TOPOLOGICAL DYNAMICS ON MODULI SPACES II

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ABSTRACT. Let M be an orientable genus $g > 0$ surface with boundary ∂M . Let Γ be the mapping class group of M fixing ∂M . The group Γ acts on $\mathcal{M}_{\mathcal{C}} = \text{Hom}_{\mathcal{C}}(\pi_1(M), \text{SU}(2))/\text{SU}(2)$, the space of $\text{SU}(2)$ -gauge equivalence classes of flat $\text{SU}(2)$ -connections on M with fixed holonomy on ∂M . We study the topological dynamics of the Γ -action and give conditions for the individual Γ -orbits to be dense in $\mathcal{M}_{\mathcal{C}}$.

1. INTRODUCTION

Let M be an orientable surface of genus g with n boundary components (circles). Let

$$\{C_1, C_2, \dots, C_n\} \subset \pi_1(M)$$

be elements in the fundamental group that correspond to these n boundary components.

The representation space $\text{Hom}(\pi_1(M), \text{SU}(2))$ has a natural topology [4, 5]. The group $\text{SU}(2)$ acts on $\text{Hom}(\pi_1(M), \text{SU}(2))$ by conjugation. Define the resulting quotient space to be

$$\mathcal{M} = \text{Hom}(\pi_1(M), \text{SU}(2))/\text{SU}(2).$$

A conjugacy class in $\text{SU}(2)$ is determined by its trace in $[-2, 2]$. To each C_i , assign a trace $-2 \leq c_i \leq 2$ and let

$$\mathcal{C} = \{c_1, c_2, \dots, c_n\}.$$

Definition 1.1. The moduli space with fixed holonomy $\mathcal{C}$ is

$$\mathcal{M}_{\mathcal{C}} = \{[\rho] \in \mathcal{M} : \text{tr}(\rho(C_i)) = c_i, 1 \leq i \leq n\}.$$

The space $\mathcal{M}_{\mathcal{C}}$ is compact, but possibly singular. The set of smooth points of $\mathcal{M}_{\mathcal{C}}$ possesses a natural symplectic structure which gives rise to a finite measure μ on $\mathcal{M}_{\mathcal{C}}$ (see [4, 5]).

Let $\text{Diff}(M, \partial M)$ be the group of diffeomorphisms fixing ∂M . The mapping class group Γ is defined to be $\pi_0(\text{Diff}(M, \partial M))$. The group Γ acts on $\pi_1(M)$. This action induces an action on $\text{Hom}(\pi_1(M), \text{SU}(2))$: $\Gamma \times \text{Hom}(\pi_1(M), \text{SU}(2)) \longrightarrow \text{Hom}(\pi_1(M), \text{SU}(2))$. If $\gamma \in \Gamma$ and $\rho \in \text{Hom}(\pi_1(M), \text{SU}(2))$, then $(\gamma \circ \rho)(X) = \rho(\gamma^{-1}(X))$. This, in turn, gives an action $\Gamma \times \mathcal{M}_{\mathcal{C}} \longrightarrow \mathcal{M}_{\mathcal{C}}$.

Theorem 1.2 (Goldman [4]). *The mapping class group Γ acts ergodically on $\mathcal{M}_{\mathcal{C}}$.*

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One may also study the topological dynamics of the mapping class group action on $\mathcal{M}_C$. The topological-dynamical problem is considerably more delicate. To begin with, not all orbits are dense in $\mathcal{M}_C$. If G is a proper closed subgroup of $\mathrm{SU}(2)$, then the image of $\mathrm{Hom}(\pi_1(M), G)$ in $\mathcal{M}_C$ is proper, closed and Γ -invariant. The case of the one-holed torus has been dealt with in [9]:

Theorem 1.3. *Suppose M is a torus with one boundary component, and that $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$ is such that $\rho(\pi_1(M))$ is dense in $\mathrm{SU}(2)$. Then the Γ -orbit of the conjugacy class $[\rho] \in \mathcal{M}_C$ is dense in $\mathcal{M}_C$.*

This paper deals with the general case of $g > 0$ and proves:

Theorem 1.4. *Suppose M is an orientable surface with boundary having genus greater than zero. Let $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$ be such that $\rho(\pi_1(M))$ is dense in $\mathrm{SU}(2)$. Then the Γ -orbit of the conjugacy class $[\rho] \in \mathcal{M}_C$ is dense in $\mathcal{M}_C$.*

The group $\mathrm{SU}(2)$ double covers the group $\mathrm{SO}(3)$. The group $\mathrm{SO}(3)$ contains closed subgroups isomorphic to $\mathrm{O}(2)$, as well as the symmetry groups of the regular polyhedra: T' (the tetrahedron), C' (the cube), D' (the dodecahedron), and their subgroups. The inverse images (by the covering map) of $\mathrm{O}(2)$, T' , C' , D' will be called $\mathrm{Pin}(2)$, T , C , D , respectively. The identity component of $\mathrm{Pin}(2)$ is called $\mathrm{Spin}(2)$. Let $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$. Theorem 1.4 implies that if $\rho(\pi_1(M))$ is not contained in a group isomorphic to C, D , or $\mathrm{Pin}(2)$, then the Γ -orbit of the conjugacy class $[\rho] \in \mathcal{M}_C$ is dense in $\mathcal{M}_C$. Theorem 1.4 covers all moduli spaces except those of the n -holed spheres.

1.1. Outline of the proof. A pants decomposition $\mathcal{P}$ of M gives rise to a smooth open dense subset $\mathcal{M}_{\mathcal{P}}$ that is an integrable system (see [7]) inside the moduli space $\mathcal{M}_C$. Hence, one obtains the following diagram [4]:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{P}} & \xrightarrow{i} & \mathcal{M}_C \\ \downarrow f_{\mathcal{P}} & & \downarrow f_{\mathcal{P}} \\ P & \xrightarrow{i} & P' \end{array}$$

where $\mathcal{M}_{\mathcal{P}}$ is a torus bundle over P , and $P' \subset [-2, 2]^N$, where $N = \frac{1}{2} \dim(\mathcal{M}_C)$. The subgroup $\Gamma_{\mathcal{P}} \subset \Gamma$ that preserves the fibres of $f_{\mathcal{P}}$ acts as rotations on each fibre with angles depending on the base coordinates [4]. Section 2 gives a brief outline of this integrable system and the decomposition of the Γ -action.

The proof of Theorem 1.4 involves two steps. Suppose that $[\rho]$ is a generic representation (i.e. $\rho(\pi_1(M))$ is dense in $\mathrm{SU}(2)$). Let $\Gamma([\rho])$ denote the Γ -orbit of $[\rho]$. The first step is to show that if $f_{\mathcal{P}}(\Gamma([\rho]))$ is dense in P , then $\Gamma([\rho])$ is dense in $\mathcal{M}_C$ (Corollary 2.4). The second step involves proving the base density theorem, i.e., the density of $f_{\mathcal{P}}(\Gamma([\rho]))$ in P .

As the problem deals with arbitrary genus, the proof involves induction with the one-holed torus as the base case. For a generic representation ρ , one first shows that there is a one-holed torus T inside M such that the restriction of ρ to $\pi_1(T)$ is generic. This is a detailed combinatorial calculation which is outlined in Section 3.

After obtaining a generic handle, we proceed to demonstrate the base density theorem for the $(n + 2g - 2)$ -holed torus. An analysis of the case of the four-holed sphere is required to get the induction process started. This is used to prove the result for the case of the two-holed torus. From there, the case of the three-holed

torus is proven, which, in turn, is used to prove the case of the $(n + 2g - 2)$ -holed torus.

To complete the proof, the $2g - 2$ holes of the $(n + 2g - 2)$ -holed torus are grouped in pairs, and each pair is glued along their boundary to obtain the original surface M with genus g and n boundary components.

1.2. Some definitions. Fix a surface M with genus $g > 0$ and n boundary components. Then M may be described as an n -holed $2g$ -gon, with appropriate identifications. More precisely, the fundamental group $\pi_1(M, O)$ is generated by $S = \{A_i\}_{i=1}^{2g+n}$, subject to the relation

$$\left(\prod_{i=1}^g [A_i, A_{i+g}]\right) \left(\prod_{i=2g+1}^{2g+n} A_i\right) = e.$$

Definition 1.5.

