rho_0 = \rho((t\xi)+,t)$ is a downward shock. Upward shocks arise in the usual way, for example as in Remark 2.3.2 above. $\square$

To finish this section we prove the inequalities used in the proof above. Let us write $\overline{\underline{\lim}}_{\varepsilon\searrow 0}$ to simultaneously include both the lim sup and the lim inf as $\varepsilon \searrow 0$.

Lemma 3.1. *Assume only that f is concave on $[0,K]$, define g by* (2.8) *and u by* (2.10)*. Then the following inequalities hold for left and right x-derivatives of $u(x,t)$ at fixed $t > 0$.*

$$-g'_+\left(\frac{x - y^-(x,t)}{t}\right) \le \overline{\underline{\lim}}_{\varepsilon\searrow 0}\, \frac{u(x-\varepsilon,t) - u(x,t)}{-\varepsilon} \le -g'_-\left(\frac{x - y^-(x,t)}{t}\right) \tag{3.3}$$

and

$$-g'_+\left(\frac{x - y^+(x,t)}{t}\right) \le \overline{\underline{\lim}}_{\varepsilon\searrow 0}\, \frac{u(x+\varepsilon,t) - u(x,t)}{\varepsilon} \le -g'_-\left(\frac{x - y^+(x,t)}{t}\right). \tag{3.4}$$

Proof. The proofs use convexity, the Hopf-Lax formula, and the right [left] continuity of $y^+(\cdot\,,t)$ and g'_+ [$y^-(\cdot\,,t)$ and g'_-]. As an illustration, here is the argument for the second inequality of (3.4). We leave the rest to the reader.

$$\begin{aligned}
&u(x+\varepsilon,t)-u(x,t)\\
&\le u_0(y^+(x+\varepsilon,t)) - tg\left(\frac{x+\varepsilon-y^+(x+\varepsilon,t)}{t}\right)\\
&\qquad\qquad - u_0(y^+(x+\varepsilon,t)) + tg\left(\frac{x-y^+(x+\varepsilon,t)}{t}\right)\\
&= -\varepsilon\cdot\left(\frac{t}{\varepsilon}\right)\cdot\left\{g\left(\frac{x-y^+(x+\varepsilon,t)}{t}+\frac{\varepsilon}{t}\right)-g\left(\frac{x-y^+(x+\varepsilon,t)}{t}\right)\right\}\\
&\le -\varepsilon\cdot g'_+\left(\frac{x-y^+(x+\varepsilon,t)}{t}\right).
\end{aligned}$$

Consider two possibilities: If $y^+(x+\varepsilon,t)=y^+(x,t)$ for small enough $\varepsilon>0$, then

$$g'_+\left(\frac{x-y^+(x+\varepsilon,t)}{t}\right)=g'_+\left(\frac{x-y^+(x,t)}{t}\right)\ge g'_-\left(\frac{x-y^+(x,t)}{t}\right).$$

The other possibility is that $y^+(x+\varepsilon,t)>y^+(x,t)$ for all $\varepsilon>0$. From right continuity of $x\mapsto y^+(x,t)$ it follows that $(x-y^+(x+\varepsilon,t))/t$ increases strictly to $(x-y^+(x,t))/t$ as $\varepsilon\searrow 0$. And then

$$\lim_{\varepsilon\searrow 0} g'_+\left(\frac{x-y^+(x+\varepsilon,t)}{t}\right)=g'_-\left(\frac{x-y^+(x,t)}{t}\right).$$

In either case we obtain the second inequality of (3.4). □

3.2. Differentiability of the flux and characteristics. In Corollary 3.1(b) below we show that differentiability of f implies $y^-(x,t)=y^+(x,t)$ for all but countably many x, the property used in part (3.2) of the Lax-Oleinik formula. In Proposition 3.1 we prove the property needed for Theorem 4(b), namely that $x^-(b;s,t)=x^+(b;s,t)$ for $0<s<t$. Example 3.1 shows how various properties fail without differentiability of f. The hypothesis of differentiability of f is used in the equivalent form that g is strictly convex on $[-1,1]$.

Let $I(x;s,t)$ denote the set of $y\in[x-(t-s),x+(t-s)]$ at which the supremum in (2.15) is achieved.

Lemma 3.2. *Suppose* $0\le t_1<t_2<t_3$, $y_2\in I(x;t_2,t_3)$, *and* $y_1\in I(y_2;t_1,t_2)$. *Then* $y_1\in I(x;t_1,t_3)$. *And either*

$$\frac{x-y_1}{t_3-t_1}=\frac{x-y_2}{t_3-t_2}=\frac{y_2-y_1}{t_2-t_1}$$

or g *is linear between* $(y_2-y_1)/(t_2-t_1)$ *and* $(x-y_2)/(t_3-t_2)$.

Proof. Modify the proof of Lemma 3.1 in [18]. □

Lemma 3.3. *Suppose* g *is strictly convex on* $[-1,1]$, $x_1<x_2$, $y_i\in I(x_i;s,t)$ *for* $i=1,2$. *Then* $y_1\le y_2$.

Proof. The assumptions $y_i\in I(x_i;s,t)$ give

$$u(y_1,s)-(t-s)g\left(\frac{x_1-y_1}{t-s}\right)\ge u(y_2,s)-(t-s)g\left(\frac{x_1-y_2}{t-s}\right)$$

and

$$u(y_2,s)-(t-s)g\left(\frac{x_2-y_2}{t-s}\right)\geq u(y_1,s)-(t-s)g\left(\frac{x_2-y_1}{t-s}\right).$$

To get a contradiction, suppose $y_1-y_2>0$. Then the above inequalities give

$$\frac{g\left(\frac{x_2-y_2}{t-s}\right)-g\left(\frac{x_2-y_1}{t-s}\right)}{\left(\frac{x_2-y_2}{t-s}\right)-\left(\frac{x_2-y_1}{t-s}\right)}\leq\frac{g\left(\frac{x_1-y_2}{t-s}\right)-g\left(\frac{x_1-y_1}{t-s}\right)}{\left(\frac{x_1-y_2}{t-s}\right)-\left(\frac{x_1-y_1}{t-s}\right)}.$$

Since $x_2-y_2>x_1-y_2$, $x_2-y_1>x_1-y_1$, and $(x_i-y_i)/(t-s)\in[-1,1]$, this contradicts the assumption on g. □

Corollary 3.1. *Suppose g is strictly convex on $[-1,1]$.*
(a) *For $x_1<x_2$, $y^+(x_1;s,t)\leq y^-(x_2;s,t)$.*
(b) *For fixed $0\leq s<t$, $y^-(x;s,t)=y^+(x;s,t)$ for all but countably many x.*
(c) $x^+(b;s,t)=\inf\{x:y^+(x;s,t)>b\}$ *and* $x^-(b;s,t)=\sup\{x:y^-(x;s,t)<b\}$.
(d) *If $x^-(b;s,t)<z<x^+(b;s,t)$, then $y^\pm(z;s,t)=b$.*

Proof. (a) is immediate from Lemma 3.3.

