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MONOTONICITY OF STABLE SOLUTIONS IN SHADOW SYSTEMS

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ABSTRACT. A shadow system appears as a limit of a reaction-diffusion system in which some components have infinite diffusivity. We investigate the spatial structure of its stable solutions. It is known that, unlike scalar reaction-diffusion equations, some shadow systems may have stable nonconstant (monotone) solutions. On the other hand, it is also known that in autonomous shadow systems any nonconstant non-monotone stationary solution is necessarily unstable. In this paper, it is shown in a general setting that any stable bounded (not necessarily stationary) solution is asymptotically homogeneous or eventually monotone in x.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the system of the form

$$\begin{aligned} &u_t = u_{xx} + f(u,v,t), \quad x \in (0,1),\ t>0,\\ &u_x(0,t) = 0 = u_x(1,t), \quad t>0, \\ &v_t = \int_0^1 g(u,v,t)dx, \qquad t>0, \end{aligned} \tag{1.1}$$

where $u = u(x,t) \in \mathbf{R}$ and $v = v(t) \in \mathbf{R}^m$. This system is closely related to the $(1+m)$-component reaction-diffusion system

$$\begin{aligned} u_t &= u_{xx} + f(u,v,t),\\ v_t &= \varepsilon^{-2} v_{xx} + g(u,v,t), \end{aligned} \tag{1.2}$$

with the homogeneous Neumann boundary conditions. In fact, the system (1.1) appears as a limit of (1.2) as $\varepsilon \downarrow 0$ and is called the shadow system of (1.2). See [9, 14] for a more precise relation between (1.1) and (1.2) concerning equilibria and the dynamics.

We assume that the nonlinearities f and g satisfy the following hypotheses:

(H1) For each $M>0$, the functions f, g, f_u, f_v, g_u and g_v are continuous and bounded in $[-M,M]^{m+1} \times [0,\infty]$.

(H2) There is an $\alpha \in (0,1)$ such that for each $M>0$ and $\tau \geq 0$, the functions f, g, f_u, f_v, g_u and g_v are Hölder continuous with exponent α in the region $[-M,M]^{m+1} \times [\tau, \tau+1]$ and their Hölder norms are bounded by a constant independent of τ.

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Under these assumptions, the system (1.1) is well-posed on $C[0,1] \times \mathbf{R}^m$. More precisely, for any $u_0(\cdot) \in C[0,1]$ and $v_0 \in \mathbf{R}^m$ there is a unique solution of (1.1) subject to the initial condition

$$u(x,0) = u_0(x), \quad v(0) = v_0,$$

and the solution enjoys the usual continuous-dependence and regularity properties of parabolic equations (see, e.g., [10]).

Our main objective is to describe the spatial structure of stable solutions of (1.1). Our investigation was motivated by earlier results on the autonomous shadow system

$$\begin{aligned} &u_t = u_{xx} + f(u,v), && x \in (0,1),\ t > 0,\\ &u_x(0,t) = 0 = u_x(1,t), && t > 0,\\ &v_t = \int_0^1 g(u,v)dx, && t > 0. \end{aligned} \tag{1.3}$$

It was shown by Nishiura [14] and Ni, Takagi and Yanagida [13] that systems of the form (1.3) may have stable stationary solutions that are spatially inhomogeneous and monotone (see also [8] for a discussion of similar results for scalar nonlocal equations). In [13], it was also shown that a time-periodic solution may appear in an autonomous shadow system through a Hopf bifurcation. A numerical computation by Fukushima and Yanagida (see the survey paper [12]) indicates that the time-periodic solution is stable under some conditions if the solution is spatially monotone. These results are in contrast to scalar reaction-diffusion equations for which any stable periodic (or almost periodic) solution must be spatially homogeneous (cf. [11, 15, 17]).

On the other hand, Nishiura proved in [14, Theorem 4.1] that except for constant solutions and monotone solutions, there are no other stable stationary solutions of (1.3). One of our results here extends this theorem to time-periodic solutions: We show that such solutions are unstable, unless they are spatially constant or monotone. This is a consequence of our main theorem in which we consider general time-dependent solutions.

As we deal with bounded solutions that are not necessarily stationary or time-periodic, some care is needed in the definition of stability. Let (u,v) be any bounded solution of (1.1) and consider the linearized equation along that solution

$$\begin{aligned} &U_t = U_{xx} + f_u(u,v,t)U + f_v(u,v,t)V, && x \in (0,1),\ t > 0,\\ &U_x(0,t) = 0 = U_x(1,t), && t > 0,\\ &V_t = \int_0^1 \{g_u(u,v,t)U + g_v(u,v,t)V\}dx, && t > 0, \end{aligned} \tag{1.4}$$

where $U = U(x,t) \in \mathbf{R}$ and $V = V(t) \in \mathbf{R}^m$. Let $T(t,s)$ denote the evolution operator of this problem on $C[0,1] \times \mathbf{R}^m$. We say that the solution (u,v) is *linearly stable* if there are positive constants C and λ such that

$$\|T(t,s)\| \le Ce^{-\lambda(t-s)} \quad (t > s \ge 0).$$

If this property holds with $\lambda \ge 0$, we say that (u,v) is at least *linearly neutrally stable*. We say a solution (u,v) of (1.1) is *linearly exponentially unstable* if there exists a solution (U, V) of (1.4) such that

$$\|U\|_{L^\infty(0,1)} + |V| \geq \exp(\lambda t)$$

with some positive number $\lambda > 0$.

