

A MONOIDAL APPROACH TO SPLITTING MORPHISMS OF BIALGEBRAS

A. ARDIZZONI, C. MENINI, AND D. ŞTEFAN

ABSTRACT. The main goal of this paper is to investigate the structure of Hopf algebras with the property that either its Jacobson radical is a Hopf ideal or its coradical is a subalgebra. Let us consider a Hopf algebra A such that its Jacobson radical J is a nilpotent Hopf ideal and $H := A/J$ is a semisimple algebra. We prove that the canonical projection of A on H has a section which is an H -colinear algebra map. Furthermore, if H is cosemisimple too, then we can choose this section to be an (H, H) -bilinear algebra morphism. This fact allows us to describe A as a ‘generalized bosonization’ of a certain algebra R in the category of Yetter–Drinfeld modules over H . As an application we give a categorical proof of Radford’s result about Hopf algebras with projections. We also consider the dual situation. Let A be a bialgebra such that its coradical is a Hopf sub-bialgebra with antipode. Then there is a retraction of the canonical injection of H into A which is an H -linear coalgebra morphism. Furthermore, if H is semisimple too, then we can choose this retraction to be an (H, H) -bilinear coalgebra morphism. Then, also in this case, we can describe A as a ‘generalized bosonization’ of a certain coalgebra R in the category of Yetter–Drinfeld modules over H .

INTRODUCTION

Let H be a Hopf algebra. The categories ${}^H_H\mathcal{YD}$ and ${}^H_H\mathcal{M}_H^H$, of Yetter–Drinfeld modules and respectively Hopf bimodules appeared, in particular, as an attempt to construct new solutions to the Yang–Baxter equation. Nowadays we can recognize their most important properties in the definition of braided categories, a very general and abstract setting useful, not only for providing new solutions to the Yang–Baxter equation, but also in many other areas of mathematics, such as the theory of quantum groups and low-dimensional topology.

Partially motivated by these applications, the theory of Hopf algebras was known in the 80’s as an outstanding development. Besides many striking results obtained since then, we would like to recall, more or less chronologically, a few of them that will play a very important role in our paper.

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- The description of the coradical filtration of a pointed coalgebra, due to Taft and Wilson [TW], that is crucial in the classification of finite-dimensional pointed Hopf algebras.

- The characterization of bialgebras with projection due to Radford [Ra1]. Later Majid [Maj1] showed that this result can be interpreted in terms of bialgebras in a braided category.

- The equivalence of braided categories ${}^H_H\mathcal{YD} \simeq {}^H_H\mathfrak{M}_H^H$ (see [Wo], [AD] and [Sch1]), and its relation with the Drinfeld double $D(H)$ [Dr].

- The classification of certain classes of pointed Hopf algebras of finite dimension. One of the used methods is the ‘lifting’ method (see [AS1], [AS2], [AS3], [AS4]). Let A be a Hopf algebra such that its coradical is a Hopf subalgebra H . Then the coradical filtration of A is a filtration of Hopf algebras, and hence $\text{gr } A$ is a graded Hopf algebra. One of the main steps of the ‘lifting’ method is to describe $\text{gr } A$, by using the second-mentioned result, as the ‘bosonization’ of a certain Hopf algebra R in ${}^H_H\mathcal{YD}$ by H . The next step is to find all Hopf algebras A having a given graded Hopf algebra $\text{gr } A$.

- Let A be a finite-dimensional Hopf algebra over a field k of characteristic zero whose coradical, say H , forms a Hopf subalgebra. Then the left H -module coalgebra A is a cosmash in the sense that there exists an H -linear coalgebra map $\gamma : A \rightarrow H$ such that $\gamma|_H = \text{Id}_H$; see [SvO]. Masuoka showed in [Mas], using a different method, that the above result still holds true without any assumption on the dimension of A and $\text{char } k$.

- For a Hopf algebra A a conjectural formula for A_1 , the first component of the coradical filtration of A , is proposed in [AS5]. This formula is proved in the same paper in the case when A is a graded Hopf algebra such that its coradical is a Hopf subalgebra of A . In [CDMM] the conjecture is proved in the ungraded case.

One of the main aims of this paper is to strengthen some of the results that we mentioned above. Our approach is based on the following results. Let A be a Hopf algebra such that its Jacobson radical J is a nilpotent Hopf ideal and $H := A/J$ is a semisimple algebra. Then the canonical projection of A on H has a section which is an H -colinear algebra map. Furthermore, if H is cosemisimple too, then we can choose this section to be an (H, H) -bilinear algebra morphism. We also consider the dual situation. Let A be a bialgebra such that its coradical is a sub-bialgebra with antipode. Then there is a retraction of the canonical injection of H into A which is an H -linear coalgebra morphism. Furthermore, if H is semisimple too, then we can choose this retraction to be an (H, H) -bilinear coalgebra morphism. These results are achieved by applying Theorem 2.12 and Theorem 2.16, that were proved in [AMS] in the framework of monoidal categories. Thus we start the first section by recalling the definition of a monoidal category. Then we present a list of the monoidal categories we are interested into, motivating why we chose to make use of this terminology.

Then, in the second section, we relate the concept of semisimple and separable algebra in the categories of (bi)comodules over H , and cosemisimple and coseparable coalgebra in the categories of (bi)modules over H by means of some suitable integrals. This will allow us to apply the above-mentioned theorems.

The main results of this section recall us [Ra1], where it is assumed that a Hopf algebra morphism $\pi : A \rightarrow H$ has a section $\sigma : H \rightarrow A$ which is a morphism of Hopf algebras. In [Ra1] it is shown that there is a bialgebra R in ${}^H_H\mathcal{YD}$ such that A is

the smash product algebra and the smash product coalgebra of R by H . It is then natural to look for a similar description of a bialgebra A , supposing that $\pi : A \rightarrow H$ has a section σ which is only a morphism of algebras in ${}^H\mathfrak{M}^H$. This will be done in the second section of the paper. The starting point is the simple observation that A becomes in a natural way an object in ${}^H_H\mathfrak{M}^H_H$. Of course the left and right comodule structures are induced by π . Since σ is a morphism of algebras, A is a bimodule over H , and the fact that σ is a morphism of bicomodules is enough to ensure the required compatibility relations. By using the equivalence

$${}^H_H\mathfrak{M}^H_H \simeq {}^H_H\mathcal{YD}$$

we have $A \simeq R \otimes H$ (isomorphism in ${}^H_H\mathfrak{M}^H_H$), where $R = A^{co(H)}$. Moreover, the multiplication of A is a morphism in ${}^H_H\mathfrak{M}^H_H$, and the unit of A is in R . Therefore R becomes an algebra in ${}^H_H\mathcal{YD}$, and A can be identified as an algebra with the smash product $R \# H$. We cannot repeat this argument for the coalgebra structure since Δ is only (H, H) -bilinear. Thus, by identifying A and $R \# H$ as algebras, the problem of describing all bialgebras A as above is equivalent to finding all coalgebras structures on $R \# H$ such that the comultiplication is a morphism of (H, H) -bicomodules. We prove that $\Delta_{R \# H}$ is uniquely determined by a pair of K -linear maps

$$\delta : R \rightarrow R \otimes R \quad \text{and} \quad \omega : H \rightarrow R \otimes R.$$

Let ε be the restriction of the counit of A to R . The properties of δ , ω and ε necessary to get a bialgebra structure on $R \# H$ are listed in Definition 3.42. The result that we obtain is stated in Theorem 3.49.