(b) Suppose x_0 is a continuity point of the nondecreasing, right-continuous function $x\mapsto y^+(x;s,t)$. Then

$$y^+(x_0;s,t)\geq y^-(x_0;s,t)\geq\lim_{x\nearrow x_0}y^+(x;s,t)=y^+(x_0;s,t),$$

where the second inequality follows from (a). Thus $y^-(x;s,t)=y^+(x;s,t)$ at all continuity points of $y^+(x;s,t)$, which includes all but countably many x.

(c) Let $w(b)=\inf\{x:y^+(x;s,t)>b\}$. We have $x^+(b;s,t)\geq w(b)$ without any new assumptions: if $x<w(b)$, then

$$y^+(x;s,t)\leq b\implies y^-(x;s,t)\leq b\implies x^+(b;s,t)\geq x.$$

For the converse we need part (a). Let $x>w(b)$. Then there exists x_1 such that $x>x_1>w(b)$.

$$y^+(x_1;s,t)>b\implies y^-(x;s,t)>b\implies x^+(b;s,t)\leq x.$$

We leave the proof of the other formula to the reader.

(d) Immediate from definition (2.18–2.19) and part (c). □

Proposition 3.1. *Suppose g is strictly convex on $[-1,1]$, and $s>0$. Then $x^-(b;s,t)=x^+(b;s,t)$ for all $b\in\mathbf{R}$ and $t>s$.*

Proof. Suppose there exists a z such that $x^-(b;s,t)<z<x^+(b;s,t)$. By Corollary (d) $y^\pm(z;s,t)=b$, and consequently for $\varepsilon>0$,

$$u(b,s)-(t-s)g\left(\frac{z-b}{t-s}\right)\geq u(b\pm\varepsilon,s)-(t-s)g\left(\frac{z-b\mp\varepsilon}{t-s}\right).$$

Letting $\varepsilon\searrow 0$ then gives

$$\limsup_{\varepsilon\searrow 0}\frac{u(b+\varepsilon,s)-u(b,s)}{\varepsilon}\leq -g'_-\left(\frac{z-b}{t-s}\right)$$

and

$$\liminf_{\varepsilon\searrow 0}\frac{u(b-\varepsilon,s)-u(b,s)}{-\varepsilon}\geq -g'_+\left(\frac{z-b}{t-s}\right).$$

Since we may vary z in an interval, and since by assumption the slope of g cannot remain constant in this interval, we conclude the strict inequality

$$\limsup_{\varepsilon\searrow 0} \frac{u(b+\varepsilon,s)-u(b,s)}{\varepsilon} < \liminf_{\varepsilon\searrow 0} \frac{u(b-\varepsilon,s)-u(b,s)}{-\varepsilon}.$$

By (3.3–3.4), this implies

$$g'_+\left(\frac{b-y^+(b,s)}{s}\right) > g'_-\left(\frac{b-y^-(b,s)}{s}\right). \tag{3.5}$$

The slope of a convex function is nondecreasing and by definition $(b-y^-(b,s))/s \geq (b-y^+(b,s))/s$. So (3.5) implies that $y^-(b,s) = y^+(b,s)$ and g has a corner at $(b-y^\pm(b,s))/s$. Since we are assuming g strictly convex in $[-1,1]$, Lemma 3.2 now forces $(z-b)/(t-s) = (b-y^\pm(b,s))/s$. This cannot hold for distinct z's. But the preceding argument is valid for all $z \in (x^-(b;s,t), x^+(b;s,t))$, so to avoid contradiction we must have $x^-(b;s,t) = x^+(b;s,t)$. □

Finally we show by example how a point of nondifferentiability in f destroys all the good properties of the characteristics.

Example 3.1. Suppose $\rho \in (0,K)$ is a point such that $f'_-(\rho) = \xi_2 > \xi_1 = f'_+(\rho)$. Then g has a linear segment of slope $-\rho$ over $[\xi_1,\xi_2]$. Apply the formulas to the constant initial profile $\rho_0(x) \equiv \rho$, with $u_0(x) = \rho x$. We get $I(x;s,t) = [x-\xi_2(t-s), x-\xi_1(t-s)]$, $y^-(x;s,t) = x-\xi_2(t-s)$, $y^+(x;s,t) = x-\xi_1(t-s)$, $x^-(b;s,t) = b+\xi_1(t-s)$, $x^+(b;s,t) = b+\xi_2(t-s)$, and $u(x,t) = \rho x - tf(\rho)$. In particular $\rho(x,t) = u_x(x,t) \equiv \rho$, so the constant profile is the solution. But the conclusions of Lemma 3.3, Corollary 3.1(a)–(b), and Proposition 3.1 have been contradicted. □

3.3. Proof of Theorem 2. We prove the uniqueness criterion of Theorem 2 under these general assumptions: The flux f is continuous and concave on an interval I, and g is defined by $g(x) = \sup_{\rho\in I}\{f(\rho)-\rho x\}$. The initial density $\rho_0(x)$ is a given locally bounded measurable function on $\mathbf{R}$, $u(x,t)$ is a locally Lipschitz function on $\mathbf{R}\times[0,\infty)$ defined by the Hopf-Lax formula (2.10), and $\rho(x,t) = u_x(x,t)$ exists for a.e. x for any fixed t. $\lambda(x,t)$ is a locally bounded measurable function on $\mathbf{R}\times[0,\infty)$ that satisfies the boundary condition $\lambda(x,0) = \rho_0(x)$ a.e. and the integral condition (2.6). The ranges of ρ and λ lie in I.

The goal is to show that, for a.e. x,

$$\int_0^t f(\lambda(x,s))ds \leq \int_0^t f(\rho(x,s))ds \tag{3.6}$$

for all $t \geq 0$. And conversely, if equality holds in (3.6) for almost all (x,t), then $\lambda(x,t) = \rho(x,t)$ for almost all (x,t).

Some technical difficulties stem from assuming only (2.6) and no regularity on λ. We cannot expect the function $x \mapsto \lambda(x,t)$ to be a sensible object for every fixed t because $\lambda(\cdot,t)$ can be redefined on any null set of t's without affecting (2.6). At $t=0$ we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon dt\int_{\mathbf{R}} \lambda(x,t)\psi(x)dx = \int_{\mathbf{R}} \rho_0(x)\psi(x)dx \tag{3.7}$$

for $\psi \in C_c^1(\mathbf{R})$. This is obtained from (2.6) by taking $\phi(x,t) = \psi(x)g(t)$ where $g(0) = 1$, $g' = -1/\varepsilon$ on $(0,\varepsilon)$, and $g(t) = 0$ for $t \geq \varepsilon$. [The C^1 test functions in (2.6) can be replaced by continuous, piecewise C^1 test functions by taking limits.]