It is well-known that linearly stable solutions are uniformly asymptotically stable (see, e.g., [10, Sect. 5.1 and Excercise 9 in Sect. 7.1]). It is also well-known that if (u, v) is an equilibrium of an autonomous system or a periodic solution of a time-periodic system, then, to be stable, it must be at least linearly neutrally stable. For general nonautonomous equations, similar instability criteria are not so easily formulated and additional conditions are involved in general (see [10, Exercise 4 in Sect. 5.1] and [2, Sect.16.2]).

In what follows we consider solutions $(u(x,t), v(t))$ of (1.1) that satisfy the following conditions:

(A1) $(u, v) = (u(x,t), v(t))$ is uniformly bounded in $(x,t) \in [0,1] \times [0, \infty)$.

(A2) $\|u_x(\cdot, t)\|_{L^\infty(0,1)}$ is bounded away from 0, that is,

$$\|u_x(\cdot, t)\|_{L^\infty(0,1)} > d \quad (t > 0)$$

for some constant $d > 0$.

Note that (A1) and parabolic regularity in particular imply that

$$\{(u(\cdot, t), v(t)) : t \geq 1\}$$

is relatively compact in $C^2[0,1] \times \mathbf{R}^m$. The role of (A2) is to guarantee that the solution stays away from the space of constant functions.

Now we state the main result of this paper.

Theorem 1.1. *Assume that* (H1) *and* (H2) *hold. Let* $(u(x,t), v(t))$*,* $t \geq 0$*, be a solution of* (1.1) *satisfying* (A1), (A2) *that is at least linearly neutrally stable. Then there is a* t_0 *such that*

$$u_x(x,t) \neq 0 \quad \textit{for all} \ \ (x,t) \in (0,1) \times [t_0, \infty).$$

We have a more specific linear instability result for solutions that have a certain symmetry property. We say that a function $v(x)$ is *k-symmetric* in $[0,1]$, $k \geq 2$, if the restriction $v(x)$, $x \in \left[\frac{i-1}{k}, \frac{i+1}{k}\right]$, is even symmetric with respect to the point $x = i/k$ for all $i = 1, 2, \ldots, k-1$, that is,

$$v(x) = v(2i/k - x) \quad \text{for all} \ \ x \in \left[\frac{i-1}{k}, \frac{i+1}{k}\right].$$

We call a solution (u, v) of (1.1) k-symmetric if $u(x,t)$ is k-symmetric for every t.

Theorem 1.2. *Assume that* (H1) *and* (H2) *hold. Let* $(u(x,t), v(t))$ *be a solution of* (1.1) *satisfying* (A1), (A2) *that is* k*-symmetric with some* $k \geq 2$*. Then* $(u(x,t), v(t))$ *is linearly exponentially unstable.*

Theorem 1.2, besides giving an additional instability property, is also the main ingredient of the proof of Theorem 1.1. Theorem 1.2 will be used in conjunction with the following result which links general solutions to k-symmetric solutions. We say that $(\phi, \xi) \in C[0,1] \times \mathbf{R}^m$ is a limit point of a solution $(u(x,t), v(t))$ if

$$(u(x,t_n), v(t_n)) \to (\phi(x), \xi) \quad \text{in} \ \ C[0,1] \times \mathbf{R}^m$$

for some sequence $\{t_n\}$ approaching ∞.

Proposition 1.3. *Assume that* (H1) *and* (H2) *hold. Let* $(u(x,t), v(t))$ *be a solution of* (1.1) *satisfying* (A1), (A2). *Then there is a positive integer* k *such that each limit point* (ϕ, ξ) *of* $(u(x,t), v(t))$ *satisfies* $\phi_x(x) \neq 0$ *for* $x \in (0, 1/k)$ *and* $\phi'(1/k) = 0$. *Moreover, if* $k \geq 2$*, then* ϕ *is* k*-symmetric.*

This proposition follows directly from [5, Theorem B] upon noting that u solves a scalar reaction diffusion equation for which all hypotheses of [5, Theorem B] are fulfilled. Also note that (A2) implies that no limit point can be constant in x.

Proposition 1.3 readily implies that every time-periodic solution which is neither spatially homogeneous nor monotone must necessarily be k-symmetric for some $k \geq 2$. As an immediate consequence of Theorem 1.2, we therefore obtain the following instability result for such periodic solutions.

Corollary 1.4. *Suppose that* $f(u,v,t)$ *and* $g(u,v,t)$ *are periodic in* t *with a common period* τ *and that they are differentiable with respect to* (u,v) *and the derivatives are locally Hölder continuous. Then any* τ*-periodic solution of* (1.1) *is linearly exponentially unstable if it is spatially inhomogeneous and non-monotone.*

The previous corollary in particular implies that non-monotone stationary solutions of autonomous shadow systems are linearly exponentially unstable. In fact, the latter result holds under weaker regularity assumptions. More precisely, it is formulated as follows (cf. [14]).

