A GEOMETRIC CHARACTERIZATION OF INTERPOLATION IN $\hat{\mathcal{E}}'(\mathbb{R})$

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ABSTRACT. We give a geometric description of the interpolating varieties for the algebra of Fourier transforms of distributions (or Beurling ultradistributions) with compact support on the real line.

1. INTRODUCTION

Let $\mathcal{E}(\mathbb{R})$ be the space of smooth functions in $\mathbb{R}$ and let $\mathcal{E}'(\mathbb{R})$ be its dual, the space of distributions with compact support on $\mathbb{R}$. It is well known that the space $\hat{\mathcal{E}}'(\mathbb{R})$ of Fourier transforms of distributions in $\mathcal{E}'(\mathbb{R})$ coincides with the algebra of entire functions f such that

$$|f(z)| \leq C(1 + |z|)^A e^{B|\operatorname{Im} z|},$$

where $A, B, C > 0$ may depend on f (see [BrGa95, Theorem 1.4.15]).

A discrete sequence $\Lambda \subset \mathbb{C}$ is called $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolating when the interpolation problem $f(\lambda) = v_\lambda$, $\lambda \in \Lambda$, has a solution $f \in \hat{\mathcal{E}}'(\mathbb{R})$ for every sequence of complex values $\{v_\lambda\}_{\lambda \in \Lambda}$ having the characteristic growth of $\hat{\mathcal{E}}'(\mathbb{R})$ on Λ (see the precise definition in Section 2).

The origin of the interest in $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolation lies in its relationship with convolution equations and, in particular, with the density of exponential families $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ in the space of solutions $g \in \mathcal{E}(\mathbb{R})$ of equations of type $\mu \star g = 0$, $\mu \in \mathcal{E}'(\mathbb{R})$. Any solution g to the convolution equation is the limit of linear combinations of $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ where Λ is the zero set of $\hat{\mu}$. If, moreover, the sequence Λ is $\hat{\mathcal{E}}'(\mathbb{R})$ -interpolating, then the series that represents g enjoys better convergence properties. For more on this relationship see [EhMa74] or [BrGa95, Chapter 6] (in particular, Theorem 6.1.11).

The space $\hat{\mathcal{E}}'(\mathbb{R})$ is a particular case of the algebras

$$A_p = \{f \in H(\mathbb{C}) : \log |f(z)| \leq A + Bp(z) \text{ for some } A, B > 0\}$$

associated to positive measurable weights p , obtained by taking

$$(1) \quad p(z) = |\operatorname{Im} z| + \log(1 + |z|^2).$$

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There exists an analytic characterization of interpolating sequences for general A_p spaces when p is subharmonic (see Theorem A below). However, a complete geometric description exists only for subharmonic weights p which are both radial ($p(z) = p(|z|)$) and doubling ($p(2z) \leq Cp(z)$ for some $C > 0$); see [BrLi95, Corollary 4.8].

For the weight (1) Ehrenpreis and Malliavin gave a necessary geometric condition which turns out to be sufficient provided that Λ is a zero sequence of a slowly decreasing function (see [EhMa74, Theorem 4] and its proof). Later Squires, probably unaware of Ehrenpreis and Malliavin's result (which was stated in terms of solutions to convolution equations), proved the same result [Sq83, Theorem 2].

In this paper we give a geometric characterization for $\mathcal{E}'(\mathbb{R})$ -interpolating sequences (Theorem 1). The characterization shows in particular that the geometric condition given by Ehrenpreis & Malliavin and Squires is also sufficient whenever the sequence is contained in the region

$$|\operatorname{Im} z| \leq C \log(1 + |z|^2).$$

In general, however, their condition alone is not sufficient.

A similar characterization is obtained for the more general Beurling weights. These weights appear naturally in the context of convolution equations when one replaces distributions with compact support with Beurling-Björck ultradistributions of compact support (see [Bj66]). They are not necessarily subharmonic, but we will prove that they are equivalent to a subharmonic weight (see Lemma 8).

The paper is structured as follows. In Section 2 we give the precise definition of interpolating variety, discuss the background for the problem and state the main result. In Section 3 we prove that the geometric conditions of Theorem 1 are necessary, while in Section 4 we show that they are also sufficient.

A final remark about notation. C will always denote a positive constant and its actual value may change from one occurrence to the next. $A = O(B)$ and $A \lesssim B$ mean that $A \leq cB$ for some $c > 0$, and $A \simeq B$ is $A \lesssim B \lesssim A$.

2. PRELIMINARIES

For the following definition and general background on the problem we refer to [BrGa95, Chapter 2].

A measurable function $p : \mathbb{C} \rightarrow \mathbb{R}_+$ is called a *weight* if for some $C, D > 0$:

- (a) $\log(1 + |z|^2) \leq Cp(z)$ for all $z \in \mathbb{C}$.
- (b) $p(\zeta) \leq Cp(z) + D$ if $|\zeta - z| \leq 1$.

The importance of these properties lies in their consequences for the ring A_p defined in the introduction: (a) implies that A_p contains all polynomials, and (b) that A_p is closed under differentiation.

The algebra A_p can be thought of as the union of the Hilbert spaces

$$A_{p,\alpha}^2 := \left\{ f \in H(\mathbb{C}) : \|f\|_{A_{p,\alpha}^2}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha p(z)} dm(z) < \infty \right\}$$

for $\alpha > 0$, as well as the union of the Banach spaces

$$A_{p,\alpha}^\infty := \left\{ f \in H(\mathbb{C}) : \|f\|_{A_{p,\alpha}^\infty} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\alpha p(z)} < \infty \right\}.$$

Also, $A_p = \bigcup_{\alpha > 0} A_{p,\alpha}^\infty$ has the structure of an (LF)-space with the topology of the inductive limit.

