THE BEST CONSTANT OF THE MOSER-TRUDINGER INEQUALITY ON $\mathbf{S}^2$

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ABSTRACT. We consider the best constant of the Moser-Trudinger inequality on $\mathbf{S}^2$ under a certain orthogonality condition. Applying Moser's calculation, we construct a counterexample to the sharper inequality with the condition.

1. INTRODUCTION

Let (M, g) be a compact smooth Riemannian manifold of dimension n . We denote by $\mathbf{H}_1^n(M)$ the Banach space obtained from the vector space of C^∞ functions φ such that $\|\nabla\varphi\|_{L^n}$ is bounded, by completion with the norm

$$\|\varphi\|_{\mathbf{H}_1^n} := \sum_{l=0}^1 \|\nabla^l \varphi\|_{L^n}.$$

It is well known as an exceptional case of the Sobolev imbedding theorem that there exist constants C , μ and ν such that all $\varphi \in \mathbf{H}_1^n(M)$ satisfy

$$(1.1) \quad \int_M e^\varphi dV \leq C \exp(\mu \|\nabla\varphi\|_{L^n}^n + \nu \|\varphi\|_{L^n}^n)$$

(Trudinger, Aubin; cf. [2], p. 63). For applications, the best constant μ in (1.1) is essential. For example, in the case of the sphere, the following result was proved by Moser [5] when $n = 2$, and Aubin [1] for general n . All $\varphi \in \mathbf{H}_1^n(\mathbf{S}^n)$ with integral equal to zero satisfy

$$(1.2) \quad \int_{\mathbf{S}^n} e^\varphi dV \leq C \exp(\mu_n \|\nabla\varphi\|_{L^n}^n),$$

where C depends only on n , and

$$\mu_n = (n-1)^{n-1} n^{1-2n} \omega_{n-1}^{-1}.$$

Here ω_n denotes the volume of $\mathbf{S}^n(1) := \{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}$. Note that ν in (1.1) equals zero in this case. In particular, when $n = 2$, (1.2) becomes

$$(1.3) \quad \log \int_{\mathbf{S}^2} e^\varphi d\mu \leq \frac{1}{16\pi} \int_{\mathbf{S}^2} |\text{grad } \varphi|^2 d\mu + \frac{1}{4\pi} \int_{\mathbf{S}^2} \varphi d\mu + C,$$

where $d\mu$ denotes the volume element of the canonical metric on $\mathbf{S}^2$. It is the best constant of the coefficient of $\int_{\mathbf{S}^2} |\text{grad } \varphi|^2 d\mu$ which is useful for Nirenberg's

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problem: Characterize all Gauss curvature functions K of Riemannian metrics $\tilde{g}$ conformal to the canonical metric g_0 , so that $\tilde{g} = e^{-\varphi}g_0$. This requires a solution of the following nonlinear elliptic partial differential equation for a given K :

$$(1.4) \quad 1 - \Delta_{g_0}\varphi = Ke^{-\varphi},$$

where $\Delta_{g_0} := -\nabla^i \nabla_i$ denotes the real Laplacian with respect to g_0 . For the variational method we define a functional $S(\varphi)$ by

$$S(\varphi) := -\log \int_{\mathbf{S}^2} Ke^{\varphi} d\mu + \frac{1}{16\pi} \int_{\mathbf{S}^2} |\text{grad } \varphi|^2 d\mu + \frac{1}{4\pi} \int_{\mathbf{S}^2} \varphi d\mu.$$