Proposition 1.5. *Suppose that* $f(u,v)$ *and* $g(u,v)$ *are of class* C^1. *Then any spatially inhomogeneous non-monotone steady state of* (1.3) *is linearly exponentially unstable.*

Finally, we remark that the above results hold also for scalar nonlocal equations of the form

$$\begin{aligned} &u_t = u_{xx} + f(u,v,t), \quad x \in (0,1),\ t > 0, \\ &u_x(0,t) = 0 = u_x(1,t), \quad t > 0, \end{aligned}$$

with

$$v = \int_0^1 g(u,t)\,dx.$$

For this equation, the stability of solutions is defined in a similar manner as for the shadow system (1.1), using the linearized equation

$$\begin{aligned} &U_t = U_{xx} + f_u(u,v,t)U + f_v(u,v,t)V, && x \in (0,1),\ t > 0, \\ &U_x(0,t) = 0 = U_x(1,t), && t > 0, \end{aligned}$$

with

$$V = \int_0^1 g_u(u,t)U\,dx.$$

The proofs of our instability results work in this case with straightforward modifications.

The paper is organized as follows. In Section 2, in order to make our strategy clear, we give a short proof of Proposition 1.5. In Section 3, we give a proof of Theorem 1.2 by generalizing the arguments of Section 2 to the time-dependent case. Then we prove Theorem 1.1.

2. Instability of non-monotone steady states

In this section we consider the autonomous shadow system (1.3) and give a proof of Proposition 1.5.

Let $(u(x), \alpha)$ be a stationary solution of (1.3), that is, $(u(x), \alpha)$ satisfies

$$\begin{aligned} & u'' + f(u, \alpha) = 0, \quad x \in (0,1), \\ & u'(0) = 0 = u'(1), \\ & \int_0^1 g(u(x), \alpha) dx = 0. \end{aligned} \tag{2.1}$$

Clearly, if $(u(x), \alpha)$ is a nonconstant non-monotone solution of (2.1), then $u(x)$ is k-symmetric with some $k \geq 2$ and monotone in $[0, 1/k]$.

Let us consider the following eigenvalue problem associated with the linearized operator around $u(x)$:

$$\begin{aligned} & \ell\varphi(x) = \varphi''(x) + f_u(u(x), \alpha)\varphi(x), \quad x \in (0,1), \\ & \varphi'(0) = 0 = \varphi'(1). \end{aligned} \tag{2.2}$$

According to the Sturm-Liouville theory, the eigenvalues of (2.2) are real numbers $\ell_0 > \ell_1 > \ell_2 > \cdots \to -\infty$, and the corresponding eigenfunctions $\varphi_0, \varphi_1, \varphi_2, \ldots,$ are characterized by the property that φ_j has exactly j zeros in $(0,1)$. We assume that these eigenfunctions are normalized in $L^2(0,1)$.

Next, let us consider the eigenvalue problem

$$\begin{aligned} & \widetilde{\ell}\widetilde{\varphi}(x) = \widetilde{\varphi}''(x) + f_u(u(x), \alpha)\widetilde{\varphi}(x), \quad x \in (0, 1/k), \\ & \widetilde{\varphi}'(0) = 0 = \widetilde{\varphi}'(1/k). \end{aligned} \tag{2.3}$$

We denote by $\widetilde{\ell}_j$ and $\widetilde{\varphi}_j$ the jth eigenvalue and corresponding eigenfunction of (2.3), respectively. We assume that the eigenfunctions are normalized in $L^2(0, 1/k)$. Since $\widetilde{\varphi}_j$ has exactly j zeros in $(0, 1/k)$, it follows from reflection and the number of zeros that

$$\widetilde{\ell}_j = \ell_{jk}, \quad \widetilde{\varphi}_j(x) \equiv \sqrt{k}\varphi_{jk}(x) \quad \text{on} \quad [0, 1/k],$$

for all $j = 0, 1, 2, \ldots$.

Lemma 2.1. *Let $w(x)$ be any k-symmetric function on $[0,1]$. Then*

$$\int_0^1 w(x)\varphi_j(x) dx = 0, \quad j \neq 0, k, 2k, \ldots.$$

Proof. Let $\langle \cdot, \cdot \rangle_{L^2(a,b)}$ denote the L^2-inner product on (a, b). By reflection, we have for $x \in (0, 1/k)$

$$\begin{aligned} w &= \sum_{j=0}^{\infty} \langle w, \widetilde{\varphi}_j \rangle_{L^2(0,1/k)} \widetilde{\varphi}_j \\ &= \sum_{j=0}^{\infty} k \langle w, \varphi_{jk} \rangle_{L^2(0,1/k)} \varphi_{jk} \\ &= \sum_{j=0}^{\infty} \langle w, \varphi_{jk} \rangle_{L^2(0,1)} \varphi_{jk}. \end{aligned}$$

Hence, again by reflection, we obtain

$$w = \sum_{j=0}^{\infty} \langle\, w, \varphi_{jk} \,\rangle_{L^2(0,1)} \varphi_{jk} \quad \text{ on } [0,1].$$

On the other hand, we can expand w as

$$w = \sum_{j=0}^{\infty} \langle\, w, \varphi_{j} \,\rangle_{L^2(0,1)} \varphi_{j} \quad \text{ on } [0,1].$$

Comparing these two expansions termwise, we obtain the conclusion. □

Lemma 2.2. *If $u(x)$ is k-symmetric, then the eigenvalues of* (2.2) *satisfy* $\ell_0 > \ell_1 > \cdots > \ell_{k-1} > 0$.