Definition. Let Λ be a discrete sequence in $\mathbb{C}$ and let $\{m_\lambda\}_{\lambda \in \Lambda}$ be a sequence of natural numbers. The pair $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is called an *interpolating variety* for the space A_p if for every sequence of values $\{v_\lambda^l\}_{\lambda \in \Lambda, l=0, \dots, m_\lambda-1}$, with

$$(2) \quad \sup_{\lambda \in \Lambda} \left(\sum_{l=0}^{m_\lambda-1} |v_\lambda^l| \right) e^{-\alpha p(\lambda)} < \infty$$

for some $\alpha > 0$, there exists $f \in A_p$ with

$$\frac{f^{(l)}(\lambda)}{l!} = v_\lambda^l, \quad \lambda \in \Lambda; \quad l = 0, \dots, m_\lambda - 1.$$

The choice of condition (2) on the values to be interpolated reflects the fact that for every $f \in A_p$ there exists $\alpha > 0$ such that

$$\sup_{z \in \mathbb{C}} \left(\sum_{l=0}^{\infty} \left| \frac{f^{(l)}(z)}{l!} \right| \right) e^{-\alpha p(z)} < \infty.$$

Thus, denoting by $A_p(X)$ the space of sequences $\{v_\lambda^l\}_{\lambda \in \Lambda, l=0, \dots, m_\lambda-1}$ satisfying (2) for some $\alpha > 0$, we can equivalently define interpolating varieties X as those such that the restriction operator

$$\begin{aligned} \mathcal{R}_X : A_p(\mathbb{C}) &\longrightarrow A_p(X) \\ f &\longmapsto \left\{ \frac{f^{(l)}(\lambda)}{l!} \right\}_{\lambda, l} \end{aligned}$$

is onto.

There exists an analytic characterization of A_p -interpolating varieties for general subharmonic weights p (see [BrLi95, Corollary 3.5]). Results for $p(z) = |z|$ and the weight (1) were previously obtained respectively by Leont'ev [Le72] and Squires [Sq81].

Given a holomorphic function f let $\mathcal{Z}(f)$ denote its zero variety, i.e., the set of pairs $(z, m_z) \in \mathbb{C} \times \mathbb{N}$ such that $f(z) = 0$ with multiplicity m_z .

Theorem A. A variety $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is A_p -interpolating if and only if there exists $f \in A_p$ such that $X \subset \mathcal{Z}(f)$ and for some constants $\delta, C > 0$,

$$\left| \frac{f^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| \geq \delta e^{-Cp(\lambda)}, \quad \lambda \in \Lambda.$$

We would like to give a geometric description of A_p -interpolating varieties for the non-isotropic Beurling weights

$$p(z) = |\operatorname{Im} z| + \omega(|z|),$$

where $\omega(t)$ is a subadditive increasing continuous function, normalized with $\omega(0) = 0$ and such that:

- (c) $\log(1+t) \lesssim \omega(t)$ for $t > 1$.
- (d) $\int_0^\infty \frac{\omega(t)}{1+t^2} dt < \infty$.

Canonical examples of such weights are given by $\omega(t) = \log(1+t^2)$ and $\omega(t) = t^\gamma$, $\gamma \in (0, 1)$.

Beurling weights satisfy the following additional properties:

- (e) For every $c > 0$ there exists $C > 0$ such that $p(\zeta) \leq Cp(z)$ if $\zeta \in D(z, cp(z))$.

- (f) For $\varepsilon > 0$ small enough, there exists $C(\varepsilon) > 0$ such that if $z \in D(\zeta, \varepsilon p(\zeta))$, then $p(\zeta) \leq C(\varepsilon)p(z)$. Also, $C(\varepsilon)$ tends to 1 as ε goes to 0.
- (g) For $x \in \mathbb{R}^+$ big enough, the function $\omega(x)$ does not oscillate too much. More precisely, for fixed $C > 0$, if $y \in (x - C\omega(x), x + C\omega(x))$, then $1/2 \leq \omega(y)/\omega(x) \leq 2$ for x big enough.

Properties (e) and (f) follow easily from the subadditivity of ω . Property (g) follows from the subadditivity and the fact that $\omega(x) = o(|x|/\log|x|)$ (see [Bj66, Lemma 1.2.8]): for any $y \in (x - C\omega(x), x + C\omega(x))$,

$$\begin{aligned} \omega(x - C\omega(x)) &\leq \omega(y) \leq \omega(x + C\omega(x)) \leq \omega(x - C\omega(x)) + \omega(2C\omega(x)) \\ &\leq \omega(x - C\omega(x)) + \omega(2Cx/\log x) \leq 2\omega(x - C\omega(x)). \end{aligned}$$

In order to state the geometric conditions on a variety X as above, we consider the counting function $n(z, r) = \sum_{\lambda \in D(z, r)} m_\lambda$ and the integrated version

$$N(z, r) = \int_0^r \frac{n(z, t) - n(z, 0)}{t} dt + n(z, 0) \log r.$$

In case we want to specify the variety X to which the functions n and N refer, we will use the notation $n(z, r, X)$ and $N(z, r, X)$ respectively.

We are ready to state our main result.

Theorem 1. *A variety $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is A_p -interpolating if and only if:*

- (i) *There is $C > 0$ such that*

$$N(\lambda, p(\lambda), X) \leq Cp(\lambda) \quad \forall \lambda \in \Lambda.$$

- (ii) *The following Carleson-type condition holds:*

$$\sup_{x \in \mathbb{R}} \sum_{\lambda: |\operatorname{Im} \lambda| > \omega(|\lambda|)} m_\lambda \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2} < \infty.$$

Since the Poisson kernel at λ in the corresponding half-plane (upper half-plane if $\operatorname{Im} \lambda > 0$ and lower half-plane when $\operatorname{Im} \lambda < 0$) is $P(\lambda, x) = \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2}$, a restatement of condition (ii) is that the measure $\sum_{\lambda: |\operatorname{Im} \lambda| > \omega(|\lambda|)} m_\lambda \delta_\lambda$ has bounded Poisson balayage.