1. A representation ρ into $SU(2)$ is generic if the image of ρ is dense in $SU(2)$.
2. A handle (A, B) consists of two simple loops $A, B \in \pi_1(M, O)$ crossing at O , but otherwise disjoint (see [3]).
3. A handle (A, B) is generic with respect to a representation ρ if the image of $\langle A, B \rangle \subset \pi_1(M)$ is dense in $SU(2)$ (cf. Theorem 1.3).
4. Suppose $G \subset SU(2)$. A representation ρ is said (resp. not) to be G if $Im(\rho)$ is (resp. not) contained in some (resp. any) isomorphic copy of G in $SU(2)$.
5. Associated to each simple loop $A \in \pi_1(M, O)$ is the Dehn twist in A , represented in Γ by a diffeomorphism of M . The action of the Dehn twist amounts to cutting M at A , twisting one of the resulting boundary circles once, and then re-identifying the two circles.
6. With a fixed representation ρ , $X \in \pi_1(M, O)$, and $\gamma \in \Gamma$, we write X for $\rho(X)$ and $\gamma(X)$ for $\gamma(\rho(X))$ when there is no ambiguity. A small letter will be used to denote the trace of the matrix represented by the corresponding capital letter. For example, we use x to denote $\text{tr}(\rho(X))$. In this setting, $\gamma(x)$ denotes $\text{tr}(\gamma(\rho(X)))$.
7. Let $\langle V, d \rangle$ be a metric space. For any $\epsilon > 0$, a subset $U \subset V$ is ϵ -dense in V if for each $v \in V$, there exists a point $u \in U$ such that $d(u, v) < \epsilon$.

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2. MODULI SPACES AS TORI BUNDLES

We begin by giving a brief description of the integrable system on the moduli space [5]. Suppose M has genus $g \geq 1$ and $n \geq 0$ boundary components. Let

$$\mathcal{C} = \{c_1, \dots, c_n\}$$

be a fixed set of conjugacy classes with $c_i \neq \pm 2$ for all i . Then the real dimension of $\mathcal{M}_{\mathcal{C}}$ is $6g - 6 + 2n$. Since the case of the torus is well understood [9], we assume throughout the remainder of the paper that $g > 1$ or $n > 0$.

There is a map

$$f_{\mathcal{P}} : \mathcal{M}_{\mathcal{C}} \longrightarrow [-2, 2]^N$$

that arises from a pants decomposition $\mathcal{P}$ of M , where $N = 3g - 3 + n$ (see [4]). Fix a pants decomposition $\mathcal{P}$ of M . This provides $3g - 3 + n$ loops $B_1, \dots, B_{3g-3+n} \in \pi_1(M, O)$. Let $[\rho] \in \mathcal{M}_{\mathcal{C}}$ be given. Then

$$f_{\mathcal{P}}([\rho]) = (b_1, \dots, b_{3g-3+n})$$

is the desired map, where $b_i = \text{tr}(\rho(B_i))$. Let $\beta = (b_1, \dots, b_{3g-3+n})$, let P' be the image of $f_{\mathcal{P}}$ and let $P = P' \setminus \partial P'$. The map $f_{\mathcal{P}}$ restricted to $\mathcal{M}_{\mathcal{P}} = f_{\mathcal{P}}^{-1}(P)$ is a submersion [4]. Denote by $\Gamma_{\mathcal{P}} \subset \Gamma$ the stabilizer of the fibres of $f_{\mathcal{P}}$.

Proposition 2.1. *The set P' consists of all $\beta \in [-2, 2]^{3g-3+n}$ that simultaneously satisfy the $2g - 2 + n - 1$ inequalities*

$$b_i^2 + b_j^2 + b_k^2 - b_i b_j b_k \leq 4,$$

where the three curves B_i, B_j and B_k bound a triply punctured sphere in the decomposition $\mathcal{P}$ of M (this includes the possibility of B_i being a boundary curve in ∂M). Moreover, for $\beta \in P$, there is a $\Gamma_{\mathcal{P}}$ -equivariant homeomorphism

$$h : f_{\mathcal{P}}^{-1}(\beta) \rightarrow T^{3g-3+n}$$

such that for all $\xi \in f_{\mathcal{P}}^{-1}(\beta)$ and $(n_1, \dots, n_{3g-3+n}) \in \mathbb{Z}^{3g-3+n}$

$$h : \tau_1^{n_1} \dots \tau_{3g-3+n}^{n_{3g-3+n}} \xi \mapsto \begin{bmatrix} e^{in_1 \theta_1} h_1 \\ \vdots \\ e^{in_{3g-3+n} \theta_{3g-3+n}} h_{3g-3+n} \end{bmatrix},$$

where

$$h(\xi) = \begin{bmatrix} h_1 \\ \vdots \\ h_{3g-3+n} \end{bmatrix},$$

$\theta_j = \cos^{-1}(b_j/2)$, and τ_i is the action of the Dehn twist in B_i .

Proof. See [4]. □

In short, $(\mathcal{M}_{\mathcal{P}}, f_{\mathcal{P}})$ is an integrable system (see [7] for details). The real dimensions of P and $\mathcal{M}_{\mathcal{P}}$ are $3g - 3 + n$ and $6g - 6 + 2n$, respectively. Denote by $\mathcal{M}_{\mathcal{C}}^s$ the subset of irreducible representations of $\mathcal{M}_{\mathcal{C}}$.

Lemma 2.2. *$\mathcal{M}_{\mathcal{C}}^s$ is smooth, open, and dense in $\mathcal{M}_{\mathcal{C}}$.*

Proof. See [4, 5]. □

Proposition 2.3. *The subset $\mathcal{M}_{\mathcal{P}}$ is open and dense in $\mathcal{M}_{\mathcal{C}}^s$.*

Proof. A direct calculation from Proposition 2.1 shows that $\mathcal{M}_{\mathcal{C}}^s \setminus \mathcal{M}_{\mathcal{P}}$ is a real algebraic subvariety with positive codimension. The result then follows from the fact that $\mathcal{M}_{\mathcal{C}}^s$ is smooth and has dimension $6g - 6 + 2n$. □

Together, Propositions 2.1 and 2.3 imply:

Corollary 2.4. *Let $[\rho] \in \mathcal{M}_{\mathcal{C}}$, and let $\Gamma([\rho])$ be the Γ -orbit of $[\rho]$. If $f_{\mathcal{P}}(\Gamma([\rho]))$ is dense in P , then $\Gamma([\rho])$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

 FIGURE 1. $(A_i, A_{g+i}A_j)$ is a handle.

3. GENERIC REPRESENTATIONS AND HANDLES

Here we adapt an idea in [3] to find a generic handle (one-holed torus) inside M .

Proposition 3.1. *For any generic $\rho \in \text{Hom}(\pi_1(M, O), \text{SU}(2))$, there exists a generic handle (A, B) .*

The proof of Proposition 3.1 is highly computational. One should first consult Section 5 and [9], as the proof involves the moduli spaces of one-holed tori and uses many ideas pertaining to those spaces.

The proof starts with a generic representation ρ , which when restricted to each handle $\langle A_i, A_{g+i} \rangle$ is $\text{Spin}(2)$, but not $\text{Spin}(2)$ on M . One shows that there exists j such that $\langle A_i, A_{g+i}A_j \rangle$ is not $\text{Spin}(2)$, with $\langle A_i, A_{g+i}A_j \rangle$ forming a handle (see Figure 1).

From there, one applies the same technique to the groups $\text{Pin}(2)$, T , C , and finally to D . This process amounts to numerous routine computations, which we omit here but which are available at <http://vortex.bd.psu.edu/~jpp/td2/> in the form of MAPLE worksheets with accompanying text.