Proof. Differentiating (2.1) by x, we obtain

$$\{u'(x)\}'' + f_u(u(x), \alpha)u'(x) = 0, \quad x \in (0,1).$$

We also have $u'(0) = u'(1) = 0$. Clearly $u'(x)$ has $k-1$ zeros in $(0,1)$ and $\varphi_j(x)$ has exactly j zeros in $(0,1)$. Then it follows from the Sturm comparison theorem (see, e.g. [7]) that $\ell_{k-1} > 0$. □

We now give a proof of Proposition 1.5.

Proof of Proposition 1.5. Let $(u(x), \alpha)$ be any spatially inhomogeneous non-monotone solution of (2.1), and consider the eigenvalue problem

$$\begin{aligned} &\lambda\Phi(x) = \Phi''(x) + f_u(u(x), \alpha)\Phi(x) + f_v(u(x), \alpha)\eta, \quad x \in (0,1), \\ &\lambda\eta = \int_0^1 \{g_u(u(x), \alpha)\Phi(x) + g_v(u(x), \alpha)\eta\}\, dx, \\ &\Phi'(0) = 0 = \Phi'(1). \end{aligned} \tag{2.4}$$

Since $g_u(u(x), \alpha)$ is k-symmetric with some $k \geq 2$, it follows from Lemma 2.1 that

$$\int_0^1 g_u(u(x), \alpha)\varphi_j(x) dx = 0 \quad \text{ for } j \neq 0, k, 2k, \ldots .$$

Hence $(\lambda, \Phi, \eta) = (\ell_j, \varphi_j, 0)$ satisfies (2.4) if $j \neq 0, k, 2k, \ldots$. This implies that $(U, V) = (e^{\ell_j t}\varphi_j(x), 0)$ satisfies (1.4) if $j \neq 0, k, 2k, \ldots$. Since $\ell_j > 0$ for $j = 1, 2, \ldots, k-1$ by Lemma 2.2, the steady state $(u(x), \alpha)$ is linearly exponentially unstable. □

3. Time-dependent case

In this section, we generalize the argument of the previous section to the time-dependent case. Our main tool to do this is a theory of Chow, Lu and Mallet-Paret [6] concerning the Floquet bundles for a linear parabolic equation (see also [16]). We briefly summarize their results in the following. We also recall basic properties of the zero number functional which plays an important role below.

For a function $v \in C[0,1]$, let $z(v)$ denote the number (possibly infinite) of sign changes of v in $(0,1)$. Specifically, $z(v)$ is the supremum of numbers n such that there are $0 < x_0 < \cdots < x_n < 1$ with $v(x_i)v(x_{i+1}) < 0$, $i = 0, \ldots, n-1$. Note that $v \mapsto z(v)$ is upper semicontinuous. If $v \in C^1[0,1]$ and all zeros of v in $(0,1)$ are simple, then $z(v)$ is equal to the number of these zeros.

Let us consider the linear parabolic equation

$$
\begin{aligned}
&\psi_t = \psi_{xx} + q(x,t)\psi, \qquad &(x,t) \in (a,b)\times \mathbf{R},\\
&\psi_x(a,t) = 0 = \psi_x(b,t), \qquad &t \in \mathbf{R},
\end{aligned}
\tag{3.1}
$$

where $q \in L^\infty((a,b)\times\mathbf{R})$. A proof of the following lemma can be found in [1, 4].

Lemma 3.1. *Let ψ be a nontrivial solution of* (3.1) *on an interval (t_0,T). Then the following properties hold:*

(i) *$z(\psi(\cdot,t))$ is a finite nonincreasing function of $t \in (t_0,T)$.*
(ii) *If for some $t_1 \in (t_0,T)$ the function $x \mapsto \psi(x,t_1)$ has a multiple zero in $[0,1]$, then $z(\psi(\cdot,t))$ decreases strictly at t_1:*

$$z(\psi(\cdot,t_2)) > z(\psi(\cdot,t_3)) \quad \textit{for any} \;\; t_2 < t_1 < t_3.$$

In particular, there are only a finite number of values t_1 for which $x \mapsto \psi(x,t_1)$ has a multiple zero.

Moreover, the same statements hold if at some (possibly each) boundary point $x_0 \in \{a,b\}$ the Neumann boundary condition is replaced by one of the following conditions:

$$\psi(x_0,t) \equiv 0 \;\; \textit{for} \;\; t\in(t_0,T) \qquad \textit{or} \qquad \psi(x_0,t) \neq 0 \;\; \textit{for} \;\; t \in (t_0,T).$$

We now recall results of [6] pertinent to our considerations. According to Corollary 5.3 of [6], for each j there is a solution ψ_j of (3.1) with $z(\psi_j(\cdot,t)) \equiv j$ ($t \in \mathbf{R}$), and the solution ψ_j is unique up to a constant multiple. Necessarily, by Lemma 3.1(ii), all zeros of $\psi_j(\cdot,t)$ in $[a,b]$ are simple for any t. In particular $\psi_j(0,t) \neq 0 \neq \psi_j(1,t)$ for any $t \in \mathbf{R}$. We normalize ψ_j by the condition $\|\psi_j(\cdot,0)\|_{L^2(a,b)} = 1$.

As shown in Proposition 5.6 of [6], ψ_j depends on $q(x,t)$ continuously in the weak* topology. More precisely, if $\{q_n\}$ is a sequence such that $q_n \to q$ in the weak* topology of $L^\infty((0,1)\times\mathbf{R})$, then for the corresponding functions ψ_j^n we have

$$\psi_j^n(\cdot,t) \to \psi_j(\cdot,t) \quad \text{in} \;\; C^1[0,1].$$

This convergence takes place for any t; in fact, due to standard continuous dependence properties, it is uniform on any compact time interval.