The existence of a solution of (1.4) is reduced to finding a minimum of $S(\varphi)$. Clearly $S(\varphi)$ is finite, but it is impossible to find a convergent subsequence of a given minimizing sequence when the best constant μ equals $\frac{1}{16\pi}$. So we consider a sharper version of the Moser-Trudinger inequality (1.3) under additional constraints on φ . For example, Aubin [1] proved that if $\varphi \in \mathbf{H}_1^n(\mathbf{S}^n)$ satisfies

$$(1.5) \quad \int_{\mathbf{S}^n} \xi e^{\varphi} dV = 0, \quad \int_{\mathbf{S}^n} \varphi dV = 0$$

for all $\xi \in \Lambda$, which denotes the first eigenspace of the Laplacian, then

$$\int_{\mathbf{S}^n} e^{\varphi} dV \leq C(\epsilon) \exp\left(\left(\frac{\mu_n}{2} + \epsilon\right) \|\nabla \varphi\|_{L^n}^n\right)$$

for any $\epsilon > 0$ and some constant $C(\epsilon)$. Also when $n = 2$, Moser [6] proved that we can lower the best constant of μ to $\frac{1}{32\pi}$ for all $\varphi \in C^\infty(\mathbf{RP}^2)$, i.e., $\varphi(\xi) = \varphi(-\xi)$ for $|\xi| = 1$. Note that under these circumstances the minimum of $S(\varphi)$ exists, but these are just only partial answers to Nirenberg's problem.

In the context of Kähler geometry, the equation (1.4) can be regarded as a 1-dimensional complex Monge-Ampère equation, since $\mathbf{S}^2$ is a 1-dimensional Kähler manifold. On this problem, similar results hold as follows. Let X be a compact m -dimensional Kähler manifold with the positive first Chern class and its Kähler form ω representing the first Chern class. We define

$$P(X, \omega) := \{\varphi \in C^\infty \mid \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\},$$

and a functional F_ω on $P(X, \omega)$ [3] by

$$F_\omega(\varphi) := J_\omega(\varphi) - \frac{1}{V} \int_X \varphi \omega^m - \log\left(\frac{1}{V} \int_X e^{h_\omega - \varphi} \omega^m\right),$$

$$\frac{1}{V} \int_X \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{k=0}^{m-1} \frac{m-k}{m+1} \omega^{m-1-k} \wedge (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^k,$$

where $V = \int_X \omega^m$, and h_ω is uniquely determined by

$$\text{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega,$$

$$\int_X (e^{h_\omega} - 1)\omega^m = 0.$$

(See [4] for the way how F_ω and Mabuchi's K-energy are related to the Futaki character.) Note that the functional $S(\varphi)$ is equivalent to $F_{\omega_{KE}}(-\varphi)$ by regarding $\mathbf{S}^2$ as $\mathbf{CP}^1$. It may be commonly known among experts in this problem that the

inequality (1.3) can be shown from the viewpoint of the existence of the Kähler-Einstein metrics. Concretely, if X admits a Kähler-Einstein metric ω_{KE} , $F_{\omega_{KE}}(-\varphi)$ is bounded from below by some absolute constant for all $-\varphi \in P(X, \omega_{KE})$. Namely

$$(1.6) \quad \frac{1}{V} \int_X e^\varphi \omega_{KE}^m \leq C \exp(J_{\omega_{KE}}(-\varphi) + \frac{1}{V} \int_X \varphi \omega_{KE}^m).$$

When the complex dimension of X equals one, so that $X = \mathbf{CP}^1 = \mathbf{S}^2$, (1.6) is equivalent to (1.3). So (1.6) can be regarded as a fully nonlinear generalization of the Moser-Trudinger inequality. In [7], Tian proved the sharper form of (1.6): if (X, ω_{KE}) is a Kähler-Einstein manifold, there are positive constants δ, C such that

$$F_{\omega_{KE}}(-\varphi) \geq \delta \frac{J_{\omega_{KE}}(-\varphi)}{(1 + \operatorname{osc}_X \varphi)^{2m+2/2m+2+\beta}} - C$$

for any $-\varphi \in P(X, \omega_{KE})$ perpendicular to Λ_1 (i.e., $\int_X \xi \cdot (-\varphi) \omega_{KE}^m = 0$ for any $\xi \in \Lambda_1$). Here $\beta = \frac{1}{4}e^{-m}$ and Λ_1 denotes the space of the eigenfunctions of the complex Laplacian Δ_{KE} with eigenvalue 1 with respect to ω_{KE} (if this space is not empty, it is equivalent to the first eigenspace). Note that δ and C may depend on the dimension of X , the first eigenvalue of Δ_{KE} which is greater than one, and the Sobolev constant with respect to ω_{KE} . Moreover, Tian posed the following conjecture (see [7], p. 20, conjecture 5.5): if X is a Kähler-Einstein manifold, there are constants $\delta > 0, C_\delta$, which may depend on δ , such that