A corresponding property holds for certain later times t. Abbreviate

$$\mathrm{Leb}(h) = \text{the set of Lebesgue points of the function } h$$

for any measurable, locally bounded function h on either $\mathbf{R}$ or $\mathbf{R} \times [0,\infty)$. (Properties of Lebesgue points can be found for example in Section 7.2 in [26].) Define

$$\mathbf{T} = \{t > 0 : (x,t) \in \mathrm{Leb}(\lambda) \text{ for almost every } x \in \mathbf{R}\}.$$

Since the complement of $\mathrm{Leb}(\lambda)$ is a null set of $\mathbf{R} \times (0,\infty)$, almost every t lies in $\mathbf{T}$. By considering test functions of the type $\phi(x,t) = \psi(x)g(t)$ for suitable g, one obtains from (2.6) a limit such as (3.7) around $t \in \mathbf{T}$, and then the following for $t \in \mathbf{T}$:

$$\int_{\mathbf{R}} \psi(x)\lambda(x,t)dx - \int_{\mathbf{R}} \psi(x)\rho_0(x)dx = \int_0^t \int_{\mathbf{R}} f(\lambda(x,s))\psi'(x)dx\,ds \tag{3.8}$$

for compactly supported C^1 functions ψ on $\mathbf{R}$. By taking limits of such functions, (3.8) is valid for compactly supported, continuous, piecewise C^1 functions ψ.

Lemma 3.4. (i) *Fix finite $A < B$ and $T > 0$. Then there exists a constant $C = C(A,B,T)$ such that, for all $A \leq a < b \leq B$ and $s,t \in \mathbf{T} \cup \{0\}$ such that $0 \leq s,t \leq T$,*

$$\left| \int_a^b \lambda(x,t)dx - \int_a^b \lambda(x,s)dx \right| \leq C \cdot |t-s|. \tag{3.9}$$

(ii) *If $t \in \mathbf{T} \cup \{0\}$ and $(a,t),(b,t) \in \mathrm{Leb}(f \circ \lambda)$, then we have a time derivative along points of $\mathbf{T}$:*

$$\lim_{\substack{s \to t \\ s \in \mathbf{T}}} \frac{1}{t-s} \left\{ \int_a^b \lambda(x,t)dx - \int_a^b \lambda(x,s)dx \right\} = f(\lambda(a,t)) - f(\lambda(b,t)). \tag{3.10}$$

Proof. In (3.8) take ψ such that $\psi = 0$ outside $[a-\varepsilon, b+\varepsilon]$, $\psi' = 1/\varepsilon$ on $(a-\varepsilon,a)$, $\psi = 1$ on $[a,b]$, and $\psi' = -1/\varepsilon$ on $(b, b+\varepsilon)$. We get

$$\begin{aligned} &\int_a^b \lambda(x,t)dx - \int_a^b \lambda(x,s)dx + O(\varepsilon) \\ &= \frac{1}{\varepsilon}\int_s^t \int_{a-\varepsilon}^a f(\lambda(x,\tau))dxd\tau - \frac{1}{\varepsilon}\int_s^t \int_b^{b+\varepsilon} f(\lambda(x,\tau))dxd\tau. \end{aligned} \tag{3.11}$$

The $O(\varepsilon)$ term contains the integrals over $[a-\varepsilon,a]$ and $[b,b+\varepsilon]$ from the left-hand side. These are bounded by $O(\varepsilon)$ uniformly over the range of a,b,s,t by the local boundedness of λ. The $1/\varepsilon$ factors on the right are the slopes of ψ. The second line in (3.11) is bounded by $C|t-s|$ in absolute value, where C is twice the supremum of $|f(\lambda(x,\tau))|$ over $[A,B] \times [0,T]$. Part (i) follows by letting $\varepsilon \searrow 0$. For part (ii) set $\varepsilon = \delta|t-s|$, divide through (3.11) by $t-s$, let first $\mathbf{T} \ni s \to t$ and then $\delta \searrow 0$. $\square$

As a locally Lipschitz function $u(x,t)$ is differentiable a.e. by Rademacher's theorem (Section 5.8.3 in [4]). And as the viscosity solution defined by the Hopf-Lax formula, $u(x,t)$ satisfies the equation $u_t + f(u_x) = 0$ a.e. (Theorem 5, Section 3.3.2 in [4]). By definition $\rho = u_x$ exists a.e., so for almost every x_0 and all $t > 0$,

$$\int_0^t f(\rho(x_0,s))ds = -\int_0^t u_t(x_0,s)ds = u_0(x_0) - u(x_0,t). \tag{3.12}$$

It is also the case that for almost every x_0, $(x_0,t) \in \mathrm{Leb}(f \circ \lambda)$ for almost every t. Fix such an x_0 for which (3.12) also holds. For $(x,t) \in \mathbf{R} \times \mathbf{T}$ define

$$v(x,t) = \int_{x_0}^x \lambda(y,t)dy - \int_0^t f(\lambda(x_0,s))ds + u_0(x_0). \tag{3.13}$$

The first integral is interpreted with a sign so that $v(x_2,t) - v(x_1,t) = \int_{x_1}^{x_2} \lambda(y,t)dy$ for $x_1 < x_2$. By Lemma 3.4(i) and local boundedness of λ and $f \circ \lambda$, v is locally Lipschitz on $\mathbf{R} \times \mathbf{T}$. Consequently v extends uniquely to a locally Lipschitz function on $\mathbf{R} \times [0,\infty)$. This extension has the correct boundary values at $t = 0$ because by Lemma 3.4(i),

$$\begin{aligned}\lim_{\mathbf{T}\ni t\to 0} v(x,t) &= \int_{x_0}^x \lambda(y,0)dy + u_0(x_0) = \int_{x_0}^x \rho_0(y)dy + u_0(x_0)\\ &= u_0(x).\end{aligned}$$

The goal is now to prove

$$v(x,t) \geq u(x,t) \text{ for all } (x,t). \tag{3.14}$$

Setting $x = x_0$ in (3.14) then gives, by (3.12) and (3.13), the conclusion (3.6) for $x = x_0$ and all $t \in \mathbf{T}$. Both sides of (3.6) are continuous in t because the integrands are locally bounded, so (3.6) follows for all t. This whole argument beginning with (3.12) can be repeated for almost every x_0. At the end of this section we argue that equality in (3.6) implies $\lambda = \rho$ a.e. First we prove (3.14).

Fix (x,t). For a fixed $y \in \mathbf{R}$ set $\xi(s) = y + (s/t)(x-y)$ for $s \in [0,t]$. As y varies we consider the collection of line segments $\{(\xi(s),s) : 0 \leq s \leq t\}$ on the plane $\mathbf{R} \times [0,\infty)$. Their union forms the "backward light cone" from the apex (x,t) to the base line $\mathbf{R} \times \{0\}$. For almost every such y, it must be that

$$(\xi(s),s) \in \mathrm{Leb}(f \circ \lambda) \cap \mathrm{Leb}(\lambda)$$

for almost every $s \in [0,t]$, because only a null set of points of $\mathbf{R} \times [0,\infty)$ fail to be Lebesgue points. Fix now such a y.

