MAXIMAL HOLONOMY OF INFRA-NILMANIFOLDS WITH 2-DIMENSIONAL QUATERNIONIC HEISENBERG GEOMETRY

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ABSTRACT. Let $\mathbf{H}_{4n-1}(\mathbb{H})$ be the quaternionic Heisenberg group of real dimension $4n - 1$ and let I_n denote the maximal order of the holonomy groups of all infra-nilmanifolds with $\mathbf{H}_{4n-1}(\mathbb{H})$ -geometry. We prove that $I_2 = 48$. As an application, by applying Kim and Parker's result, we obtain that the minimum volume of a 2-dimensional quaternionic hyperbolic manifold with k cusps is at least $\frac{\sqrt{2}k}{720}$.

1. INTRODUCTION

The complex Heisenberg group $\mathbf{H}_{2n-1}(\mathbb{C})$ is

$$\mathbf{H}_{2n-1}(\mathbb{C}) = \mathbb{R} \tilde{\times} \mathbb{C}^{n-1}$$

with group operation given by

$$(s, \mathbf{z})(t, \mathbf{z}') = (s + t + 2\operatorname{Im}\{\mathbf{z}\bar{\mathbf{z}}'\}, \mathbf{z} + \mathbf{z}'),$$

where $\operatorname{Im}(\mathbf{z}\bar{\mathbf{z}}')$ is the imaginary part of the complex number $z_1\bar{z}'_1 + z_2\bar{z}'_2 + \cdots + z_{n-1}\bar{z}'_{n-1}$ for $\mathbf{z} = (z_1, z_2, \dots, z_{n-1})$, $\mathbf{z}' = (z'_1, z'_2, \dots, z'_{n-1}) \in \mathbb{C}^{n-1}$. Then $\mathbf{H}_{2n-1}(\mathbb{C})$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathbf{H}_{2n-1}(\mathbb{C})) = \mathbb{R}$. Let M be an infra-nilmanifold with $\mathbf{H}_{2n-1}(\mathbb{C})$ -geometry; that is, $M = \Pi \backslash \mathbf{H}_{2n-1}(\mathbb{C})$, where Π is a torsion free, discrete, cocompact subgroup of $\mathbf{H}_{2n-1}(\mathbb{C}) \rtimes C$ for some compact subgroup C of $\operatorname{Aut}(\mathbf{H}_{2n-1}(\mathbb{C}))$. It is well known (see for example [4]) that Π contains a cocompact lattice of $\mathbf{H}_{2n-1}(\mathbb{C})$ with index bounded above by a universal constant I . That is, I is the maximal order of the holonomy groups. Note that the analogue in the Euclidean case is a consequence of a theorem of Bieberbach which states that there are only finitely many flat manifolds in each dimension. It is shown in [8] that when $n = 3$, $I = 24$. As a consequence ([8, Corollary]), the minimum volume of a complex hyperbolic 3-manifold M of finite volume with k cusps is $k/9$, and k is at most $-24\pi^3\chi(M)$.

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The main concern of this paper is the quaternionic Heisenberg group

$$\begin{aligned}\mathfrak{H}_n &= \mathbf{H}_{4n-1}(\mathbb{H}) = \{(s, \mathbf{x}) \mid s \in \text{Im}(\mathbb{H}), \mathbf{x} \in \mathbb{H}^{n-1}\} \\ &\approx \mathbb{R}^3 \tilde{\times} \mathbb{H}^{n-1}\end{aligned}$$

with group operation given by

$$(s, \mathbf{p})(t, \mathbf{q}) = (s + t + 2\text{Im}\{\mathbf{q}^* \cdot \mathbf{p}\}, \mathbf{p} + \mathbf{q}),$$

for the column vectors $\mathbf{p} = (p_1, p_2, \dots, p_{n-1})^t$, $\mathbf{q} = (q_1, q_2, \dots, q_{n-1})^t \in \mathbb{H}^{n-1}$, where $\text{Im}\{\mathbf{q}^* \cdot \mathbf{p}\}$ is the imaginary part of the quaternion $\bar{q}_1 p_1 + \bar{q}_2 p_2 + \dots + \bar{q}_{n-1} p_{n-1}$ seen as an element of $\mathbb{R}^3$. Then $\mathfrak{H}_n$ is a simply connected 2-step nilpotent Lie group with the center $\mathcal{Z}(\mathfrak{H}_n) = \mathbb{R}^3$.

Let M be an infra-nilmanifold with $\mathfrak{H}_n$ -geometry; that is, $M = \Pi \backslash \mathfrak{H}_n$, where $\Pi \subset \mathfrak{H}_n \rtimes C$ is a torsion free, discrete subgroup with compact quotient, where C is a compact subgroup of $\text{Aut}(\mathfrak{H}_n)$. Such a group Π is called an *almost Bieberbach group* (=AB-group). Since $\Pi \cap \mathcal{Z}(\mathfrak{H}_n) \cong \mathbb{Z}^3$ is a lattice of $\mathcal{Z}(\mathfrak{H}_n)$, M fits $T^3 \rightarrow M \rightarrow N$, a Seifert 3-torus “bundle” over a $4(n-1)$ -dimensional flat orbifold. When there is no singular point, it is a genuine bundle over the base space N which is a flat Riemannian $4(n-1)$ -manifold.

It is remarkable that the group of isometries of $\mathfrak{H}_2$ as the boundary of the Siegel domain model of $\mathbb{H}_{\mathbb{H}}^2$ (that is, the nilpotent factor of the Iwasawa decomposition of the isometry group $P\text{Sp}(2, 1)$ of $\mathbb{H}_{\mathbb{H}}^2$) contains a maximal subgroup of automorphisms of $\mathfrak{H}_2$ as a real nilpotent Lie group (forgetting quaternionic structure). Therefore, the group of isometries for the natural Hermitian metric is isomorphic to the group of isometries for any left invariant metric on $\mathfrak{H}_2$.

Let I_n be the maximal order of the holonomy groups of all infra-nilmanifolds with $\mathfrak{H}_n$ -geometry. Then the main result of this paper is the following.

Main Theorem. $I_2 = 48$.

The number I_2 is significant in its own right, but here is another application. According to a recent work of Kim and Parker [5, Corollary 5.3], it is related to the minimum volume of quaternionic hyperbolic orbifolds. More precisely, let M be an n -dimensional quaternionic hyperbolic orbifold with k cusps. Then the volume of M is

$$\text{vol}(M) \geq \frac{2^{n-1}k}{3^{n-1}(2n+1)m\sqrt{2}},$$

where m is the maximal index of a lattice in any of the subgroups of $\pi_1(M)$ stabilizing a cusp. When $n = 2$, it is shown in [5, Proposition 5.8] that $m = 576$. We note that this result is still true for manifolds, and in this case $m = I_n$. Hence we have a very sharp result for manifolds; namely, the minimum volume of a 2-dimensional quaternionic hyperbolic manifold with k cusps is at least $\frac{\sqrt{2}k}{15I_2} = \frac{\sqrt{2}k}{720}$.

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2. THE GROUP OF ISOMETRIES AND AFFINE STRUCTURE OF $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$

The group

$$\mathfrak{H}_n = \{(s, \mathbf{x}) \mid s \in \text{Im}(\mathbb{H}), \mathbf{x} \in \mathbb{H}^{n-1}\}$$

is the nilpotent part of the Iwasawa decomposition KAN of $P\text{Sp}(n, 1)$. It is the boundary of the Siegel domain $\mathfrak{H}_n \times \mathbb{R}^+$. The following theorem is known (see [5]).

Theorem 2.1. *With a natural Hermitian metric on $P\mathrm{Sp}(n, 1)$, the Heisenberg isometry group is $\mathrm{Isom}_0(\mathfrak{H}_n) = \mathfrak{H}_n \rtimes (\mathrm{Sp}(n-1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1))$.*

For $n = 2$, this is $\mathfrak{H}_2 \rtimes (\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1))$. We shall explain how this acts on $\mathfrak{H}_2$ below.

We embed $\mathbb{H}$ into $\mathfrak{gl}(4, \mathbb{R})$ as a division algebra as follows:

$$\zeta : \mathbb{H} \longrightarrow \mathfrak{gl}(4, \mathbb{R})$$

defined by

$$(2-1) \quad \zeta(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix}.$$

Note that if $\mathbf{q} = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ corresponds to the first column, then the second, third and fourth columns correspond to $\mathbf{qi}$, $\mathbf{qj}$, $\mathbf{qk}$, respectively. Then ζ is an injective homomorphism (preserving addition and multiplication).

Observe that:

- (1) $\zeta(\bar{\mathbf{q}}) = \zeta(\mathbf{q})^t$, the transpose of $\zeta(\mathbf{q})$, and
- (2) $\det(\zeta(\mathbf{q})) = |\mathbf{q}|^4$.

Using the map $\zeta : \mathbb{H} \longrightarrow \mathfrak{gl}(4, \mathbb{R})$, we can embed the group $\mathfrak{H}_2$ into the affine group $\mathrm{Aff}(\mathbb{R}^7)$ as follows: For $s = s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k}$, $\mathbf{x} = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$,

$$(s, \mathbf{x}) \longrightarrow \begin{bmatrix} I_3 & -2\bar{\xi}(\mathbf{x}) & S \\ 0 & I_4 & X \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aff}(\mathbb{R}^7) \subset \mathrm{GL}(8, \mathbb{R}),$$

where I_3, I_4 are identity matrices of size 3 and 4 respectively,

$$S = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

and $\bar{\xi}(\mathbf{x})$ is obtained by removing the first row from $\zeta(\mathbf{x})^t$ (so it is a real 3×4 matrix). By abuse of notation, we shall use s and $\mathbf{x}$ for S and X .

Then the product in the affine group is

$$\begin{bmatrix} I_3 & -2\bar{\xi}(\mathbf{x}) & s \\ 0 & I_4 & \mathbf{x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 & -2\bar{\xi}(\mathbf{y}) & t \\ 0 & I_4 & \mathbf{y} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_3 & -2\bar{\xi}(\mathbf{x} + \mathbf{y}) & s + t - 2\bar{\xi}(\mathbf{x})\mathbf{y} \\ 0 & I_4 & \mathbf{x} + \mathbf{y} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\bar{\xi}(\mathbf{x})\mathbf{y} = \mathrm{Im}(\bar{\mathbf{x}}\mathbf{y}) = -\mathrm{Im}(\bar{\mathbf{y}}\mathbf{x})$,

$$s + t - 2\bar{\xi}(\mathbf{x})\mathbf{y} = s + t + 2\mathrm{Im}(\bar{\mathbf{y}}\mathbf{x}),$$

and the above equality shows that our map $\mathfrak{H}_2 \longrightarrow \mathrm{Aff}(\mathbb{R}^7)$ is an embedding.

Throughout the paper, σ denotes the matrix

$$(2-2) \quad \sigma = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $\sigma \in O(4) - \mathrm{SO}(4)$.

The group of isometries of $\mathfrak{N}_2$ fixing the identity is

$$(2-3) \quad \mathrm{Sp}(1) \times \mathrm{Sp}(1) = S^3 \times S^3 \subset \mathbb{H}^* \times \mathbb{H}^* \approx \mathrm{GL}(1, \mathbb{H}) \times \mathbb{H}^*$$

which acts on the base space of the bundle $\mathbb{R}^3 \rightarrow \mathfrak{N}_2 \rightarrow \mathbb{H}$ as left and right translations:

$$(2-4) \quad (\mathrm{GL}(1, \mathbb{H}) \times \mathbb{H}^*) \times \mathbb{H} \longrightarrow \mathbb{H},$$

$$((c, \lambda), \mathbf{x}) \mapsto c\mathbf{x}\lambda^{-1}.$$

The action of this group on $\mathfrak{N}_2$ can be described as follows. The left multiplication by c and right multiplication by $d = \lambda^{-1}$ (assuming the modulus of c and d are +1) correspond to conjugation by

$$\begin{bmatrix} I_3 & 0 & 0 \\ 0 & \zeta(c) & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \widehat{D} & 0 & 0 \\ 0 & D & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$(2-5) \quad D = \sigma\zeta(\bar{d})\sigma = \begin{bmatrix} d_1 & -d_2 & -d_3 & -d_4 \\ d_2 & d_1 & d_4 & -d_3 \\ d_3 & -d_4 & d_1 & d_2 \\ d_4 & d_3 & -d_2 & d_1 \end{bmatrix}$$

and

$$(2-6) \quad \widehat{D} = \begin{bmatrix} d_1^2 + d_2^2 - d_3^2 - d_4^2 & 2(d_2d_3 + d_1d_4) & -2(d_1d_3 - d_2d_4) \\ 2(d_2d_3 - d_1d_4) & d_1^2 - d_2^2 + d_3^2 - d_4^2 & 2(d_1d_2 + d_3d_4) \\ 2(d_1d_3 + d_2d_4) & -2(d_1d_2 - d_3d_4) & d_1^2 - d_2^2 - d_3^2 + d_4^2 \end{bmatrix}.$$

[This matrix appears in [5] already.] Notice that $D \in \mathrm{SO}(4)$ and $\widehat{D} \in \mathrm{SO}(3)$. Thus,

$$\begin{bmatrix} I_3 & 0 & 0 \\ 0 & \zeta(c) & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I & -2\bar{\xi}(\mathbf{x}) & s \\ 0 & I & \mathbf{x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 & 0 & 0 \\ 0 & \zeta(c) & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} I & -2\bar{\xi}(\mathbf{x})\zeta(c)^{-1} & s \\ 0 & I & \zeta(c)\mathbf{x} \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \widehat{D} & 0 & 0 \\ 0 & D & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I & -2\bar{\xi}(\mathbf{x}) & s \\ 0 & I & \mathbf{x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{D} & 0 & 0 \\ 0 & D & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} I & -2\widehat{D}\bar{\xi}(\mathbf{x})D^{-1} & s \\ 0 & I & D\mathbf{x} \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that

$$\bar{\xi}(\mathbf{x})\zeta(c)^{-1} = \bar{\xi}(\zeta(c)\mathbf{x}) \quad \text{and} \quad \widehat{D}\bar{\xi}(\mathbf{x})D^{-1} = \bar{\xi}(D\mathbf{x})$$

so that both S^3 's in the group (2-3) are automorphisms of $\mathfrak{N}_2$. If $\widehat{D} = I_3$, then $D = \pm I_4$. Therefore the two S^3 's intersect at $\mathbb{Z}_2 = \{\pm I_4\}$.