Then we also prove the dual result, namely that a Hopf algebra A , having the coradical a semisimple and cosemisimple Hopf subalgebra, is as a Hopf algebra, not only as a coalgebra, a kind of smash product; see Theorem 3.66. We expect that this last result is strongly connected with the lifting method introduced by N. Andruskiewitsch and H.J. Schneider. Theorem 3.66 can probably be used to get direct information about a Hopf algebra A with the property that its coradical is a subalgebra, skipping the step when the associated graded Hopf algebra $gr A$ is investigated.

We conclude the paper with Theorem 3.71. There we prove that if H is a cosemisimple Hopf algebra and (C, Δ, ε) is a coalgebra in \mathfrak{M}_H such that the coradical C_0 of C is H , then the first term C_1 of the coradical filtration of C verifies (89). In [AS5, Lemma 4.2] it was shown that (89) holds true for $C := gr A$, where A is a Hopf algebra such that A_0 , the coradical of A , is a subalgebra, while in [CDMM, Remark 3.2] it was pointed out that the proof of (89), given in [AS5] for $gr A$, also works in the case $C := A$ is a Hopf algebra such that A_0 , the coradical of A , is a subalgebra.

1. INTEGRAL, SEPARABILITY AND COSEPARABILITY

1.1. A *monoidal category* means a category \mathcal{M} that is endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$, an object $\mathbf{1} \in \mathcal{M}$ and functorial isomorphisms: $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \rightarrow X$ and $r_X : X \otimes \mathbf{1} \rightarrow X$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom*,

that is, the diagram

$$\begin{array}{ccc}
 U \otimes ((V \otimes W) \otimes X) & \xrightarrow{U \otimes a_{V,W,X}} & U \otimes (V \otimes (W \otimes X)) \\
 \nearrow a_{U,V \otimes W,X} & & \nwarrow a_{U,V,W \otimes X} \\
 (U \otimes (V \otimes W)) \otimes X & & (U \otimes V) \otimes (W \otimes X) \\
 \nwarrow a_{U,V,W \otimes X} & & \nearrow a_{U \otimes V,W,X} \\
 ((U \otimes V) \otimes W) \otimes X & &
 \end{array}$$

is commutative for every U, V, W, X in \mathcal{M} . The morphisms l and r are called the *unit constraints*, and they are assumed to satisfy the *Triangle Axiom*, that is, the diagram

$$\begin{array}{ccc}
 (V \otimes \mathbf{1}) \otimes W & \xrightarrow{a_{V,\mathbf{1},W}} & V \otimes (\mathbf{1} \otimes W) \\
 \searrow r_V \otimes W & & \swarrow V \otimes l_W \\
 & V \otimes W &
 \end{array}$$

is commutative. The object $\mathbf{1}$ is called the *unit* of \mathcal{M} .

For details on monoidal categories we refer to [Ka, Chapter XI] and [Maj2]. A monoidal category is called *strict* if the associativity constraint and unit constraints are the corresponding identity morphisms.

1.2. As it is noted in [Maj2, p. 420], the Pentagon Axiom solves the consistency problem that appears, because there are two ways to go from $((U \otimes V) \otimes W) \otimes X$ to $U \otimes (V \otimes (W \otimes X))$. The coherence theorem, due to S. Mac Lane, solves the similar problem for the tensor product of an arbitrary number of objects in \mathcal{M} . Accordingly with this theorem, we can always omit all brackets and simply write $X_1 \otimes \cdots \otimes X_n$ for any object obtained from X_1, \dots, X_n by using \otimes and brackets. Also as a consequence of the coherence theorem, the morphisms a, l, r take care of themselves, so they can be omitted in any computation involving morphisms in \mathcal{M} .

Let $(H, m_H, u_H, \Delta_H, \varepsilon_H, S)$ be a Hopf algebra over field K . Basically we are interested in the following examples of monoidal categories.

- The category ${}_H\mathfrak{M} = ({}_H\mathfrak{M}, \otimes_K, K)$ of all left modules over H . The tensor $V \otimes W$ of two left H -modules is an object in ${}_H\mathfrak{M}$ via the diagonal action; the unit is K regarded as a left H -module via ε_H .

- The category ${}_H\mathfrak{M}_H = ({}_H\mathfrak{M}_H, \otimes_K, K)$ of all two-sided modules over H . The tensor $V \otimes W$ of two (H, H) -bimodules carries, on both sides, the diagonal action; the unit is K regarded as an (H, H) -bimodule via ε_H .

We can dualize all the structures given for modules in order to obtain categories of comodules.

- The category ${}^H\mathfrak{M} = ({}^H\mathfrak{M}, \otimes_K, K)$ of all left comodules over H . The tensor product $V \otimes W$ of two left H -comodules is an object in ${}^H\mathfrak{M}$ via the diagonal coaction; the unit is K regarded as a left H -comodule via the map $k \mapsto 1_H \otimes k$.

- The category ${}^H\mathfrak{M}^H = ({}^H\mathfrak{M}^H, \otimes_K, K)$ of all two-sided comodules over H . The tensor $V \otimes W$ of two (H, H) -bicomodules carries, on both sides, the diagonal

coaction; the unit is K regarded as an H -bicomodule via the maps $k \mapsto 1_H \otimes k$ and $k \mapsto k \otimes 1_H$.

• The category ${}^H_H\mathcal{YD} = ({}^H_H\mathcal{YD}, \otimes_K, K)$ of left Yetter-Drinfeld modules over H . Recall that an object V in ${}^H_H\mathcal{YD}$ is a left H -module and a left H -comodule satisfying, for any $h \in H, v \in V$, the compatibility condition

$$\sum ({}^{h(1)}v)_{\langle -1 \rangle} h_{(2)} \otimes ({}^{h(1)}v)_{\langle 0 \rangle} = \sum h_{(1)} v_{\langle -1 \rangle} \otimes {}^{h(2)}v_{\langle 0 \rangle}$$

or, equivalently,

$$\rho(hv) = \sum h_{(1)} v_{\langle -1 \rangle} S(h_{(3)}) \otimes {}^{h(2)}v_{\langle 0 \rangle},$$

where for the module structure on V we used the notation ${}^h v$. For Yetter-Drinfeld modules we shall keep this notation throughout the paper.