The function $\gamma(s) = v(\xi(s),s)$, $0 \leq s \leq t$, is Lipschitz, and hence $\gamma'(s)$ exists for a.e. s and

$$\gamma(t) - \gamma(0) = \int_0^t \gamma'(s)ds. \tag{3.15}$$

Fix $s_0 \in [0,t]$ such that all this holds: $s_0 \in \mathbf{T}$, $\gamma'(s_0)$ exists, $(x_0,s_0), (\xi(s_0),s_0) \in \mathrm{Leb}(f \circ \lambda) \cap \mathrm{Leb}(\lambda)$, and $s_0 \in \mathrm{Leb}\{f(\lambda(x_0,\cdot))\}$. These conditions are satisfied by a.e. s_0. For such an s_0 we prove that

$$\gamma'(s_0) = \frac{x-y}{t}\lambda(\xi(s_0),s_0) - f(\lambda(\xi(s_0),s_0)). \tag{3.16}$$

To prove (3.16) we calculate $\gamma'(s_0)$ as the limit

$$\lim_{\substack{s\searrow s_0\\ s\in\mathbf{T}}} \frac{\gamma(s)-\gamma(s_0)}{s-s_0}.$$

So consider $s \in \mathbf{T}$, $s > s_0$.

$$
\begin{aligned}
&\frac{\gamma(s)-\gamma(s_0)}{s-s_0} = \frac{v(\xi(s),s)-v(\xi(s_0),s_0)}{s-s_0}\\
&= \frac{1}{s-s_0}\int_{\xi(s_0)}^{\xi(s)} \lambda(y,s)dy + \frac{1}{s-s_0}\left\{\int_{x_0}^{\xi(s_0)} \lambda(y,s)dy - \int_{x_0}^{\xi(s_0)} \lambda(y,s_0)dy\right\}\\
&\quad - \frac{1}{s-s_0}\int_{s_0}^{s} f(\lambda(x_0,\tau))d\tau\\
&\equiv A_1 + A_2 - A_3.
\end{aligned}
$$

Term by term: For A_1 first use $s - s_0 = (\xi(s)-\xi(s_0))t/(x-y)$. By the assumption $s \in \mathbf{T}$ and Lemma 3.4(i), we can replace s by an average over $[s_0, s] \cap \mathbf{T}$ at the expense of an $O(s-s_0)$ error term. Since $\mathbf{T}$ covers almost all of $[s_0, s]$ we can ignore the restriction to $\mathbf{T}$ in the average. These steps give

$$
\begin{aligned}
A_1 &= \frac{1}{s-s_0}\int_{\xi(s_0)}^{\xi(s)} \lambda(y,s)dy\\
&= \frac{x-y}{t}\cdot\frac{1}{(s-s_0)(\xi(s)-\xi(s_0))}\int_{s_0}^{s} d\tau \int_{\xi(s_0)}^{\xi(s)} \lambda(y,\tau)dy + O(s-s_0)\\
&\longrightarrow \frac{x-y}{t}\lambda(\xi(s_0),s_0)
\end{aligned}
$$

as $s \searrow s_0$ along points of $\mathbf{T}$, by the assumption $(\xi(s_0), s_0) \in \mathrm{Leb}(\lambda)$. Next,

$$A_2 \longrightarrow f(\lambda(x_0,s_0)) - f(\lambda(\xi(s_0),s_0))$$

as $s \searrow s_0$ along points of $\mathbf{T}$, by the assumptions $s_0 \in \mathbf{T}$ and $(x_0, s_0), (\xi(s_0), s_0) \in \mathrm{Leb}(f \circ \lambda)$ and by Lemma 3.4(ii). Finally,

$$A_3 \longrightarrow f(\lambda(x_0,s_0))$$

by the assumption $s_0 \in \mathrm{Leb}\{f(\lambda(x_0,\cdot))\}$. Altogether we have proved (3.16) for a.e. s_0.

By (2.8), (3.16) implies $\gamma'(s) \geq -g((x-y)/t)$ for a.e. s. Since $v(y,0) = u_0(y)$, we can rewrite (3.15) as

$$v(x,t) = u_0(y) + \int_0^t \gamma'(s)ds \geq u_0(y) - tg((x-y)/t).$$

This argument is valid for a.e. y, and consequently by the Hopf-Lax formula (2.10), $v(x,t) \geq u(x,t)$. We have proved (3.14), and thereby (3.6) for $x = x_0$. The choice of x_0 was such that it covers a.e. point in $\mathbf{R}$.

For the converse, suppose

$$\int_0^t f(\lambda(x,s))ds = \int_0^t f(\rho(x,s))ds \qquad \text{for a.e. } (x,t). \tag{3.17}$$

(3.8) is valid for both λ and ρ for a.e. t. Consider such a t for which (3.17) also holds for a.e. x. The right-hand side of (3.8) is $\int_{\mathbf{R}} dx\, \psi(x) \int_0^t f(\lambda(x,s))ds$, so by (3.17) its value is not changed by replacing λ with ρ. We conclude that for a.e. t, $\int_{\mathbf{R}} \lambda(x,t)\psi(x)dx = \int_{\mathbf{R}} \rho(x,t)\psi(x)dx$ for all test functions ψ. This implies $\lambda = \rho$ a.e.

4. Second class particles in the variational coupling

4.1. Construction and variational coupling. First we construct the process. Let $\{D_i : i \in \mathbf{Z}\}$ be a collection of mutually independent rate 1 Poisson jump time processes on the time line $(0, \infty)$, defined on some probability space $(\Omega, \mathcal{F}, P)$. We construct the process $\eta(t) = (\eta(i,t) : i \in \mathbf{Z})$ on Ω in terms of "current particles." These form a process $z(t) = (z(i,t) : i \in \mathbf{Z})$ of labeled particles that move on $\mathbf{Z}$. The location of the ith particle at time t is $z(i,t)$, and these satisfy

$$0 \le z(i+1,t) - z(i,t) \le K\,. \tag{4.1}$$

In the graphical construction, $z(i)$ attempts to jump 1 step to the *left* at epochs of D_i. If the jump violates (4.1), it is suppressed. We can summarize the jump rule like this:

$$\begin{aligned}&\text{If } t \text{ is an epoch of } D_i, \text{ then}\\ &z(i,t) = \max\{z(i,t-) - 1, z(i-1,t-), z(i+1,t-) - K\}.\end{aligned} \tag{4.2}$$

To construct $z(\cdot)$ from rule (4.2) we exclude an exceptional null set of realizations of $\{D_i\}$. For the remainder of this section, fix a realization $\{D_i\}$ with these properties:

(4.3a) There are no simultaneous jump attempts.
(4.3b) Each D_i has only finitely many epochs in every bounded time interval.
(4.3c) Given any $t_1 > 0$, there are arbitrarily faraway indices $i_0 << 0 << i_1$ such that D_{i_0} and D_{i_1} have no epochs in the time interval $[0, t_1]$.

These properties are satisfied almost surely, so the evolution $z(t)$, $0 \le t < \infty$, is well-defined almost everywhere on Ω. Then the process $\eta(t)$ is defined by

$$\eta(i,t) = z(i,t) - z(i-1,t)\,. \tag{4.4}$$

It should be clear that $\eta(\cdot)$ satisfies the rules described in Section 2.