$$F_{\omega_{KE}}(-\varphi) \geq \delta \cdot J_{\omega_{KE}}(-\varphi) - C_\delta,$$

namely

$$(1.7) \quad \frac{1}{V} \int_X e^\varphi \omega_{KE}^m \leq C_\delta \exp((1 - \delta)J_{\omega_{KE}}(-\varphi) + \frac{1}{V} \int_X \varphi \omega_{KE}^m)$$

for any $-\varphi \in P(M, \omega_{KE})$ perpendicular to Λ_1 . When $X = \mathbf{S}^2$, (1.7) is equivalent to

$$(1.8) \quad \log \int_{\mathbf{S}^2} e^\varphi d\mu \leq \frac{1}{16\pi}(1 - \delta) \int_{\mathbf{S}^2} |\operatorname{grad} \varphi|^2 d\mu + \frac{1}{4\pi} \int_{\mathbf{S}^2} \varphi d\mu + C_\delta.$$

In a word, Tian's conjecture suggests that the perpendicularity to Λ_1 could make the best constant of the Moser-Trudinger inequality on $\mathbf{S}^2$ smaller than $\frac{1}{16\pi}$. If this conjecture is true, it will give a new partial answer to Nirenberg's problem. Note that this perpendicularity is different from Aubin's one (1.5) and is essential. In fact, it is known (Ding [3]) that we cannot make δ in (1.7) larger than zero without the perpendicularity if X admits a nonzero holomorphic vector field (i.e., Λ_1 is not empty).

The purpose of this paper is to construct a counterexample to (1.8) for $\delta > 0$ even if φ is perpendicular to Λ . Note that Λ is equivalent to Λ_1 . In other words, the perpendicularity to Λ cannot make the best constant of the Moser-Trudinger inequality smaller than $\frac{1}{16\pi}$. Our counterexample is based on Moser's proof in [5].

Remark 1.1. Our example is not a counterexample to Tian's conjecture, because it does not satisfy the Kähler condition.

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2. CONSTRUCTION OF THE COUNTEREXAMPLE

As mentioned, our example closely follows the calculations of [5]. It suffices to find a sequence of functions $\{\varphi\}$ on $\mathbf{S}^2$ such that for any $\beta > 1$

$$(2.1) \quad \log \int_{\mathbf{S}^2} e^\varphi d\mu - \frac{1}{16\pi\beta} \int_{\mathbf{S}^2} |\operatorname{grad} \varphi|^2 d\mu + \frac{1}{4\pi} \int_{\mathbf{S}^2} \varphi d\mu$$

tends to infinity and that φ is perpendicular to Λ . Theorem 2 in [5] implies that if φ is a smooth function defined on $\mathbf{S}^2$ satisfying

$$\int_{\mathbf{S}^2} |\operatorname{grad} \varphi|^2 d\mu \leq 1, \quad \int_{\mathbf{S}^2} \varphi d\mu = 0,$$

then there exists an absolute constant C such that

$$\int_{\mathbf{S}^2} e^{4\pi\varphi^2} d\mu \leq C.$$

First, we will construct a sequence of functions $\{\varphi\}$ such that

- (i) $\int_{\mathbf{S}^2} |\operatorname{grad} \varphi|^2 d\mu = 1,$
- (ii) $\int_{\mathbf{S}^2} \varphi d\mu = 0,$
- (iii) φ is perpendicular to $\Lambda,$
- (iv) $\int_{\mathbf{S}^2} e^{4\pi\beta\varphi^2} d\mu$ tends to infinity for any $\beta > 1.$