Remark 2. Notice that for sequences Λ within the region $|\operatorname{Im} z| \leq \omega(|z|)$, condition (i) (shown to be necessary by Ehrenpreis & Malliavin and Squires) provides a complete characterization. However, this is not the case in general, i.e. condition (ii) does not follow from (i), as it is shown in the following example. Take the sequence Λ contained in the sector $\mathcal{A} = \{z \in \mathbb{C}; |\operatorname{Re} z| < \operatorname{Im} z\}$ and having in each segment $\{\operatorname{Im} z = 2^n\} \cap \mathcal{A}$ exactly 2^n equispaced points. Then Λ satisfies condition (i) (basically $n(\lambda, t) \leq t$ for $t \leq p(\lambda)$), but it does not satisfy (ii) (it is not even a Blaschke sequence).

3. NECESSARY CONDITIONS

A standard feature of the spaces A_p is that the interpolation can be performed in a stable way. This consequence of the open mapping theorem for (LF)-spaces applied to the restriction mapping $\mathcal{R}_X$ defined in Section 2 (see [BrGa95, Lemma 2.2.6]) is stated precisely in the following lemma.

Lemma 3. *If X is an interpolating variety, there exist $C > 0$, $M \in \mathbb{N}$ such that for every $\lambda \in \Lambda$ there are functions $f_\lambda, g_\lambda \in A_p$ with bounded norms $\|f_\lambda\|_{A_{p,M}^\infty}, \|g_\lambda\|_{A_{p,M}^\infty} \leq C$ and*

$$\begin{aligned} f_\lambda^{(l)}(\lambda')/l! &= \delta_{\lambda\lambda'}\delta_{l0}, \\ g_\lambda^{(l)}(\lambda')/l! &= \delta_{\lambda\lambda'}\delta_{l(m_\lambda-1)} \quad \forall \lambda, \lambda' \in \Lambda, \quad 0 \leq l \leq m_\lambda. \end{aligned}$$

An application of Jensen's Formula to the functions f_λ, g_λ in the disk $D(\lambda, p(\lambda))$ gives the following result (see [EhMa74, Theorem 4] or [Sq83, Theorem 1]).

Theorem 4. *If X is A_p -interpolating, then condition (i) of Theorem 1 holds.*

The necessity of condition (ii) is an immediate consequence of the following result. Assume that $\Lambda \cap \mathbb{R} = \emptyset$; otherwise move the horizontal line so that it does not touch any of the points in Λ . Let $\mathbb{H}$ denote the upper half-plane.

Proposition 5. *Let X be A_p -interpolating. There exist $C > 0$ such that*

$$\sum_{\substack{\lambda' \in \Lambda \cap \mathbb{H} \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1} \leq Cp(\lambda) \quad \text{for all } \lambda \in \Lambda \cap \mathbb{H}.$$

Of course, an analogous result could be given for any upper ($\{z : \text{Im} z > a\}$) or lower ($\{z : \text{Im} z < a\}$) half-plane.

Before giving the proof of Proposition 5, we show that it implies condition (ii) of Theorem 1. Define $\Lambda_+ = \Lambda \cap \{\text{Im} z > \omega(|z|)\}$. Given $x \in \mathbb{R}$ consider $\lambda \in \Lambda_+$ such that $|x - \lambda| = \inf_{\Lambda_+} |x - \lambda|$. Then

$$|\lambda - \bar{\lambda}'| \leq |\lambda - x| + |x - \bar{\lambda}'| = |\lambda - x| + |x - \lambda'| \leq 2|x - \lambda'|,$$

and therefore

$$\sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\text{Im } \lambda'|}{|x - \bar{\lambda}'|^2} \leq 2 \sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\text{Im } \lambda'|}{|\lambda - \bar{\lambda}'|^2}.$$

The estimate $\log t^{-1} \geq 1 - t$ for $t \in (0, 1)$ shows that

$$\sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \neq \lambda}} m_{\lambda'} \frac{|\text{Im } \lambda| |\text{Im } \lambda'|}{|\lambda - \bar{\lambda}'|^2} \leq \sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1}.$$

Since $p(\lambda) \simeq |\text{Im } \lambda|$ for $\lambda \in \Lambda_+$, it is clear that this implies condition (ii) of Theorem 1.

Remark 6. The necessary condition of Proposition 5 can be seen as a Carleson-type condition; it can be rewritten as

$$|B_\lambda(\lambda)| \geq \delta e^{-Cp(\lambda)}, \quad \lambda \in \Lambda \cap \mathbb{H},$$

where B denotes the Blaschke product in $\mathbb{H}$ of $\{(\lambda, m_\lambda)\}_{\lambda \in \Lambda \cap \mathbb{H}}$, and

$$B_\lambda(z) = B(z) \left(\frac{z - \bar{\lambda}}{z - \lambda} \right)^{m_\lambda}.$$

It can also be seen as density conditions for the counting function associated to the hyperbolic metric in the half-plane. Letting $\nu = \sum_{\lambda \in \Lambda \cap \mathbb{H}} m_\lambda \delta_\lambda$ and using the distribution function we have

$$\sum_{\lambda \in \Lambda \cap \mathbb{H}} m_\lambda \log \left| \frac{z - \lambda}{z - \bar{\lambda}} \right|^{-1} = \int_{\mathbb{H}} \log \left| \frac{z - \zeta}{z - \bar{\zeta}} \right|^{-1} d\nu(\zeta) = \int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt,$$

where

$$D_{\mathbb{H}}(z, t) = \left\{ \zeta : \left| \frac{z - \zeta}{z - \bar{\zeta}} \right| < t \right\}, \quad \text{and} \quad n_{\mathbb{H}}(z, t) := \nu(D_{\mathbb{H}}(z, t))$$

is the number of points of Λ in the pseudohyperbolic disk of “center” z and “radius” t (actually the true disk of center $\operatorname{Re} z + i \frac{1+t^2}{1-t^2} \operatorname{Im} z$ and radius $\frac{2t}{1-t^2} \operatorname{Im} z$).