Define a family $\{w^k : k \in \mathbf{Z}\}$ of auxiliary processes on Ω. Each $w^k(t) = (w^k(i,t) : i \in \mathbf{Z})$ is a process of the same type as $z(t)$, so (4.1) is in force for all $w^k(i,t)$. The initial configuration of w^k depends on the initial position $z(k,0)$:

$$w^k(i,0) = \begin{cases} z(k,0)\,, & i \ge 0,\\ z(k,0) + Ki\,, & i < 0.\end{cases}$$

The process w^k reads its jump commands from the same Poisson processes $\{D_i\}$, but with a translated index: at epochs t of D_{i+k},

$$w^k(i,t) = \max\{w^k(i,t-) - 1, w^k(i-1,t-), w^k(i+1,t-) - K\}.$$

We prove a number of pathwise properties for the processes z and $\{w^k\}$ that are valid under the fixed realization $\{D_i\}$ that satisfies (4.3a–c). Since (4.3a–c) are valid almost surely, the properties we derive are almost sure properties, but we leave the modifier "almost surely" out of the statements that follow.

The coupling of the processes z and $\{w^k\}$ preserves ordering. One consequence is the following inequality, valid for all $k < l$, all i and all t:

$$w^l(i-l,t) \le w^k(i-k,t) + [z(l,0) - z(k,0)]. \tag{4.5}$$

The usefulness of the family of processes $\{w^k\}$ lies in this "variational coupling" lemma:

Lemma 4.1. *For all $i \in \mathbf{Z}$ and $t \geq 0$,*

$$z(i,t) = \sup_{k \in \mathbf{Z}} w^k(i-k,t). \tag{4.6}$$

In the case $K = \infty$ the supremum in (4.6) is over $\{k : k \leq i\}$ because $w^k(i-k,t) = -\infty$ for $k > i$. To prove Lemma 4.1, observe that due to properties (4.3), it suffices to consider finitely many jump attempts in a finite lattice interval (i_0, i_1) and time interval $[0, t_1]$. (4.6) is true at $t = 0$ by construction. Through case-by-case analysis one shows that if (4.6) holds before a jump attempt, (4.6) continues to hold after the jump attempt. For details, see Lemma 4.2 in [24].

For some purposes it is convenient to modify the processes w^k so that they start at level zero and advance in the positive direction. So define processes $\{\xi^k\}$ by

$$\xi^k(i,t) = z(k,0) - w^k(i,t).$$

The process ξ^k does not depend on $z(k,0)$, and depends on the superscript k only through a translation of the i-index of the Poisson processes $\{D_i\}$. Initially

$$\xi^k(i,0) = \begin{cases} 0\,, & i \geq 0, \\ -Ki\,, & i < 0\,. \end{cases}$$

Dynamically, at epochs t of D_{i+k},

$$\xi^k(i,t) = \min\{\xi^k(i,t-) + 1, \xi^k(i-1,t-), \xi^k(i+1,t-) + K\}.$$

These jumps preserve the inequalities

$$\xi^k(i,t) \leq \xi^k(i-1,t) \qquad \text{and} \qquad \xi^k(i,t) \leq \xi^k(i+1,t) + K.$$

Now (4.6) can be written as

$$z(i,t) = \sup_{k \in \mathbf{Z}} \{z(k,0) - \xi^k(i-k,t)\}. \tag{4.7}$$

This variational formula is a microscopic analogue of the Hopf-Lax formula (2.10), and the basis for the proof of the hydrodynamic limit. To prove Theorem 1, one shows that the right-hand side of (4.7) converges to the right-hand side of (2.10). More details about the construction, the variational coupling lemma and the proofs appear in [24].

4.2. The second class particle. We investigate the position $X(t)$ of a second class particle in the variational coupling. Fix again a sample point so that we can regard $\{D_i\}$, $(\eta(i,0))$, and $X(0)$ as deterministic, with (4.3a–c) satisfied by $\{D_i\}$.

As defined in Section 2.2, $X(t)$ is the location of the unique discrepancy between two processes η and $\widetilde{\eta}$ that satisfy initially $\widetilde{\eta}(X(0),0) = \eta(X(0),0)+1$ and $\widetilde{\eta}(i,0) = \eta(i,0)$ for $i \neq X(0)$. In particular, the second class particle is not counted in the η-variables and by the K-exclusion rule always $\eta(X(t),t) \leq K-1$.

We define η and $\widetilde{\eta}$ by (4.4), in terms of processes z and $\widetilde{z}$ that initially satisfy

$$\widetilde{z}(i,0) = z(i,0) \text{ for } i \leq X(0) - 1 \text{ and } \widetilde{z}(i,0) = z(i,0) + 1 \text{ for } i \geq X(0).$$

Processes z and $\widetilde{z}$ follow the same Poisson processes $\{D_i\}$, so this is the basic coupling. The location $X(t)$ is always uniquely defined as the location of the discrepancy, as shown in this lemma:

Lemma 4.2. *For all $t \geq 0$ there is a finite location $X(t)$ such that*

$$\widetilde{z}(i,t) = z(i,t) \text{ for } i \leq X(t) - 1 \text{ and } \widetilde{z}(i,t) = z(i,t) + 1 \text{ for } i \geq X(t). \tag{4.8}$$

Proof. Given any finite time horizon t_0, find indices $\cdots < i_{-1} < i_0 < i_1 < \cdots$ such that D_{i_m} has no jump epochs in the time interval $(0, t_0]$ for all m, $i_m \to \pm\infty$ as $m \to \pm\infty$, and $i_0 < X(0) < i_1$. Particles $z(i_m)$ and $\widetilde{z}(i_m)$ do not move up to time t_0. In each finite segment (i_{m-1}, i_m) there are only finitely many jump epochs up to time t_0. Arrange these finitely many jump epochs in temporal order, and then prove by induction that the lemma holds after each jump time. □

(4.8) defines $X(t)$ in our framework. Next we give $X(t)$ a representation that is a microscopic analogue of the characteristics.

Proposition 4.1. *We have the formulas*

$$X(t) = \inf\{i \in \mathbf{Z} : z(i,t) = z(k,0) - \xi^k(i-k,t) \text{ for some } k \geq X(0)\} \tag{4.9}$$

and

$$X(t) = 1 + \sup\{i \in \mathbf{Z} : \widetilde{z}(i,t) = \widetilde{z}(k,0) - \xi^k(i-k,t) \text{ for some } k < X(0)\}. \tag{4.10}$$

Proof. Consider these four statements:

$$\text{If } i < X(t), \text{ then } z(i,t) > z(k,0) - \xi^k(i-k,t) \text{ for all } k \geq X(0). \tag{4.11}$$

$$\text{If } i \geq X(t), \text{ then } z(i,t) = z(k,0) - \xi^k(i-k,t) \text{ for some } k \geq X(0). \tag{4.12}$$

$$\text{If } i < X(t), \text{ then } \widetilde{z}(i,t) = \widetilde{z}(k,0) - \xi^k(i-k,t) \text{ for some } k < X(0). \tag{4.13}$$

$$\text{If } i \geq X(t), \text{ then } \widetilde{z}(i,t) > \widetilde{z}(k,0) - \xi^k(i-k,t) \text{ for all } k < X(0). \tag{4.14}$$

These statements imply that the infimum and supremum in the formulas (4.9–4.10) are well-defined and finite, and also prove the formulas themselves.