4. THE THREE-HOLED SPHERE

Suppose M is a three-holed sphere. Then $\pi_1(M)$ has a presentation:

$$\langle A, B, C : ABC = I \rangle,$$

where A, B , and C represent the homotopy classes of the three boundaries of M . A direct computation, together with results from [4] and [9], shows

Proposition 4.1.

1. *A representation ρ on a three-holed sphere is a $\text{Spin}(2)$ -representation if and only if $a^2 + b^2 + c^2 - abc - 4 = 0$.*

2. A representation ρ on a three-holed sphere is $\text{Pin}(2)$ and not $\text{Spin}(2)$ if and only if $a^2 + b^2 + c^2 - abc - 4 \neq 0$ and at least two of the three values a, b, c are zero.

5. THE ONE-HOLED TORUS

We briefly summarize some relevant results that appear in [4] and [9]. Suppose that M is a one-holed torus. The fundamental group $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle X, Y, K : K = XYX^{-1}Y^{-1} \rangle,$$

where K represents the element corresponding to the boundary component. Let

$$E = \text{Hom}(\pi_1(M), \text{SU}(2)) / \text{SU}(2).$$

A representation class $[\rho] \in E$ is determined by

$$x = \text{tr}(\rho(X)), \quad y = \text{tr}(\rho(Y)), \quad z = \text{tr}(\rho(XY)).$$

Therefore, there is a global coordinate chart:

$$[\rho] \xrightarrow{F} (x, y, z).$$

In addition, $k = \text{tr}(\rho(K))$ is given by the formula

$$k = \text{tr}(\rho(K)) = x^2 + y^2 + z^2 - xyz - 2.$$

Let

$$E_k = \{(x, y, z) \in [-2, 2]^3 : x^2 + y^2 + z^2 - xyz - 2 = k\};$$

then

$$E = \bigcup_{k \in [-2, 2]} E_k.$$

For $-2 < k < 2$, the set E_k is a smooth two-sphere; the set E_2 is a singular sphere, and $E_{-2} = (0, 0, 0)$.

The mapping class group Γ is generated by the maps τ_X and τ_Y induced by the Dehn twists in $X, Y \in \pi_1(M)$. These act on the fundamental group as

$$\tau_X(X) = X \text{ and } \tau_X(Y) = YX^{-1}$$

and

$$\tau_Y(X) = XY^{-1} \text{ and } \tau_Y(Y) = Y.$$

The effect on a representation ρ is as follows:

$$(\tau_X \circ \rho)(X) = \rho(X) \text{ and } \tau_X(\rho(Y)) = \rho(YX)$$

and

$$(\tau_Y \circ \rho)(X) = \rho(XY) \text{ and } \tau_Y(\rho(Y)) = \rho(Y).$$

Remark 5.1. The reader should be aware that when Y actually stands for $\rho(Y)$ (Definition 1.5.6), $\tau_X(Y) = YX$.

The induced action of Γ on E preserves E_k . The actions can be described explicitly [4]:

$$\tau_X(x, y, z) = (x, z, xz - y),$$

$$\tau_Y(x, y, z) = (z, y, yz - x).$$

The action of τ_X fixes x and k , and preserves the ellipse

$$X_k(x) = \{x\} \times \{(y, z) : \frac{2-x}{4}(y+z)^2 + \frac{2+x}{4}(y-z)^2 = 2+k-x^2\}.$$

A change of coordinates transforms $X_k(x)$ into the circle

$$X_k(x) = \{x\} \times \{(\tilde{y}, \tilde{z}) : \tilde{y}^2 + \tilde{z}^2 = 2+k-x^2\}$$

(see [4, 9]). In this new coordinate system, τ_X acts as a rotation by $\cos^{-1}(x/2)$. In short, the sphere E_k is the union of circles

$$E_k = \bigcup_x X_k(x),$$

and τ_X rotates (up to a coordinate transformation) each level set $X_k(x)$ by an angle of $\cos^{-1}(x/2)$. Similarly, there is a coordinate transformation so that τ_Y acts as a rotation of $Y_k(y)$ by an angle of $\cos^{-1}(y/2)$.

Proposition 5.2. *The space of $\text{Spin}(2)$ representation classes consists precisely of E_2 . The $\text{Pin}(2)$ representation classes consist of E_2 and the intersections of the three coordinate axes with E . For each $-2 < k < 2$, there are exactly six points corresponding to $\text{Pin}(2)$ representation classes in E_k . Moreover, a representation class $(x, y, z) \in E_k$ with $-2 < k < 2$ and $x \neq 0$ is $\text{Pin}(2)$ if and only if $k = x^2 - 2$.*

Proof. See [9]. □

Remark 5.3. This is the first explicit example of Proposition 2.1 with $g = 1$ and $n = 1$. One obtains a pair of pants by cutting along X (resp. Y). The important property is that if $X_k(x)$ (resp. $Y_k(y)$) is a non-degenerate circle, then τ_X (resp. τ_Y) acts on the fibre $X_k(x)$ (resp. $Y_k(y)$) as a rotation with an angle depending solely on x (resp. y) and is independent of either k or y (resp. x). In particular, if a representation ρ is not $\text{Pin}(2)$, then neither τ_X nor τ_Y fixes $[\rho]$.

Remark 5.4. Let ρ be generic. By Theorem 1.3, the $\langle \tau_X, \tau_Y \rangle$ -orbit $\mathcal{O}$ of $[\rho]$ is dense in E_k . Hence, there is a number $r > 0$ such that the set $\{(x, y) : (x, y, z) \in \mathcal{O}\}$ is dense in $R = [-r, r]^2$. In particular, by Dehn twisting in τ_X and τ_Y , one can always assume that x and y are simultaneously off of any finite set of values and are arbitrarily close to zero.

Remark 5.5. In addition to Theorem 1.3, it was shown in [9] that the k -values for the surjective T, C, D representations are in the set

$$\mathcal{S} = \left\{ \frac{1 \pm \sqrt{5}}{2}, 0, 1 \right\}.$$

In particular, if (x, y, z) is not $\text{Pin}(2)$ and $k \notin \mathcal{S}$, then (x, y, z) is generic.

6. THE FOUR-HOLED SPHERE

We first review some results that appear in [1] and [4]. Suppose M is a four-holed sphere. Then the fundamental group $\pi_1(M, O)$ admits a presentation

$$\langle A, B, C, D : ABCD = I \rangle.$$

FIGURE 2. The four-holed sphere

6.1. The moduli space $\mathcal{M}$. If one of $a, b, c, d = \pm 2$, the moduli space can be identified with a moduli space of a three-holed sphere, so we assume that $a, b, c, d \neq \pm 2$. The moduli space $\mathcal{M}$ for the four-holed sphere is six-dimensional. Let $g : \mathcal{M} \rightarrow [-2, 2]^4$ be the map defined by $g([\rho]) = (a, b, c, d)$.

Set $X = AB$, $Y = BC$, and $Z = CA$ (see Figure 2). With the (x, y, z) -global coordinates, the moduli space $E_{(a,b,c,d)} = g^{-1}(a, b, c, d)$ is the level set defined by the equation

$$x^2 + y^2 + z^2 + xyz = (ab + cd)x + (ad + bc)y + (ac + bd)z - (a^2 + b^2 + c^2 + d^2 + abcd - 4).$$

Generically, $E_{(a,b,c,d)}$ is a topological two-sphere.

Let

$$I_{a,b} = \left[\frac{ab - \sqrt{(a^2 - 4)(b^2 - 4)}}{2}, \frac{ab + \sqrt{(a^2 - 4)(b^2 - 4)}}{2} \right].$$

For any $\kappa = (a, b, c, d) \in (-2, 2)^4$ with $x \in I_{a,b} \cap I_{c,d}$, the x -level sets $X_\kappa(x) \subset E_\kappa$ are ellipses (possibly degenerate) in y and z given by

$$\begin{aligned} & \frac{2+x}{4} \left[(y+z) - \frac{(a+b)(d+c)}{2+x} \right]^2 + \frac{2-x}{4} \left[(y-z) - \frac{(a-b)(d-c)}{2-x} \right]^2 \\ &= \frac{(x^2 - abx + a^2 + b^2 - 4)(x^2 - cdx + c^2 + d^2 - 4)}{4 - x^2}. \end{aligned}$$

If $x \notin I_{a,b} \cap I_{c,d}$, then the x -level sets are empty. There are similar descriptions for the y - and z -level sets $Y_\kappa(y)$ and $Z_\kappa(z)$ respectively (see [4]).