The tensor product $V \otimes W$ of two Yetter-Drinfeld modules is an object in ${}^H_H\mathcal{YD}$ via the diagonal action and the diagonal coaction; the unit in ${}^H_H\mathcal{YD}$ is K regarded as a left H -comodule via the map $x \mapsto 1_H \otimes x$ and as a left H -module via ε_H .

Analogously one defines the category \mathcal{YD}_H^H .

In this paper we shall always assume that \mathcal{M} is an *abelian category* and that, for every $M \in \mathcal{M}$, both the functors $M \otimes (-) : \mathcal{M} \rightarrow \mathcal{M}$ and $(-) \otimes M : \mathcal{M} \rightarrow \mathcal{M}$ are *right exact*. The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. For more details, see [AMS]. Given an algebra A in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, we can construct the monoidal category of (A, A) -bimodules $({}_A\mathcal{M}_A, \otimes_A, A)$, which by the above assumptions is an abelian category, too.

• Let us consider the monoidal category $\mathcal{M} := ({}^H\mathfrak{M}, \otimes_K, K)$ of left H -comodules. Algebras in \mathcal{M} are exactly *left H -comodule algebras* and coalgebras in \mathcal{M} are left H -comodule coalgebras.

Let A be a left H -comodule algebra. The category of all (A, A) -bimodules in \mathcal{M} will be denoted by ${}_A^H\mathfrak{M}_A$. An object M in ${}_A^H\mathfrak{M}_A$ is a left H -comodule which is also an (A, A) -bimodule such that $\mu_l : A \otimes M \rightarrow M$ and $\mu_r : M \otimes A \rightarrow M$ are morphisms of left H -comodules. Here μ_l and μ_r define the module structures on M , and both $A \otimes M$ and $M \otimes A$ are left H -comodules via the diagonal coaction. $({}_A^H\mathfrak{M}_A, \otimes_A, A)$ is a monoidal category with the usual tensor product of two (A, A) -bimodules $(-) \otimes_A (-)$. If $V, W \in {}_A^H\mathfrak{M}_A$, then the left structures on $V \otimes_A W$ are given by

$$(1) \quad a(v \otimes_A w) = av \otimes_A w,$$

$$(2) \quad \rho_{V \otimes_A W}^l(v \otimes_A w) = \sum v_{\langle -1 \rangle} w_{\langle -1 \rangle} \otimes (v_{\langle 0 \rangle} \otimes_A w_{\langle 0 \rangle}).$$

The right A -module structure is analogous to the left one. The unit in ${}_A^H\mathfrak{M}_A$ is A .

For $A = K$ with trivial H -comodule structures we get the category of left H -comodules. Also for the trivial Hopf algebra $H = K$ we get that A is just a K -algebra and ${}_A^H\mathfrak{M}_A = {}_A\mathfrak{M}_A$.

• Let us consider the monoidal category $\mathcal{M} := ({}^H\mathfrak{M}^H, \otimes_K, K)$ of (H, H) -bicomodules. An algebra in \mathcal{M} is an algebra A which is an (H, H) -bicomodule such that A is a left and a right H -comodule algebra. We shall say that A is an *H -bicomodule algebra*.

Let A be an H -bicomodule algebra. The category of all (A, A) -bimodules in \mathcal{M} will be denoted by ${}_A^H\mathfrak{M}_A^H$. An object M in ${}_A^H\mathfrak{M}_A^H$ in an (H, H) -bicomodule which is also an (A, A) -bimodule such that $\mu_l : A \otimes M \rightarrow M$ and $\mu_r : M \otimes A \rightarrow M$ are

morphisms of (H, H) -bicomodules. Here μ_l and μ_r define the module structures on M , and both $A \otimes M$ and $M \otimes A$ are (H, H) -bicomodules via the diagonal coactions. $({}_A^H\mathfrak{M}_A^H, \otimes_A, A)$ is a monoidal category with the usual tensor product of two (A, A) -bimodules $(-) \otimes_A (-)$. If $V, W \in {}_A^H\mathfrak{M}_A^H$, then the left structures on $V \otimes_A W$ are given by (1) and (2). The right structures are defined similarly. The unit in ${}_A^H\mathfrak{M}_A^H$ is A .

For $A = K$ with trivial H -comodule structures we get the category of (H, H) -bicomodules. Also for the trivial Hopf algebra $H = K$ we get that A is just a K -algebra and ${}_A^H\mathfrak{M}_A^H = {}_A\mathfrak{M}_A$. Another interesting particular case is obtained by taking $A := H$. The category of (A, A) -bimodules we get in this case is $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$, that is, the category of two-sided Hopf-modules.

All the definitions above can be dualized. Given a coalgebra C in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, we can construct the monoidal category of C -bicomodules $({}^C\mathcal{M}^C, \square_C, C)$.

- Let us consider the monoidal category $\mathcal{M} = ({}_H\mathfrak{M}, \otimes_K, K)$ of left H -modules. Algebras in \mathcal{M} are left H -module algebras and coalgebras in \mathcal{M} are exactly left H -module coalgebras.

Let D be a left H -module coalgebra. The category of all (D, D) -bicomodules in \mathcal{M} will be denoted by ${}_H^D\mathfrak{M}^D$. An object M in ${}_H^D\mathfrak{M}^D$ is a left H -module which is also a (D, D) -bicomodule such that $\rho^l : M \rightarrow D \otimes M$ and $\rho^r : M \rightarrow M \otimes D$ are morphisms of left H -modules. Here ρ^l and ρ^r define the comodule structures on M and both $D \otimes M$ and $M \otimes D$ are left H -modules via the diagonal actions. $({}_H^D\mathfrak{M}^D, \square_D, D)$ is a monoidal category with respect to the tensor product given by $(-) \square_D (-)$, the cotensor product of two (D, D) -bicomodules. If $V, W \in {}_H^D\mathfrak{M}^D$, then the left structures on $V \square_D W$ are given by

$$(3) \quad h(v \square_D w) = \sum h_{(1)} v \square_D h_{(2)} w,$$

$$(4) \quad \rho_{V \otimes_H W}^l(v \square_D w) = \sum v_{(-1)} \otimes (v_{(0)} \square_D w).$$

The right D -module structure is analogous to the left one. The unit in ${}_H^D\mathfrak{M}^D$ is D .