To contradict (4.11), suppose $i < X(t)$ and

$$z(i,t) = z(k,0) - \xi^k(i-k,t) \tag{4.15}$$

for some $k \geq X(0)$. Then by (4.8)

$$\widetilde{z}(i,t) = \widetilde{z}(k,0) - \xi^k(i-k,t) - 1$$

which contradicts the variational formula (4.7) for process $\widetilde{z}$. This contradiction proves (4.11), and also implies that there must exist some $k < X(0)$ such that (4.15) holds. By (4.8), (4.15) then turns into

$$\widetilde{z}(i,t) = \widetilde{z}(k,0) - \xi^k(i-k,t),$$

and this proves (4.13).

(4.12) and (4.14) are proved by a similar argument that starts by contradicting (4.14). □

The second class particle jumps both left and right, but a certain monotonicity can be found from the representation (4.9):

Proposition 4.2. *Let*

$$k_1(t) = \sup\{k \geq X(0) : z(X(t),t) = w^k(X(t)-k,t)\}$$

denote the maximal k that satisfies the requirement in formula (4.9)*, or infinity. Then $k_1(t)$ is nondecreasing as a function of time. If there are initially arbitrarily large indices j such that $\eta(j,0) \leq K-1$, then $k_1(t)$ is finite for all t.*

We shall not prove Proposition 4.2 for we make no use of it in this paper.

5. Proofs of the laws of large numbers

Now assume we are in the setting described in the paragraph preceding Theorem 1. The initial particle configurations $(\eta_n(i,0) : i \in \mathbf{Z})$, $n \in \mathbf{N}$, and the Poisson jump time processes $\{D_i\}$ are defined on a probability space $(\Omega, \mathcal{F}, P)$. Define initial configurations $(z_n(i,0) : i \in \mathbf{Z})$ by setting $z_n(0,0) = 0$ and using (4.4). Construct the processes $z_n(\cdot)$ as in Section 4.1, and then define the processes $\eta_n(\cdot)$ by (4.4) in terms of $z_n(\cdot)$. The location of the second class particle is given by the variational representation in Proposition 4.1: for the nth process

$$\begin{aligned} X_n(t) &= \inf\{i \in \mathbf{Z} : z_n(i,t) = w_n^k(i-k,t) \text{ for some } k \geq X_n(0)\} \\ &= \inf\{i \in \mathbf{Z} : z_n(i,t) = z_n(k,0) - \xi^k(i-k,t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } k \geq X_n(0)\}. \end{aligned}$$

Theorems 3 and 4 rely on the basic hydrodynamic limits, which we summarize here from [24] in terms of the z-variables. The following statements all hold almost surely:

$$\lim_{n\to\infty} \frac{1}{n} z_n([ny],0) = u_0(y) \quad \text{for all } y \in \mathbf{R}; \tag{5.1}$$

$$\lim_{n\to\infty} \frac{1}{n} \xi^{[nr]}([nx],nt) = tg(x/t) \quad \text{for all } x, r \in \mathbf{R} \text{ and } t > 0 \tag{5.2}$$

and

$$\lim_{n\to\infty} \frac{1}{n} z_n([nx],nt) = u(x,t) \quad \text{for all } x \in \mathbf{R} \text{ and } t > 0. \tag{5.3}$$

The function u_0 is defined by (2.7) from the given initial profile ρ_0. (5.1) is a restatement of assumption (2.3). (5.2) is proved by subadditivity, and this limit defines g. Then $u(x,t)$ is defined in terms of u_0 and g by the Hopf-Lax formula (2.10). Next (5.3) is proved, by (5.1), (5.2), and the variational coupling (4.7). Lastly, f is defined as the conjugate of g, and $u(x,t)$ is identified as the viscosity solution of (2.11).

The following estimate ([24, Lemma 6.1]) reduces the range of indices needed in the variational formula. It will be used several times in the proofs.

Lemma 5.1. *Define*

$$\zeta_n(i,l,t) = \max_{i-l \leq k \leq i+l} \{z_n(k,0) - \xi^k(i-k,t)\}.$$

Fix $r > t$. Then there exists a finite positive constant $C = C(r,t)$ such that for all $i \in \mathbf{Z}$ and all $n \geq 1$,

$$P\big(z_n(i,nt) \neq \zeta_n(i,nr,nt)\big) \leq e^{-Cn}. \tag{5.4}$$

First we prove Theorem 3, so recall assumption (2.20):

$$\lim_{n\to\infty} \frac{1}{n} X_n(0) = b \quad \text{a.s.} \tag{5.5}$$

Proposition 5.1. *Assume* (2.3) *and* (2.20). *Then*

$$\limsup_{n\to\infty} n^{-1} X_n(nt) \leq x^+(b,t) \qquad \textit{almost surely.}$$

Proof. Fix $x_1 > x^+(b,t)$. We shall show that $\limsup_{n\to\infty} n^{-1}X_n(nt) \le x_1$ almost surely. By definition $x^+(b,t) = \sup\{x : y^-(x,t) \le b\}$, so $y^-(x_1,t) > b$. Choose c_1 so that $b < c_1 < y^-(x_1,t)$.

Let Γ be the event on which

$$X_n(nt) > nx_1 \text{ for infinitely many } n, \tag{5.6}$$

assumptions (4.3a–c) of Section 4.1 hold so that we can use variational formulas (4.7) and (4.9), the laws of large numbers (5.1)–(5.3) and (5.5) hold, and

$$z_n([nx_1], nt) = w_n^k([nx_1] - k, nt) \tag{5.7}$$

for some $k \ge [nc_0]$ for all large enough n, for some fixed finite c_0. Except for (5.6) the conditions of Γ hold with probability one, so it suffices to show that Γ is empty.

Suppose Γ is not empty. Fix a sample point in Γ, and suppose n is such that $X_n(nt) > nx_1$. Then $z_n([nx_1], nt) \ne w_n^k([nx_1] - k, nt)$ for all $k \ge X_n(0)$, so there must exist a $k < X_n(0)$ such that (5.7) holds. Since limit (5.5) holds on Γ, $X_n(0) \le [nc_1]$ for large enough n. Thus for a subsequence of n's there exists a k in (5.7) such that $[nc_0] \le k \le [nc_1]$.