For x in the interior of $I_{a,b} \cap I_{c,d}$ the level set $X_\kappa(x)$ is an ellipse centered at

$$(1) \quad \begin{cases} y_c(x) &= \frac{2[2(ad+bc)-x(ac+bd)]}{4-x^2}, \\ z_c(x) &= \frac{2[2(ac+bd)-x(ad+bc)]}{4-x^2}. \end{cases}$$

Note that

$$E_\kappa = \bigcup_{x \in I_{a,b} \cap I_{c,d}} X_\kappa(x)$$

(see Proposition 2.1 and [1]). By symmetry, similar constructs exist for the y - and z -coordinates.

6.2. The mapping class action. The Dehn twist along X acts on $\pi_1(M)$ as follows:

$$\tau_X(A) = A, \quad \tau_X(B) = B, \quad \tau_X(C) = X^{-1}CX, \quad \tau_X(D) = X^{-1}DX.$$

This action effects the representation ρ as

$$\tau_X(\rho(A)) = \rho(A), \quad \tau_X(\rho(B)) = \rho(B), \quad \tau_X(\rho(C)) = \rho(XCX^{-1}),$$

$$\tau_X(\rho(D)) = \rho(XDX^{-1}).$$

We define τ_Y and τ_Z similarly. In local coordinates, this implies that the actions of τ_X, τ_Y, τ_Z are

$$\begin{aligned} \begin{bmatrix} y \\ z \end{bmatrix} &\xrightarrow{\tau_X} \begin{bmatrix} ad + bc - x(ac + bd - xy - z) - y \\ ac + bd - xy - z \end{bmatrix}, \\ \begin{bmatrix} z \\ x \end{bmatrix} &\xrightarrow{\tau_Y} \begin{bmatrix} bd + ca - y(ba + cd - yz - x) - z \\ ba + cd - yz - x \end{bmatrix}, \\ \begin{bmatrix} x \\ y \end{bmatrix} &\xrightarrow{\tau_Z} \begin{bmatrix} cd + ab - z(cb + ad - zx - y) - x \\ cb + ad - zx - y \end{bmatrix}. \end{aligned}$$

These actions preserve the ellipses $X_\kappa(x) \subset E_\kappa$, $Y_\kappa(y) \subset E_\kappa$, and $Z_\kappa(z) \subset E_\kappa$, respectively. After coordinate transformations, these are rotations by the angles $2 \cos^{-1}(x/2)$, $2 \cos^{-1}(y/2)$, and $2 \cos^{-1}(z/2)$, respectively [4].

Remark 6.1. This is the second explicit example of Proposition 2.1 with $g = 0$ and $n = 4$. One obtains two pairs of pants by cutting along X (resp. Y or Z). The important property is that if $X_\kappa(x)$ (resp. $Y_\kappa(y)$ or $Z_\kappa(z)$) is a non-degenerate circle, then τ_X (resp. τ_Y or τ_Z) acts on the fibre $X_\kappa(x)$ (resp. $Y_\kappa(x)$ or $Z_\kappa(z)$) as a rotation with an angle depending only on x (resp. y or z) and is independent of κ .

Let d be the metric on $\mathbb{R}^3$ given by

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}.$$

The metric d generates the usual topology on $\mathcal{M}_C$. For a fixed κ , the coordinates provide an embedding of E_κ into $\mathbb{R}^3$.

6.3. Filtration on the level sets. We introduce a filtration that is analogous to the one introduced in [9] for the one-holed torus. The Dehn twist τ_Y acts on the (transformed) subsets $Y_\kappa(y)$ via a rotation of angle $2 \cos^{-1}(y/2)$. Thus there is a filtration of the y -coordinates that yields finite orbits under τ_Y as follows:

Let $Y_n \subset (-2, 2)$ be such that $y \in Y_n$ if and only if the τ_Y -action on non-fixed points $(x, y, z) \in E_\kappa$ is periodic with period less than or equal to n . This gives a filtration

$$\{0\} = Y_2 \subset Y_3 \subset \dots \subset Y_n \subset \dots$$

For example: $Y_2 = \{0\}$, $Y_3 = \{0, 1, -1\}$, $Y_4 = \{0, 1, -1, \sqrt{2}, -\sqrt{2}\}$, etc. Note that the filtration is independent of the choice of κ and that Y_n is a finite set for every n . By symmetry, there are similar filtrations X_n and Z_n , with $X_n = Y_n = Z_n$ as sets. The following lemmas, though proven for the Y_n filtration, apply equally to the other filtrations.

Lemma 6.2. *For $\epsilon > 0$ there exists $N(\epsilon) > 0$ so that if $y \notin Y_{N(\epsilon)}$, then the τ_Y -orbit of (x, y, z) is ϵ -dense in $Y_\kappa(y)$ for any (x, y, z) in any E_κ (see part 7 of Definition 1.5).*

Proof. Since the ellipses $Y_\kappa(y)$ are (possibly degenerate) of uniformly bounded circumferences, there exists $N(\epsilon) > 1$ such that for any $y \notin Y_{N(\epsilon)}$, the τ_Y -orbit is ϵ -dense in $Y_\kappa(y)$. $\square$

Throughout the remainder of the paper, the moduli spaces of four-holed spheres with $\kappa = (a, b, c, d)$, having small $|c|$ and $|d|$, play an important role. We first analyze the case of E_κ for $\kappa = (a, b, 0, 0)$.

Lemma 6.3. *Suppose $(a, b, 0, 0) = \kappa \in (-2, 2)^4$. Then $X_\kappa(x) \subset E_\kappa$ is an ellipse (possibly degenerate) centered at $(x, 0, 0)$. Thus, $Y_\kappa(0)$ and $Z_\kappa(0)$ intersect every ellipse $X_\kappa(x)$.*

Proof. This result follows directly from equation (1). $\square$

Lemma 6.4. *Let $a, b \in (-2, 2)$ and $\epsilon > 0$. Then there exists $\delta > 0$ so that for any $\kappa = (a, b, c, d)$ with $|c|, |d| < \delta$ the following hold:*

1. *The set*

$$\bigcap_{|y| < \delta} \{x : (x, y, z) \in Y_\kappa(y)\}$$

is ϵ -dense inside $\{x : (x, y, z) \in E_\kappa\}$, i.e., for all $|y| < \delta$, the set of x -coordinates of $Y_\kappa(y)$ is ϵ -dense in the set of all x -coordinates of points in E_κ .

2. *For all $x \in \{x : (x, y, z) \in E_\kappa\}$, either $\{y : (x, y, z) \in X_\kappa(x)\} \subset [-\delta, \delta]$ or $[-\frac{\delta}{2}, \frac{\delta}{2}] \subset \{y : (x, y, z) \in X_\kappa(x)\}$, i.e., the set of y -coordinates of any given $X_\kappa(x)$ either contains $[-\frac{\delta}{2}, \frac{\delta}{2}]$ or is contained in $[-\delta, \delta]$;*
3. *For all $x \in \{x : (x, y, z) \in E_\kappa\}$, either $\{z : (x, y, z) \in X_\kappa(x)\} \subset [-\delta, \delta]$ or $[-\frac{\delta}{2}, \frac{\delta}{2}] \subset \{z : (x, y, z) \in X_\kappa(x)\}$.*

Proof. The result holds by the continuous dependence of E_κ on c and d , by the continuous dependence of $X_\kappa(x)$ on x , by the geometry of E_κ for $\kappa = (a, b, 0, 0)$ as described in Lemma 6.3, and by the fact that $a, b \neq \pm 2$. $\square$

7. THE TWO-HOLED TORUS

In this section, we prove Theorem 1.4 for the case of $g = 1$ and $n = 2$. The fundamental group $\pi_1(M, O)$ has a presentation

$$\langle X, Y, A, B : XYX^{-1}Y^{-1} = BA \rangle,$$

where A and B represent the boundary components. Let

$$K = XYX^{-1}Y^{-1}, \quad W = AX, \quad W' = XB, \quad Z = XY$$

(see Figure 3).