For $D = K$ with the trivial H -module structures we get the categories of left H -bimodules. Also for the trivial Hopf algebra $H = K$ we get that D is just a K coalgebra and ${}_H^D\mathfrak{M}^D = {}^D\mathfrak{M}^D$.

- Let us consider the monoidal category $\mathcal{M} = ({}_H\mathfrak{M}_H, \otimes_K, K)$ of (H, H) -bimodules. A coalgebra in \mathcal{M} is a coalgebra D which is an (H, H) -bimodule such that D is a left and a right H -module coalgebra. We shall say that D is an H -bimodule coalgebra.

Let D be an H -bimodule coalgebra. The category of all (D, D) -bicomodules in \mathcal{M} will be denoted by ${}_H^D\mathfrak{M}_H^D$. An object M in ${}_H^D\mathfrak{M}_H^D$ is an (H, H) -bimodule which is also a (D, D) -bicomodule such that $\rho^l : M \rightarrow D \otimes M$ and $\rho^r : M \rightarrow M \otimes D$ are morphism of (H, H) -bimodules. Here ρ^l and ρ^r define the comodule structures on M , and both $D \otimes M$ and $M \otimes D$ are (H, H) -bimodules via the diagonal actions. $({}_H^D\mathfrak{M}_H^D, \square_D, D)$ is a monoidal category with respect to the tensor product given by $(-) \square_D (-)$, the cotensor product of two (D, D) -bicomodules. If $V, W \in {}_H^D\mathfrak{M}_H^D$, then the left structures on $V \square_D W$ are given by (3) and (4). The right structures are defined similarly. The unit in ${}_H^D\mathfrak{M}_H^D$ is D .

For $D = K$ with the trivial H -module structures we get the categories of (H, H) -bimodules. Also for the trivial Hopf algebra $H = K$ we get that D is just a K

coalgebra and ${}^D_H\mathfrak{M}_H^D = {}^D\mathfrak{M}^D$. Note that, for $D := H$, an object in the category of (D, D) -bicomodules is $({}^H_H\mathcal{M}_H^H, \square_H, H)$, the category of two-sided Hopf-modules.

1.3. A *monoidal functor* between two monoidal categories $(\mathcal{M}, \otimes, \mathbf{1}, a, l, r)$ and $(\mathcal{M}', \otimes', \mathbf{1}', a', l', r')$ is a triple (F, ϕ_0, ϕ_2) , where $F : \mathcal{M} \rightarrow \mathcal{M}'$ is a functor, $\phi_0 : \mathbf{1}' \rightarrow F(\mathbf{1})$ is an isomorphism such that the diagrams

$$\begin{array}{ccc} \mathbf{1}' \otimes' F(U) & \xrightarrow{l'_{F(U)}} & F(U) \\ \phi_0 \otimes F(U) \downarrow & & \uparrow F(l_U) \\ F(\mathbf{1}) \otimes F(U) & \xrightarrow{\phi_2(\mathbf{1}, U)} & F(\mathbf{1} \otimes U) \end{array} \quad \begin{array}{ccc} F(U) \otimes' \mathbf{1}' & \xrightarrow{r'_{F(U)}} & F(U) \\ F(U) \otimes \phi_0 \downarrow & & \uparrow F(r_U) \\ F(U) \otimes F(\mathbf{1}) & \xrightarrow{\phi_2(U, \mathbf{1})} & F(U \otimes \mathbf{1}) \end{array}$$

are commutative, and $\phi_2(U, V) : F(U) \otimes' F(V) \rightarrow F(U \otimes V)$ is a family of functorial isomorphisms such that the diagram

$$\begin{array}{ccccc} (F(U) \otimes' F(V)) \otimes' F(W) & \xrightarrow{\phi_2(U, V) \otimes' F(W)} & F(U \otimes' V) \otimes' F(W) & \xrightarrow{\phi_2(U \otimes' V, W)} & F((U \otimes' V) \otimes W) \\ \downarrow a'_{F(U), F(V), F(W)} & & & & \downarrow F(a_{U, V, W}) \\ F(U) \otimes' (F(V) \otimes' F(W)) & \xrightarrow{F(U) \otimes' \phi_2(V, W)} & F(U) \otimes' F(V \otimes W) & \xrightarrow{\phi_2(U, V \otimes W)} & F(U \otimes (V \otimes W)) \end{array}$$

is commutative. A monoidal functor (F, ϕ_0, ϕ_2) is called *strict* if both ϕ_0 and ϕ_2 are identities.

Let (F, ϕ_0, ϕ_2) and $(L, \lambda_0, \lambda_2)$ be two monoidal functors, where $F : (\mathcal{M}, \otimes, \mathbf{1}) \rightarrow (\mathcal{M}', \otimes', \mathbf{1}')$ and $L : (\mathcal{M}', \otimes', \mathbf{1}') \rightarrow (\mathcal{M}'', \otimes'', \mathbf{1}'')$. Then the composition $T = LF$ again has the structure of a monoidal functor (T, τ_0, τ_2) , where

$$\begin{aligned} \tau_2(U, V) &:= T(U) \otimes'' T(V) \xrightarrow{\lambda_2(F(U), F(V))} L[F(U) \otimes' F(V)] \xrightarrow{L[\phi_2(U, V)]} T(U \otimes V), \\ \tau_0 &:= \mathbf{1}'' \xrightarrow{\lambda_0} L(\mathbf{1}') \xrightarrow{L(\phi_0)} T(\mathbf{1}). \end{aligned}$$

A functorial morphism $\xi : F \rightarrow L$ between two monoidal functors $F, L : (\mathcal{M}, \otimes, \mathbf{1}) \rightarrow (\mathcal{M}', \otimes', \mathbf{1}')$ is said to be monoidal if

$$\begin{array}{ccc} F(U) \otimes' F(V) & \xrightarrow{\xi_U \otimes' \xi_V} & L(U) \otimes' L(V) \\ \phi_2(U, V) \downarrow & & \downarrow \lambda_2(U, V) \\ F(U \otimes V) & \xrightarrow{\xi_{U \otimes V}} & L(U \otimes V) \end{array} \quad \begin{array}{ccc} F(\mathbf{1}) & \xrightarrow{\xi_{\mathbf{1}}} & L(\mathbf{1}) \\ \swarrow \phi_0 & & \searrow \lambda_0 \\ & \mathbf{1}' & \end{array}$$

We include the following useful result.