Proof of Proposition 5. Let $z = x + iy$ and consider the Poisson transform of $\omega(|t|)$:

$$u(z) := P[\omega](z) = \int_{\mathbb{R}} \frac{y \omega(|t|)}{(x-t)^2 + y^2} dt,$$

which converges by (d). Define $H = \exp(u + i\tilde{u})$, where $\tilde{u}$ is a harmonic conjugate of u .

Given $\lambda \in \Lambda \cap \mathbb{H}$, take the function f_{λ} given by Lemma 3 and define

$$h_{\lambda}(z) = \frac{f_{\lambda}(z) e^{iM_1 z}}{(H(z))^{M_2}},$$

with M_1, M_2 to be chosen. It is clear that h_{λ} is holomorphic in $\mathbb{H}$. On the other hand, for all z in the upper half-plane $|\log |H(z)| - \omega(|\operatorname{Re} z|)| \leq A + B|\operatorname{Im} z|$; see [Bj66, Lemma 1.3.11]. Moreover, $|\omega(|\operatorname{Re} z|) - \omega(|z|)| \leq \omega(|\operatorname{Im} z|) \leq A + B|\operatorname{Im} z|$, thus $|\log |H(z)| - \omega(|z|)| \leq A + B|\operatorname{Im} z|$. Therefore, if M_1 and M_2 are big enough, h_{λ} is bounded in $\mathbb{H}$ by a constant which does not depend on λ :

$$|h_{\lambda}(z)| \leq C e^{Mp(z) - M_1 \operatorname{Im} z - M_2 \log |H(z)|} \lesssim 1.$$

Also,

$$|h_{\lambda}(\lambda)| = e^{-M_1 \operatorname{Im} \lambda - M_2 \log |H(\lambda)|} \geq e^{-Cp(\lambda)}.$$

Now apply Jensen’s formula in the half-plane to the function h_{λ} :

$$\log |h_{\lambda}(\lambda)| = \int_{\mathbb{R}} P(\lambda, x) \log |h_{\lambda}(x)| dx - \int_{\mathbb{H}} G(\lambda, \zeta) \Delta \log |h_{\lambda}(\zeta)|,$$

where $P(\lambda, x)$ denotes the Poisson kernel and $G(\lambda, \zeta) = \log \left| \frac{\lambda - \zeta}{\lambda - \bar{\zeta}} \right|^{-1}$ is the Green function in $\mathbb{H}$ with pole in λ .

Since h_{λ} vanishes on $\Lambda \setminus \{\lambda\}$, Jensen’s Formula and the estimates above yield

$$\sum_{\substack{\lambda' \in \Lambda \cap \mathbb{H} \\ \lambda' \neq \lambda}} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \bar{\lambda}'} \right|^{-1} \leq \sup_{\mathbb{R}} \log |h_{\lambda}| - \log |h_{\lambda}(\lambda)| \lesssim p(\lambda).$$

□

4. SUFFICIENT CONDITIONS

We split the sequence into three pieces, according to the non-isotropy of the weight p . Consider the regions

$$\Omega_0 = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \omega(|z|)\},$$

$$\Omega_+ = \{z \in \mathbb{C} : \operatorname{Im} z > \omega(|z|)\},$$

$$\Omega_- = \{z \in \mathbb{C} : \operatorname{Im} z < -\omega(|z|)\},$$

and define $\Lambda_0 = \Lambda \cap \Omega_0$, $\Lambda_+ = \Lambda \cap \Omega_+$ and $\Lambda_- = \Lambda \cap \Omega_-$. Let also $X_0 = \{(\lambda, m_{\lambda})\}_{\lambda \in \Lambda_0}$, $X_+ = \{(\lambda, m_{\lambda})\}_{\lambda \in \Lambda_+}$ and $X_- = \{(\lambda, m_{\lambda})\}_{\lambda \in \Lambda_-}$.

It is enough to prove that each piece X_+ , X_- , X_0 of the variety X is A_p -interpolating. This is so because X is weakly separated (see Lemma 7(i) below), and

a weakly separated union of a finite number of A_p -interpolating varieties is also A_p -interpolating [Ou03, Theorem II.1]. It is also clear that the varieties X^+ and X^- can be dealt with similarly.

We start with some easy consequences of condition (i) of Theorem 1.

Lemma 7. *If condition (i) in Theorem 1 holds, then*

- (i) *X is weakly separated: there exist $\delta, C > 0$ such that the disks $D_\lambda = D(\lambda, \delta e^{-C \frac{p(\lambda)}{m_\lambda}})$ are pairwise disjoint, i.e.*

$$|\lambda - \lambda'| \geq 2\delta \max[e^{-C \frac{p(\lambda)}{m_\lambda}}, e^{-C \frac{p(\lambda')}{m_{\lambda'}}}], \quad \lambda \neq \lambda'.$$

- (ii) *There exist $\varepsilon, C > 0$ such that $n(z, \varepsilon p(z), X) \leq Cp(z)$, $\forall z \in \mathbb{C}$.*

Proof. (i) If there exists λ' such that $|\lambda' - \lambda| < 1$, then

$$N(\lambda, p(\lambda), X) \geq \int_{|\lambda' - \lambda|}^1 \frac{n(\lambda, t) - m_\lambda}{t} dt \geq \int_{|\lambda' - \lambda|}^1 \frac{m_{\lambda'}}{t} dt = \log\left(\frac{1}{|\lambda' - \lambda|}\right)^{m_{\lambda'}}.$$

Using condition (i) in Theorem 1 and reversing the roles of λ and λ' we obtain the desired estimate.