3. THE GROUP OF AUTOMORPHISMS OF $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$

In this section, we shall calculate $\mathrm{Aut}(\mathfrak{N}_2)$, the group of automorphisms of the real Lie group $\mathfrak{N}_2$ (no quaternionic structure). From now on, we shall use $\mathfrak{N}_2 = \mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ rather than $\mathbb{R}^3 \tilde{\times} \mathbb{H}$. Using the identification

$$(3-1) \quad \rho : \mathbb{H} \longrightarrow \mathbb{R}^4,$$

$$\rho(x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}) = [x_1, x_2, x_3, x_4]^t,$$

we identify $\mathbb{R}^3 \tilde{\times} \mathbb{H}$ with $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ by

$$(s, \mathbf{q}) \longleftrightarrow (s, \rho(\mathbf{q})).$$

Accordingly, we introduce a new notation for $\text{Im}\{\bar{\mathbf{y}}\mathbf{x}\}$. For

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix},$$

define

$$(3-2) \quad \mathcal{I}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} -\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \\ -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 \\ x_4 & y_4 \end{vmatrix} \\ -\begin{vmatrix} x_1 & y_1 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \end{bmatrix}.$$

We shall represent $\mathcal{I}(\mathbf{x}, \mathbf{y})$ by matrix products as follows. Using the function $\zeta : \mathbb{H} \longrightarrow \mathbb{R}^4$ in (2-1), let

$$K_1 = \zeta(-\mathbf{i}), \quad K_2 = \zeta(-\mathbf{j}), \quad K_3 = \zeta(-\mathbf{k}).$$

Then

$$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Let σ be the matrix in (2-2) and

$$(3-3) \quad J_i = \sigma^{-1} K_i \sigma, \quad i = 1, 2, 3,$$

so that

$$J_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, if $\mathbf{x} = \rho(\mathbf{q})$, we have

$$\begin{aligned} J_1 \mathbf{x} &= \rho(\mathbf{qi}), & J_2 \mathbf{x} &= \rho(\mathbf{qj}), & J_3 \mathbf{x} &= \rho(\mathbf{qk}), \\ -K_1 \mathbf{x} &= \rho(\mathbf{iq}), & -K_2 \mathbf{x} &= \rho(\mathbf{jq}), & -K_3 \mathbf{x} &= \rho(\mathbf{kq}). \end{aligned}$$

The definition of $\mathcal{I}$ in (3-2) becomes

$$(3-4) \quad \mathcal{I}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^t J_1 \mathbf{y}, \mathbf{x}^t J_2 \mathbf{y}, \mathbf{x}^t J_3 \mathbf{y})^t.$$

Clearly $\mathcal{I}(\mathbf{x}, \mathbf{y}) = -\mathcal{I}(\mathbf{y}, \mathbf{x})$ and $\mathcal{I}(\mathbf{x}, \mathbf{y})$ corresponds to $\text{Im}\{\bar{\mathbf{y}}\mathbf{x}\}$. Thus the group operation in $\mathfrak{N}_2 = \mathbb{R}^3 \tilde{\times} \mathbb{R}^4$ becomes

$$(s, \mathbf{x})(t, \mathbf{y}) = (s + t + 2\mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{x} + \mathbf{y}).$$

Since $\mathcal{I}(\mathbf{x}, \pm \mathbf{x}) = 0$, we see easily that

$$(s, \mathbf{x})^{-1} = (-s, -\mathbf{x}).$$

Thus we have

$$[(s, \mathbf{x}), (t, \mathbf{y})] = (s, \mathbf{x})^{-1}(t, \mathbf{y})^{-1}(s, \mathbf{x})(t, \mathbf{y}) = (4\mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{0}).$$

Clearly J_1, J_2, J_3 together with K_1, K_2, K_3 form a linear basis for the vector space $\mathfrak{so}(4, \mathbb{R})$ of the skew-symmetric matrices. We denote the two subspaces of $\mathfrak{so}(4, \mathbb{R})$ as follows:

$$\begin{aligned}\mathfrak{so}(4, \mathbb{R})_\ell &= \langle J_1, J_2, J_3 \rangle, \\ \mathfrak{so}(4, \mathbb{R})_r &= \langle K_1, K_2, K_3 \rangle.\end{aligned}$$

Since $J_i K_j = K_j J_i$ for all $i, j = 1, 2, 3$, $\mathfrak{so}(4, \mathbb{R})_\ell$ and $\mathfrak{so}(4, \mathbb{R})_r$ are ideals of $\mathfrak{so}(4, \mathbb{R})$ and

$$\mathfrak{so}(4, \mathbb{R}) = \mathfrak{so}(4, \mathbb{R})_\ell \oplus \mathfrak{so}(4, \mathbb{R})_r.$$

For any $C \in \mathrm{GL}(4, \mathbb{R})$ and $V \in \mathfrak{so}(4, \mathbb{R})$,

$$J_C(V) = C^t V C$$

defines a linear isomorphism $J_C : \mathfrak{so}(4, \mathbb{R}) \longrightarrow \mathfrak{so}(4, \mathbb{R})$. In fact, with respect to the basis $\{J_1, J_2, J_3, K_1, K_2, K_3\}$, it turns out that $\det(J_C) = (\det(C))^3$.

In order to calculate the group of automorphisms of $\mathfrak{N}_2$, first we define

$$O(\mathbf{J}, 4) = \{C \in \mathrm{GL}(4, \mathbb{R}) \mid C^t J_i C \in \mathfrak{so}(4, \mathbb{R})_\ell \text{ for } i = 1, 2, 3\}.$$

Then $C \in O(\mathbf{J}, 4)$ if and only if the map J_C leaves the subspace $\mathfrak{so}(4, \mathbb{R})_\ell$ spanned by J_1, J_2, J_3 invariant. Therefore,

$$(3-5) \quad C^t J_i C = \lambda_{i1} J_1 + \lambda_{i2} J_2 + \lambda_{i3} J_3, \quad \lambda_{ij} \in \mathbb{R},$$

for $i = 1, 2, 3$. Then it turns out that the matrix $\lambda = (\lambda_{ij})$ is non-singular.

Now we form the column vector

$$\mathbf{J} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix}$$

with entries the matrices J_1, J_2, J_3 . With some abuse of notation, we can write

$$O(\mathbf{J}, 4) = \{C \in \mathrm{GL}(4, \mathbb{R}) \mid C^t \mathbf{J} C = \lambda \mathbf{J}, \lambda \in \mathrm{GL}(3, \mathbb{R})\}.$$

Clearly $O(\mathbf{J}, 4)$ is a closed subgroup of $\mathrm{GL}(4, \mathbb{R})$. For $C \in O(\mathbf{J}, 4)$, let $\widehat{C} \in \mathrm{GL}(3, \mathbb{R})$ denote the non-singular 3×3 matrix λ which satisfies $C^t \mathbf{J} C = \lambda \mathbf{J}$. Therefore,

$$C^t \mathbf{J} C = \widehat{C} \mathbf{J}.$$

Then $C \mapsto \widehat{C}$ defines a homomorphism $\epsilon : O(\mathbf{J}, 4) \rightarrow \mathrm{GL}(3, \mathbb{R})$ by

$$\epsilon(C) = \widehat{C}$$

so that $C^t \mathbf{J} C = \epsilon(C) \mathbf{J}$.

Lemma 3.1. (1) ϵ is a homomorphism.

(2) $C \mapsto (\frac{1}{\sqrt{|\det(C)|}} \widehat{C}, \sqrt{|\det(C)|})$ defines a homomorphism $\widehat{} : O(\mathbf{J}, 4) \longrightarrow$

$O(3) \times \mathbb{R}^+$. Moreover, $\det(\widehat{C})^2 = |\det(C)|^3$.

(3) $O(\mathbf{J}, 4)$ is closed under transpose and the above homomorphism ϵ commutes with transpose.

Proof. For $C, D \in O(\mathbf{J}, 4)$,

$$\begin{aligned} (CD)^t \mathbf{J}(CD) &= D^t (C^t \mathbf{J} C) D \\ &= D^t (\widehat{C} \mathbf{J}) D \\ &= \widehat{C} \cdot D^t \mathbf{J} D \text{ (since } \widehat{C} \text{ is a "scalar" matrix)} \\ &= \widehat{C} \widehat{D} \mathbf{J} \end{aligned}$$

shows $\epsilon(CD) = \epsilon(C)\epsilon(D)$.

Now assume that $C \in O(\mathbf{J}, 4)$ and $\det(C) = \pm 1$. Let $\widehat{C} = (\lambda_{ij})$. Then the equalities (3–5) are satisfied. A direct calculation from these equalities shows that

$$|\det(C)| = \lambda_{i1}^2 + \lambda_{i2}^2 + \lambda_{i3}^2$$

for all $i = 1, 2, 3$. Let $\mathbf{c}_i$ be the i th column of the matrix C . Then

$$\mathcal{I}(\mathbf{c}_i, \mathbf{c}_j) = \begin{bmatrix} \mathbf{c}_i^t J_1 \mathbf{c}_j \\ \mathbf{c}_i^t J_2 \mathbf{c}_j \\ \mathbf{c}_i^t J_3 \mathbf{c}_j \end{bmatrix} = \begin{bmatrix} (C^t J_1 C)_{(i,j)} \\ (C^t J_2 C)_{(i,j)} \\ (C^t J_3 C)_{(i,j)} \end{bmatrix}.$$

From the equalities (3–5), we have

$$\begin{aligned} -(C^t J_1 C)_{(1,2)} &= \lambda_{11} = (C^t J_1 C)_{(3,4)}, \\ -(C^t J_2 C)_{(1,2)} &= \lambda_{21} = (C^t J_2 C)_{(3,4)}, \\ -(C^t J_3 C)_{(1,2)} &= \lambda_{31} = (C^t J_3 C)_{(3,4)}. \end{aligned}$$

Consequently, we have

$$-\mathcal{I}(\mathbf{c}_1, \mathbf{c}_2) = \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \end{bmatrix} = \mathcal{I}(\mathbf{c}_3, \mathbf{c}_4).$$

Similarly, the second and third columns of $\widehat{C} = (\lambda_{ij})$ are

$$\begin{aligned} -\mathcal{I}(\mathbf{c}_1, \mathbf{c}_3) &= -\mathcal{I}(\mathbf{c}_2, \mathbf{c}_4), \\ -\mathcal{I}(\mathbf{c}_1, \mathbf{c}_4) &= \mathcal{I}(\mathbf{c}_2, \mathbf{c}_3) \end{aligned}$$

so that

$$(3-6) \quad \widehat{C} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} -\mathcal{I}(\mathbf{c}_1, \mathbf{c}_2) & -\mathcal{I}(\mathbf{c}_1, \mathbf{c}_3) & -\mathcal{I}(\mathbf{c}_1, \mathbf{c}_4) \end{bmatrix}.$$

For the sake of calculation, we set the second matrix to (λ'_{ij}) , so

$$(3-7) \quad \widehat{C} = \begin{bmatrix} \lambda'_{11} & \lambda'_{12} & \lambda'_{13} \\ \lambda'_{21} & \lambda'_{22} & \lambda'_{23} \\ \lambda'_{31} & \lambda'_{32} & \lambda'_{33} \end{bmatrix} = \begin{bmatrix} \mathcal{I}(\mathbf{c}_3, \mathbf{c}_4) & -\mathcal{I}(\mathbf{c}_2, \mathbf{c}_4) & \mathcal{I}(\mathbf{c}_2, \mathbf{c}_3) \end{bmatrix}.$$

Of course, then $\lambda_{ij} = \lambda'_{ij}$ for all i, j . For $j \neq k$, using the equalities (3–6) and (3–7), it can be shown that

$$\begin{aligned} &2(\lambda_{j1}\lambda_{k1} + \lambda_{j2}\lambda_{k2} + \lambda_{j3}\lambda_{k3}) \\ &= (\lambda_{j1}\lambda'_{k1} + \lambda_{j2}\lambda'_{k2} + \lambda_{j3}\lambda'_{k3}) + (\lambda'_{j1}\lambda_{k1} + \lambda'_{j2}\lambda_{k2} + \lambda'_{j3}\lambda_{k3}) \\ &= 0. \end{aligned}$$

Thus any two row vectors of $\widehat{C}$ are orthogonal. Consequently if $C \in O(\mathbf{J}, 4)$ with $\det(C) = \pm 1$, then $\widehat{C} \in O(3)$.