Cutting M along K yields a one-holed torus with boundary K and a three-holed sphere with boundaries A, B, K . If $a = \pm 2$ (resp. $b = \pm 2$), then $A = \pm I$ (resp.

FIGURE 3. Two-holed torus.

$B = \pm I$). In this case the moduli space for M can be identified with the moduli space of a one-holed torus (with $K = BA$), and so Theorem 1.3 applies. Thus we may assume that $a, b \neq \pm 2$.

Assume that $\rho \in \text{Hom}(\pi_1(M, O), \text{SU}(2))$ is generic. By Proposition 3.1, we may assume without loss of generality that $\langle X, Y \rangle$ is a generic handle. If we cut M along X (resp. Y), then we obtain a four-holed sphere. For $\kappa = (b, a, x, x)$ (resp., $\kappa = (b, a, y, y)$), we obtain the moduli space E_κ of a four-holed sphere.

The defining equation of the moduli space of this 4-holed sphere, E_κ , is

$$\begin{aligned} & w^2 + (w')^2 + k^2 + kw' \\ & = k(ab + x^2) + wx(a + b) + w'(a + b) - a^2 - b^2 - 2x^2 - abx^2 + 4, \end{aligned}$$

and the formulas for the actions $\tau_K, \tau_W, \tau_{W'}$ of the Dehn twists on the associated moduli space of this four-holed sphere become

$$\begin{aligned} \begin{bmatrix} w \\ w' \end{bmatrix} & \xrightarrow{\tau_K} \begin{bmatrix} x(a + b) - k(x(a + b) - kw - w') - w \\ x(a + b) - kw - w' \end{bmatrix}, \\ \begin{bmatrix} w' \\ k \end{bmatrix} & \xrightarrow{\tau_W} \begin{bmatrix} x(a + b) - w(x^2 + ab - ww' - k) - w' \\ x^2 + ab - ww' - k \end{bmatrix}, \\ \begin{bmatrix} k \\ w \end{bmatrix} & \xrightarrow{\tau_{W'}} \begin{bmatrix} ab + x^2 - w'(x(a + b) - w'k - w) - k \\ x(a + b) - w'k - w \end{bmatrix}. \end{aligned}$$

Note that $I_{x,x} = [x^2 - 2, 2]$ and that the actions τ_W and $\tau_{W'}$ do not (in general) preserve y or z (see Figure 3), while the action τ_K preserves x, y , and z .

Lemma 7.1. *For the representation ρ , there exists a $\gamma \in \Gamma$ such that at least one of the following has non-zero trace: $\gamma(AX)$, $\gamma(XB)$, $\gamma(AXY)$, $\gamma(XYB)$, $\gamma(AY)$, $\gamma(YB)$, $\gamma(AYX)$, $\gamma(YXB)$, with $\langle \gamma(X), \gamma(Y) \rangle$ generic.*

Proof. Suppose that $AX, XB, AXY, XYB, AY, YB, AYX$, and YXB all have zero trace. If τ_K preserves $w = w' = 0$, then $x(a + b) = 0$ (if not, note that the twist in τ_K preserves x, y and k , so $\langle \tau_K(X), \tau_K(Y) \rangle$ remains generic). By Remark 5.4, x can be assumed non-zero. Hence $a = -b$. The equation for $E_{(-a, a, x, x)}$ is

$$(2) \quad k^2 + ka^2 + 2a^2 - 4 = x^2(k - 2 + a^2).$$

By Remark 5.4 and the fact that the left-hand side of equation (2) is invariant under $\langle \tau_X, \tau_Y \rangle$, it must be that $k - 2 - a^2 = 0$. The formula $\text{tr}(AB^{-1}) + \text{tr}(AB) = \text{tr}(A)\text{tr}(B)$ (see [6, 8]) and the fact that $K = BA$ give $\text{tr}(AB^{-1}) = -2$. This implies that $A = -B$ as matrices.

Suppose $A = -B$. Conjugating by an element in $\text{SU}(2)$, we may assume that

$$X = \begin{bmatrix} x_1 + y_1 i & 0 \\ 0 & x_1 - y_1 i \end{bmatrix}, \quad Y = \begin{bmatrix} x_2 + y_2 i & z_2 \\ -z_2 & x_2 - y_2 i \end{bmatrix},$$

and

$$A = \begin{bmatrix} a_1 + b_1 i & c_1 + d_1 i \\ -c_1 + d_1 i & a_1 - b_1 i \end{bmatrix}.$$

The equations $\text{tr}(AX) = 0$, $\text{tr}(AY) = 0$, $\text{tr}(AXY) = 0$, $\text{tr}(AYX) = 0$, together with the matrix equation $K = -A^2 = XYX^{-1}Y^{-1}$ and $\det(A) = \det(X) = \det(Y) = 1$ lead to one of the following: $x_1 = 0, \pm 1$, $a_1 = 0$ or $a_1 = \pm\sqrt{1-x_1^2}$. By Remark 5.4, we have $a_1 \neq \sqrt{1-x_1^2}$ and $x_1 \neq 0, \pm 1$. This leaves us with the case $a_1 = 0$, which implies that $K = I$, contradicting the fact that $\langle X, Y \rangle$ is generic. $\square$

Combining Proposition 5.2 and Lemma 7.1, one obtains

Corollary 7.2. *For a generic ρ , we may assume that not all of $\langle AX, Y \rangle$, $\langle XB, Y \rangle$, $\langle AY, X \rangle$, and $\langle YB, X \rangle$ are $\text{Pin}(2)$.*

By Remark 5.3 and Corollary 7.2, we assume without loss of generality that the action τ_Y does not fix $w = \text{tr}(AX)$.

Proposition 7.3. *Suppose $g = 1$, $n = 2$, and ρ is a generic representation. Then the Γ -orbit $\Gamma([\rho])$ is dense in $\mathcal{M}_{\mathcal{C}}$.*

Proof. Let $\epsilon > 0$. Let ρ be a generic representation. By Proposition 3.1, M has a generic handle $\langle X, Y \rangle$ (we adopt the notation presented in Figure 3). By Proposition 2.3, it is enough to show that for any $[\rho_0] \in \mathcal{M}_{\mathcal{P}}$, there exists $\gamma \in \Gamma$ such that $d(\gamma([\rho]), [\rho_0]) < \epsilon$. Let $x_0 = \text{tr}(\rho_0(X))$, $k_0 = \text{tr}(\rho_0(K))$, $w_0 = \text{tr}(\rho_0(W))$, etc.

Cutting along K and X gives a pants decomposition of M . Hence, by Corollary 2.4, we only need to show that there exists $\gamma \in \Gamma$ such that $\gamma([\rho])$ satisfies $|\gamma(k) - k_0| < \epsilon$ and $|\gamma(x) - x_0| < \epsilon$.

The strategy is summarized as follows: Obtain a τ_Y -orbit with a sufficiently large number of points that have the following properties:

1. x -coordinates that are sufficiently close to zero so that $E_{(b,a,x,x)}$ (the moduli space of the four-holed sphere obtained by cutting M at X) contains points with k -coordinates equal to k_0 ;
2. each of the points has sufficiently many points on its respective τ_W -orbits (in $E_{(b,a,x,x)}$) so that one of these points can further be moved via τ_K and τ_W to get a point with k -coordinate within ϵ of k_0 .

Along the way, there will be a certain finite list of values to avoid, e.g. $\mathcal{S}$, to guarantee that the handle $\langle X, Y \rangle$ remains generic.

One potential obstacle is the possibility that the points on the τ_Y -orbit themselves have τ_W -actions that fix $(k, w, w') \in E_{(b,a,x,x)}$. If this occurs, we will show that we can resort to the $\tau_{W'}$ -action. This amounts to showing that $\langle Y, W' \rangle$ itself is not $\text{Pin}(2)$ and that the corresponding $\tau_{W'}$ -actions will not fix $(k, w, w') \in E_{(b,a,x,x)}$.