- (ii) When $z = \lambda \in \Lambda$, this is immediate from the estimate

$$\int_{1/2p(\lambda)}^{p(\lambda)} \frac{n(\lambda, 1/2p(\lambda)) - 1}{t} dt \leq N(\lambda, p(\lambda)).$$

When $z \notin \Lambda$, then let $\varepsilon > 0$ be such that $\zeta \in D(z, \varepsilon p(z))$ implies

$$D(z, \varepsilon p(z)) \subset D(\zeta, 1/2p(\zeta)),$$

which exists by property (f) of the weight. Take $\lambda \in D(z, \varepsilon p(z))$ (if there is no such λ the estimate is obviously true). Then, by the previous case and property (e) of the weight

$$n(z, \varepsilon p(z)) \leq n(\lambda, 1/2p(\lambda)) \lesssim p(\lambda) \lesssim p(z).$$

□

4.1. Case Λ_0 . We would like to prove that $X_0 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_0}$ is A_p -interpolating using a $\bar{\partial}$ -scheme. This is easier if we can regularize the weight in the following way.

Lemma 8. *There exists $\tilde{p}$ subharmonic in $\mathbb{C}$ such that $p(z) \simeq \tilde{p}(z)$ and*

$$(3) \quad 1/\tilde{p}(z) \lesssim \Delta \tilde{p}(z) \quad \text{if} \quad |\operatorname{Im} z| \leq 2\omega(|z|).$$

The fact that $p \simeq \tilde{p}$ clearly implies that $A_p = A_{\tilde{p}}$ and the interpolating varieties for A_p and $A_{\tilde{p}}$ are the same.

Proof. We will construct $\tilde{p}(z) = |\operatorname{Im} z| + r(z)$, where r satisfies the following properties:

- (i) $r \geq 0$ and $\tilde{p}$ is subharmonic in $\mathbb{C}$.
- (ii) $r(z) = 0$ if $|\operatorname{Im} z| \geq 10\omega(|z|)$.
- (iii) $1/p(z) \lesssim \Delta \tilde{p}(z)$ and $r(z) \simeq \omega(|z|)$ if $|\operatorname{Im} z| \leq 2\omega(|z|)$.

In order to construct r , we partition the real line into intervals I_n defined in the following way.

Let $x_1 > 1$, $x_{n+1} = x_n + \omega(x_n)$ for $n \geq 1$ and $x_n = -x_{-n}$ for $n \leq -1$. Set $I_0 = [x_{-1}, x_1]$, $I_n = [x_n, x_{n+1}]$ for $n \geq 1$ and $I_n = [x_{n-1}, x_n]$ for $n \leq -1$. Denote by ω_n the length of I_n .

We consider two measures in $\mathbb{C}$. The first one is the usual length measure $d\nu$ in $\mathbb{R}$, which we split $d\nu = \sum_n d\nu_n$, with $d\nu_n = dx|_{I_n}$. The second one is defined as a sum of convolutions of the $d\nu_n$'s: let

$$d\mu_n(z) = \left(\frac{1}{100\pi\omega_n^2} \int_{I_n} \chi_{D_n}(z-x) dx \right) dm(z),$$

where $D_n = D(0, 10\omega_n)$, and define $d\mu = \sum_n d\mu_n$.

Notice that when z is at a distance of I_n smaller than $2\omega_n$, we can use property (g) of the Beurling weights to deduce that $d\mu(z) \simeq 1/\omega(|z|) \simeq 1/p(z)$. Hence $d\mu(z) \simeq dm(z)/p(z)$.

Define

$$r(z) = \int_{\mathbb{C}} \log |z-w| (d\mu(w) - d\nu(w)).$$

Since $\Delta|\operatorname{Im} z| = d\nu$, we have $\Delta\tilde{p} = d\mu \geq 0$.

Let S_n denote the support of μ_n . Let

$$r_n(z) := \int_{\mathbb{C}} \log |z-w| (d\mu_n(w) - d\nu_n(w)) = \int_{S_n} \log |z-w| d\mu_n(w) - \int_{I_n} \log |z-x| dx.$$

Using the definition of μ_n and reversing the order of integration we get

$$r_n(z) = \int_{I_n} M(x) dx,$$

where

$$M(x) = \frac{1}{100\pi\omega_n^2} \int_{D(x, 10\omega_n)} \log |z-w| dm(w) - \log |z-x| \geq 0.$$

In particular, r is non-negative in $\mathbb{C}$.

If $z \notin S_n$ and $x \in I_n$, $\log |z-w|$ is harmonic in $D(x, 10\omega_n)$, hence $r_n(z) = 0$.

Suppose now $z \in D(x_n, 3\omega_n)$. Then, for each $x \in I_n$, $|z-x| \leq 4\omega_n$ and

$$M(x) \geq \frac{1}{100\pi\omega_n^2} \int_{9\omega_n \leq |w-x| \leq 10\omega_n} \log \frac{|z-w|}{|z-x|} dm(w) \gtrsim 1.$$

Thus, $r_n(z) \gtrsim \omega_n \gtrsim \omega(|z|)$.

If $z \in S_n$, using that μ_n and ν_n have the same mass $\omega(x_n)$, we obtain

$$\begin{aligned} \int_{\mathbb{C}} \log |z-w| (d\mu_n(w) - d\nu_n(w)) &\leq \int_{\mathbb{C}} \left| \log \frac{|z-w|}{\omega(x_n)} \right| (d\mu_n(w) + d\nu_n(w)) \\ &\lesssim \int_{\mathbb{C}} \left| \log \frac{|x_n-w|}{\omega(x_n)} \right| (d\mu_n(w) + d\nu_n(w)) \lesssim \omega(|z|). \end{aligned}$$

Since $|\operatorname{Im} z| \leq 2\omega(|z|)$, z belongs at most to a finite number of S_n 's and at least to one $D(x_n, 10\omega_n)$, by property (g) of the Beurling weights, we are done. $\square$

Let us prove now that X_0 is $A_{\tilde{p}}$ -interpolating. In view of Lemma 8, we assume that $1/p \lesssim \Delta\tilde{p}$ on $|\operatorname{Im} z| \leq 2\omega(|z|)$.

Consider the separation radii $\delta_\lambda := \delta e^{-Cp(\lambda)}$ given by Lemma 7(i).