Suppose $C \in O(\mathbf{J}, 4)$. Let $d = \frac{1}{\sqrt[4]{|\det(C)|}}$ and $D = dI \in \text{GL}(4, \mathbb{R})$, a diagonal matrix. Then $D \in O(\mathbf{J}, 4)$ and $dC = DC \in O(\mathbf{J}, 4)$ with $\det(dC) = \pm 1$. By the above argument, $\widehat{dC} \in O(3)$ and $\widehat{dC} = \widehat{D}\widehat{C} = d^2\widehat{C}$. Hence $\frac{1}{\sqrt{|\det(C)|}}\widehat{C} \in O(3)$. This shows that

$$\wedge: O(\mathbf{J}, 4) \longrightarrow O(3) \times \mathbb{R}^+, \quad C \mapsto \left(\frac{1}{\sqrt{|\det(C)|}}\widehat{C}, \sqrt{|\det(C)|} \right)$$

is a homomorphism and $\det(\widehat{C})^2 = |\det(C)|^3$.

Next suppose $C \in O(\mathbf{J}, 4)$. Let $\widehat{C} = (\lambda_{ij})$. From the equalities (3-5), we have $J_1^2 = -I_4$, $J_1J_2 = -J_2J_1$, etc. Therefore,

$$(\lambda_{i1}J_1 + \lambda_{i2}J_2 + \lambda_{i3}J_3)^2 = -|\det(C)| I_4.$$

Taking inverse matrices of the equalities (3-5), we get

$$\begin{aligned} -C^{-1}J_i(C^{-1})^t &= (C^tJ_iC)^{-1} \\ &= (\lambda_{i1}J_1 + \lambda_{i2}J_2 + \lambda_{i3}J_3)^{-1} \\ &= -\frac{1}{|\det(C)|}(\lambda_{i1}J_1 + \lambda_{i2}J_2 + \lambda_{i3}J_3). \end{aligned}$$

This shows that $(C^{-1})^t$ is also an element of $O(\mathbf{J}, 4)$ with

$$((C^{-1})^t)^\wedge = \frac{1}{|\det(C)|}\widehat{C}.$$

Suppose $\det(C) = \pm 1$. Then, from the above equality, we have

$$\begin{aligned} (C^t)^\wedge &= (C^{-1})^\wedge \\ &= (C^\wedge)^{-1} \text{ (since } \epsilon \text{ is a homomorphism)} \\ &= (C^\wedge)^t \text{ (since } \widehat{C} \in O(3)). \end{aligned}$$

One can now easily see the equality $(C^t)^\wedge = (C^\wedge)^t$ for any $C \in O(\mathbf{J}, 4)$. This proves the lemma. $\square$

Let $S^3 \subset \mathbb{H}^*$ be the group of unit quaternions

$$S^3 = \{x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

Using $\zeta: \mathbb{H} \longrightarrow \mathfrak{gl}(4, \mathbb{R})$ of (2-1), we obtain two copies of S^3 in $O(\mathbf{J}, 4)$:

$$\begin{aligned} S_\ell^3 &= \zeta(S^3), \\ (3-8) \quad S_r^3 &= \sigma\zeta(S^3)\sigma. \end{aligned}$$

Then

Proposition 3.2. (1) ϵ has image $\epsilon(S_r^3 \times \mathbb{R}) = \text{SO}(3) \times \mathbb{R}^+$.
 (2) ϵ has kernel S_ℓ^3 so that $O(\mathbf{J}, 4) = \text{SO}(4) \times \mathbb{R}^+ = (S_\ell^3 \times_{\mathbb{Z}_2} S_r^3) \times \mathbb{R}^+$.

Proof. For $\mathbf{q} = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$,

$$(3-9) \quad \begin{aligned} \zeta(\mathbf{q}) &= \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \in S_\ell^3, \\ \sigma\zeta(\overline{\mathbf{q}})\sigma &= \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & x_4 & -x_3 \\ x_3 & -x_4 & x_1 & x_2 \\ x_4 & x_3 & -x_2 & x_1 \end{bmatrix} \in S_r^3 \end{aligned}$$

with

$$(3-10) \quad \begin{aligned} \zeta(\mathbf{q})^\wedge &= I_3, \\ (\sigma\zeta(\overline{\mathbf{q}})\sigma)^\wedge &= \begin{bmatrix} x_1^2 + x_2^2 - x_3^2 - x_4^2 & 2(x_2x_3 + x_1x_4) & -2(x_1x_3 - x_2x_4) \\ 2(x_2x_3 - x_1x_4) & x_1^2 - x_2^2 + x_3^2 - x_4^2 & 2(x_1x_2 + x_3x_4) \\ 2(x_1x_3 + x_2x_4) & -2(x_1x_2 - x_3x_4) & x_1^2 - x_2^2 - x_3^2 + x_4^2 \end{bmatrix}. \end{aligned}$$

See the equalities (2-1), (2-5) and (2-6).

Then $S_\ell^3 \times S_r^3$ corresponds to $S^3 \times S^3 \subset \mathbb{H}^* \times \mathbb{H}^* = \mathrm{GL}(1, \mathbb{H}) \times \mathbb{H}^*$ in (2-4) acting on $\mathbb{H}$, by left and right multiplications. Clearly, $S_\ell^3 \cap S_r^3 = \mathbb{Z}_2 = \{\pm I_4\}$ and

$$S_\ell^3 \times_{\mathbb{Z}_2} S_r^3 = \mathrm{SO}(4) \subset O(\mathbf{J}, 4).$$

From the equalities (3-10), the map

$$\epsilon : O(\mathbf{J}, 4) \rightarrow O(3) \times \mathbb{R}^+$$

maps S_r^3 onto $\mathrm{SO}(3)$ (with kernel $\mathbb{Z}_2 = \{\pm I_4\}$) and S_ℓ^3 trivially.

Next we observe that the matrix $\mathrm{diag}\{1, 1, -1\}$ is not in the image of the homomorphism. Otherwise, there would be $C \in O(\mathbf{J}, 4)$ so that

$$C^t J_1 C = J_1, \quad C^t J_2 C = J_2, \quad C^t J_3 C = -J_3.$$

Since $J_1 J_2 = -J_3$, we would have

$$\begin{aligned} J_1 J_2 &= -J_3 = C^t J_3 C = -C^t J_1 J_2 C \\ &= -C^t J_1 C \cdot C^{-1} (C^t)^{-1} \cdot C^t J_2 C \\ &= -J_1 (C^t C)^{-1} J_2, \end{aligned}$$

and $C^t C = -I$, and hence the $(1, 1)$ -entry would be $\sum_i c_{i1}^2 = -1$, which is impossible. Thus $\mathrm{diag}\{1, 1, -1\}$ is not in the image of ϵ . Now suppose there exists $C \in O(\mathbf{J}, 4)$ so that $\widehat{C} \in O(3) \setminus \mathrm{SO}(3)$. Then we can write $\widehat{C} = \mathrm{diag}\{1, 1, -1\} \widehat{E}$ for some $E \in O(\mathbf{J}, 4)$. Hence $\widehat{C} E^{-1} = \mathrm{diag}\{1, 1, -1\}$. But the right-hand side is not in the image of the homomorphism. This proves that

$$\epsilon : O(\mathbf{J}, 4) \longrightarrow \mathrm{SO}(3) \times \mathbb{R}^+$$

is a surjective homomorphism. (Recall, for $s \neq 0$, $sI_4 \in O(\mathbf{J}, 4)$ with $(sI)^\wedge = s^2 I_3$.)

Suppose that $C \in \ker(\epsilon)$. Then $\widehat{C} = I$ and thus $C^t J_i C = J_i$ for $i = 1, 2, 3$. Since $J_1 J_2 = -J_3$, we have

$$\begin{aligned} J_1 J_2 &= -J_3 = -C^t J_3 C = C^t J_1 J_2 C = C^t J_1 C \cdot C^{-1} (C^t)^{-1} \cdot C^t J_2 C \\ &= J_1 (C^t C)^{-1} J_2, \end{aligned}$$

$C^t C = I$ and $C \in O(4)$. On the other hand, $C^t J_1 C = J_1$ and $C^t J_2 C = J_2$ immediately induce that C is of the form $\zeta(\mathbf{q})$ for some $\mathbf{q} \in S_\ell^3$. Since we know that S_ℓ^3 is already in the kernel of ϵ , we conclude that

$$\ker(\epsilon) = S_\ell^3.$$

Thus we have obtained an exact sequence

$$1 \rightarrow S_\ell^3 \rightarrow O(\mathbf{J}, 4) \rightarrow \mathrm{SO}(3) \times \mathbb{R}^+ \rightarrow 1$$

and equalities

$$\begin{aligned} O(\mathbf{J}, 4) &= (S_\ell^3 \times_{\mathbb{Z}_2} S_r^3) \times \mathbb{R}^+ \\ &= \mathrm{SO}(4) \times \mathbb{R}^+. \end{aligned}$$

It is clear that S_ℓ^3 and S_r^3 and $\mathbb{R}^+$ commute with each other. $\square$

Corollary 3.3. *If $C \in O(\mathbf{J}, 4)$, then*

$$\det(\hat{C}) = \det(C)^{3/2}.$$

Proof. Follows from Lemma 3.1 and Proposition 3.2. $\square$

Lemma 3.4. *Every matrix in $S_\ell^3 \cup S_r^3$ except $\pm I$ has eigenvalues $a + b\mathbf{i}$, $a + b\mathbf{i}$, $a - b\mathbf{i}$, $a - b\mathbf{i}$ with $b \neq 0$ (two complex double roots). Conversely, suppose $A \in \mathrm{SO}(4)$ has eigenvalues $a \pm b\mathbf{i}$ with $b \neq 0$ of multiplicity 2. Then either $A \in S_\ell^3$ or $A \in S_r^3$.*

Proof. Let $A \in \mathrm{SO}(4)$ have multiple eigenvalues $a \pm b\mathbf{i}$ with $b \neq 0$. Looking at the standard maximal torus of $\mathrm{SO}(4)$, there is $P \in \mathrm{SO}(4)$ such that $PAP^{-1} = \begin{bmatrix} R(\theta) & 0 \\ 0 & R(\theta') \end{bmatrix}$, where $R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. Since $A \in \mathrm{SO}(4)$ has multiple eigenvalues $a \pm b\mathbf{i}$ with $b \neq 0$, either $\theta = \theta'$ or $\theta = -\theta'$. If $PAP^{-1} = \begin{bmatrix} R(\theta) & 0 \\ 0 & R(\theta) \end{bmatrix}$, then $PAP^{-1} \in S_\ell^3$; if $PAP^{-1} = \begin{bmatrix} R(\theta) & 0 \\ 0 & R(-\theta) \end{bmatrix}$, then $PAP^{-1} \in S_r^3$. Since S_ℓ^3 and S_r^3 are normal subgroups in $\mathrm{SO}(4)$, $A \in S_r^3$ or $A \in S_\ell^3$. $\square$

Since the center, $\mathcal{Z}(\mathfrak{N}_2) = \mathbb{R}^3$, is a characteristic subgroup of $\mathfrak{N}_2$, every automorphism of $\mathfrak{N}_2$ restricts to an automorphism of $\mathbb{R}^3$. Consequently an automorphism of $\mathfrak{N}_2$ induces an automorphism on the quotient group $\mathbb{R}^4$. Thus there is a natural homomorphism $\mathrm{Aut}(\mathfrak{N}_2) \rightarrow \mathrm{Aut}(\mathbb{R}^3) \times \mathrm{Aut}(\mathbb{R}^4)$, $\theta \mapsto (\hat{\theta}, \bar{\theta})$.