Lemma 7.4. *Suppose that $x, y, w \neq 0$ and $\tau_W(k, w, w') = (k, w, w') \in E_{(b, a, x, x)}$. Then we have the following:*

1. *If $\langle W', Y \rangle$ is $\text{Pin}(2)$, then*

$$2k = ab + x^2.$$

2. *If $\tau_{W'}(k, w, w') = (k, w, w') \in E_{(b, a, x, x)}$, then*

$$x^2 = \frac{2k - ab}{1 - \left(\frac{a+b}{k+2}\right)^2}.$$

Proof. Case 1: $\langle W', Y \rangle$ is $\text{Pin}(2)$. This implies that $w' = 0$. Hence,

$$(3) \quad 2k = ab + x^2.$$

Case 2: Suppose that $\tau_{W'}(k, w, w') = (k, w, w') \in E_{(b, a, x, x)}$. Then

$$\begin{cases} 2k &= ab + x^2 - ww', \\ 2w' &= x(a+b) - wk, \\ 2w &= x(a+b) - w'k. \end{cases}$$

This leads to $2(w' - w) = k(w' - w)$. Since $k \neq \pm 2$, we have $w = w' = \frac{x(a+b)}{k+2}$. Moreover, since $w \neq 0$, we have $a \neq -b$. This implies that

$$(4) \quad x^2 \left[1 - \left(\frac{a+b}{k+2} \right)^2 \right] = 2k - ab.$$

If $a+b = \pm(k+2)$ and $k = \frac{ab}{2}$, then either $a = \pm 2$ or $b = \pm 2$. This contradicts our assumptions on a and b . So, we must have

$$(5) \quad x^2 = \frac{2k - ab}{1 - \left(\frac{a+b}{k+2} \right)^2}.$$

□

Since $\langle X, Y \rangle$ is generic, Lemma 7.1 implies that we may readily find points with τ_Y -orbits having coordinates that satisfy $w \neq 0$ and $y \neq 0$, while violating both equations (3) and (5).

The following lemma follows immediately from Remark 5.4, Corollary 7.2 and the fact that the τ_X and τ_Y actions fix k .

Lemma 7.5. *For any integer $J > 0$, there is $\gamma \in \Gamma$ such that the τ_Y -orbit of $\gamma(\rho)$ has at least J points satisfying the following conditions:*

1. *The x -coordinates of these J points have $|x|$ small enough so that $k_0 \pm \frac{\epsilon}{2} \in I_{a,b} \cap I_{x,x}$ and $|x| \leq \delta$, where δ is provided in Lemma 6.4 for $E_{(b, a, x, x)}$.*
2. *These J points do not belong to the real subvarieties defined by equations (3), (5), $y = 0$, and $w = 0$.*

Consider the handle $\langle Y, W \rangle$. Choose J sufficiently large in Lemma 7.5 so that one of the J points on the τ_Y -orbit of $\gamma(\rho)$, denoted $\gamma_1(\rho)$, has a τ_W -orbit (resp. $\tau_{W'}$ -orbit) with at least

$$N\left(\frac{\delta}{N(\frac{\epsilon}{2}) + 3}\right) + 3$$

points having distinct k -coordinates inside $E_{(b, a, \gamma_1(x), \gamma_1(x))}$.

FIGURE 4. Three-holed torus.

Since $|\gamma_1(x)| \leq \delta$ and $k_0 \pm \epsilon \in I_{a,b} \cap I_{\gamma_1(x), \gamma_1(x)}$, the sphere E_κ (with $\kappa = (b, a, \gamma_1(x), \gamma_1(x))$) has points with k -coordinates inside $[k_0 - \frac{\epsilon}{2}, k_0 + \frac{\epsilon}{2}]$. Since the τ_W -orbit (resp. $\tau_{W'}$) of $\gamma_1(\rho)$ has at least $N(\frac{\delta}{N(\frac{\epsilon}{2})+3}) + 3$ points with distinct k -coordinates, One such point $\gamma_2(\rho)$, has k -coordinate not in $K_{N(\frac{\delta}{N(\frac{\epsilon}{2})+3})}$ (where K_n denotes the filtration in K) with non-degenerate ellipse $K_\kappa(\gamma_2(k)) \subset E_\kappa$. Thus, the τ_K orbit of $\gamma_2(w)$ is $\frac{\delta}{N(\frac{\epsilon}{2})+3}$ -dense. The τ_K -orbit of $\gamma_2(\rho)$ has at least $N(\frac{\epsilon}{2}) + 3$ points with w -coordinates inside $(-\delta, \delta)$. One such point, $\gamma_3(\rho)$ has w -coordinate not in $W_{N(\frac{\epsilon}{2})}$ with non-degenerate ellipse $W_\kappa(\gamma_3(w))$. Thus, the τ_W -orbit of $\gamma_3(\rho)$ is $\frac{\epsilon}{2}$ -dense in $W_\kappa(\gamma_3(w))$. This fact, together with the properties of δ provided in Lemma 6.4, implies that at least one point, $\gamma_4([\rho])$, in the τ_W -orbit of $\gamma_3(\rho)$ has k -coordinate $\gamma_4(k)$ that comes within ϵ of k_0 .

The restriction to the one-holed torus $\langle \gamma_4(X), \gamma_4(Y) \rangle$ is generic so long as $\gamma_4(k) \notin \mathcal{S}$ and $\gamma_4(k) \neq (\gamma_4(x))^2 - 2$ (see Proposition 5.2 and Remark 5.5). This can be accomplished by replacing ϵ by $\frac{\epsilon}{20}$ at the start of the argument. Since $\langle \gamma_4(X), \gamma_4(Y) \rangle$ is generic, we next apply τ_X and τ_Y to obtain $\gamma_5 \in \Gamma$ so that the x -coordinate $\gamma_5(x)$ of $\gamma_5([\rho])$ is within ϵ of x_0 . Note that both τ_X and τ_Y fix $\gamma_4(k)$. Thus $\langle \gamma_5(X), \gamma_5(Y) \rangle$ remains generic. The result now follows from Corollary 2.4. $\square$

8. THE n -HOLED TORUS

The conjugacy classes that correspond to $c_i = \pm 2$ are central. Thus, for $n > 2$, one may assume that $C_i \neq \pm I$ for all i , or that $n = 1$ and $C_1 = \pm I$.

Proposition 8.1. *Let M be an n -holed torus and ρ a generic representation with generic handle $\langle X, Y \rangle$, which is part of a pants decomposition $\mathcal{P}$, i.e., both $K = XYX^{-1}Y^{-1} \in \mathcal{P}$ and $X \in \mathcal{P}$. Then there is $\gamma \in \Gamma$ so that $\gamma(P) \neq \pm I$ for all $P \in \mathcal{P}$. Moreover, the handle $\langle \gamma(X), \gamma(Y) \rangle$ is generic.*

Proof. We first treat the case of $n = 3$. Let C and D be two boundary loops separated from $\langle X, Y \rangle$ by $B = CD$. Let A be the remaining boundary loop (see

Figure 4). Since $\langle X, Y \rangle$ is generic, we have that $K, X \neq \pm I$. The goal, therefore, is to find $\gamma \in \Gamma$ such that $\gamma(B) \neq \pm I$.

Suppose $B = \pm I$. Then $C = \pm D^{-1}$ and $A = \pm K^{-1}$. If $AD = E \neq \pm I$, then we may apply Proposition 7.3 to the two-holed torus bounded by E and C . Hence, if $E \neq \pm I$, there exists an element $\gamma \in \Gamma$ that fixes a with $\gamma(k) \neq \pm a$ such that $\langle \gamma(X), \gamma(Y) \rangle$ is generic. This implies $\gamma(B) \neq \pm I$. The same conclusion can be drawn if $AC \neq \pm I$. Thus, we assume that $AC = \pm I$ and $AD = \pm I$. Since $A = \pm K^{-1}$, we have that $C, D = \pm K$. Hence, $B = CD = \pm K^2 = \pm I$. This is only possible if $k = 0$ or $K = \pm I$. The latter case is ruled out by the generic assumption on $\langle X, Y \rangle$. Hence $k = 0$. We will show that in this special case, we can obtain $\gamma(B) \neq \pm I$.