Given a sequence of values $\{v_\lambda^\ell\}_{\lambda,\ell}$ satisfying (2), define the smooth interpolating function

$$F(z) = \sum_{\lambda \in \Lambda_0} p_\lambda(z) \mathcal{X}\left(\frac{|z - \lambda|^2}{\delta_\lambda^2}\right),$$

where $p_\lambda(z) = \sum_{l=0}^{m_\lambda-1} v_\lambda^l (z - \lambda)^l$ and $\mathcal{X}$ is a smooth cut-off function with $|\mathcal{X}'| \lesssim 1$, $\mathcal{X}(x) = 1$ if $|x| \leq 1$ and $\mathcal{X}(x) = 0$ if $|x| \geq 2$.

It is clear that $F^{(l)}(\lambda)/l! = v_\lambda^l$, and that F has the characteristic growth of A_p functions; the support of F is contained in $\bigcup_\lambda D_\lambda$ and for $z \in D_\lambda$,

$$|F(z)| \leq \sum_{l=0}^{m_\lambda-1} |v_\lambda^l| \leq C e^{\alpha p(\lambda)} \lesssim e^{Kp(z)}.$$

There is also a good estimate on $\bar{\partial}F$. Its support is the union of the annuli

$$C_\lambda = \{z \in \mathbb{C} : \delta_\lambda \leq |z - \lambda| \leq 2\delta_\lambda\},$$

and for $z \in C_\lambda$,

$$\left| \frac{\partial F}{\partial \bar{z}}(z) \right| \lesssim \sum_{l=0}^{m_\lambda-1} |v_\lambda^l| |\mathcal{X}'| \frac{1}{\delta_\lambda} \lesssim e^{Cp(\lambda)} \lesssim e^{Kp(z)},$$

for K big enough.

Altogether, there exists $\gamma > 0$ such that

$$(4) \quad \int_{\mathbb{C}} |F(z)|^2 e^{-\gamma p(z)} < \infty, \quad \int_{\mathbb{C}} |\bar{\partial}F(z)|^2 e^{-\gamma p(z)} < \infty.$$

Now, when looking for a holomorphic interpolating function of the form $f = F - u$, we are led to the $\bar{\partial}$ -problem

$$\bar{\partial}u = \bar{\partial}F,$$

which we solve using Hörmander's theorem [Ho94, Theorem 4.2.1]: given a (pluri)-subharmonic function ψ in $\mathbb{C}$, there exists a solution u to the above equation such that

$$2 \int_{\mathbb{C}} |u|^2 \frac{e^{-\psi}}{(1 + |z|^2)^2} dm \leq \int_{\mathbb{C}} |\bar{\partial}F|^2 e^{-\psi} dm.$$

We apply Hörmander's theorem with

$$\psi_\beta(z) = \beta p(z) + v(z),$$

where $\beta > 0$ will be chosen later on and

$$v(z) = \sum_{\lambda \in \Lambda_0} m_\lambda \left[\log |z - \lambda|^2 - \frac{1}{\pi \varepsilon^2 p^2(\lambda)} \int_{D(\lambda, \varepsilon p(\lambda))} \log |z - \zeta|^2 dm(\zeta) \right].$$

Here ε is a fixed small constant to be determined later on.

Integrating by parts the equality

$$\int_0^{2\pi} \log |a - re^{i\theta}|^2 \frac{d\theta}{2\pi} = \begin{cases} \log |a|^2 & \text{if } |a| > r, \\ \log r^2 & \text{if } |a| \leq r, \end{cases}$$

one sees that for $a \in \mathbb{C}$ and $r > 0$,

$$\log |a|^2 - \frac{1}{\pi r^2} \int_{D(a,r)} \log |\zeta|^2 dm(\zeta) = \begin{cases} \log \left| \frac{a}{r} \right|^2 + 1 - \left| \frac{a}{r} \right|^2 & \text{if } |a| \leq r, \\ 0 & \text{if } |a| > r. \end{cases}$$

Thus

$$v(z) = \sum_{\lambda: |\lambda-z| \leq \varepsilon p(\lambda)} m_\lambda \left[\log \frac{|z-\lambda|^2}{\varepsilon^2 p^2(\lambda)} + 1 - \frac{|z-\lambda|^2}{\varepsilon^2 p^2(\lambda)} \right].$$

In particular, $v \leq 0$ and $\Delta v(z) = 0$ if $z \notin \bigcup_\lambda D(\lambda, \varepsilon p(\lambda))$. For $z \in \bigcup_\lambda D(\lambda, \varepsilon p(\lambda))$ we have $|\operatorname{Im} z| \leq 2\omega(|z|)$ and

$$\Delta v(z) \geq \sum_{\lambda: |\lambda-z| \leq \varepsilon p(\lambda)} \frac{-m_\lambda}{\varepsilon^2 p^2(\lambda)} \gtrsim \sum_{\lambda: |\lambda-z| \leq C(\varepsilon)p(z)} \frac{-m_\lambda}{p^2(z)} = -\frac{n(z, C(\varepsilon)p(z))}{p^2(z)}.$$

As observed in Lemma 7(ii), with ε small enough $n(z, C(\varepsilon)p(z)) \lesssim p(z)$, thus $\Delta v(z) \gtrsim -1/p(z)$. This and (3) show that ψ_β is subharmonic if β is chosen big enough.