Pick a partition $c_0 = y_0 < y_1 < \cdots < y_m = c_1$ of $[c_0, c_1]$ with mesh $\Delta y = \max(y_{i+1} - y_i)$. Fix i so that for infinitely many n in the subsequence picked above, some k in (5.7) satisfies $[ny_i] \le k \le [ny_{i+1}]$. By (4.5) we have, for infinitely many n,

$$\begin{aligned} z_n([nx_1], nt) &= w_n^k([nx_1] - k, nt) \\ &\le w_n^{[ny_i]}([nx_1] - [ny_i], nt) + \big\{z_n([ny_{i+1}], 0) - z_n([ny_i], 0)\big\}. \end{aligned} \tag{5.8}$$

Divide by n and let $n \to \infty$ along the appropriate subsequence of n's. Since the laws of large numbers hold for our fixed sample point, we get

$$u(x_1, t) \le u_0(y_i) - tg\bigg(\frac{x_1 - y_i}{t}\bigg) + \big\{u_0(y_{i+1}) - u_0(y_i)\big\}. \tag{5.9}$$

Take the mesh $\Delta y \to 0$. By continuity and compactness, we conclude that the Hopf-Lax formula (2.10) for $u(x_1,t)$ has a maximizer in $[c_0, c_1]$, so that $y^-(x_1,t) \le c_1$. This contradiction with the choice of c_1 implies that Γ is empty. $\square$

Proposition 5.2. *Assume* (2.3) *and* (2.20). *Then*

$$\liminf_{n\to\infty} n^{-1}X_n(nt) \ge x^-(b,t) \quad \textit{almost surely.}$$

Proof. Fix $x_1 < x^-(b,t)$. Consider the event Γ on which $X_n(nt) < nx_1$ for infinitely many n, assumptions (4.3a–c) hold, and the laws of large numbers (5.1)–(5.3), and (5.5) hold. We assume that Γ has positive probability, and derive a contradiction.

As in Section 4.2, we have the processes $\widetilde{z}_n$, coupled with z_n through the Poisson processes $\{D_i\}$, that satisfy

$$\widetilde{z}_n(i, nt) = z_n(i, nt) \text{ for } i < X_n(nt) \tag{5.10}$$

and

$$\widetilde{z}_n(i, nt) = z_n(i, nt) + 1 \text{ for } i \ge X_n(nt). \tag{5.11}$$

(4.10) and $X_n(nt) < nx_1$ imply that

$$\widetilde{z}_n([nx_1], nt) = \widetilde{z}_n(k, 0) - \xi^k([nx_1] - k, nt) \tag{5.12}$$

for some $k \geq X_n(0)$, and (5.12) cannot hold for any $k < X_n(0)$. By applying Lemma 5.1 to the processes $\widetilde{z}_n$ and by shrinking Γ by no more than a null set, we may assume that for large enough n the k in (5.12) must lie in the range $X_n(0) \leq k \leq [nc_1]$ for a constant c_1. Then by (5.5) we conclude that for large n, $[nc_0] \leq k \leq [nc_1]$ where c_0 can be taken arbitrarily close to b. By left-continuity of $x^-(\cdot, t)$ and the assumption $x_1 < x^-(b, t)$, we may choose c_0 so that $x_1 < x^-(c_0, t)$.

Now starting from (5.12) repeat the argument around (5.8)–(5.9) for $\widetilde{z}_n$. By (5.10)–(5.11) $\widetilde{z}_n$ satisfies the limits (5.1) and (5.3). As in the step after (5.9), we get that $u(x_1, t)$ has a maximizer in $[c_0, c_1]$ so that $y^+(x_1, t) \geq c_0$. By definition (2.18) this implies $x^-(c_0, t) \leq x_1$ and contradicts the choice of c_0. □

Propositions 5.1–5.2 prove Theorem 3.

Proof of Theorem 4(a). Assume $x(y, t) = x^{\pm}(y, t)$ for all $y \in \mathbf{R}$. Let Γ be the event on which (4.3a–c) and (5.1)–(5.3) hold, and also for large enough n

$$-nb_0 \leq X_n(0) \leq nb_0 \tag{5.13}$$

for some constant b_0. We prove that

$$\lim_{n\to\infty} \left| \frac{X_n(nt)}{n} - x\left(\frac{X_n(0)}{n}\,,\, t\right) \right| = 0 \tag{5.14}$$

almost surely on Γ. By assumption (2.23), Theorem 4(a) follows by repeating this for a sequence of b_0's increasing to ∞.

Since $x^-(y, t)$ is left-continuous and $x^+(y, t)$ right-continuous in y, the assumed unique characteristic $x(y, t) = x^{\pm}(y, t)$ is continuous in y. Given $\varepsilon > 0$, choose a partition $-b_0 = r_0 < r_1 < \cdots < r_h = b_0$ so that $x(r_{j+1}, t) < x(r_j, t) + \varepsilon/2$. Set $x_j = x(r_j, t) + \varepsilon$. Shrink Γ by a null set to guarantee that a.s. for every $j = 0, \ldots, h$, for large enough n there exists a $k \geq nc_0$ such that

$$z_n([nx_j], nt) = w_n^k([nx_j] - k, nt).$$

This can be done by Lemma 5.1 if c_0 is chosen small enough.

To contradict (5.14), suppose Γ contains a sample point for which

$$X_n(nt) > nx(n^{-1}X_n(0), t) + n\varepsilon \tag{5.15}$$

for infinitely many n. Restrict this subsequence of n's further so that for some fixed j,

$$nr_j \leq X_n(0) \leq nr_{j+1} \tag{5.16}$$

for all these n. By the monotonicity of $x(\cdot, t)$, (5.15–5.16) imply that $X_n(nt) > nx_j$ for infinitely many n. For each such n there exists $k < X_n(0)$ and no $k \geq X_n(0)$ such that $z_n([nx_j], nt) = w_n^k([nx_j] - k, nt)$. By the assumptions made above, $nc_0 \leq k \leq nr_{j+1}$. Repeat the partitioning and limiting argument around (5.8)–(5.9) to conclude that $y^-(x_j, t) \leq r_{j+1}$, and then by definition $x^+(r_{j+1}, t) \geq x_j$. Now we have contradicted $x_j = x(r_j, t) + \varepsilon > x(r_{j+1}, t) + \varepsilon/2$.

It remains to show that assuming $X_n(nt) < nx(n^{-1}X_n(0), t) - n\varepsilon$ for infinitely many n also leads to a contradiction. Combine this with (5.16) to get

$$X_n(nt) < nx(n^{-1}X_n(0), t) - n\varepsilon \leq nx(r_{j+1}, t) - n\varepsilon \equiv nx'_j.$$

As in the proof of Proposition 5.2, switch to the $\widetilde{z}_n$ processes to get

$$\widetilde{z}_n([nx'_j], nt) = \widetilde{z}_n(k, 0) - \xi^k([nx'_j] - k, nt)$$

for some k such that $nr_j \le X_n(0) \le k \le nc_1$, where c_1 is a fixed constant. From this conclude that $y^+(x'_j,t) \ge r_j$, which implies $x(r_j,t) = x^-(r_j,t) \le x'_j = x(r_{j+1},t) - \varepsilon$, a contradiction. □

To prove Theorem 4(b), first a lemma.