Lemma 3.5. $\mathrm{Image}\{\mathrm{Aut}(\mathfrak{N}_2) \rightarrow \mathrm{Aut}(\mathbb{R}^4)\} = O(\mathbf{J}, 4)$. *Moreover, the exact sequence $\mathrm{Aut}(\mathfrak{N}_2) \rightarrow O(\mathbf{J}, 4) \rightarrow 1$ splits.*

Proof. Let $\theta \in \mathrm{Aut}(\mathfrak{N}_2)$. Then $(\hat{\theta}, \bar{\theta}) \in \mathrm{Aut}(\mathbb{R}^3) \times \mathrm{Aut}(\mathbb{R}^4)$. Since $[(s, \mathbf{x}), (t, \mathbf{y})] = (4\mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{0})$,

$$\theta[(s, \mathbf{x}), (t, \mathbf{y})] = \theta(4\mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{0}) = (\hat{\theta}(4\mathcal{I}(\mathbf{x}, \mathbf{y})), \mathbf{0}) = (4\hat{\theta}(\mathcal{I}(\mathbf{x}, \mathbf{y})), \mathbf{0})$$

and

$$[\theta(s, \mathbf{x}), \theta(t, \mathbf{y})] = [(*, \bar{\theta}(\mathbf{x})), (*, \bar{\theta}(\mathbf{y}))] = (4\mathcal{I}(\bar{\theta}(\mathbf{x}), \bar{\theta}(\mathbf{y})), \mathbf{0})$$

yield

$$\mathcal{I}(\bar{\theta}(\mathbf{x}), \bar{\theta}(\mathbf{y})) = \hat{\theta}(\mathcal{I}(\mathbf{x}, \mathbf{y})),$$

or, equivalently,

$$(\bar{\theta}(\mathbf{x})^t J_1 \bar{\theta}(\mathbf{y}), \bar{\theta}(\mathbf{x})^t J_2 \bar{\theta}(\mathbf{y}), \bar{\theta}(\mathbf{x})^t J_3 \bar{\theta}(\mathbf{y}))^t = \hat{\theta} \cdot (\mathbf{x}^t J_1 \mathbf{y}, \mathbf{x}^t J_2 \mathbf{y}, \mathbf{x}^t J_3 \mathbf{y})^t$$

for all $\mathbf{x}, \mathbf{y}$. This happens if and only if $\bar{\theta}^t \mathbf{J} \bar{\theta} = \hat{\theta} \mathbf{J}$.

Conversely, suppose that $\bar{\theta} \in O(\mathbf{J}, 4)$, i.e., $\bar{\theta}^t \mathbf{J} \bar{\theta} = \lambda \mathbf{J}$ is satisfied for some $\lambda \in \text{GL}(3, \mathbb{R})$. We define $\theta \in \text{Aut}(\mathfrak{N}_2)$ by

$$\theta(s, \mathbf{x}) = (\lambda \cdot s, \bar{\theta}(\mathbf{x})).$$

Then

$$\begin{aligned} \theta((s, \mathbf{x}) \cdot (t, \mathbf{y})) &= \theta(s + t + 2\mathcal{I}(\mathbf{x}, \mathbf{y}), \mathbf{x} + \mathbf{y}) \\ &= (\lambda \cdot (s + t + 2\mathcal{I}(\mathbf{x}, \mathbf{y})), \bar{\theta}(\mathbf{x} + \mathbf{y})) \\ &= (\lambda \cdot s + \lambda \cdot t + \lambda \cdot 2\mathcal{I}(\mathbf{x}, \mathbf{y}), \bar{\theta}(\mathbf{x} + \mathbf{y})), \\ \theta(s, \mathbf{x}) \cdot \theta(t, \mathbf{y}) &= (\lambda \cdot s, \bar{\theta}(\mathbf{x})) \cdot (\lambda \cdot t, \bar{\theta}(\mathbf{y})) \\ &= (\lambda \cdot s + \lambda \cdot t + 2\mathcal{I}(\bar{\theta}(\mathbf{x}), \bar{\theta}(\mathbf{y})), \bar{\theta}(\mathbf{x}) + \bar{\theta}(\mathbf{y})). \end{aligned}$$

Now the condition $\bar{\theta}^t \mathbf{J} \bar{\theta} = \lambda \mathbf{J}$ guarantees that

$$\lambda \cdot \mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\bar{\theta}(\mathbf{x}), \bar{\theta}(\mathbf{y})).$$

Thus θ is an automorphism of $\mathfrak{N}_2$. Moreover, this defines a split homomorphism $O(\mathbf{J}, 4) \rightarrow \text{Aut}(\mathfrak{N}_2)$. $\square$

Theorem 3.6 (Structure of $\text{Aut}(\mathfrak{N}_2)$).

$$\begin{aligned} \text{Aut}(\mathfrak{N}_2) &= \text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(\mathbf{J}, 4) \\ &= \text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes ((S_\ell^3 \times_{\mathbb{Z}_2} S_r^3) \times \mathbb{R}^+), \end{aligned}$$

where an element $(\eta, A) \in \text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(\mathbf{J}, 4)$ acts by

$$(\eta, A)(s, \mathbf{x}) = (\hat{A}s + \eta(\mathbf{x}), A\mathbf{x}).$$

Proof. Let $\theta \in \text{Aut}(\mathfrak{N}_2)$. Then we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathfrak{N}_2 & \longrightarrow & \mathbb{R}^4 \longrightarrow 1 \\ & & \downarrow \hat{\theta} & & \downarrow \theta & & \downarrow \bar{\theta} \\ 1 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathfrak{N}_2 & \longrightarrow & \mathbb{R}^4 \longrightarrow 1 \end{array}$$

Thus $\theta(s, \mathbf{x}) = (\hat{\theta}(s) + \eta(s, \mathbf{x}), \bar{\theta}(\mathbf{x}))$ for $(s, \mathbf{x}) \in \mathfrak{N}_2$, where $\eta : \mathfrak{N}_2 \rightarrow \mathbb{R}^3$. Since θ is a homomorphism, one can show that η is a homomorphism, i.e.,

$$\eta((s, \mathbf{x})(t, \mathbf{y})) = \eta(s, \mathbf{x}) + \eta(t, \mathbf{y}).$$

In particular, $(\hat{\theta}(s), \mathbf{0}) = \theta(s, \mathbf{0}) = (\hat{\theta}(s) + \eta(s, \mathbf{0}), \mathbf{0})$ implies that $\eta(s, \mathbf{0}) = \mathbf{0}$ for all $s \in \mathbb{R}^3$, and thus $\eta(s, \mathbf{x}) = \eta((s, \mathbf{0})(0, \mathbf{x})) = \eta(s, \mathbf{0}) + \eta(0, \mathbf{x}) = \eta(0, \mathbf{x})$. Hence $\eta \in \text{Hom}(\mathbb{R}^4, \mathbb{R}^3)$.

Let us find out the kernel of the surjective homomorphism of Lemma 3.5:

$$\text{Aut}(\mathfrak{N}_2) \rightarrow O(\mathbf{J}, 4), \quad \theta \mapsto \bar{\theta}.$$

Suppose that $\theta \in \text{Aut}(\mathfrak{N}_2)$ with $\bar{\theta} = \text{id}_{\mathbb{R}^4}$. Then $\hat{\theta} = \text{id}_{\mathbb{R}^3}$ and thus $\theta(s, \mathbf{x}) = (s + \eta(\mathbf{x}), \mathbf{x})$ for some $\eta \in \text{Hom}(\mathbb{R}^4, \mathbb{R}^3)$. Conversely given $\eta \in \text{Hom}(\mathbb{R}^4, \mathbb{R}^3)$, define $\theta \in \text{Aut}(\mathfrak{N}_2)$ by $\theta(s, \mathbf{x}) = (s + \eta(\mathbf{x}), \mathbf{x})$. Clearly this θ lies in the kernel of the homomorphism. Hence we have a short exact sequence

$$1 \rightarrow \text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rightarrow \text{Aut}(\mathfrak{N}_2) \rightarrow O(\mathbf{J}, 4) \rightarrow 1.$$

By Lemma 3.5, this sequence is split. $\square$

Note that $\text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(\mathbf{J}, 4)$ is sitting in $\text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes (\text{GL}(3, \mathbb{R}) \times O(\mathbf{J}, 4))$ as $(\eta, (\hat{A}, A))$, and the action of $O(\mathbf{J}, 4)$ on $\text{Hom}(\mathbb{R}^4, \mathbb{R}^3)$ is ${}^A\eta(\mathbf{x}) = \hat{A} \cdot \eta(A^{-1}\mathbf{x})$. The group operation on $\mathfrak{N}_2 \rtimes O(\mathbf{J}, 4)$ is given by

$$\begin{aligned} ((s, \mathbf{x}), A)((t, \mathbf{y}), B) &= ((s, \mathbf{x}) \cdot {}^A(t, \mathbf{y}), AB) \\ &= ((s, \mathbf{x}) \cdot (\hat{A}t, A\mathbf{y}), AB) \\ &= ((s + \hat{A}t + 2\mathcal{I}(\mathbf{x}, A\mathbf{y}), \mathbf{x} + A\mathbf{y}), AB). \end{aligned}$$

4. THE STRUCTURE OF AB-GROUPS FOR $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4$

Let $\Pi \subset \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2)$ be an AB-group. Then it is well known that $\Gamma = \Pi \cap \mathfrak{N}_2$, the pure translations in Π , is the maximal normal nilpotent subgroup, and $\Phi = \Pi/\Gamma$, the holonomy group of Π , is finite. Since Γ is a lattice of $\mathfrak{N}_2$, $\Gamma \cap \mathcal{Z}(\mathfrak{N}_2)$ is a lattice of $\mathcal{Z}(\mathfrak{N}_2) = \mathbb{R}^3$, and $\Gamma/\Gamma \cap \mathcal{Z}(\mathfrak{N}_2)$ is a lattice of $\mathfrak{N}_2/\mathcal{Z}(\mathfrak{N}_2) = \mathbb{R}^4$. Thus $\Gamma \cap \mathcal{Z}(\mathfrak{N}_2) \cong \mathbb{Z}^3$ and $\Gamma/\Gamma \cap \mathcal{Z}(\mathfrak{N}_2) \cong \mathbb{Z}^4$.

Consider the following natural commutative diagram:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{R}^3 & \xrightarrow{=} & \mathbb{R}^3 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathfrak{N}_2 & \longrightarrow & \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2) & \longrightarrow & \text{Aut}(\mathfrak{N}_2) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \rtimes \text{Aut}(\mathfrak{N}_2) & \longrightarrow & \text{Aut}(\mathfrak{N}_2) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

Recall from Theorem 3.6 that an element $(\eta, A) \in \text{Aut}(\mathfrak{N}_2) = \text{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(\mathbf{J}, 4)$ acts on $(s, \mathbf{x}) \in \mathfrak{N}_2$ by

$$(\eta, A)(s, \mathbf{x}) = (\hat{A}s + \eta(\mathbf{x}), A\mathbf{x}).$$

Thus $O(\mathbf{J}, 4)$ acts on $\mathbb{R}^3$ via the homomorphism $\wedge : O(\mathbf{J}, 4) \rightarrow \text{GL}(3, \mathbb{R})$, and $O(\mathbf{J}, 4)$ acts on $\mathbb{R}^4$ by matrix multiplication $O(\mathbf{J}, 4) \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$.

Let $Q = \Pi/\mathbb{Z}^3$. Then the previous diagram induces the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{Z}^3 & \xrightarrow{\quad} & \mathbb{Z}^3 & & \\
 & & \downarrow & & \downarrow & & \\
 (4-1) & 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow \Phi \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow = \\
 & 1 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & Q & \longrightarrow \Phi \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & &
 \end{array}$$

Here $\Phi \subset O(\mathbf{J}, 4)$ acts on $\mathbb{Z}^4$ by matrix multiplication, and on $\mathbb{Z}^3$ via the homomorphism $\hat{\cdot}: O(\mathbf{J}, 4) \rightarrow \mathrm{GL}(3, \mathbb{R})$.

Lemma 4.1. *Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be a basis of $\mathbb{R}^4$ with rational entries. Then for each $i = 1, 2, 3, 4$, the set $\{\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \dots, \hat{i}, \dots, 4\}$ forms a lattice of $\mathbb{R}^3$.*

Proof. Let V be the 4×4 -matrix whose column vectors are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and let W_i be the 3×3 -matrix whose column vectors are $\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j)$ with $j = 1, \dots, \hat{i}, \dots, 4$. Then we can show that

$$\det(W_1) = \det[\mathcal{I}(\mathbf{v}_1, \mathbf{v}_2), \mathcal{I}(\mathbf{v}_1, \mathbf{v}_3), \mathcal{I}(\mathbf{v}_1, \mathbf{v}_4)] = -|\mathbf{v}_1|^2 \det[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4].$$

Similarly, $\det(W_i) = \pm |\mathbf{v}_i|^2 \det[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4]$. Thus if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is a basis of $\mathbb{R}^4$, then the set $\{\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \dots, \hat{i}, \dots, 4\}$ spans $\mathbb{R}^3$. The map $\mathcal{I}(\mathbf{x}, \mathbf{y})$ consists of polynomial functions of the entries of $\mathbf{x}$ and $\mathbf{y}$. Therefore the group generated by $\{\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \dots, \hat{i}, \dots, 4\}$ is discrete so that it forms a lattice of $\mathbb{R}^3$. $\square$

This lemma tells us that the lattice $\mathbb{Z}^4$ of $\mathbb{R}^4$ generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ yields a lattice generated by $\{4\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \dots, \hat{i}, \dots, 4\}$, which must be contained in $\mathbb{Z}^3$ in the above diagram. However, the $\mathbb{Z}^3$ in the diagram can be finer than the lattice generated by $\{4\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \mid j = 1, \dots, \hat{i}, \dots, 4\}$.

Recall from [6, Proposition 2] that a virtually free abelian group $1 \rightarrow \mathbb{Z}^4 \rightarrow Q \rightarrow \Phi \rightarrow 1$ is a crystallographic group if and only if the centralizer of $\mathbb{Z}^4$ in Q has no torsion elements. Since Φ acts effectively on $\mathbb{Z}^4$, it follows that Q is naturally a 4-dimensional crystallographic group.