Also, we deduce from (c) that for any $\beta' > \beta$,

$$\int_{\mathbb{C}} |u|^2 e^{-\beta' p} dm \lesssim \int_{\mathbb{C}} |u|^2 \frac{e^{-\psi_\beta}}{(1+|z|^2)^2} dm \lesssim \int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-\psi_\beta} dm.$$

We need to control ψ_β on the support of $\bar{\partial} F$. For $z \in C_\lambda$,

$$\begin{aligned} |\psi_\beta(z) - \beta p(z)| &\leq \sum_{\lambda: |\lambda-z| \leq \varepsilon p(\lambda)} m_\lambda \log \frac{\varepsilon^2 p^2(\lambda)}{|z-\lambda|^2} \\ &\simeq m_\lambda \log \frac{\varepsilon^2 p^2(\lambda)}{|z-\lambda|^2} + \sum_{\substack{\lambda': |z-\lambda'| \leq \varepsilon p(\lambda') \\ \lambda' \neq \lambda}} m_{\lambda'} \log \frac{\varepsilon^2 p^2(\lambda')}{|z-\lambda'|^2} \\ &\lesssim p(\lambda) + \sum_{\substack{\lambda': |\lambda'-z| \leq C(\varepsilon)p(z) \\ \lambda' \neq \lambda}} m_{\lambda'} \log \frac{C(\varepsilon)^2 p^2(z)}{|z-\lambda'|^2} \\ &\lesssim p(z) + N(z, C(\varepsilon)p(z)). \end{aligned}$$

Claim 9. For ε small enough $N(z, C(\varepsilon)p(z)) \lesssim p(z)$ for all $z \in \operatorname{supp}(\bar{\partial} F)$.

Assuming the claim we have $|\psi_\beta(z) - \beta p(z)| \leq Kp(z)$ on $\operatorname{supp}(\bar{\partial} F)$. Therefore, for β big enough

$$\int_{\mathbb{C}} |u|^2 e^{-\beta' p} dm \lesssim \int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-\psi_\beta} dm \leq \int_{\mathbb{C}} |\bar{\partial} F|^2 e^{-\gamma p} dm < \infty.$$

This shows that $f := F - u \in A_p$. Since $e^{-\psi_\beta} \simeq |z-\lambda|^{-2m_\lambda}$ around each λ , also $u^{(l)}(\lambda) = 0$ for all $\lambda \in \Lambda$, $l = 0, \dots, m_\lambda - 1$, and therefore $f^{(l)}(\lambda)/l! = F^{(l)}(\lambda)/l! = v_\lambda^l$, as required.

Proof of the claim. Assume $z \in C_\lambda$ and observe that $n(z, t) = 0$ for $t < \delta_\lambda$ and that $n(z, t) \leq m_\lambda$ for $\delta_\lambda \leq t < 2\delta_\lambda$. Since $D(z, t) \subset D(\lambda, t + 2\delta_\lambda)$ and $|z| < |\lambda| + 2\delta_\lambda$, we have (changing into $s = t + 2\delta_\lambda$)

$$\begin{aligned} N(z, C(\varepsilon)p(z)) &\leq \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t} dt + \int_{2\delta_\lambda}^{C(\varepsilon)p(z)} \frac{n(z, t) - m_\lambda}{t} dt \\ &\leq p(\lambda) + \int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda} \frac{n(\lambda, s) - m_\lambda}{s - 2\delta_\lambda} ds \\ &\lesssim p(\lambda) + \int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda} \frac{n(\lambda, s) - m_\lambda}{s/2} ds \lesssim p(\lambda) + N(\lambda, C'(\varepsilon)p(\lambda)). \end{aligned}$$

From the properties of the weight and the hypothesis we have finally that for ε small $N(z, C(\varepsilon)p(z)) \lesssim p(\lambda) \lesssim p(z)$.

4.2. Case Λ^+ . According to Theorem A, it is enough to construct a function $G \in A_p$ such that $X_+ \subset \mathcal{Z}(G)$ and

$$\frac{|G^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq \varepsilon e^{-Kp(\lambda)}, \quad \lambda \in \Lambda_+,$$

for some constants $\varepsilon, k > 0$. In fact, the hypotheses of Theorem A require the weight p to be subharmonic, and our weights are not necessarily so. Nevertheless, by Lemma 8, there exists a subharmonic weight $\tilde{p}$ equivalent to p , and we may apply Theorem A to $\tilde{p}$.

Take any entire function F such that $\mathcal{Z}(F) = X_+$. Since the necessary conditions imply that X_+ satisfies the Blaschke condition in $\mathbb{H}$, we can consider also the Blaschke product

$$B(z) = \prod_{\lambda \in \Lambda_+} \left(\frac{z - \lambda}{z - \bar{\lambda}} \right)^{m_\lambda}, \quad z \in \mathbb{H}.$$

Define

$$\phi(z) = \begin{cases} \log \left| \frac{F(z)}{B(z)} \right|, & \operatorname{Im} z > 0, \\ \log |F(z)|, & \operatorname{Im} z \leq 0. \end{cases}$$

Lemma 10. *ϕ is harmonic outside the real axis, subharmonic on $\mathbb{C}$ and its Laplacian is uniformly bounded.*

Proof. It is clear, by definition, that ϕ is harmonic on $\mathbb{C} \setminus \mathbb{R}$. In order to prove that ϕ is subharmonic on $\mathbb{C}$, it is enough to check the mean inequality for $x \in \mathbb{R}$. We have

$$\phi(x) = \log |F(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(x + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(x + re^{i\theta}) d\theta.$$

Since $\Delta \log |F| \equiv 0$ around $\mathbb{R}$, it is enough to compute the Laplacian of

$$\psi(z) = \begin{cases} \log \frac{1}{|B(z)|}, & \operatorname{Im} z > 0, \\ 0, & \operatorname{Im} z \leq 0. \end{cases}$$

Being

$$\log \frac{1}{|B(z)|} = \frac{1}{2} \sum_{\lambda \in \Lambda^+} m_\lambda \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2,$$

it will be enough to compute the Laplacian of each term

$$\psi_\lambda(z) = \begin{cases} \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2, & \operatorname{Im} z > 0, \\ 0, & \operatorname{Im} z \leq 0. \end{cases}$$

It is clear that $\partial \psi_\lambda / \partial x = 0$ on $\mathbb{R}$, hence $\Delta \psi_\lambda = \partial^2 \psi_\lambda / \partial y^2$. Since ψ_λ is continuous around $\mathbb{R}$, this Laplacian has a magnitude equivalent to the jump of the first derivative of ψ_λ . The derivative of the Green function on the half-plane with respect to the normal direction y is the Poisson kernel:

$$\frac{\partial}{\partial y} \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2 \Big|_{y=0} = \frac{4 \operatorname{Im} \lambda}{|x - \lambda|^2}.$$