Proposition 1.4. *Let \mathcal{M} and \mathcal{M}' be monoidal categories. Let (F, ϕ_0, ϕ_2) , be a monoidal functor between the categories \mathcal{M} and \mathcal{M}' , and assume that $F : \mathcal{M} \rightarrow \mathcal{M}'$ is an equivalence of categories. Let $G : \mathcal{M}' \rightarrow \mathcal{M}$ be a right adjoint of F and denote by $\epsilon : FG \rightarrow Id_{\mathcal{M}'}$ the counit and by $\eta : Id_{\mathcal{M}} \rightarrow GF$ the unit of the adjunction. Let γ_0 denote the composition of*

$$\mathbf{1} \xrightarrow{\eta_{\mathbf{1}}} GF(\mathbf{1}) \xrightarrow{G(\phi_0^{-1})} G(\mathbf{1}')$$

and let $\gamma_2(U, V)$ be the composition of

$$\begin{aligned} G(U) \otimes G(V) &\xrightarrow{\eta_{G(U) \otimes G(V)}} GF[G(U) \otimes G(V)] \xrightarrow{G[\phi_2^{-1}(GU, GV)]} G[FG(U) \otimes' FG(V)] \\ &\xrightarrow{G(\epsilon_U \otimes' \epsilon_V)} G(U \otimes' V). \end{aligned}$$

Then (G, γ_0, γ_2) defines on G a structure of a monoidal functor between the categories \mathcal{M}' and \mathcal{M} . Moreover this is the unique monoidal structure on G such that ϵ and η are monoidal isomorphisms.

Proof. See [SR, Proposition 4.4.2] and [Sch3, Section 2]. \square

The following proposition states that the image of an algebra (resp. coalgebra) through a monoidal functor carries a natural algebra (resp. coalgebra) structure.

Proposition 1.5. *Let \mathcal{M} and \mathcal{M}' be monoidal categories. Let (F, ϕ_0, ϕ_2) be a monoidal functor between the categories \mathcal{M} and \mathcal{M}' . Then:*

1) *If (A, m, u) is an algebra in \mathcal{M} , then $(F(A), m_{F(A)}, u_{F(A)})$ is an algebra in \mathcal{M}' , where*

$$\begin{aligned} m_{F(A)} &:= F(A) \otimes' F(A) \xrightarrow{\phi_2(A, A)} F(A \otimes A) \xrightarrow{F(m)} F(A), \\ u_{F(A)} &:= \mathbf{1}' \xrightarrow{\phi_0} F(\mathbf{1}) \xrightarrow{F(u)} F(A). \end{aligned}$$

2) *If (C, Δ, ϵ) is a coalgebra in \mathcal{M} , then $(F(C), \Delta_{F(C)}, \epsilon_{F(C)})$ is a coalgebra in \mathcal{M}' , where*

$$\begin{aligned} \Delta_{F(C)} &:= F(C) \xrightarrow{F(\Delta)} F(C \otimes C) \xrightarrow{\phi_2^{-1}(C, C)} F(C) \otimes' F(C), \\ \epsilon_{F(C)} &:= F(C) \xrightarrow{F(\epsilon)} F(\mathbf{1}) \xrightarrow{\phi_0^{-1}} \mathbf{1}'. \end{aligned}$$

Proof. Follows directly from the definitions. \square

Proposition 1.6. *Let \mathcal{M} and \mathcal{M}' be monoidal categories. Let $\xi : F \rightarrow L$ be a monoidal morphism between two monoidal functors (F, ϕ_0, ϕ_2) and $(L, \lambda_0, \lambda_2)$, where $F, L : (\mathcal{M}, \otimes, \mathbf{1}) \rightarrow (\mathcal{M}', \otimes', \mathbf{1}')$. We have that:*

1) *if A is an algebra in \mathcal{M} , then $\xi_A : F(A) \rightarrow L(A)$ is an algebra homomorphism (where $F(A)$ and $L(A)$ carry the algebra structures induced by F and L);*

2) *if C is a coalgebra in \mathcal{M} , then $\xi_C : F(C) \rightarrow L(C)$ is a coalgebra homomorphism (where $F(C)$ and $L(C)$ carry the coalgebra structures induced by F and L).*

Proof. Follows directly from the definitions. \square

Corollary 1.7. *Let \mathcal{M} and \mathcal{M}' be monoidal categories. Let (F, ϕ_0, ϕ_2) be a monoidal functor between the categories \mathcal{M} and \mathcal{M}' , and assume that $F : \mathcal{M} \rightarrow \mathcal{M}'$ is an equivalence of categories. Let $G : \mathcal{M}' \rightarrow \mathcal{M}$ be a right adjoint of F , and denote by $\epsilon : FG \rightarrow Id_{\mathcal{M}'}$ the counit and by $\eta : Id_{\mathcal{M}} \rightarrow GF$ the unit of the adjunction. Endow G with the monoidal functor structure (G, γ_0, γ_2) as in Proposition 1.4. We have that:*

1) *if A' is an algebra (resp. coalgebra) in \mathcal{M}' , then $\epsilon_{A'} : FG(A') \rightarrow A'$ is an algebra (resp. coalgebra) isomorphism, where $FG(A')$ carries the algebra (resp. coalgebra) structure induced by FG ;*

2) *if A is an algebra (resp. coalgebra) in \mathcal{M} , then $\eta_A : A \rightarrow GF(A)$ is an algebra (resp. coalgebra) isomorphism, where $GF(A)$ carries the algebra (resp. coalgebra) structure induced by GF .*

Proof. Apply Proposition 1.4 and Proposition 1.6. \square

1.8. Let H be a Hopf algebra with antipode S over a field K . The celebrated result by Sweedler establishes that $F : \mathfrak{M}_K \longrightarrow \mathfrak{M}_H^H$ is an equivalence of categories, where for every $V \in \mathfrak{M}_K$

$$F(V) = V \otimes H.$$

The right adjoint of F is $G : \mathfrak{M}_H^H \longrightarrow \mathfrak{M}_K$, which is defined by

$$G(M) = M^{coH} := \{x \in M \mid \rho_M(x) = x \otimes 1_H\}.$$

Let $\epsilon : FG \rightarrow \text{Id}_{\mathfrak{M}_H^H}$ be the counit of the adjunction (F, G) and ϵ^{-1} its inverse. Then, for every $M \in \mathfrak{M}_H^H$, we have

$$(5) \quad \epsilon_M : M^{Co(H)} \otimes H \rightarrow M, \quad \epsilon_M(v \otimes h) = vh,$$

$$(6) \quad \epsilon_M^{-1} : M \rightarrow M^{Co(H)} \otimes H, \quad \epsilon_M^{-1}(x) = \sum x_{(0)} Sx_{(1)} \otimes x_{(2)}.$$

Let $\eta : \text{Id}_{\mathfrak{M}_K} \rightarrow GF$ be the unit of the adjunction (F, G) and η^{-1} its inverse. Then, for $V \in \mathfrak{M}_K$, we have

$$(7) \quad \eta_V : V \rightarrow (V \otimes H)^{Co(H)}, \quad \eta_V(v) = v \otimes 1_H,$$

$$(8) \quad \eta_V^{-1} : (V \otimes H)^{Co(H)} \rightarrow V, \quad \eta_V^{-1}(\sum v_i \otimes h_i) = \sum v_i \varepsilon_H(h_i).$$