Suppose that X commutes with K . If $\tau_Y(X) = XY$ also commutes with K , then X, Y , and K all belong to the same one-parameter subgroup of $SU(2)$. Hence X and Y commute, contradicting the fact that $\langle X, Y \rangle$ is generic. Suppose that $XKX^{-1}K^{-1} = -I$ (X and K anti-commute). Then, $X = K(-X)K^{-1}$, which implies that $x = 0$. Again, since $\langle X, Y \rangle$ is generic, we may assume that $x \neq 0$. To summarize, since $\langle X, Y \rangle$ is generic, one may arrange that X neither commutes nor anti-commutes with K (i.e. $XKX^{-1}K^{-1} \neq \pm I$).

Now consider the curve $Z = XC$. The Dehn twist in Z preserves $E = \pm I$ and c . Hence it fixes $k = 0$ (consider the pants bounded by C, E , and K). On the four-holed sphere bounded by C, D, X and W , the action of τ_Z on the matrix B is $\tau_Z(B) = CZDZ^{-1} = \pm CXCD C^{-1}X^{-1}$. Since $C, D = \pm K$ and $K^2 = -I$, $\tau_Z(B) = \pm KXK^{-1}X^{-1} \neq \pm I$. Hence there exists $\gamma \in \Gamma$ with $\gamma(B) \neq \pm I$. By Remark 5.4, we may assume that x is not in $\{0, \pm 1, \pm \frac{1 \pm \sqrt{5}}{2}, \pm \sqrt{2}\}$. Since τ_Z preserves k and x , the handle $\langle \tau_Z(X), \tau_Z(Y) \rangle$ is generic, as $k = 0$ and $x \notin \{\pm \sqrt{2}, 0\}$. This proves the case $n = 3$.

The above argument may be repeated iteratively, starting with loops in $\mathcal{P}$ that bound two boundary loops and working inward towards K . We demonstrate using the case $n = 5$, with notation provided in Figure 5.

First consider the three-holed torus bounded by E, F , and $H = ACD$. Since $\langle X, Y \rangle$ is generic, if $B_1 = \pm I$, then $ACD \neq \pm I$ (otherwise $K = \pm I$). Use the previous argument for $n = 3$ to arrange $B_1 \neq \pm I$. Next, the three-holed torus bounded by B_1, D , and AC is used to make $B_2 \neq \pm I$, etc. $\square$

Proposition 8.2. *Suppose that $g = 1$, $n = 3$ and ρ is a generic representation. Then the Γ -orbit $\Gamma([\rho])$ is dense in $\mathcal{M}_C$.*

Proof. The proof is similar in nature to the proof of Proposition 7.3. Consider the notation suggested in Figure 4.

Cutting along X, K and B yields a pants decomposition. Hence, by Proposition 7.3 and Corollary 2.4, we need to show that there exists $\gamma \in \Gamma$ such that $\gamma([\rho])$ satisfies $|\gamma(b) - b_0| < \epsilon$, with $\langle \gamma(X), \gamma(Y) \rangle$ generic. We first move x and w near zero so that the moduli space of the four-holed sphere bounded by X, W, C , and D contains b -coordinates near b_0 . We use Propositions 8.1 and 7.3 to do this.

As in the proof of Proposition 7.3, we wish to obtain a τ_Y -orbit with a sufficiently large number of points that have the following properties:

1. x - and w -coordinates that are sufficiently near zero so that the moduli space of the four-holed sphere bounded by X, W, C , and D contains points with b -coordinate equal to b_0 ;

2. each of these points has sufficiently many points on their respective τ_Z -orbits (in $E_{(x,w,c,d)}$) so that one such point can further be moved via τ_B and τ_Z on $E_{(x,w,c,d)}$ to get a point with a b -coordinate within ϵ of b_0 .

For this strategy to work, we must first have that $\langle Y, Z \rangle$ or $\langle Y, Z' \rangle$ is not $\text{Pin}(2)$, where $Z' = DX$. If both $\langle Y, Z \rangle$ and $\langle Y, Z' \rangle$ are $\text{Pin}(2)$, then $z = z' = 0$ and $\text{tr}(YZ') = \text{tr}(YZ) = 0$. Note that the application of τ_B fixes the moduli space of the two-holed torus bounded by A and B , i.e., τ_B fixes a, b, x, y, k , and w . If the Dehn twist τ_B fixes (b, z, z') with $z = z' = 0$, then we have

$$(6) \quad wd + xc = 0$$

and

$$(7) \quad xd + wc = 0.$$

In the special case where $c = d = 0$, the defining equation for $E_{(x,w,c,d)}$ yields

$$(8) \quad b^2 = xwb - x^2 - w^2 + 4.$$

However, by Proposition 7.3 and the fact that b, c, d are fixed by mapping class actions on the two-holed torus bounded by B and A , we may assume that the x, w -coordinates do not satisfy any of the above equations. Hence, we may assume that $\langle Y, Z \rangle$ is not $\text{Pin}(2)$.

We consider the possibility that the points on the τ_Y -orbit themselves have τ_Z -actions that fix $(b, z, z') \in E_{(x,w,c,d)}$. As before, in such cases we will rely on the $\tau_{Z'}$ -action.

If τ_Z fixes $(b, z, z') \in E_{(x,w,c,d)}$ and $\langle Y, Z' \rangle$ is $\text{Pin}(2)$, then

$$(9) \quad 2b = xw + cd.$$

If neither $\langle Y, Z \rangle$ nor $\langle Y, Z' \rangle$ is $\text{Pin}(2)$ but both τ_Z and $\tau_{Z'}$ fix (b, z, z') , then

$$\begin{cases} 2b &= xw + cd - zz', \\ 2z &= xc + wd - bz, \\ 2z' &= xd + wc - bz'. \end{cases}$$

This implies that τ_B also fixes (b, z, z') . As neither $\langle Y, Z \rangle$ nor $\langle Y, Z' \rangle$ is $\text{Pin}(2)$, we may assume that $z, z' \neq \pm 2$. Hence $B_\kappa(b)$, $Z_\kappa(z)$ and $Z'_\kappa(z')$ are single points. This leads to $I_{x,w} \cap I_{c,d} = \{b\}$. However, as $b \neq \pm 2$, one can use Proposition 7.3 to guarantee that the x, w -coordinates on sufficiently many points of a τ_Y -orbit do not satisfy equation (9) and are close enough to zero so that b is not an endpoint of $I_{x,w}$.

The argument now follows similarly to the one presented in Proposition 7.3 to produce $\gamma \in \Gamma$ satisfying $|\gamma(b) - b_0| < \epsilon$. $\square$

Proposition 8.3. *Suppose that M is a genus one surface with $n \geq 1$ boundary components. Let ρ be a generic representation. Then the Γ -orbit $\Gamma([\rho])$ is dense in $\mathcal{M}_C$.*

Proof. The cases $n = 1, 2$, and 3 have previously been established. For $n \geq 4$, the argument follows by an induction process similar to that in the proof of Proposition 8.1.

 FIGURE 5. Getting rid of $\pm I$ on $\mathcal{P}$ for $n = 5$.

Let ρ be a generic representation. By Proposition 3.1, M has a generic handle $\langle X, Y \rangle$. We demonstrate how to proceed in the case $n = 5$ (see Figure 5). Let $[\rho_0] \in \mathcal{M}_{\mathcal{P}}$.