We introduce longitude ϕ and latitude θ on $\mathbf{S}^2$ so that the canonical metric is given by

$$ds^2 = d\theta^2 + \cos^2 \theta d\phi^2, \quad |\theta| < \frac{\pi}{2},$$

and $\theta = \pm\pi/2$ corresponds to the two poles. Let $\varphi(\theta)$ be a radially symmetric function, that is, independent of ϕ . Moreover, introducing the variable t and the functions $w(t), \rho(t)$ by

$$\begin{aligned} e^{t/2} &:= \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right), \\ w(t) &:= (4\pi)^{1/2} \varphi(\theta), \\ \rho(t) &:= \frac{1}{e^t + e^{-t} + 2}, \end{aligned}$$

it suffices to find a sequence of functions such that (see [5])

- (i') $\int_{-\infty}^{+\infty} \dot{w}^2 dt = 1,$
- (ii') $\int_{-\infty}^{+\infty} w \rho dt = 0,$
- (iii) φ is perpendicular to $\Lambda,$
- (iv') $\int_{-\infty}^{+\infty} e^{\beta w^2} \rho dt$ tends to infinity for any $\beta > 1.$

Regarding $\mathbf{S}^2$ as $\{(x, y, z) \in \mathbf{R}^3 | x^2 + y^2 + z^2 = 1\}$, the basis of the first eigenfunctions is given by the coordinate functions x, y and z . Since $\varphi(\theta)$ is radially symmetric, we may assume that $\varphi(\theta)$ is perpendicular to both x and y . Then, we

have only to require φ to be perpendicular to z . Since z is the eigenfunction of eigenvalue 1 (we are considering the complex Laplacian),

$$0 = \int_{\mathbf{S}^2} \varphi \cdot z d\mu = \int_{\mathbf{S}^2} \varphi \cdot \Delta z d\mu = \int_{\mathbf{S}^2} \nabla^i \varphi \nabla_i z d\mu.$$

Note also that the condition (iii) reduces to

$$(iii') \quad \int_{-\infty}^{+\infty} \dot{w}(t) \dot{z}(t) dt = 0.$$

Here we regard z as a function of t .

Now we define $w_\sigma(t)$ as follows ($\sigma > 0$):

$$w_\sigma(t) = \begin{cases} \varepsilon, & t < -\varepsilon, \\ -t, & -\varepsilon \leq t < 0, \\ t\sqrt{\frac{1-\varepsilon}{\sigma}}, & 0 \leq t < \sigma, \\ \sqrt{(1-\varepsilon)\sigma}, & \sigma \leq t. \end{cases}$$

$0 < \varepsilon < 1$ is determined by σ as indicated below.

It is obvious that $w_\sigma(t)$ as above satisfies (i'). To fulfil (iii'), $w_\sigma(t)$ has to satisfy

$$\int_{-\varepsilon}^0 -\dot{z}(t) dt + \int_0^\sigma \sqrt{\frac{1-\varepsilon}{\sigma}} \dot{z}(t) dt = 0,$$

hence

$$z(-\varepsilon) = \sqrt{\frac{1-\varepsilon}{\sigma}} z(\sigma).$$

Note that $\varepsilon \rightarrow 0$ ($\sigma \rightarrow +\infty$), since $0 < \varepsilon < 1$ and $z(\sigma)$ tends to 1 as $\sigma \rightarrow +\infty$. Let A_σ and $\tilde{w}_\sigma(t)$ denote $\int_{-\infty}^{+\infty} w_\sigma(t) \cdot \rho(t) dt$ and $w_\sigma(t) - A_\sigma$. Therefore $\tilde{w}_\sigma(t)$ satisfies (i'), (ii'), (iii'). Note that A_σ tends to 0 as $\sigma \rightarrow +\infty$, since

$$0 \leq A_\sigma \leq \varepsilon \int_{-\infty}^0 \rho dt + \sqrt{\frac{1-\varepsilon}{\sigma}} \int_0^\sigma t e^{-t} dt + \sqrt{(1-\varepsilon)\sigma} \int_\sigma^{+\infty} e^{-t} dt.$$