1.9. It is well known that this equivalence induces a monoidal category equivalence

$$({}_H\mathfrak{M}, \otimes, K) \xrightarrow{F} ({}_H\mathfrak{M}_H^H, \otimes_H, H) \xrightarrow{G} ({}_H\mathfrak{M}, \otimes, K)$$

between $({}_H\mathfrak{M}, \otimes, K)$ and the category $({}_H\mathfrak{M}_H^H, \otimes_H, H)$, where, for $M \in {}_H\mathfrak{M}_H^H$, the left H -module structure of $G(M) = M^{Co(H)}$ is given by the left adjoint action

$$(9) \quad {}^h v = \sum h_{(1)} v S h_{(2)}.$$

Conversely, if $V \in {}_H\mathfrak{M}$, then $F(V) = V \otimes H$ becomes an object in ${}_H\mathfrak{M}_H^H$ with the canonical right structures (coming from H) and with diagonal left action

$$(10) \quad {}^h(x \otimes l) = \sum {}^{h_{(1)}}x \otimes h_{(2)}l, \quad \forall x \in W, \quad \forall h, l \in H.$$

The counit $\epsilon : FG \rightarrow \text{Id}_{\mathfrak{M}_H^H}$, the unit $\eta : \text{Id}_{\mathfrak{M}_K} \rightarrow GF$ and their inverses are the same. The monoidal functor structure (F, ϕ_0, ϕ_2) of F is given by

$$\begin{aligned} \phi_0 & : H \rightarrow F(K) : h \mapsto 1 \otimes h, \\ \phi_2(U, V) & : F(U) \otimes_H F(V) \rightarrow F(U \otimes V) : (u \otimes h) \otimes_H (v \otimes l) \\ & \mapsto \sum u \otimes h_{(1)} v \otimes h_{(2)} l, \end{aligned}$$

while their inverses are given by

$$\begin{aligned} \phi_0^{-1} & : F(K) \rightarrow H : k \otimes h \mapsto kh, \\ \phi_2^{-1}(U, V) & : F(U \otimes V) \rightarrow F(U) \otimes_H F(V) : u \otimes v \otimes l \mapsto (u \otimes 1_H) \otimes_H (v \otimes l). \end{aligned}$$

We endow G with the monoidal functor structure (G, γ_0, γ_2) as in Proposition 1.4:

$$\begin{aligned} \gamma_0 & : K \rightarrow G(H) : k \mapsto k1_H, \\ \gamma_2(M, N) & : G(M) \otimes G(N) \rightarrow G(M \otimes_H N) : x \otimes y \mapsto x \otimes_H y, \end{aligned}$$

while their inverses are given by

$$\begin{aligned}\gamma_0^{-1} &: G(H) \rightarrow K : h \mapsto \varepsilon_H(h), \\ \gamma_2^{-1}(U, V) &: G(M \otimes_H N) \rightarrow G(M) \otimes G(N) : \sum x_i \otimes_H y_i \\ &\mapsto \sum x_{i\langle 0 \rangle} Sx_{i\langle 1 \rangle} \otimes x_{i\langle 2 \rangle} y_i.\end{aligned}$$

1.10. It is also well known that the Sweedler equivalence induces a monoidal category equivalence

$$({}^H\mathfrak{M}, \otimes, K) \xrightarrow{F} ({}^H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H\mathfrak{M}, \otimes, K)$$

between $({}^H\mathfrak{M}, \otimes, K)$ and the category $({}^H\mathfrak{M}_H^H, \square_H, H)$, where, for $M \in {}^H\mathfrak{M}_H^H$, the left H -comodule structure of $G(M) = M^{Co(H)}$ is given by the restriction of the left comodule structure of M

$$(11) \quad \rho = \rho_M^l|_{M^{Co(H)}}.$$

Conversely if $V \in {}^H\mathfrak{M}$, then $F(V) = V \otimes H$ becomes an object in ${}^H\mathfrak{M}_H^H$ with the canonical right structures (coming from H) and with diagonal left coaction:

$$(12) \quad \rho^l(x \otimes h) = \sum x_{\langle -1 \rangle} h_{\langle 1 \rangle} \otimes x_{\langle 0 \rangle} \otimes h_{\langle 2 \rangle}, \quad \forall x \in W, \quad \forall h \in H.$$

The counit $\epsilon : FG \rightarrow \text{Id}_{{}^H\mathfrak{M}_H^H}$, the unit $\eta : \text{Id}_{{}^H\mathfrak{M}} \rightarrow GF$ and their inverses are the same. The monoidal functor structure (F, ϕ_0, ϕ_2) of F is given by

$$\begin{aligned}\phi_0 &: H \rightarrow F(K) : h \mapsto 1 \otimes h, \\ \phi_2(U, V) &: F(U) \square_H F(V) \rightarrow F(U \otimes V) : (u \otimes h) \square_H (v \otimes l) \mapsto u \otimes \varepsilon_H(h) v \otimes l,\end{aligned}$$

while their inverses are given by

$$\begin{aligned}\phi_0^{-1} &: F(K) \rightarrow H : k \otimes h \mapsto kh, \\ \phi_2^{-1}(U, V) &: F(U \otimes V) \rightarrow F(U) \square_H F(V) : u \otimes v \otimes l \\ &\mapsto \sum (u \otimes v_{\langle -1 \rangle} l_{\langle 1 \rangle}) \square_H (v_{\langle 0 \rangle} \otimes l_{\langle 2 \rangle}).\end{aligned}$$

We endow G with the monoidal functor structure (G, γ_0, γ_2) as in Proposition 1.4:

$$\begin{aligned}\gamma_0 &: K \rightarrow G(H) : k \mapsto k1_H, \\ \gamma_2(M, N) &: G(M) \otimes G(N) \rightarrow G(M \square_H N) : x \otimes y \mapsto \sum xy_{\langle -1 \rangle} \square_H y_{\langle 0 \rangle},\end{aligned}$$

while their inverses are given by

$$\begin{aligned}\gamma_0^{-1} &: G(H) \rightarrow K : h \mapsto \varepsilon_H(h), \\ \gamma_2^{-1}(U, V) &: G(M \square_H N) \rightarrow G(M) \otimes G(N) : \sum x_i \square_H y_i \\ &\mapsto \sum x_{i\langle 0 \rangle} Sx_{i\langle 1 \rangle} \otimes y_i.\end{aligned}$$

1.11. The most remarkable result (see [Sch1] and [AD]) is that the Sweedler equivalence gives rise to a monoidal category equivalence

$$({}_H^H\mathcal{YD}, \otimes, K) \xrightarrow{F} ({}_H^H\mathfrak{M}_H^H, \otimes_H, H) \xrightarrow{G} ({}_H^H\mathcal{YD}, \otimes, K)$$

between the category of Yetter–Drinfeld modules $({}_H^H\mathcal{YD}, \otimes, K)$ and the category $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$. The structures making $G(M) = M^{Co(H)}$ a left Yetter–Drinfeld module are the left adjoint action (9) and the restriction of the left comodule structure of M (11).