The finite group Φ must be in a maximal compact subgroup $\mathrm{SO}(4)$ of $\mathrm{Aut}(\mathfrak{N}_2) = \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes (\mathrm{SO}(4) \times \mathbb{R}^+)$. Note that this coincides with the compact subgroup $\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$ in $\mathrm{Isom}_0(\mathfrak{N}_2) = \mathfrak{N}_2 \rtimes (\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)) = S^3 \times_{\mathbb{Z}_2} S^3 = \mathfrak{N}_2 \rtimes \mathrm{SO}(4)$. See Theorem 2.1.

Construction of an AB-group Π from Q . For each 4-dimensional crystallographic group Q , we shall check if there exists an AB-group Π constructed from Q ; that is, a torsion free $\Pi \subset \mathfrak{N}_2 \rtimes \mathrm{SO}(4) \subset \mathfrak{N}_2 \rtimes \mathrm{Aut}(\mathfrak{N}_2)$ fitting the short exact sequence

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow \Pi \longrightarrow Q \longrightarrow 1.$$

This is the key notion for our arguments and construction. We have a complete classification of 4-dimensional crystallographic groups (Q 's in the above statement). We shall use the representations of the 4-dimensional crystallographic groups given in the book [1]. Every Q has an explicit representation $Q \longrightarrow \mathbb{R}^4 \rtimes \mathrm{GL}(4, \mathbb{Z})$ (not into $\mathbb{R}^4 \rtimes O(4)$) in this book.

5. ALGEBRAIC CRITERIA FOR EXISTENCE OF LIFTING

Corollary 5.1. *For a 4-dimensional crystallographic group Q to have an AB-group Π , its holonomy group Φ must be in $\mathrm{SO}(4)$. Therefore, if Φ contains a matrix of determinant -1 , there is no AB-group Π from Q .*

Proof. The holonomy group Φ is a finite subgroup of $O(\mathbf{J}, 4) = \mathrm{SO}(4) \times \mathbb{R}^+$ (by Proposition 3.2) so that $\Phi \subset \mathrm{SO}(4)$. $\square$

Our goal is to determine which Q will give rise to a *torsion free* Π that fits the diagram (4-1). When Q is torsion free, then Π will be automatically torsion free, but when Q contains a torsion subgroup Q_0 , we need to check whether the lift Q_0 to Π will be torsion free.

Let Π_0 be the lift of Q_0 to Π . Thus $1 \rightarrow \mathbb{Z}^3 \rightarrow \Pi_0 \rightarrow Q_0 \rightarrow 1$ is exact. The finite subgroup $Q_0 \subset Q$ injects into the holonomy group (of Q) $\Phi \subset \mathrm{SO}(4)$ so that we can view $Q_0 \subset \mathrm{SO}(4)$. Recall $\mathrm{SO}(4) = S_\ell^3 \times_{\mathbb{Z}_2} S_r^3$, and S_ℓ^3 is the kernel of the homomorphism

$$\epsilon : \mathrm{SO}(4) \longrightarrow \mathrm{SO}(3).$$

Let $Q_1 = Q_0 \cap S_\ell^3$, and $Q_2 = Q_0/Q_1$ so that

$$1 \rightarrow Q_1 \rightarrow Q_0 \rightarrow Q_2 \rightarrow 1$$

is exact. Consider the pullbacks of the extension $1 \rightarrow \mathbb{Z}^3 \rightarrow \Pi \rightarrow Q \rightarrow 1$ by the inclusions $Q_1 \hookrightarrow Q_0 \hookrightarrow Q$. Then we have the following commutative diagram:

$$(5-1) \quad \begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Pi_1 & \longrightarrow & Q_1 \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \Pi_0 & \longrightarrow & Q_0 \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \\ & & & & Q_2 & \xrightarrow{=} & Q_2 \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array}$$

Since Q_1 acts trivially on $\mathbb{Z}^3$, $\Pi_1 = C_{\Pi_0}(\mathbb{Z}^3)$, the centralizer of $\mathbb{Z}^3$ in Π_0 , is torsion free. Hence by [6, Proposition 2], $\Pi_1 = C_{\Pi_0}(\mathbb{Z}^3) \cong \mathbb{Z}^3$ and $Q_1 = \Pi_1/\mathbb{Z}^3 \subset S^3$ is abelian. But every abelian subgroup of S^3 is cyclic. Hence Q_1 is cyclic. Therefore, Q_0 is a central extension of Q_1 by Q_2 .

Since Q_2 acts effectively on $\mathbb{Z}^3$, Π_0 is naturally a 3-dimensional orientable Bieberbach group, and thus Q_2 must be a holonomy group, i.e., $Q_2 \cong 1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In particular, we have shown the following.

Lemma 5.2. *Any finite subgroup of Q is an extension of a cyclic group by an orientable 3-dimensional holonomy group $1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, where the cyclic group can be conjugated into S_ℓ^3 .*

Lemma 5.3. *Q cannot contain a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Suppose $Q \subset \mathbb{R}^4 \rtimes \mathrm{SO}(4)$ contains a finite subgroup Q_0 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then Q_0 inject into Φ . By Lemma 5.2, Q_0 is an extension of Q_1 by Q_2 , where Q_0, Q_1, Q_2 fit into the commutative diagram (5–1), $Q_1 \cong \mathbb{Z}_2$ acts trivially on $\mathbb{Z}^3$ and $Q_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts effectively on $\mathbb{Z}^3$. By conjugation we may assume that Π_0 is the group $\mathfrak{G}_6$ of [9, Theorem 3.5.5]. Thus Q_0 is a finite group acting freely on the 3-torus T so that T/Q_0 is homeomorphic to $\mathbb{R}^3/\mathfrak{G}_6$. Since Q_0 is an abelian group, by [7, Theorem 6.1] or [3, Corollary 5.3], Q_0 must be $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$. This contradicts the assumption that $Q_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

The following is crucial in eliminating the possibility of constructing a Bieberbach group Π from a crystallographic group Q .

Proposition 5.4. *Let Q be an abstract orientable crystallographic group (not yet embedded into $\mathbb{R}^4 \rtimes \mathrm{SO}(4)$) with holonomy group Φ . Then Φ contains unique subgroups Φ_ℓ and Φ_r (up to interchanging) with the property that*

- (1) $\Phi_\ell \cap \Phi_r \subset \{\pm I\}$.
- (2) Elements from Φ_ℓ and Φ_r commute with each other.
- (3) Suppose Q_0 in the diagram (5–1) is obtained from Q . Then either $Q_1 = Q_0 \cap \Phi_\ell$ or $Q_1 = Q_0 \cap \Phi_r$.

Proof. Recall from Lemma 3.4 that $S_\ell^3 \cup S_r^3$ is exactly the subset of $\mathrm{SO}(4)$ consisting of elements with eigenvalues $a + bi, a + bi, a - bi, a - bi$ with $b \neq 0$, together with $\pm I$. In fact, $\pm I \in S_\ell^3 \cap S_r^3$. Assume Φ is embedded in $\mathrm{SO}(4)$ in any way. Then let

$$\begin{aligned}\Phi_\ell &= \Phi \cap S_\ell^3, \\ \Phi_r &= \Phi \cap S_r^3.\end{aligned}$$

The choice of an embedding does not change the sets Φ_ℓ and Φ_r except interchanging, because conjugation leaves the eigenvalues unchanged (this happens if $\Phi \subset \mathrm{SO}(4)$ is conjugated by the matrix σ in (2–2)). The last claim is clear because the kernel of $\epsilon : O(\mathbf{J}, 4) \rightarrow \mathrm{SO}(3) \times \mathbb{R}^+$ is S_ℓ^3 . $\square$

Procedure 5.5 (of finding Φ_ℓ and Φ_r from Q). We assume Φ lies in $\mathrm{GL}(4, \mathbb{R})$. Look at those elements of Φ which are not $\pm I$ (since $\pm I$ must belong to both Φ_ℓ and Φ_r). Suppose $A \in \Phi$ has the eigenvalues $a \pm bi$, with $b \neq 0$, of multiplicity 2. Then put A into Φ_ℓ . For the next $B \in \Phi$ with the right eigenvalues, we look at the product AB . If this has the right eigenvalues, then put B into Φ_ℓ . Otherwise put B into Φ_r . Proceed to the next element.

Corollary 5.6. *Suppose Q contains a non-abelian torsion subgroup Q_0 of order > 12 , with all the matrices of Q_0 lying completely inside Φ_ℓ or Φ_r . Then Q does not yield a Bieberbach group Π .*

Proof. If $Q_0 \subset \Phi_\ell$, then $Q_0 = Q_1$. But this is impossible since Q_0 is not abelian. Thus we must have $Q_0 \subset \Phi_r$. Then Q_1 is at most $\mathbb{Z}_2$ (or trivial) so that Q_2 has order > 6 . But the order of Q_2 must be ≤ 6 by Lemma 5.2, a contradiction. $\square$

Corollary 5.7. *Suppose Q contains a non-abelian torsion subgroup Q_0 of order p , and let the highest order of holonomy matrices in $\Phi_\ell \cup \Phi_r$ be r . If $p/r > 6$, then Q does not yield a Bieberbach group Π .*

Proof. Since the kernel Q_1 of the map ϵ must be cyclic coming from Φ_ℓ or Φ_r , Q_2 will have order > 6 . Now apply Lemma 5.2. $\square$

In what follows, the bold-faced numbers associated to the 4-dimensional crystallographic groups refer to the numbering in the book [1].

Corollary 5.8. *There is no AB-group $\Pi \subset \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2)$ constructed from Q given in **32/12/02/004** (holonomy group of order 64).*

Proof. Let Q_0 be the subgroups of Q

$$Q_0 = \langle t_2(a, A)(b, B)(d, D), (a, A)^7(b, B)(c, C)(d, D) \rangle,$$

a non-abelian group of order 16. By looking at the eigenvalues, we find that $\Delta = Q_0 \cap (\Phi_\ell \cup \Phi_r)$ is a non-cyclic group of order 8. Further, Δ is completely in one side of $\Phi_\ell \cup \Phi_r$ (again from the eigenvalues). Since Δ is not cyclic, it cannot be in the kernel of ϵ . Therefore, $Q_1 = \{\pm I_4\}$ and Q_2 is a group of order 8, which is impossible by Lemma 5.2. $\square$

Recall that any almost Bieberbach group Π for $\mathfrak{N}_2$ is a torsion free extension of $\mathbb{Z}_3$ by a 4-dimensional crystallographic group Q . There are 4783 4-dimensional crystallographic groups up to isomorphism. However, not all 4-dimensional crystallographic groups are qualified here. By Corollary 5.1, we eliminate about half of those 4-dimensional crystallographic groups. By Lemmas 5.2, 5.3, and Corollaries 5.6 and 5.7 together with Lemma 3.4, we can eliminate most of the remaining 4-dimensional crystallographic groups with holonomy order ≤ 48 , except for a few cases.

6. ELIMINATION OF SPECIAL CASES

In this section, we shall prove that there is no construction of AB-groups from the crystallographic groups of type **33/08** (and **33/12**), **33/09** and **33/10**. This requires special effort.

Given a 4-dimensional crystallographic group Q , it may happen that there is no AB-group Π constructed from Q itself, but there may be one from a conjugate of Q . But, it turns out that we need to check only two cases of different representations of Q into $E_0(4) = \mathbb{R}^4 \rtimes \text{SO}(4)$ in this special case.

First, we present an example of Q , which has two distinct (non-isomorphic) constructions depending on the representations:

Example 6.1. Two affinely conjugate representations of the same group into $E_0(4)$ as crystallographic groups can result in two non-isomorphic constructions of Π .

Consider the 4-dimensional crystallographic group Q given by

$$Q = \langle (\mathbf{e}_1, I), (\mathbf{e}_2, I), (\mathbf{e}_3, I), (\mathbf{e}_4, I), (\mathbf{0}, -K_1) \rangle.$$

This is the group given in **10/01**. Conjugation by σ (see (2-2)) sends $(\mathbf{e}_1, I)$ to $(-\mathbf{e}_1, I)$, $(\mathbf{e}_i, I)$ to $(\mathbf{e}_i, I)$ for $i = 2, 3, 4$. Also it sends $-K_1$ to $-J_1$. Thus, the conjugation maps Q to

$$Q' = \langle (\mathbf{e}_1, I), (\mathbf{e}_2, I), (\mathbf{e}_3, I), (\mathbf{e}_4, I), (\mathbf{0}, -J_1) \rangle.$$

Now consider the two groups

$$\begin{aligned} \Pi &= \langle (0, \mathbf{e}_1, I), (0, \mathbf{e}_2, I), (0, \mathbf{e}_3, I), (0, \mathbf{e}_4, I), (a, \mathbf{0}, -K_1) \rangle, \\ \Pi' &= \langle (0, \mathbf{e}_1, I), (0, \mathbf{e}_2, I), (0, \mathbf{e}_3, I), (0, \mathbf{e}_4, I), (a, \mathbf{0}, -J_1) \rangle, \end{aligned}$$

where $a = [1, 0, 0]^t$. It is quite easy to see that the subgroup Γ generated by the 4 translations is a lattice of the Heisenberg group (of rank 7), and is the Fitting subgroup (the maximal normal nilpotent subgroup) in both Π and Π' . Further,

$$\Gamma \cap \mathbb{R}^3 \cong \mathbb{Z}^3$$

is generated by the 3 vectors $[4, 0, 0]^t$, $[0, 4, 0]^t$, $[0, 0, 4]^t$. Now the action of the holonomy group on this characteristic subgroup $\mathbb{Z}^3$ is via

$$\widehat{J}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \widehat{K}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, it is easy to see now that both Π and Π' are torsion free. In fact, in both cases,

$$(a, \mathbf{0}, -J_1)^4 = ([4, 0, 0]^t, \mathbf{0}, I) = (a, \mathbf{0}, -K_1)^4.$$

Thus, they are AB-groups. However, they cannot be isomorphic because the action of the holonomy groups on the center of the Fitting subgroups Γ are distinct as shown above. Recall that Q' was conjugate to Q by an element of $O(4)$.