By Proposition 8.1, we have that $B_1 \neq \pm I$. Assume that Proposition 8.3 is true for the four-holed torus bounded by B_1, A, C and D . We use this four-holed torus to arrange for x and w (see Figure 5) to have near zero traces. This ensures that the b_1 -coordinate of $[\rho_0]$ is accessible. We then arrange $H \neq \pm I$. Next, cut M at H and use Proposition 8.2 to get the b_1 -coordinate of $\gamma([\rho])$ within ϵ of $(b_1)_0$. Next, we get the b_2 -coordinate of $\gamma([\rho])$ within ϵ of $(b_2)_0$, by using the four-holed torus obtained by cutting at B_1 . Finally, Proposition 8.2 and Corollary 2.4 yield the result. The situation is identical for any $n > 3$. $\square$

9. GENUS g WITH n BOUNDARY COMPONENTS

Suppose M is a surface with $g > 1$ and $n \geq 0$. Again, we assume that $C_i \neq \pm I$ for all boundary curves $C_i \in \partial M$, unless $n = 1$ and $C_1 = \pm I$.

Proposition 9.1. *Let M have generic $\langle X, Y \rangle$ that is part of a pants decomposition $\mathcal{P}$. Then there is $\gamma \in \Gamma$ so that $\gamma(\rho(P)) \neq \pm I$, for all $P \in \mathcal{P}$ with $\langle \gamma(X), \gamma(Y) \rangle$ generic. (In the special case $n = 1$ and $C_1 = \pm I$, we have $\gamma(\rho(P)) \neq \pm I$ for all $P \in \mathcal{P}$ except C_1 .) Furthermore, $\langle \gamma(P), \gamma(Q) \rangle$ is generic, for all $P \in \mathcal{P}$ and $PQP^{-1}Q^{-1} = R \in \mathcal{P}$ that bound a three-holed sphere which forms a one-holed torus in M .*

Proof. We first show that there is $\gamma \in \Gamma$ so that $\langle \gamma(P), \gamma(Q) \rangle$ is generic for all $P \in \mathcal{P}$ and $PQP^{-1}Q^{-1} = R \in \mathcal{P}$ that bound a three-holed sphere which forms a one-holed torus in M . Consider P, Q , and R as in Figure 6, where $P, R \in \mathcal{P}$ bound a pants that forms a one-holed torus in M .

Case 1: Suppose that $R \neq \pm I$. In this case, at least one of p, q or $\text{tr}(PQ)$ is not in $\{0, \pm 2\}$. Without loss of generality, assume that $p \notin \{0, \pm 2\}$. Thus, we

FIGURE 6. A punctured genus two inside M .

may cut M at P and A and apply Proposition 8.2 to the resulting three-holed torus to obtain $r \notin \mathcal{S}$ with $r \neq p^2 + 2$. Note that in the special case $A = \pm I$, the corresponding three-holed torus can be identified to that of a two-holed torus, to which Proposition 7.3 can be applied. Thus the handle $\langle \gamma(P), \gamma(Q) \rangle$ is generic.

Case 2: Suppose that $R = -I$. Then both P and Q are not $\pm I$. Moreover, $A \neq \pm I$ since $K \neq \pm I$. Thus, we may cut M at P and A , and then apply Proposition 8.2 to reduce the situation to Case 1.

Case 3: Suppose that $R = I$. Then, necessarily, $A = \pm K^{-1} \neq \pm I$, since $\langle X, Y \rangle$ is generic. If either $P \neq \pm I$ or $Q \neq \pm I$, we may cut at P (or Q) and apply Proposition 8.2 as in case 2. If both $P = \pm I$ and $Q = \pm I$, then $\tau_{PA}(Q) = Q(PA) = \pm K \neq \pm I$. Note that the action of τ_{PA} does not affect the generic handle $\langle X, Y \rangle$. We now cut at Q and apply Proposition 8.2 as in case 2. This argument can be applied independently to each of the $g - 1$ handles of M .

Having obtained that all handles are generic, we now apply Proposition 8.1 to the remaining curves in $\mathcal{P}$ that are interior to the $(n + g - 1)$ -holed torus obtained by cutting off each of the $g - 1$ handles. In the special case $n = 1$ and $C_1 = \pm I$, Proposition 8.1 applies to those curves of $\mathcal{P}$ that are separated from C_1 by the curve KC_1 . Moreover, $KC_1 \neq \pm I$ since $\langle X, Y \rangle$ is assumed generic. $\square$

Now we prove Theorem 1.4. Let ρ be a generic representation. By Proposition 3.1, M has a generic handle $\langle X, Y \rangle$ (we adopt the notation presented in Figure 7). By Proposition 2.3, without loss of generality we may take $[\rho_0] \in \mathcal{M}_{\mathcal{P}}$ and show that there exists $\gamma \in \Gamma$ such that $d(\gamma([\rho]), [\rho_0]) < \epsilon$.

By Proposition 9.1, we may assume that $\langle P_i, Q_i \rangle$ is generic and that $p_i \notin \{0, \pm 2\}$ for all $1 \leq i \leq g - 1$.

Dehn twist each generic handle $\langle P_i, Q_i \rangle$ to obtain p_i arbitrarily close to zero. Since x and $z_i = \text{tr}(R_i X)$ can be made arbitrarily close to zero by Proposition 8.3 (one can do this also in the exceptional case $n = 1$, $C_1 = \pm I$, and $g > 1$), we are ensured that the target value r_i -coordinate of ρ_0 will be inside the ϵ -neighborhood

 FIGURE 7. Decomposition of M .

of $I_{p_i, p_i} \cap I_{x, z_i}$. Thus, there are points on the moduli space of the four-holed sphere $E_{(p_i, p_i, x, z_i)}$ with r_i -coordinates within ϵ of $(r_i)_0$, the r_i -coordinate of ρ_0 .

We next apply Proposition 8.3 to the $(n+2g-2)$ -holed torus obtained by cutting M at each of the P_i to move each r_i within ϵ of $(r_i)_0$, keeping $\langle \gamma(P_i), \gamma(Q_i) \rangle$ generic. Again, this is also possible in the special case of $n = 1$, $C_1 = \pm I$, and $g > 1$, as the moduli space of the $(1+2g-2)$ -holed torus reduces to that of a generic $(2g-2)$ -holed torus.

Next, we twist on each of the $g-1$ generic handles to get p_i within ϵ of $(p_i)_0$. Finally, we cut off each handle at R_i and apply Proposition 8.3 to the resulting $(n+g-1)$ -holed torus with generic $\langle \gamma(X), \gamma(Y) \rangle$.

Theorem 1.4 now follows from Corollary 2.4.

REFERENCES

- [1] Benedetto, R. L., and Goldman, W. M., The topology of the relative character varieties of a quadruply-punctured sphere *Experimental Mathematics*, 8 (1) (1999), 85-104. MR **2000c**:57028
- [2] Conway, J. H. and Jones, A. J., Trigonometric diophantine equations (On vanishing sums of roots of unity) *Acta Arithmetica*, Vol. XXX, (1976), 229-240. MR **54**:10141
- [3] Gallo, D., Kapovich, M., Marden, A., The Monodromy groups of Schwarzian equations on closed Riemann surfaces, *Ann. of Math.* (2) 151 (2000), 625-704.
- [4] Goldman, W. M., Ergodic theory on moduli spaces *Ann. of Math.*, Vol. 146, (1997), 475-507. MR **99a**:58024
- [5] Goldman, W. M., The symplectic nature of fundamental groups of surfaces, *Adv. Math.*, Vol. 54, (1984), 200-225. MR **86i**:32042
- [6] Horowitz, R., Characters of free groups represented in the two-dimensional special linear group, *Comm. Pure Appl. Math.*, Vol. 25, (1972), 635-649. MR **47**:4542

- [7] Jeffrey, L., Weitsman, J., Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula, *Comm. Math. Phys.*, Vol. 150, (1992), 595-630. MR **94g**:58085
- [8] Magnus, W., Rings of Fricke characters and automorphism groups of free groups, *Mathematische Zeitschrift*, Vol. 170, (1980), 91-103. MR **81a**:20043
- [9] Previte, J. P., Xia, E. Z., Topological dynamics on moduli spaces I, *Pacific Journal of Mathematics* Vol 193, No 2, (2000), 397-418. MR **2001e**:53090
- [10] Seshadri, C., Fibrés vectoriels sur les courbes algébriques, *Astérisque*, 96 (1982). MR **85b**:14023

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