Conversely, if $V \in {}^H_H\mathcal{YD}$, then $F(V) = V \otimes H$ becomes an object in ${}^H_H\mathfrak{M}_H^H$ with the canonical right structures (coming from H) and with diagonal left action (10) and coaction (12).

The counit $\epsilon : FG \rightarrow \text{Id}_{{}^H_H\mathfrak{M}_H^H}$, the unit $\eta : \text{Id}_{{}^H_H\mathcal{YD}} \rightarrow GF$ and their inverses are the same as in 1.10. The functors F and G carry the same monoidal functor structures discussed in 1.9.

1.12. Sweedler equivalence gives rise to another monoidal category equivalence (see [Sch1] and [AD])

$$({}^H_H\mathcal{YD}, \otimes, K) \xrightarrow{F} ({}^H_H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H_H\mathcal{YD}, \otimes, K)$$

between the category of Yetter–Drinfeld modules $({}^H_H\mathcal{YD}, \otimes, K)$ and the category $({}^H_H\mathfrak{M}_H^H, \square_H, H)$. As an equivalence this is the same of 1.11. The functors F and G carry the same monoidal functor structures discussed in 1.10.

1.13. In an analogous way the category of Yetter–Drinfeld modules \mathcal{YD}_H^H can be introduced, and one has (see [Sch1] and [AD]) a monoidal category equivalence between ${}^H_H\mathfrak{M}_H^H$ and \mathcal{YD}_H^H .

1.14. Let us remark that H can be regarded as an object in ${}^H_H\mathcal{YD}$ in two different ways, namely:

$$\begin{aligned} (13) \quad \mu_H^l &= m, \quad \rho_H^l(h) = \sum h_{(1)} Sh_{(3)} \otimes h_{(2)}, \\ (14) \quad \bar{\mu}_H^l(h \otimes x) &= \sum h_{(1)} x Sh_{(2)}, \quad \bar{\rho}_H^l(h) = \Delta. \end{aligned}$$

2. INTEGRALS VERSUS (CO)SEPARABILITY IN SOME MONOIDAL CATEGORIES

In this section we relate the concept of semisimple and separable algebra in the categories of (bi)comodules over H , and cosemisimple and coseparable coalgebra in the categories of (bi)modules over H by means of some suitable integrals. This will allow us to apply the results that we obtained in the previous section.

For future reference, let us recall the definition of integrals.

Definition 2.1. Let H be a Hopf algebra.

a) An element $t \in H$ is called a *left (resp. right) integral in H* if $ht = \varepsilon(h)t, \forall h \in H$ (resp. $th = \varepsilon(h)t, \forall h \in H$).

b) An element $\lambda \in H^*$ is a *left (resp. right) integral in H^** if $\sum h_{(1)} \lambda(h_{(2)}) = \lambda(h)1_H, \forall h \in H$ (resp. $\sum \lambda(h_{(1)})h_{(2)} = \lambda(h)1_H, \forall h \in H$).

Lemma 2.2. Let H be a Hopf algebra. Let $t \in H$ and let $\lambda \in H^*$. Then:

1) t is a left integral in H if and only if

$$\sum ht_{(1)} \otimes St_{(2)} = \sum t_{(1)} \otimes St_{(2)}h, \forall h \in H.$$

2) t is a right integral in H if and only if

$$\sum St_{(1)} \otimes t_{(2)}h = \sum hSt_{(1)} \otimes t_{(2)}, \forall h \in H.$$

3) λ is a left integral in H^* if and only if

$$\sum x_{(1)} \lambda(x_{(2)} Sh) = \sum \lambda(x Sh_{(1)}) h_{(2)}, \forall h, x \in H.$$

4) λ is a right integral in H^* if and only if

$$\sum \lambda(Shx_{(1)}) x_{(2)} = \sum h_{(1)} \lambda(Sh_{(2)}x), \forall h, x \in H.$$

Proof. 1) Assume that $ht = \varepsilon(h)t, \forall h \in H$. Then

$$\begin{aligned} \sum ht_1 \otimes S(t_2) &= \sum h_1 t_1 \otimes S(h_2 t_2) h_3 = \sum \varepsilon_H(h_1) t_1 \otimes S(t_2) h_2 \\ &= \sum t_1 \otimes S(t_2) h, \forall h \in H. \end{aligned}$$

Conversely, by applying $H \otimes \varepsilon$ to this equality, we get $ht = \varepsilon(h)t, \forall h \in H$.

2) Assume that $th = \varepsilon(h)t, \forall h \in H$. Then

$$\begin{aligned} \sum St_{(1)} \otimes t_{(2)} h &= \sum h_{(1)} S(t_{(1)} h_{(2)}) \otimes t_{(2)} h_{(3)} = \sum h_{(1)} St_{(1)} \otimes t_{(2)} \varepsilon(h_{(2)}) \\ &= \sum h St_{(1)} \otimes t_{(2)}, \forall h \in H. \end{aligned}$$

Conversely, by applying $\varepsilon \otimes H$ to this equality, we get $th = \varepsilon(h)t, \forall h \in H$.

3) Assume that $\sum h_{(1)} \lambda(h_{(2)}) = \lambda(h)1_H, \forall h \in H$. Then

$$\begin{aligned} \sum x_{(1)} \lambda(x_{(2)}) Sh &= \sum x_{(1)} Sh_{(2)} \lambda(x_{(2)} Sh_{(1)}) h_{(3)} \\ &= \sum x_{(1)} (Sh_{(1)})_1 \lambda[x_{(2)} (Sh_{(1)})_2] h_{(2)} \\ &= \sum \lambda(x Sh_{(1)}) h_{(2)}, \forall h, x \in H. \end{aligned}$$

Conversely, by applying this equality in the case when $h = 1_H$, we get $\sum x_{(1)} \lambda(x_{(2)}) = \lambda(x)1_H, \forall x \in H$.