Proposition 6.2. *Let $Q \subset \mathbb{R}^4 \rtimes \text{SO}(4)$ be a crystallographic group with holonomy group Φ . Suppose that any symmetric matrix that commutes with every element of Φ has the form λI , where $\lambda \neq 0$. Then there is an AB-group $\Pi \subset \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2)$ constructed from some embedding Q into $\mathbb{R}^4 \rtimes \text{SO}(4)$ if and only if there is an AB-group Π constructed from Q or $(\mathbf{0}, \sigma)^{-1}Q(\mathbf{0}, \sigma)$.*

Proof. Suppose that there is an AB-group Π constructed from some Q' , an embedding of Q into $\mathbb{R}^4 \rtimes \text{SO}(4)$. Then, by the second Bieberbach Theorem, Q' is a conjugate to Q by an element of $\mathbb{R}^4 \rtimes \text{GL}(4, \mathbb{R})$, say, $(\mathbf{x}, X)$. Then

$$\begin{aligned} (\mathbf{x}, X)^{-1}Q(\mathbf{x}, X) &= Q' \subset \mathbb{R}^4 \rtimes \text{SO}(4) \\ \implies X^{-1}AX &\in \text{SO}(4) \text{ for all } A \in \Phi \\ \iff (X^{-1}AX)(X^{-1}AX)^t &= I_4 \text{ for all } A \in \Phi \\ \iff ASA^t &= S \text{ for all } A \in \Phi, \end{aligned}$$

where $S = XX^t$. By the assumption, $S = kI_4$ with $k > 0$. Hence $X \in O(4) \times \mathbb{R}^+$.

If $X \in \text{SO}(4) \times \mathbb{R}^+$, then conjugation of Π by $(0, \mathbf{x}, X)^{-1}$ is an AB-group constructed from Q because $(\mathbf{x}, X)Q'(\mathbf{x}, X)^{-1} = Q$.

If $X \notin \text{SO}(4) \times \mathbb{R}^+$, then $X = \sigma Y$ for some $Y \in \text{SO}(4) \times \mathbb{R}^+$. Then

$$(0, \sigma)(\sigma \mathbf{x}, Y)Q'(\sigma \mathbf{x}, Y)^{-1}(0, \sigma)^{-1} = (\mathbf{x}, X)Q'(\mathbf{x}, X)^{-1} = Q$$

so that

$$(\sigma \mathbf{x}, Y)Q'(\sigma \mathbf{x}, Y)^{-1} = (0, \sigma)^{-1}Q(0, \sigma).$$

Therefore $(\sigma \mathbf{x}, Y)Q'(\sigma \mathbf{x}, Y)^{-1}$ is an AB-group Π constructed from $(\mathbf{0}, \sigma)^{-1}Q(\mathbf{0}, \sigma)$. The converse is trivial. $\square$

Proposition 6.3. *Let $Q \subset \mathbb{R}^4 \rtimes \mathrm{SO}(4)$ be a crystallographic group generated by*

$$(\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I), (\mathbf{a}_1, A_1), (\mathbf{a}_2, A_2), \dots, (\mathbf{a}_p, A_p),$$

where the subgroup $\langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I) \rangle$ is the maximal normal free abelian of rank 4. Then there exists an AB-group obtained from Q if and only if the group $\Pi_{(s_1, s_2, \dots, s_p)}$ generated by

$$(0, \mathbf{v}_1, I), (0, \mathbf{v}_2, I), (0, \mathbf{v}_3, I), (0, \mathbf{v}_4, I), \\ (s_1, \mathbf{a}_1, A_1), (s_2, \mathbf{a}_2, A_2), \dots, (s_p, \mathbf{a}_p, A_p)$$

(for some $s_1, s_2, \dots, s_p \in \mathbb{R}^3$) is an AB-group.

Proof. Any AB-group $\Pi \subset \mathfrak{N}_2 \rtimes \mathrm{Aut}(\mathfrak{N}_2)$ obtained from Q is generated by

$$(v_1, \mathbf{0}, I), (v_2, \mathbf{0}, I), (v_3, \mathbf{0}, I), \\ (w_1, \mathbf{v}_1, I), (w_2, \mathbf{v}_2, I), (w_3, \mathbf{v}_3, I), (w_4, \mathbf{v}_4, I), \\ (t_1, \mathbf{a}_1, A_1), (t_2, \mathbf{a}_2, A_2), \dots, (t_p, \mathbf{a}_p, A_p)$$

for some $v_1, v_2, v_3, w_1, w_2, w_3, w_4, t_1, t_2, \dots, t_p \in \mathbb{R}^3$. Since the group generated by

$$(w_1, \mathbf{v}_1, I), (w_2, \mathbf{v}_2, I), (w_3, \mathbf{v}_3, I), (w_4, \mathbf{v}_4, I)$$

becomes a lattice of $\mathfrak{N}_2$ already (see Lemma 4.1), the group generated by

$$(w_1, \mathbf{v}_1, I), (w_2, \mathbf{v}_2, I), (w_3, \mathbf{v}_3, I), (w_4, \mathbf{v}_4, I), \\ (t_1, \mathbf{a}_1, A_1), (t_2, \mathbf{a}_2, A_2), \dots, (t_p, \mathbf{a}_p, A_p)$$

(without $(v_1, \mathbf{0}, I), (v_2, \mathbf{0}, I), (v_3, \mathbf{0}, I)$) will be a subgroup of Π of finite index, and is an AB-group. Let us call this smaller group Π' .

Let $(s, \mathbf{x}, \eta, E) \in \mathbb{R}^3 \tilde{\times} \mathbb{R}^4 \rtimes (\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \rtimes O(\mathbf{J}, 4)) = \mathfrak{N}_2 \rtimes \mathrm{Aut}(\mathfrak{N}_2)$. Then for $(0, \mathbf{v}_i) \in \mathbb{R}^3 \tilde{\times} \mathbb{R}^4 = \mathfrak{N}_2$,

$$(s, \mathbf{x}, \eta, E)(0, \mathbf{v}_i)(s, \mathbf{x}, \eta, E)^{-1} = (\eta(\mathbf{v}_i) + 4\mathcal{I}(\mathbf{x}, E\mathbf{v}_i), E\mathbf{v}_i).$$

Thus given $w_1, w_2, w_3, w_4 \in \mathbb{R}^3$, it is obvious that there is $\eta \in \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3)$ such that $\eta(\mathbf{v}_i) = w_i$ for $i = 1, 2, 3, 4$. Hence $\tilde{\eta} = (0, \mathbf{0}, \eta, I) \in \mathfrak{N}_2 \rtimes \mathrm{Aut}(\mathfrak{N}_2)$ and $\mu(\tilde{\eta})(w_i, \mathbf{v}_i, I) = (0, \mathbf{v}_i, I)$ for all $i = 1, 2, 3, 4$. Consequently, the group Π'' obtained by conjugating Π' by $\tilde{\eta}$ is another AB-group, satisfying the condition $w_1 = \dots = w_4 = 0$. [In the course of conjugation, t_i will change.]

Consequently, we have obtained an AB-group $\Pi'' = \Pi_{(s_1, s_2, \dots, s_p)} \subset \mathfrak{N}_2 \rtimes \mathrm{Aut}(\mathfrak{N}_2)$ generated by

$$(0, \mathbf{v}_1, I), (0, \mathbf{v}_2, I), (0, \mathbf{v}_3, I), (0, \mathbf{v}_4, I), \\ (s_1, \mathbf{a}_1, A_1), (s_2, \mathbf{a}_2, A_2), \dots, (s_p, \mathbf{a}_p, A_p)$$

for some $s_1, s_2, \dots, s_p \in \mathbb{R}^3$. □

The proposition says that if there is an AB-group Π constructed from Q , then $\Pi_{(s_1, s_2, \dots, s_p)}$ is a subgroup of Π of finite index, which is another AB-group constructed from Q . Conversely, if $\Pi_{(s_1, s_2, \dots, s_p)}$ is an AB-group (i.e., is torsion free), then we are done. Therefore, the existence/non-existence of construction is solely determined by the group $\Pi_{(s_1, s_2, \dots, s_p)}$ described above.

Let $Q \subset \mathbb{R}^4 \rtimes \text{SO}(4)$ be a crystallographic group

$$\begin{aligned} Q &= \langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I), (\mathbf{a}_1, A_1), (\mathbf{a}_2, A_2), \dots, (\mathbf{a}_p, A_p) \rangle \\ &= \langle t_1, t_2, t_3, t_4, \bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_p \rangle. \end{aligned}$$

We assume that the holonomy group $\Phi = \langle A_1, A_2, \dots, A_p \rangle$ satisfies

$$\langle A_1 \rangle \triangleleft \langle A_1, A_2 \rangle \triangleleft \dots \triangleleft \langle A_1, A_2, \dots, A_p \rangle = \Phi.$$

For $s_1, s_2, \dots, s_p \in \mathbb{R}^3$, denote

$$\begin{aligned} t_1 &= (0, \mathbf{v}_1, I), & t_2 &= (0, \mathbf{v}_2, I), & t_3 &= (0, \mathbf{v}_3, I), & t_4 &= (0, \mathbf{v}_4, I), \\ \alpha_1 &= (s_1, \mathbf{a}_1, A_1), & \alpha_2 &= (s_2, \mathbf{a}_2, A_1), & \dots, & \alpha_p &= (s_p, \mathbf{a}_p, A_p). \end{aligned}$$

Let Π be the group generated by $t_1, t_2, t_3, t_4, \alpha_1, \alpha_2, \dots, \alpha_p$. That is,

$$\Pi = \Pi_{(s_1, s_2, \dots, s_p)} = \langle t_1, t_2, t_3, t_4, \alpha_1, \alpha_2, \dots, \alpha_p \rangle.$$

Proposition 6.4 (Description of $\Pi \cap \mathbb{R}^3$). *Every element of Π can be written as*

$$t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4} \alpha_1^{q_1} \alpha_2^{q_2} \dots \alpha_p^{q_p}$$

(not uniquely). A generating set for $Z = \Pi \cap \mathbb{R}^3$ (the pure translations of the group Π in the central direction $\mathbb{R}^3$) is obtained by:

- (1) Take $[t_i, t_j]$ for all $i, j = 1, \dots, 4$;
- (2) Let $\omega = \alpha_1^{q_1} \alpha_2^{q_2} \dots \alpha_p^{q_p}$. If $\omega^r = (w, V, I)$ for some $r > 0$, then write V as $V = n_1 \mathbf{v}_1 + n_2 \mathbf{v}_2 + n_3 \mathbf{v}_3 + n_4 \mathbf{v}_4$, and take $t_1^{-n_1} t_2^{-n_2} t_3^{-n_3} t_4^{-n_4} \omega^r$.

Proof. Observe that

$$\begin{aligned} \langle t_1, t_2, t_3, t_4 \rangle &\triangleleft \langle t_1, t_2, t_3, t_4, \bar{\alpha}_1 \rangle \triangleleft \langle t_1, t_2, t_3, t_4, \bar{\alpha}_1, \bar{\alpha}_2 \rangle \triangleleft \dots \\ &\triangleleft \langle t_1, t_2, t_3, t_4, \bar{\alpha}_1, \dots, \bar{\alpha}_{p-1} \rangle \triangleleft \langle t_1, t_2, t_3, t_4, \bar{\alpha}_1, \dots, \bar{\alpha}_{p-1}, \bar{\alpha}_p \rangle, \end{aligned}$$

which implies that every element of Π can be written as

$$\omega = t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4} \alpha_1^{q_1} \alpha_2^{q_2} \dots \alpha_p^{q_p}$$

with $n_i \in \mathbb{Z}$, and $0 \leq q_i < \text{order of } A_i$, etc.