Next, let us consider the adjoint equation of (3.1)

$$
\begin{aligned}
&-\psi_t^* = \psi_{xx}^* + q(x,t)\psi^*, \qquad &(x,t) \in (a,b)\times \mathbf{R},\\
&\psi_x^*(a,t) = 0 = \psi_x^*(b,t), \qquad &t \in \mathbf{R}.
\end{aligned}
\tag{3.2}
$$

Note that the time reversal brings this equation to the form (3.1), with $q(x,t)$ replaced by $q(x,-t)$, hence [6] applies to (3.2). Let ψ_j^* be a solution of (3.2) with $z(\psi_j^*(\cdot,t)) \equiv j$. We normalize ψ_j^* by the condition $\|\psi_j^*(\cdot,0)\|_{L^2(a,b)} = 1$.

The functions ψ_j and ψ_j^*, as introduced above, are called the normalized *Floquet solutions* of (3.1) and (3.2), respectively.

It was proved in Proposition 6.3 of [6] that any function $v \in L^2(a,b)$ can be expanded as a convergent (in $L^2(a,b)$) Fourier series

$$w(x) = \sum_{j=0}^{\infty} \frac{\langle\, w, \psi_j(\cdot,0)\,\rangle_{L^2(a,b)}}{\langle\, \psi_j(\cdot,0), \psi_j^*(\cdot,0)\,\rangle_{L^2(a,b)}} \psi_j^*(x,0),$$

where $\langle \cdot, \cdot \rangle_{L^2(a,b)}$ denotes the L^2-inner product on (a, b). A translation of time shows that the same expansion is valid with $\psi_j(x, 0)$, $\psi_j^*(x, 0)$ replaced by $\psi_j(x, t)$, $\psi_j^*(x, t)$, respectively.

To apply the above results to (1.4), consider a solution (u, v) of (1.1) satisfying (A1). Define a continuous bounded function q by

$$q(x,t) := \begin{cases} f_u(u(x,0), v(0), 0) & \text{for } t < 0, \\ f_u(u(x,t), v(t), t) & \text{for } t \geq 0. \end{cases} \tag{3.3}$$

Then the Floquet solution $\psi = \psi_j(x, t)$ of (3.1) with $(a, b) = (0, 1)$ satisfies

$$\begin{aligned} &\psi_t = \psi_{xx} + f_u(u(x,t), v(t), t)\,\psi, && (x,t) \in (0,1) \times \mathbf{R}, \\ &\psi_x(0,t) = 0 = \psi_x(1,t), && t \in \mathbf{R}. \end{aligned} \tag{3.4}$$

We use $\{\psi_j\}$ to construct a solution of (1.4). For this end, we first extend Lemma 2.1 to the time-dependent case.

Lemma 3.2. *Let* (u, v) *be a* k*-symmetric solution of* (1.1) *satisfying* (A1)*, and* $\psi = \psi_j(x, t)$ *be the normalized Floquet solution of* (3.4)*. Then for any* k*-symmetric function* $w \in L^2(0, 1)$*, the equalities*

$$\int_0^1 w(x)\psi_j(x,t)dx = 0, \quad j \neq 0, k, 2k, \dots,$$

hold for all $t > 0$.

Proof. In the whole proof, q is as in (3.3), and $\psi = \psi_j(x, t)$, $\psi^* = \psi_j^*(x, t)$ are the normalized Floquet solutions of (3.4) and its adjoint equation, respectively. We denote by $\widetilde{\psi}_j$, $\widetilde{\psi}_j^*$ the normalized Floquet solutions of (3.4) and its adjoint equation, respectively, defined on $(a, b) = (0, 1/k)$. Since $q(x, t)$ is k-symmetric, it follows from reflection and the number of zeros that

$$\widetilde{\psi}_j \equiv \sqrt{k}\psi_{jk}, \quad (x,t) \in [0, 1/k] \times \mathbf{R},$$

and

$$\widetilde{\psi}_j^* \equiv \sqrt{k}\psi_{jk}^*, \quad (x,t) \in [0, 1/k] \times \mathbf{R},$$

for all $j = 0, 1, 2, \dots$.

Using the Fourier series and reflection, we obtain the following identities on the space interval $(0, 1/k)$ (for any fixed t):

$$\begin{aligned} w &= \sum_{j=0}^{\infty} \frac{\langle w, \widetilde{\psi}_j \rangle_{L^2(0,1/k)}}{\langle \widetilde{\psi}_j, \widetilde{\psi}_j^* \rangle_{L^2(0,1/k)}} \widetilde{\psi}_j^* \\ &= \sum_{j=0}^{\infty} \frac{\langle w, \psi_{jk} \rangle_{L^2(0,1/k)}}{\langle \psi_{jk}, \psi_{jk}^* \rangle_{L^2(0,1/k)}} \psi_{jk}^* \\ &= \sum_{j=0}^{\infty} \frac{\langle w, \psi_{jk} \rangle_{L^2(0,1)}}{\langle \psi_{jk}, \psi_{jk}^* \rangle_{L^2(0,1)}} \psi_{jk}^*. \end{aligned}$$