Therefore,

$$\Delta\phi(x) = 4 \sum_{\lambda \in \Lambda^+} m_\lambda \frac{\operatorname{Im} \lambda}{|x - \lambda|^2} dx,$$

which is bounded by the hypothesis. $\square$

Define

$$\Psi(z) = N|\operatorname{Im} z| - \phi(z).$$

Observe that $\Delta\Psi(z) = N dx - \Delta\phi(x) dx$, thus according to the previous lemma $\Delta\Psi \simeq dx$ when $N \in \mathbb{N}$ is big enough. In this situation, according to [OrSe99, Lemma 3], there exists a multiplier associated to Ψ , i.e., an entire function h such that:

- (a) $\mathcal{Z}(h)$ is a separated sequence contained in $\mathbb{R}$.
- (b) Given any $\varepsilon > 0$, $|h(z)| \simeq \exp(\Psi(z))$ for all points z such that $d(z, \mathcal{Z}(h)) > \varepsilon$.

Define now $G = hF$. It is clear that $G \in A_p$:

$$|G(z)| \lesssim e^{\Psi(z) + \log |F(z)|} \leq e^{\Psi(z) + \phi(z)} \leq e^{Np(z)}, \quad z \in \mathbb{C}.$$

It is also clear that $X_+ \subset \mathcal{Z}(G)$, since $X_+ \subset \mathcal{Z}(F)$.

In order to prove that there exist $\varepsilon, C > 0$ such that

$$(5) \quad \left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| \geq \varepsilon e^{-Cp(\lambda)}$$

consider then the disjoint disks $D_\lambda = D(\lambda, \delta_\lambda)$, $\delta_\lambda = \delta e^{-C \frac{p(\lambda)}{m_\lambda}}$ given by Lemma 7(i). Since Λ_+ is far from $\mathcal{Z}(h)$, the estimate

$$|G(z)| = |h(z)|e^{\phi(z)}|B(z)| \simeq e^{N|\operatorname{Im} z|}|B(z)|, \quad z \in \partial D_\lambda,$$

holds.

Claim 11. *There exists $C > 0$ such that $|B(z)| \geq \varepsilon e^{-Cp(z)}$, $z \in \partial D_\lambda$.*

Assuming this we have $|G(z)| \gtrsim e^{-Cp(z)}$ for all $z \in \partial D_\lambda$. Define then $g(z) = G(z)/(z - \lambda)^{m_\lambda}$. It is clear that g is holomorphic, non-vanishing in D_λ , and $|g(z)| \gtrsim e^{-cp(\lambda)}$ for $z \in \partial D_\lambda$. By the minimum principle

$$\left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| = |g(0)| \gtrsim e^{-cp(\lambda)},$$

as desired.

Proof of the claim. As observed in Remark 6(b), the estimate we want to prove is equivalent to

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim p(z), \quad z \in \partial D_\lambda.$$

This is proved like Claim 9, except we replace the Euclidean disks by the hyperbolic ones. We have

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t} dt + \int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z, t) - m_\lambda}{t} dt.$$

The first term is controlled by $p(\lambda)$. In order to control the second term observe that $D_{\mathbb{H}}(z, t) \subset D_{\mathbb{H}}(\lambda, \frac{t+\delta_\lambda}{1+t\delta_\lambda})$; hence changing the variable into $s = \frac{t+\delta_\lambda}{1+t\delta_\lambda}$ we get

$$\int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z, t) - m_\lambda}{t} dt \leq \int_{\frac{3\delta_\lambda}{1+2\delta_\lambda^2}}^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s - \delta_\lambda} \frac{1 - \delta_\lambda^2}{(1 - \delta_\lambda)^2} ds.$$

There is no restriction in assuming that $\delta_\lambda < 1/2$. Then $\frac{3\delta_\lambda}{1+2\delta_\lambda^2} > 2\delta_\lambda$ and therefore $s - \delta_\lambda > s/2$. With this and condition (ii) in Theorem 1 we obtain

$$\int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t} dt \lesssim p(\lambda) + \int_0^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds.$$

Since $p(\lambda) \lesssim p(z)$, we will be done as soon as we prove that

$$\int_0^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds \lesssim p(\lambda).$$

There exists $\delta > 0$ (independent of λ) such that $D_{\mathbb{H}}(\lambda, \delta) \subset D(\lambda, p(\lambda))$. Then

$$\begin{aligned} \int_0^\delta \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds &= \sum_{0 < |\frac{\lambda - \lambda'}{\lambda - \lambda'}| < \delta} m_{\lambda'} \log \frac{\delta}{|\frac{\lambda - \lambda'}{\lambda - \lambda'}|} \leq \sum_{0 < |\frac{\lambda - \lambda'}{\lambda - \lambda'}| < \delta} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \\ &\lesssim \sum_{0 < |\lambda - \lambda'| < p(\lambda)} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \leq N(\lambda, p(\lambda)) \lesssim p(\lambda). \end{aligned}$$

For the remaining part we use condition (ii) in Theorem 1 and the estimate $\log t^{-1} \simeq 1 - t$ for $\delta < t < 1$. Taking $x = \operatorname{Re} \lambda$ we have

$$\begin{aligned} \int_\delta^1 \frac{n_{\mathbb{H}}(\lambda, s) - m_\lambda}{s} ds &\lesssim \sum_{\lambda \neq \lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda| |\operatorname{Im} \lambda'|}{|\lambda - \bar{\lambda}'|^2} \\ &\lesssim \sum_{\lambda \neq \lambda'} m_{\lambda'} \frac{|\operatorname{Im} \lambda| |\operatorname{Im} \lambda'|}{|x - \lambda'|^2} \lesssim |\operatorname{Im} \lambda| \simeq p(\lambda). \end{aligned}$$

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