Clearly the elements in (1) and (2) represent the identity element in the quotient group Q . Conversely, an arbitrary element of Γ (the lattice of $\mathfrak{N}_2$, see the diagram (4-1)) is a product of $t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4}$ and a power of $\alpha_1^{q_1} \alpha_2^{q_2} \dots \alpha_p^{q_p}$ whose matrix part is the identity. An element of Γ lies in Z if and only if the 4-dimensional vector component is zero. It may not be true that $(0, V, I) \in \Gamma$, but we know $(0, \mathbf{v}_i, I) \in \Gamma$ for $i = 1, 2, 3, 4$. We express V in terms of $\mathbf{v}_i$'s, and multiply such $t_i = (0, \mathbf{v}_i, I)$'s in Γ to kill V . Thus the elements in (1) and (2) form a complete set of generators for Z . $\square$

We now can take care of the troublesome case: Let Q be an affine 4-dimensional crystallographic group given in **33/08**, **33/09** and **33/10** (holonomy group of order 96). The following matrix P will be used to conjugate the given abstract crystallographic group Q (which is embedded into $\mathbb{R}^4 \rtimes \text{GL}(4)$) into $\mathbb{R}^4 \rtimes \text{SO}(4)$:

$$(6-1) \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

That is, $(\mathbf{0}, P) \in \mathbb{R}^4 \rtimes \mathrm{GL}(4, \mathbb{R})$ conjugates Q into a genuine crystallographic group. If S is a symmetric matrix which commutes with every element of the holonomy matrices of $(\mathbf{0}, P)Q(\mathbf{0}, P)^{-1}$, it is easy to see that $S = \lambda I$ for some $\lambda \neq 0$. Therefore the condition in Proposition 6.2 is satisfied. Denote

$$(6-2) \quad R_0 = \mu(\mathbf{0}, P)(Q), \quad R_1 = \mu(\mathbf{0}, \sigma P)(Q),$$

and denote their holonomy groups by Φ_0 and Φ_1 . We need to check only *two specific embeddings* R_0 and $R_1 = (\mathbf{0}, \sigma)R_0(\mathbf{0}, \sigma)^{-1}$.

Let $R = R_0$ or R_1 . Write

$$R = \langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I), (\mathbf{a}, A), (\mathbf{b}, B), (\mathbf{c}, C), (\mathbf{d}, D) \rangle,$$

and denote, for some $s, t, u, v \in \mathbb{R}^3$,

$$\begin{aligned} t_1 &= (0, \mathbf{v}_1, I), & t_2 &= (0, \mathbf{v}_2, I), & t_3 &= (0, \mathbf{v}_3, I), & t_4 &= (0, \mathbf{v}_4, I), \\ \alpha &= (s, \mathbf{a}, A), & \beta &= (t, \mathbf{b}, B), & \gamma &= (u, \mathbf{c}, C), & \delta &= (v, \mathbf{d}, D). \end{aligned}$$

By Proposition 6.3, there is no AB-group from R , if and only if for any $s, t, u, v \in \mathbb{R}^3$, the group Π generated by $t_1, t_2, t_3, t_4, \alpha, \beta, \gamma, \delta$,

$$(6-3) \quad \Pi_{(s,t,u,v)} = \langle t_1, t_2, t_3, t_4, \alpha, \beta, \gamma, \delta \rangle,$$

has a non-trivial torsion element. Now let $Z = \Pi \cap \mathbb{R}^3$. Then there is no AB-group from R if and only if the following holds: For any $s, t, u, v \in \mathbb{R}^3$, there exists a non-trivial torsion element of the form

$$zt_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4}\alpha^p\beta^q\gamma^r\delta^o,$$

where $z \in Z$ and $n_i, p, q, r, o, k \in \mathbb{Z}$ with $0 \leq p < \text{order of } A$, etc.

Proposition 6.5. *There is no AB-group $\Pi \subset \mathfrak{N}_2 \rtimes \mathrm{Aut}(\mathfrak{N}_2)$ constructed from Q given in 33/08 (holonomy group of order 96).*

Proof. We work with the group $\Pi = \Pi_{(s,t,u,v)}$ in (6-3) which is to be constructed from $R = R_0$ or R_1 . We shall show the following: For any $s, t, u, v \in \mathbb{R}^3$, there exist $z \in Z$, and $n_i, p, q, r, o, k \in \mathbb{Z}$ with $0 \leq p < 2, 0 \leq q < 4, 0 \leq r < 2, 0 \leq o < 12$ such that

$$zt_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4}\alpha^p\beta^q\gamma^r\delta^o$$

is a non-trivial torsion element.

Case $R = R_0$. The finite subgroup $Q_0 = \langle (b, B), (d, D)^3 \rangle$ in R_0 forms a quaternion group (non-cyclic group of order 8) with $Q_1 = Q_0$. This contradicts Lemma 5.2.

Case $R = R_1$. Our R_1 has a representation

$$R_1 = \langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I), (\mathbf{a}, A), (\mathbf{b}, B), (\mathbf{c}, C), (\mathbf{d}, D) \rangle,$$

where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$

$$\begin{aligned}
(\mathbf{a}, A) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right), \\
(\mathbf{b}, B) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right), \\
(\mathbf{c}, C) &= \left(\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right), \\
(\mathbf{d}, D) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \right).
\end{aligned}$$

For $s, t, u, v \in \mathbb{R}^3$, the group $\Pi = \Pi_{(s, t, u, v)}$ as in (6–3) has a generating set

$$\begin{aligned}
t_1 &= (0, \mathbf{v}_1, I), & t_2 &= (0, \mathbf{v}_2, I), & t_3 &= (0, \mathbf{v}_3, I), & t_4 &= (0, \mathbf{v}_4, I), \\
\alpha &= (s, \mathbf{a}, A), & \beta &= (t, \mathbf{b}, B), & \gamma &= (u, \mathbf{c}, C), & \delta &= (v, \mathbf{d}, D).
\end{aligned}$$

Then

$$Z = \Pi \cap \mathbb{R}^3$$

is a lattice of $\mathbb{R}^3$. Let

$$z = \beta^{-4} = \left(\begin{bmatrix} 0 \\ 0 \\ -4t_3 \end{bmatrix}, \mathbf{0}, I \right).$$

Then z is an element of Z , and

$$z(\beta\delta^2)^4 = \beta^{-4}(\beta\delta^2)^4 = (0, \mathbf{0}, (BD^2)^4)$$

is a torsion element of order 3 (for any choice of s, t, u, v). $\square$

Corollary 6.6. *There is no AB-group $\Pi \subset \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2)$ constructed from Q given in **33/12** (holonomy group of order 192).*

Proof. Consider the subgroup S of Q generated by the following elements:

$$(\mathbf{e}_1, I), (\mathbf{e}_2, I), (\mathbf{e}_3, I), (\mathbf{e}_4, I), (\mathbf{a}, A), (\mathbf{b}, B), (\mathbf{c}, C), (\mathbf{d}, D)$$

[we removed (e, E)]. Then S will be a 4-dimensional crystallographic group, and its holonomy group is generated by A, B, C, D . Since the holonomy group of Q has order 192, the holonomy group of S must have order 96, and is normal in Q . If we prove that there is no AB-group constructed from any 4-dimensional crystallographic group with holonomy group of order 96 (in particular, there is no AB-group constructed from our S), then we will have proved that there is no AB-group constructed from Q .

In fact we can see that our S is one in the list already: The subgroup S of Q given above is actually isomorphic to the group Q' given in **33/08** with generators

$$(\mathbf{e}_1, I), (\mathbf{e}_2, I), (\mathbf{e}_3, I), (\mathbf{e}_4, I), (\mathbf{a}', A'), (\mathbf{b}', B'), (\mathbf{c}', C'), (\mathbf{d}', D').$$

Take

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

Then

$$\begin{aligned} \mu(\mathbf{v}, I)(\mathbf{a}, A) &= (\mathbf{e}_3, I)(\mathbf{a}', A'), \\ \mu(\mathbf{v}, I)(\mathbf{b}, B) &= (\mathbf{e}_3, I)(\mathbf{b}', B'), \\ \mu(\mathbf{v}, I)(\mathbf{c}, C) &= (\mathbf{e}_2, I)^{-1}(\mathbf{c}', C'), \\ \mu(\mathbf{v}, I)(\mathbf{d}, D) &= (\mathbf{e}_2, I)(\mathbf{e}_4, I)(\mathbf{b}', B')(\mathbf{d}', D')^{-1}. \end{aligned}$$

This shows that $\mu(\mathbf{v}, I) : S \rightarrow Q'$ is an isomorphism. $\square$

Proposition 6.7. *There is no AB-group $\Pi \subset \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2)$ constructed from Q given in **33/10** (holonomy group of order 96).*

Proof. We proceed as in the case of **33/08**. Let Q be the group given in **33/10**. Consider two embeddings R_0 and R_1 of Q into $\mathbb{R}^4 \rtimes \text{SO}(4)$ by conjugating by $(\mathbf{0}, P)$ and $(\mathbf{0}, \sigma P)$, respectively. Let

$$R_0 = \mu(\mathbf{0}, P)(Q), \quad R_1 = \mu(\mathbf{0}, \sigma P)(Q),$$

as before.

Case $R = R_0$. Our R_0 has a representation

$$R_0 = \langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I), (\mathbf{a}, A), (\mathbf{b}, B), (\mathbf{c}, C), (\mathbf{d}, D) \rangle,$$

where

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \\ (\mathbf{a}, A) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \right), \\ (\mathbf{b}, B) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \right), \\ (\mathbf{c}, C) &= \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \right), \\ (\mathbf{d}, D) &= \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right). \end{aligned}$$

For $s, t, u, v \in \mathbb{R}^3$, the group $\Pi = \Pi_{(s,t,u,v)}$ as in (6–3) has a generating set

$$\begin{array}{llll} t_1 = (0, \mathbf{v}_1, I), & t_2 = (0, \mathbf{v}_2, I), & t_3 = (0, \mathbf{v}_3, I), & t_4 = (0, \mathbf{v}_4, I), \\ \alpha = (s, \mathbf{a}, A), & \beta = (t, \mathbf{b}, B), & \gamma = (u, \mathbf{c}, C), & \delta = (v, \mathbf{d}, D). \end{array}$$

Then

$$Z = \Pi \cap \mathbb{R}^3$$

is a lattice of $\mathbb{R}^3$. Let

$$z = ([t_1, t_4]^3 (\gamma \delta)^8)^{-1} = \left(\begin{bmatrix} 0 & & \\ & 0 & \\ 76 - 8u_3 - 8v_3 & & \end{bmatrix}, \mathbf{0}, I \right).$$

Then z is an element of Z , and

$$z \cdot (\gamma^2 \delta^2)^4 = ([t_1, t_2]^3 (\gamma \delta)^8)^{-1} \cdot (\gamma^2 \delta^2)^4$$

is a torsion element of order 3 (for any choice of s, t, u, v).

Case $R = R_1$. The finite subgroup

$$Q_0 = \langle (a, A)(b, B)^3(c, C)(d, D), t_2^{-1}t_4^{-1}(b, B)^2(c, C) \rangle$$

in R_1 forms a group of order 24 with $Q_0 = \mathbb{Z}_4$, and $Q_2 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ a non-cyclic group of order 6. This contradicts Lemma 5.2. $\square$

Proposition 6.8. *There is no AB-group $\Pi \subset \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2)$ constructed from Q given in **33/09** (holonomy group of order 96).*

Proof. Let Q be the group given in **33/09**. Consider two embeddings R_0 and R_1 of Q into $\mathbb{R}^4 \rtimes \text{SO}(4)$ by conjugating by $(\mathbf{0}, P)$ and $(\mathbf{0}, \sigma P)$, respectively. Let

$$(6-4) \quad R_0 = \mu(\mathbf{0}, P)(Q), \quad R_1 = \mu(\mathbf{0}, \sigma P)(Q),$$

as before.

Case $R = R_0$. Our R_0 has a representation

$$R_1 = \langle (\mathbf{v}_1, I), (\mathbf{v}_2, I), (\mathbf{v}_3, I), (\mathbf{v}_4, I), (\mathbf{a}, A), (\mathbf{b}, B), (\mathbf{c}, C), (\mathbf{d}, D) \rangle,$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix},$$

$$\begin{aligned}
(\mathbf{a}, A) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right), \\
(\mathbf{b}, B) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \right), \\
(\mathbf{c}, C) &= \left(\frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \right), \\
(\mathbf{d}, D) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right).
\end{aligned}$$

For $s, t, u, v \in \mathbb{R}^3$, the group $\Pi = \Pi_{(s, t, u, v)}$ as in (6–3) has a generating set

$$\begin{aligned}
t_1 &= (0, \mathbf{v}_1, I), & t_2 &= (0, \mathbf{v}_2, I), & t_3 &= (0, \mathbf{v}_3, I), & t_4 &= (0, \mathbf{v}_4, I), \\
\alpha &= (s, \mathbf{a}, A), & \beta &= (t, \mathbf{b}, B), & \gamma &= (u, \mathbf{c}, C), & \delta &= (v, \mathbf{d}, D).
\end{aligned}$$

Then

$$Z = \Pi \cap \mathbb{R}^3$$

is a lattice of $\mathbb{R}^3$, and one can show that Z is generated by the following 12 elements (using row notation instead of column notation):

$$\begin{aligned}
w_1 &= (-1 + s_1, 1 + s_2, s_3), \\
w_2 &= (-1 + t_1, t_2, -t_3), \\
w_3 &= (v_1 + v_2, v_1 + v_2, 0), \\
w_4 &= (-t_2, 1 - t_1, -t_3), \\
w_5 &= (-1 + t_1, t_2, t_3), \\
w_6 &= (0, 0, 2s_3), \\
w_7 &= (-s_1 - s_2, -s_1 - s_2, 0), \\
w_8 &= (s_1 - s_2, -s_1 + s_2, -2), \\
w_9 &= (1 - s_1 + s_2 + t_1 + t_2, 1 + s_1 - s_2 + t_1 + t_2, 0), \\
w_{10} &= (-s_1 + s_2, s_1 - s_2, -2), \\
w_{11} &= (s_1 - s_2 + u_1 - u_2 - v_1 + v_2, -s_1 + s_2 - u_1 + u_2 + v_1 - v_2, 0), \\
w_{12} &= (0, 0, 1 + 6u_3)
\end{aligned}$$

[This, as well as other parts of calculations, was done by the program MATHEMATICA [10] and hand-checked.]