Hence, again by reflection, we obtain

$$w = \sum_{j=0}^{\infty} \frac{\langle w, \psi_{jk} \rangle_{L^2(0,1)}}{\langle \psi_{jk}, \psi_{jk}^* \rangle_{L^2(0,1)}} \psi_{jk}^* \quad \text{on } [0,1].$$

On the other hand, we have

$$w = \sum_{j=0}^{\infty} \frac{\langle w, \psi_j \rangle_{L^2(0,1)}}{\langle \psi_j, \psi_j^* \rangle_{L^2(0,1)}} \psi_j^* \quad \text{ on } [0,1].$$

Comparing these two expansions termwise, we obtain

$$\langle w, \psi_j \rangle_{L^2(0,1)} = \int_0^1 w(x)\psi_j(x,t)dx = 0, \quad t \in \mathbf{R},$$

for $j \neq 0, k, 2k, \ldots$. □

Next we extend Lemma 2.2 to the time-dependent case.

Lemma 3.3. *Let* (u, v) *be a* k*-symmetric solution of* (1.1) *with some* $k \geq 2$ *satisfying* (A1), (A2). *Then for any* $j \in \{0, 1, \ldots, k-1\}$, *the Floquet solution* ψ_j *satisfies*

$$\|\psi_j(\cdot, t)\|_{L^\infty(0,1)} > C_j \exp(\lambda_j t) \quad \textit{for all } t > 0$$

with some $C_j, \lambda_j > 0$.

Proof. Note that u_x satisfies a linear parabolic equation and Dirichlet boundary conditions. Using Lemma 3.1, there exists t_0 such that $u_x(\cdot, t)$ has only simple zeros in $[0, 1]$ for any $t \geq t_0$. In particular, $z(u_x(\cdot, t))$ is constant on $[t_0, \infty)$, and $u_{xx}(0, t) \neq 0 \neq u_{xx}(1, t)$ for $t \geq t_0$. Further, since $\psi_j(0, t) \neq 0$ for any t, replacing ψ_j by $-\psi_j$ if necessary, we may assume that

$$u_{xx}(0,t)\psi_j(0,t) > 0 \quad (t \geq t_0). \tag{3.5}$$

Fix any $j \in \{0, 1, \ldots, k-1\}$ and define

$$\sigma(t) := \inf\{c > 0 : z(c\psi_j(\cdot, t) - u_x(\cdot, t)) \leq j\}.$$

Since the zeros of $\psi_j(x, t)$ are all simple, $z(c\psi_j(\cdot, t) - u_x(\cdot, t)) = j$ for sufficiently large $c > 0$. Hence $\sigma(t)$ is well-defined and is finite. On the other hand, by symmetry, $u_x(x, t)$ has at least $k - 1$ zeros in $(0, 1)$ and these zeros are simple for $t \geq t_0$. This and (3.5) imply that $z(c\psi_j(\cdot, t) - u_x(\cdot, t)) \geq k > j$ for sufficiently small c. Therefore $\sigma(t) > 0$ for all $t \geq t_0$. Thus it is shown that $\sigma(t) \in (0, \infty)$ for all $t \geq t_0$.

Note that the upper semicontinuity of $v \mapsto z(v)$ and the definition of $\sigma(t)$ imply

$$z(\sigma(t)\psi_j(\cdot, t) - u_x(\cdot, t)) \leq j. \tag{3.6}$$

Since ψ_j and u_x satisfy the same linear equation and $c\psi_j(x, t) - u_x(x, t) \neq 0$ for $x = 0, 1$, $t \geq 0$ and $c > 0$, we can apply Lemma 3.1 to $c\psi_j - u_x$. Then $z(c\psi_j(\cdot, t) - u_x(\cdot, t))$ is nonincreasing in t so that $\sigma(t)$ is a nonincreasing function. We show that $\sigma(t)$ is strictly decreasing as a matter of fact. Assume it is not. Then there exist $t'' > t' > t_0$ such that $\sigma(t) \equiv \sigma_0$ on $[t', t'']$ (σ_0 is a positive constant). In this interval we choose t_1 such that $\sigma_0\psi_j(\cdot, t_1) - u_x(\cdot, t_1)$ has only simple zeros in $[0, 1]$ (cf. Lemma 3.1). But then for any $c \approx \sigma_0$ we have

$$z(c\psi_j(\cdot, t_1) - u_x(\cdot, t_1)) = z(\sigma_0\psi_j(\cdot, t_1) - u_x(\cdot, t_1)).$$

This clearly contradicts the definition of $\sigma(t_1) = \sigma_0$, showing that $\sigma(t)$ is strictly decreasing, as claimed.