In what follows, we will prove that

$$\int_{-\infty}^{+\infty} e^{\beta \tilde{w}_\sigma^2} \rho dt \rightarrow \infty$$

as $\sigma \rightarrow +\infty$. For $\sigma > 0$ we have

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{\beta \tilde{w}_\sigma^2} \rho dt &\geq \frac{1}{4} \int_\sigma^{+\infty} \exp(\beta \tilde{w}_\sigma^2 - t) dt \\ &= \frac{1}{4} \exp(\beta(\sqrt{(1-\varepsilon)\sigma} - A_\sigma)^2 - \sigma) \\ (2.2) \quad &= \frac{1}{4} \exp(\sigma(\beta(\sqrt{1-\varepsilon} - \frac{A_\sigma}{\sqrt{\sigma}})^2 - 1)). \end{aligned}$$

Since $\beta > 1$ and $A_\sigma \rightarrow 0$ as $\sigma \rightarrow +\infty$, there is a positive constant c such that

$$\beta(\sqrt{1-\varepsilon} - \frac{A_\sigma}{\sqrt{\sigma}})^2 - 1 > c$$

for σ large enough. Then $\int_{-\infty}^{+\infty} e^{\beta \tilde{w}_\sigma^2} \rho dt$ tends to infinity as $\sigma \rightarrow +\infty$.

Next we will construct a counterexample by making use of $w_\sigma(t)$, or rather

$$\varphi_\sigma := (4\pi)^{-1/2} \tilde{w}_\sigma(t).$$

Let us fix $\beta > \beta' > 1$. Now applying $a_\sigma \varphi_\sigma$ to (2.1) where a_σ is a positive constant depending only on σ , what we should show is that by taking a_σ appropriately,

$$(2.3) \quad \int_{\mathbf{S}^2} \exp(a_\sigma \varphi_\sigma - \frac{1}{16\pi\beta} a_\sigma^2) d\mu \rightarrow \infty$$

as $\sigma \rightarrow +\infty$. Let $M(\sigma)$ be the maximum of φ_σ . Now we choose a_σ such that $\frac{a_\sigma}{8\pi\beta'} = M(\sigma)$ for given σ . Since

$$4\pi\beta' \varphi_\sigma^2 - a_\sigma \varphi_\sigma + \frac{1}{16\pi\beta} a_\sigma^2 = 4\pi\beta' (\varphi_\sigma - \frac{1}{8\pi\beta'} a_\sigma)^2 + \frac{a_\sigma^2}{16\pi} (\frac{1}{\beta} - \frac{1}{\beta'}),$$

we deduce

$$(2.4) \quad a_\sigma \varphi_\sigma - \frac{1}{16\pi\beta} a_\sigma^2 \geq 4\pi\beta' \varphi_\sigma^2$$

on $\{p \in \mathbf{S}^2 \mid \varphi_\sigma(p) = M(\sigma)\}$. From (2.4) we obtain

$$(2.5) \quad \begin{aligned} \int_{\mathbf{S}^2} \exp(a_\sigma \varphi_\sigma - \frac{1}{16\pi\beta} a_\sigma^2) d\mu &\geq \int_{\{\varphi_\sigma = M(\sigma)\}} \exp(4\pi\beta' \varphi_\sigma^2) d\mu \\ &= \int_\sigma^{+\infty} \exp(\beta' \tilde{w}_\sigma(t)^2) \rho(t) dt \geq \frac{1}{4} \int_\sigma^{+\infty} \exp(\beta' \tilde{w}_\sigma(t)^2 - t) dt. \end{aligned}$$

Since $\beta' > 1$, the right hand side of (2.5) tends to infinity as $\sigma \rightarrow +\infty$ by the same argument as in (2.2). So (2.3) is proved. Thus we conclude that $\{a_\sigma \varphi_\sigma\}$ are the counterexample we want. Consequently we find that the best constant of the Moser-Trudinger inequality cannot be lowered only by requiring the perpendicularity to Λ .

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