We note that

$$\begin{aligned}
(0, 0, 4) &= -w_8 - w_{10}, \\
(4, 4, 0) &= -2w_2 + 2w_4 + w_8 + 2w_9 - w_{10}, \\
(2, 2, 2) &= -w_2 + w_4 + w_9 - w_{10}
\end{aligned}$$

are elements of Z .

Let $(2x_0, 2x_0, 0) \in Z$ be a generator of the subgroup of Z consisting of the elements of the form $(x, x, 0)$. Since $(4, 4, 0) \in Z$, x_0 is a rational number. Note

also that $w_3 = (v_1 + v_2, v_1 + v_2, 0) \in Z$. Therefore, there exist integers m and p such that

$$v_1 + v_2 = m \cdot 2x_0, \quad 4 = p \cdot 2x_0.$$

(1) Suppose $p = 2\ell$, even. Then $4 = 2\ell \cdot 2x_0$ shows $2 = \ell \cdot 2x_0$, and hence $(2, 2, 0) = \ell(2x_0, 2x_0, 0) \in Z$. Since $(2, 2, 2) \in Z$, we have $(0, 0, 2) \in Z$. Consider $z = -w_{12} + 5(0, 0, 2) \in Z$. It is easy to see that $z\gamma^6 \in \Pi$ is a torsion element of order 2.

(2) Suppose $p = 2\ell + 1$, odd. Then $4 = (2\ell + 1) \cdot 2x_0$ yields

$$\begin{aligned} (2, 2, 0) &= (2\ell + 1)(x_0, x_0, 0) \\ &= \ell(2x_0, 2x_0, 0) + (x_0, x_0, 0). \end{aligned}$$

Therefore,

$$(2, 2, 2) = \ell(2x_0, 2x_0, 0) + (x_0, x_0, 2),$$

which shows $(x_0, x_0, 2) \in Z$. Let

$$z = -m(x_0, x_0, 2) \in Z.$$

Then it is easy to see that $z\delta \in \Pi$ is a torsion element of order 2.

Case $R = R_1$. The finite subgroup $Q_0 = \langle (c, C)^4, (a, A)(d, D) \rangle$ in R_1 forms a non-commutative holonomy group of order 6. This contradicts Lemma 5.2. $\square$

7. A CONSTRUCTION OF AN AB-GROUP WITH HOLONOMY ORDER 48

Theorem 7.1 (Existence). *There exists an almost Bieberbach group $\Pi \subset \mathfrak{N}_2 \rtimes \text{Aut}(\mathfrak{N}_2)$ whose holonomy group has order 48.*

Proof. Consider the 4-dimensional crystallographic group Q (Case **33/05/01/003**) generated by the following elements:

$$(\mathbf{e}_1, I), \quad (\mathbf{e}_2, I), \quad (\mathbf{e}_3, I), \quad (\mathbf{e}_4, I), \quad (\mathbf{a}, A), \quad (\mathbf{b}, B), \quad (\mathbf{c}, C),$$

where I is the identity matrix, $\mathbf{e}_i$ is the unit vector with 1 in the i th coordinate, and

$$\begin{aligned} (\mathbf{a}, A) &= \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \right), \\ (\mathbf{b}, B) &= \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} \right), \\ (\mathbf{c}, C) &= \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \right). \end{aligned}$$

Note that

$$A^4 = I, \quad B^2 = A^2, \quad C^6 = A^2, \quad [A, B] = A^2, \quad [A, C] = BA^2, \quad [B, C] = BA^3$$

so that Q has the holonomy group of order 48.

Now we conjugate this group by the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then the new group has the following representation inside $\mathbb{R}^4 \rtimes \mathrm{SO}(4)$:

$$\begin{aligned} & (\mathbf{v}_1, I), \quad (\mathbf{v}_2, I), \quad (\mathbf{v}_3, I), \quad (\mathbf{v}_4, I), \\ & (\mathbf{a}, A) = \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right), \\ & (\mathbf{b}, B) = \left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \right), \\ & (\mathbf{c}, C) = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \right), \end{aligned}$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

Next we consider the subgroup Π of $\mathbb{R}^3 \tilde{\times} \mathbb{R}^4 \rtimes \mathrm{SO}(4)$ generated by

$$\begin{aligned} & t_1 = (0, \mathbf{v}_1, I), \quad t_2 = (0, \mathbf{v}_2, I), \quad t_3 = (0, \mathbf{v}_3, I), \quad t_4 = (0, \mathbf{v}_4, I), \\ & \alpha = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{a}, A \right), \quad \beta = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}, B \right), \quad \gamma = \left(\begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{6} \\ \frac{1}{6} \end{bmatrix}, \mathbf{c}, C \right). \end{aligned}$$

Let $Z = \Pi \cap \mathbb{R}^3$. We already have a detailed description of Z in Proposition 6.4. By calculation, we see that Z is actually generated by the vectors

$$z_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad z_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

In particular, Z is discrete and hence Π is discrete.

Next we claim that Π is torsion free. Every element of Π can be written as

$$zt_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4}\alpha^p\beta^q\gamma^r,$$

where $z \in Z$, and n_1, n_2, n_3, p, q, r are integers such that $0 \leq p < 4, 0 \leq q < 4$ and $0 \leq r < 12$. If $zt_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4}\alpha^p\beta^q\gamma^r$ is a torsion element of order g , then $(A^p B^q C^r)^g = I$. Thus it suffices to consider all the torsion elements of Φ whose

order is a prime dividing 48. The following shows the list of prime ordered elements of Φ :

Order 2 elements: $A^2, AC^3, ABC^3, A^3C^3, A^2BC^3, A^3BC^3, BC^3$.

Order 3 elements: $A^3BC^2, A^3BC^4, AC^2, A^2C^2, AC^4, BC^2, BC^4, C^4$.

It can be shown that none of the corresponding 15 equations

$$(z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4} \alpha^2 \beta^0 \gamma^0)^2 = (0, \mathbf{0}, I),$$

$$(z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4} \alpha^1 \beta^0 \gamma^3)^2 = (0, \mathbf{0}, I),$$

$$(z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4} \alpha^1 \beta^1 \gamma^3)^2 = (0, \mathbf{0}, I),$$

...

$$(z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3} t_1^{n_1} t_2^{n_2} t_3^{n_3} t_4^{n_4} \alpha^0 \beta^0 \gamma^4)^3 = (0, \mathbf{0}, I)$$

has any integral solutions for $\ell_1, \ell_2, \ell_3, n_1, n_2, n_3, n_4$. This proves that Π is indeed torsion free, and thus is an AB-group. $\square$

8. NON-EXISTENCE

Now a case-by-case study of the table from [1] shows that all 4-dimensional crystallographic groups with holonomy group of order greater than 48 satisfy at least one of the disqualification criteria proved so far. Using notations from [1], we list them below. All the calculations were done using the program MATHEMATICA [10] and hand-checked.

If the holonomy group of Q contains a matrix of determinant -1 , then Q is excluded by Corollary 5.1. Also if Q contains $(\mathbb{Z}_2)^3$, then Q is excluded by Lemma 5.3.

• 1152:

$$33/16: \det(G) = -1.$$

• 576:

$$33/14: \det(F) = -1,$$

$$33/15: \langle A, B^2, F \rangle \cong (\mathbb{Z}_2)^3.$$

• 384:

$$32/21: \det(F) = -1.$$

• 288:

$$30/13: \det(D) = -1,$$

$$33/13: \langle BD^3, C, D^6 \rangle \cong (\mathbb{Z}_2)^3.$$

• 240:

$$31/07: \det(C) = -1.$$

• 192:

$$32/18: \det(E) = -1,$$

$$32/19: \det(F) = -1,$$

$$32/20/01, 03: \langle (a, A)^2, (b, B), (c, C) \rangle \cong (\mathbb{Z}_2)^3,$$

$$32/20/02: \langle (a, A)^2(b, B), (a, A)(c, C), t_2(b, B) \rangle \cong (\mathbb{Z}_2)^3,$$

$$33/12: \text{Corollary 6.6.}$$

- 144:
 - 23/11, 29/09, 30/11, 30/12: $\det(C) = -1$,
 - 30/10: $\langle A, C^3, D \rangle \cong (\mathbb{Z}_2)^3$,
 - 33/11: The non-abelian group $\langle AC, B \rangle$ of order 24 lies completely inside Φ_ℓ or Φ_r . Apply Corollary 5.6.
- 128:
 - 32/17: $\det(D) = -1$.
- 120:
 - 31/04, 31/05: $\det(A) = -1$,
 - 31/06: $\langle A, B, D^5 \rangle \cong (\mathbb{Z}_2)^3$.
- 96:
 - 20/22: $\det(D) = -1$,
 - 25/11: $\det(E) = -1$,
 - 32/16/01, 32/16/03: $\langle A^2, C, D \rangle \cong (\mathbb{Z}_2)^3$,
 - 32/16/02: $\langle t_2^{-1}t_4^{-1}(d, D), t_4^{-1}(c, D), t_1t_2^{-1}(a, A)^2 \rangle \cong (\mathbb{Z}_2)^3$,
 - 33/08: Proposition 6.5,
 - 33/09: Proposition 6.8,
 - 33/10: Proposition 6.7.
- 72:
 - 22/11, 23/09, 23/10: $\det(B) = -1$,
 - 23/07, 29/06, 29/08: $\det(C) = -1$,
 - 29/07: $\det(D) = -1$,
 - 23/08, 29/05: $\langle A^3, B^3, C \rangle \cong (\mathbb{Z}_2)^3$,
 - 30/07: $\langle B, BC \rangle$ generates a non-abelian group of order 72 with the highest period of the elements in $\Phi_\ell \cup \Phi_r$ 6. Apply Corollary 5.7,
 - 30/08: $\langle AB^4, BC \rangle$ induces a non-commutative holonomy of order 6, Lemma 5.2,
 - 30/09: $\langle AB^4, C \rangle$ induces a non-commutative holonomy of order 6, Lemma 5.2,
 - 33/07: $\langle A, C^2 \rangle$ generates a non-abelian group of order 72 with the highest period of the elements in $\Phi_\ell \cup \Phi_r$ 6. Apply Corollary 5.7.
- 64:
 - 19/06 (C), 32/13 (D), 32/14 (D), 32/15 (C): $\det = -1$,
 - 32/12/01/002: $\langle t_3(a, A)^4(c, C), (b, B), (c, C) \rangle \cong (\mathbb{Z}_2)^3$,
 - 32/12/01/003: $\langle t_2^{-1}(a, A)^4(b, B), (b, B), (d, D) \rangle \cong (\mathbb{Z}_2)^3$,
 - 32/12/01/004: $\langle t_1(a, A)^6(d, D), t_3^{-1}(a, A)^2(d, D), (b, B) \rangle \cong (\mathbb{Z}_2)^3$,
 - 32/12/02/002: $\langle (a, A)^4, (b, B)(c, C), t_2^{-1}(c, C) \rangle \cong (\mathbb{Z}_2)^3$,
 - 32/12/02/003: $\langle t_3^{-1}(a, A)^2(c, C), t_2^{-1}(a, A)^6(c, C), t_2t_3^{-2}(a, A)^6(b, B)(c, C) \rangle \cong (\mathbb{Z}_2)^3$,
 - 32/12/02/004: Proposition 5.8,
 - 32/12/02/005: $\langle t_2(a, A)^6(d, D), t_4^{-1}(a, A)^2(d, D), t_2(b, B) \rangle \cong (\mathbb{Z}_2)^3$,
 - 32/12/02/006: $\langle t_1^{-1}t_2^2(a, A)^6(d, D), t_2t_3(a, A)^4(b, B), t_2(b, B) \rangle \cong (\mathbb{Z}_2)^3$.

- 60:
 - 31/03/01: $\langle t_1^{-1}(d, D)^3, t_1^{-1}t_2^{-1}(b, B)(b, B) \rangle$
induces a non-commutative holonomy of order 10, Lemma 5.2,
 - 31/03/02: $\langle (b, B), (d, D) \rangle$
induces a non-commutative holonomy of order 10, Lemma 5.2.
- 48:
 - 33/05/01/003: CONSTRUCTION! (Theorem 7.1).

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