Lemma 5.2. *Let c be a finite constant.*

(a) *Suppose that $X_n(0) \ge nc$ for large enough n, almost surely. Then for any $x < c - t$, $X_n(nt) \ge nx$ for large enough n, almost surely.*

(b) *Suppose that $X_n(0) \le nc$ for large enough n, almost surely. Then for any $x > c + t$, $X_n(nt) \le nx$ for large enough n, almost surely.*

Proof. (a) Pick $\varepsilon > 0$ so that $x < c - t - 2\varepsilon$, and set $c_1 = c - \varepsilon$.

As an intermediate claim, we prove that almost surely, for large enough n,

$$\xi^k([nx]-k, nt) = -K \cdot ([nx]-k) \quad \text{for all } k \ge [nx] + [n(t+\varepsilon)]. \tag{5.17}$$

In other words, the claim is that $\xi^k([nx]-k,\cdot)$ has not jumped by time nt. The time of the first jump of $\xi^k([nx]-k,\cdot)$ is distributed as $\sum_{j=0}^{k-[nx]} Y_j$ where $\{Y_j\}$ are i.i.d. rate 1 exponential random variables. The estimate $P(\sum_{j=1}^{N} Y_j \le Ns) \le \exp[-N\kappa(s)]$ is valid for $0 < s < 1$. The rate function is $\kappa(s) = s - 1 - \log s$ for $s > 0$, which satisfies $\kappa'(s) < 0 < \kappa(s)$ for $0 < s < 1$. Thus

$$\begin{aligned} P\{(5.17) \text{ fails}\} &\le \sum_{k \ge [nx]+[n(t+\varepsilon)]} P\left(\sum_{j=0}^{k-[nx]} Y_j \le nt\right) \\ &\le \sum_{i \ge [n(t+\varepsilon)]} \exp\left[-i\kappa\left(nt/i\right)\right] \le C_1 e^{-C_2 n} \end{aligned}$$

for finite positive constants C_1, C_2. This proves the claim, and (5.17) holds for large enough n a.s.

Since $\eta_n(X_n(0),0) \le K - 1$ and $[nc_1] < X_n(0)$,

$$z_n(k,0) - z_n([nc_1],0) = \sum_{i=[nc_1]+1}^{k} \eta_n(i,0) \le K(k - [nc_1]) - 1$$

for any $k \ge X_n(0)$. For the fixed x any $k \ge X_n(0)$ falls within the range in (5.17) for large enough n. So for large enough n,

$$\begin{aligned} &z_n(k,0) - \xi^k([nx]-k,nt) = z_n(k,0) + K([nx]-k) \\ &\le z_n([nc_1],0) + K([nx]-[nc_1]) - 1 = z_n([nc_1],0) - \xi^{[nc_1]}([nx]-[nc_1],nt) - 1. \end{aligned}$$

The last equality comes from (5.17) again because $[nc_1]$ also falls within the range of k's in (5.17) for large enough n. The string of inequalities shows that no $k \ge X_n(0)$ can give the supremum in the microscopic variational formula (4.7) for $z_n([nx],nt)$, because each $k \ge X_n(0)$ is inferior to $[nc_1]$. But if $X_n(nt) < [nx]$, by (4.12) some $k \ge X_n(0)$ would be a maximizer. This contradiction shows that $X_n(nt) \ge [nx]$ for large enough n, a.s.

The proof of part (b) is similar, but instead of (5.17) use that $\xi^k([nx]-k,nt) = 0$ for all $k \le [nx] - [n(t+\varepsilon)]$. □

Proof of Theorem 4(b). The goal is to show that

$$\lim_{n\to\infty}\left|\frac{X_n(nt)}{n}-x\left(\frac{X_n(ns)}{n}\,;s,\,t\right)\right|=0\quad\text{a.s.}\tag{5.18}$$

Let $\bar{u}_0(x)=u(x,s)$, $\bar{u}(x,\tau)=u(x,s+\tau)$ for $\tau>0$, and let $\bar{x}(b;0,\tau)$ denote characteristics for the solution $\bar{u}$. Then $x^{\pm}(b;s,t)=\bar{x}^{\pm}(b;0,t-s)$. From the assumptions and Proposition 3.1 we know $\bar{u}$ has unique characteristics: $\bar{x}(b;0,\tau)=\bar{x}^{\pm}(b;0,\tau)$.

Let $\bar{z}_n(i,t)=z_n(i,ns+t)$ be the nth process restarted at time ns. Let $\bar{X}_n(t)$ be the location of a second class particle in the process $\bar{z}_n$, with initial location $\bar{X}_n(0)=X_n(ns)$. In restarted terms, (5.18) is the same as

$$\lim_{n\to\infty}\left|\frac{\bar{X}_n(n(t-s))}{n}-\bar{x}\left(\frac{\bar{X}_n(0)}{n}\,;0,\,t-s\right)\right|=0\quad\text{a.s.}\tag{5.19}$$

(5.19) follows from Theorem 4(a) if we check that the ingredients of the proof are valid for the processes $\bar{z}_n$ and $\bar{X}_n$.

(i) We can obtain the variational formulas (4.7) and (4.9) for $\bar{z}_n$ and $\bar{X}_n$ in a form where the ξ-part depends on n through the time shift. For any fixed restarting time $\tau\geq 0$, let θ_τ denote the restarting operation on the probability space of the Poisson processes $\{D_i\}$: if the epochs of D_i in $(0,\infty)$ are $t_1<t_2<t_3<\cdots$ with $t_{j-1}\leq\tau<t_j$, then the epochs of $\theta_\tau D_i$ in $(0,\infty)$ are $t_j-\tau<t_{j+1}-\tau<t_{j+2}-\tau<\cdots$.

The same proofs that originally gave Lemma 4.1 and Proposition 4.1 also yield, for $t>\tau$,

$$z(i,t)=\sup_k\{z(k,\tau)-\xi^k(i-k,t-\tau)\circ\theta_\tau\}\tag{5.20}$$

and

$$X(t)=\inf\{i:z(i,t)=z(k,\tau)-\xi^k(i-k,t-\tau)\circ\theta_\tau\text{ for some }k\geq X(\tau)\}.\tag{5.21}$$

Now for process z_n take $\tau=ns$, and (5.20)–(5.21) give for $t>0$

$$\bar{z}_n(i,t)=\sup_k\{\bar{z}_n(k,0)-\xi^k(i-k,t)\circ\theta_{ns}\}$$

and

$$\bar{X}_n(t)=\inf\{i:\bar{z}_n(i,t)=\bar{z}_n(k,0)-\xi^k(i-k,t)\circ\theta_{ns}\text{ for some }k\geq X_n(ns)\}.$$

(ii) Limits (5.1) and (5.3) are true by definition of $\bar{z}_n$ and $\bar{u}$. Limits (5.2) for $\xi^{[nr]}([nx],nt)\circ\theta_{ns}$ are true by the proofs in [24] because these limits are derived from summable deviation estimates and the Borel-Cantelli lemma. The distribution of $\xi^{[nr]}$ is not changed by the time shift, so the same deviation estimates are valid.

(iii) Finally assumption (2.25) for $\bar{X}_n(0)=X_n(ns)$ is a consequence of (2.25) for $X_n(0)$ and Lemma 5.2. ∎

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