For the exponential growth of ψ_j, it is sufficient to prove that

$$\limsup_{t\to\infty} \sigma(t+1)/\sigma(t) < 1. \tag{3.7}$$

Indeed, assume for a while this is the case. Then there exist $c_j > 0$ and $\lambda_j > 0$ such that $\sigma(t) < c_j \exp(-\lambda_j t)$ for all $t > 0$. We prove the relation

$$\liminf_{t\to\infty} \sigma(t)\|\psi_j(\cdot,t)\|_{L^\infty(0,1)} > 0, \tag{3.8}$$

which gives the desired exponential growth

$$\|\psi_j(\cdot,t)\|_{L^\infty(0,1)} > C_j \exp(\lambda_j t)$$

with some $C_j > 0$. We prove (3.8) by contradiction. Suppose that it does not hold, that is, for a sequence $t_n \to \infty$ we have

$$\lim_{n\to\infty} \sigma(t_n)\|\psi_j(\cdot,t_n)\|_{L^\infty(0,1)} = 0. \tag{3.9}$$

Passing to a subsequence, we may assume that

$$u(\cdot,t_n) \to \phi \quad \text{in} \quad C^2[0,1]$$

for some ϕ. Clearly, ϕ is k-symmetric as u is, and $\phi_x \not\equiv 0$ by (A2). It follows from Proposition 1.3 that ϕ_x changes sign near any of the symmetry points i/k, $i = 1,\dots,k-1$. Hence, by (3.9), $\sigma(t_n)\psi_j(\cdot,t_n) - u_x(\cdot,t_n)$ changes sign near these points if n is large enough. Furthermore, again by Proposition 1.3, $\phi_x \neq 0$ on an interval $(0,\varepsilon)$. Therefore, by (3.5), $\sigma(t_n)\psi_j(\cdot,t_n) - u_x(\cdot,t_n)$ also changes sign near 0 for large n. Thus we conclude that for large n, we have $z(\sigma(t_n)\psi_j(\cdot,t_n) - u_x(\cdot,t_n)) \geq k > j$, in contradiction to (3.6). This shows that (3.7) implies the exponential growth of ψ_j.

It remains to prove (3.7). We proceed by contradiction. Suppose that there is a sequence $\{t_i\}$ such that $t_i \to \infty$ and

$$\lim_{i\to\infty} \sigma(t_i+1)/\sigma(t_i) = 1.$$

Since $\sigma(t)$ is monotone decreasing, we obtain

$$\sigma(t_i+s)/\sigma(t_i) \to 1 \text{ uniformly in } s \in [0,1]. \tag{3.10}$$

Using hypotheses (A1), (A2) and Schauder estimates, one shows (cf. [5, Proof of Lemma 3.7]) that passing to a subsequence, we have

$$u(x,t_i+s) \to \widehat{u}(x,s) \quad \text{in} \quad C^2[0,1]$$

for some $\widehat{u}(x,s)$ and

$$f_u(u(x,t_i+s), v(t_i+s), t_i+s) \to \widehat{q}(x,s) \quad \text{in} \quad C^2[0,1]$$

for some $\widehat{q}(x,s)$. In both cases the convergence is uniform in any compact interval of $s \in \mathbf{R}$. Then $w := \widehat{u}_x$ satisfies

$$\begin{aligned} &w_s = w_{xx} + \widehat{q}(x,s)w, && (x,s) \in (0,1)\times\mathbf{R},\\ &w(0,s) = w(1,s) = 0, && s \in \mathbf{R}. \end{aligned}$$

Using the continuous dependence of ψ_j on q, as formulated above, we further obtain

$$\frac{\psi_j(x,t_i+s)}{\|\psi_j(\cdot,t_i)\|_{C^1(0,1)}} \to \widehat{\psi}(x,s) \quad \text{in} \quad C^1[0,1]$$

uniformly in any compact interval of $s \in \mathbf{R}$, where $\widehat{\psi}$ satisfies

$$\begin{aligned} &\widehat{\psi}_s = \widehat{\psi}_{xx} + \widehat{q}(x,s)\widehat{\psi}, && (x,s) \in (0,1) \times \mathbf{R}, \\ &\widehat{\psi}_x(0,s) = \widehat{\psi}_x(1,s) = 0, && s \in \mathbf{R}, \\ &\|\widehat{\psi}(\cdot,0)\|_{C^1} = 1. \end{aligned}$$

By Lemma 3.1, it is not difficult to see that $z(\widehat{\psi}(\cdot,s)) \equiv j$.

For simplicity, we set $\sigma_i = \sigma(t_i)$ and $\gamma_i = \|\psi_j(\cdot,t_i)\|_{C^1[0,1]}$. We claim that

$$0 < K_1 \le \sigma_i \gamma_i \le K_2 \quad \text{for all } i$$

with some positive constants K_1 and K_2. In fact, if there exists a subsequence with $\sigma_i\gamma_i \to \infty$ as $i \to \infty$, then for any constant $\beta > 0$

$$\begin{aligned} z(\beta\sigma_i\psi_j(\cdot,t_i) - u_x(\cdot,t_i)) &= z\left(\frac{\beta\psi_j(\cdot,t_i)}{\gamma_i} - \frac{u_x(\cdot,t_i)}{\sigma_i\gamma_i}\right) \\ &= z(\beta\widehat{\psi}(\cdot,0)) \\ &= j \end{aligned}$$

for sufficiently large i, because $\widehat{\psi}$ has only simple zeros. If $0 < \beta < 1$, this contradicts the definition of $\sigma(t_i)$. Hence $\sigma_i\gamma_i$ is bounded above.

Similarly, if $\sigma_i\gamma_i \to 0$ along a subsequence, we rewrite

$$z(\sigma_i\psi_j(\cdot,t_i) - u_x(\cdot,t_i)) = z\left(\frac{\sigma_i\gamma_i\psi_j(\cdot,t_i)}{\gamma_i} - u_x(\cdot,t_i)\right).$$

Repeating the arguments used earlier in ruling out (3.9), we obtain

$$z\left(\frac{\sigma_i\gamma_i\psi_j(\cdot,t_i)}{\gamma_i} - u_x(\cdot,t_i)\right) \ge k > j$$

for sufficiently large i. Thus we have a contradiction to (3.6), showing that $\sigma_i\gamma_i$ is bounded away from 0.

Now we can take a subsequence such that $\sigma_i\gamma_i \to b$ as $i \to \infty$ for some positive constant b. Then it follows from (3.10) that

$$\frac{\psi_j(x,t_i+s)}{\gamma_i} - \frac{u_x(x,t_i+s)}{\gamma_i\sigma(t_i+s)} \to \zeta(x,s) := \widehat{\psi}(x,s) - \frac{w(x,s)}{b} \quad \text{in } C^1[0,1]$$

uniformly in $s \in (0,1)$. Clearly, $\zeta(x,s)$ satisfies the linear equation

$$\zeta_s = \zeta_{xx} + \widehat{q}(x,t)\zeta, \quad (x,t) \in (0,1) \times \mathbf{R}.$$

Furthermore, we have $\widehat{\psi}(x,s) \ne 0$ for $x = 0,1$, by the simplicity of zeros, and $w(0,s) = w(1,s) = 0$. Thus Lemma 3.1 applies to ζ and we can find an $s \in [0,1]$ such that $\zeta(x,s)$ has only simple zeros. But then the same is true for the function

$$\frac{\psi_j(x,t_i+s)}{\gamma_i} - \frac{u_x(x,t_i+s)}{\gamma_i\sigma(t_i+s)},$$

for i sufficiently large. However, this is not possible by the definition of $\sigma(t_i+s)$. Thus (3.7) is proved. □

Now we are in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.2. Let $\psi_j(x,t)$ be the solution of (3.4) given as above. Since $g_u(u,v,t)$ is k-symmetric if u is k-symmetric, it follows from Lemma 3.2 that $(U,V)=(\psi_j(x,t),0)$ satisfies (1.4). Then it follows from Lemma 3.3 that (u,v) is linearly exponentially unstable. $\square$

Proof of Theorem 1.1. As in the proof of Theorem 1.2, we use (H1), (H2) and Schauder estimates to find a sequence $t_i\to\infty$ such that

$$\begin{aligned} &u(\cdot,t+t_i)\to\widehat{u}(\cdot,t), && v(t+t_i)\to\widehat{v}(t),\\ &f(u,v,t+t_i)\to\widehat{f}(u,v,t), && g(u,v,t+t_i)\to\widehat{g}(u,v,t),\end{aligned}$$

for some $\widehat{u}$, $\widehat{v}$, $\widehat{f}$, $\widehat{g}$ with

$$\begin{aligned} &\widehat{u}_t=\widehat{u}_{xx}+\widehat{f}(\widehat{u},\widehat{v},t), \quad x\in(0,1),\ t>0,\\ &\widehat{u}_x(0,t)=0=\widehat{u}_x(1,t), \quad t>0,\\ &\widehat{v}_t=\int_0^1\widehat{g}(\widehat{u},\widehat{v},t)dx, \quad t>0.\end{aligned}$$

Also the hypotheses (H1), (H2), (A1), (A2) hold for the new functions (the Hölder exponent in (H2) may have to be made smaller). Furthermore, if $T(t,s)$ is the evolution operator of (1.4), then $T(t+t_i,s+t_i)$ converges in the operator norm of $C[0,1]\times\mathbf{R}^m$ to the evolution operator $\widehat{T}(t,s)$ of

$$\begin{aligned} &U_t=U_{xx}+\widehat{f}_u(\widehat{u},\widehat{v},t)U+\widehat{f}_v(\widehat{u},\widehat{v},t)V, && x\in(0,1),\ t>0,\\ &V_t=\int_0^1\{\widehat{g}_u(\widehat{u},\widehat{v},t)U+\widehat{g}_v(\widehat{u},\widehat{v},t)V\}dx, && t>0,\\ &U_x(0,t)=0=U_x(1,t), && t>0.\end{aligned}$$

Since (u,v) is assumed to be at least linearly neutrally stable, $(\widehat{u},\widehat{v})$ also is at least linearly neutrally stable.

Now, by Proposition 1.3, there is an integer $k\geq 1$ such that $\widehat{u}_x(x,t)\neq 0$ for any $x\in(0,1/k)$ and $t\in\mathbf{R}$, and if $k\geq 2$ then $\widehat{u}$ is k-symmetric. The case $k\geq 2$ is immediately ruled out, since $(\widehat{u},\widehat{v})$ would then be linearly exponentially unstable by Theorem 1.2. We thus have $k=1$ and $\widehat{u}_x(x,t)\neq 0$ for every $x\in(0,1)$. Since $\widehat{u}_x$ satisfies a linear parabolic equation and Dirichlet boundary conditions, there is a t' such that $\widehat{u}_{xx}(0,t')\neq 0\neq\widehat{u}_{xx}(1,t')$. Since the convergence $u(\cdot,t'+t_i)\to\widehat{u}(\cdot,t')$ takes place in $C^2[0,1]$, we have, for some t_i, $u_x(x,t'+t_i)\neq 0$ for every $x\in(0,1)$. Consequently, $u_x(x,t)\neq 0$ for every $x\in(0,1)$ and for every $t\geq t'+t_i$.

This completes the proof of Theorem 1.1. $\square$

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