4) Assume that $\sum \lambda(h_{(1)}) h_{(2)} = \lambda(h)1_H, \forall h \in H$. Then

$$\begin{aligned} \sum \lambda(Shx_{(1)}) x_{(2)} &= \sum h_{(1)} \lambda[Sh_{(3)} x_{(1)}] Sh_{(2)} x_{(2)} \\ &= \sum h_{(1)} \lambda[(Sh_{(2)})_{(1)} x_{(1)}] (Sh_{(2)})_{(2)} x_{(2)} \\ &= \sum h_{(1)} \lambda(Sh_{(2)} x), \forall h, x \in H. \end{aligned}$$

Conversely, by applying this equality in the case when $h = 1_H$, we get $\sum \lambda(x_{(1)}) x_{(2)} = \lambda(x)1_H, \forall x \in H$. \square

Lemma 2.3. *Let H be a Hopf algebra. The following are equivalent:*

(1) *The multiplication $m : H \otimes H \rightarrow H$ has a section in ${}_H\mathfrak{M}_H^H$, where $H \otimes H$ is regarded as a right comodule through the diagonal coaction and as a bimodule through the canonical left and right structures coming from H .*

(2) *The counit $\varepsilon : H \rightarrow K$ has a section in ${}_H\mathfrak{M}$.*

(3) *There exists a left integral t in H such that $\varepsilon(t) = 1$.*

Proof. (1) \Leftrightarrow (2) By the quoted equivalence between $({}_H\mathfrak{M}_H^H, \otimes_H, H)$ and $({}_H\mathfrak{M}, \otimes, K)$, using the canonical isomorphisms $\eta_H^{-1} : (H \otimes H)^{coH} \rightarrow H$ (see (8)) and $\eta_K^{-1} : H^{coH} \rightarrow K$, the morphism $m : H \otimes H \rightarrow H$ corresponds to the counit $\varepsilon : H \rightarrow K$.

(2) \Leftrightarrow (3) A K -linear map $\mu : K \rightarrow H$ is uniquely determined by $t := \mu(1)$. Since the left H -action on K is defined by ε , it is easy to see that μ is an H -linear section of ε if and only if t satisfies the conditions from (3). \square

In an analogous way one gets:

Lemma 2.4. *Let H be a Hopf algebra. The following are equivalent:*

(1) *The comultiplication $\Delta : H \rightarrow H \otimes H$ has a retraction in ${}^H\mathfrak{M}_H^H$, where $H \otimes H$ is regarded as a right module through the diagonal action and as a bicomodule through the canonical left and right structures coming from H .*

- (2) The unit $u : K \rightarrow H$ has a retraction in ${}^H\mathfrak{M}$.
 (3) There exists a left integral λ in H^* such that $\lambda(1_H) = 1$.

Definition 2.5. Let H be a Hopf algebra with antipode S over any field K and let $t \in H$.

t will be called an *ad-coinvariant integral* if it satisfies:

- cad1) $ht = \varepsilon_H(h)t = th$ for all $h \in H$ (i.e. t is a left and a right integral in H);
 cad2) $\sum t_{(1)}St_{(3)} \otimes t_{(2)} = 1_H \otimes t$ and $\sum t_{(2)} \otimes St_{(1)}t_{(3)} = t \otimes 1_H$;
 cad3) $\varepsilon_H(t) = 1_K$.

Proposition 2.6. Let H be a Hopf algebra. The following are equivalent:

- (1) The multiplication $m : H \otimes H \rightarrow H$ has a section in ${}^H_H\mathfrak{M}_H^H$, where $H \otimes H$ is regarded as a bicomodule through the diagonal coactions and as a bimodule through the canonical left and right structures coming from H .
 (2) The counit $\varepsilon : H \rightarrow K$ has a section in ${}^H_H\mathcal{YD}$, where H is regarded as an object in ${}^H_H\mathcal{YD}$ as in (13).
 (3) There exists a left integral t in H such that $\sum t_{(1)}St_{(3)} \otimes t_{(2)} = 1_H \otimes t$ and $\varepsilon(t) = 1$.
 (4) The counit $\varepsilon : H \rightarrow K$ has a section in \mathcal{YD}_H^H , where H is regarded as an object in \mathcal{YD}_H^H through the right analogues of the structures (13).
 (5) There exists a right integral t in H such that $\sum t_{(2)} \otimes St_{(1)}t_{(3)} = t \otimes 1_H$ and $\varepsilon(t) = 1$.
 (6) There exists an ad-coinvariant integral $t \in H$.

Moreover, if these conditions are satisfied, an element $t \in H$ satisfies (3) iff it satisfies (5) iff it is an ad-coinvariant integral. Such an element is unique.

Proof. (1) \Leftrightarrow (2) Through the quoted equivalence between $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and $({}^H_H\mathcal{YD}, \otimes, K)$, using the canonical isomorphisms $\eta_H^{-1} : (H \otimes H)^{coH} \rightarrow H$ (see (8)), where H is regarded as an object in ${}^H_H\mathcal{YD}$ as in (13) and $\eta_K^{-1} : H^{coH} \rightarrow K$, the morphism $m : H \otimes H \rightarrow H$ corresponds to the counit $\varepsilon : H \rightarrow K$.

(2) \Leftrightarrow (3) Since the structure of the left H -module of K is the one defined by ε , (2) \Leftrightarrow (3) follows by a direct computation.

(1) \Leftrightarrow (4) and (4) \Leftrightarrow (5) follow in an analogous way.

(6) \Rightarrow (3) is trivial.

(3) \Rightarrow (6) Let $t \in H$ be an element as in (3). Since (3) and (5) are equivalent, there is an $l \in H$ as in (5). Since

$$t = \varepsilon(l)t = lt = l\varepsilon(t) = l,$$

it follows that t is an ad-coinvariant integral. The uniqueness of ad-coinvariant integrals is obvious. \square

Definition 2.7. Let H be a Hopf algebra with antipode S over a field K and let $\lambda \in H^*$. λ will be called an *ad-invariant integral* if it satisfies:

- ad1) $\sum h_{(1)}\lambda(h_{(2)}) = 1_H\lambda(h) = \sum \lambda(h_{(1)})h_{(2)}, \forall h \in H$ (i.e. λ is a left and right integral in H^*);
 ad2) $\sum \lambda(h_{(1)}xSh_{(2)}) = \varepsilon(h)\lambda(x) = \sum \lambda(Sh_{(1)}xh_{(2)}), \forall h, x \in H$;
 ad3) $\lambda(1_H) = 1_K$.

Proposition 2.8. Let H be a Hopf algebra. The following are equivalent:

- (1) The comultiplication $\Delta : H \rightarrow H \otimes H$ has a retraction in ${}^H_H\mathfrak{M}_H^H$, where $H \otimes H$ is regarded as a bimodule through the diagonal actions and as a bicomodule through the canonical left and right structures coming from H .

(2) The unit $u : K \rightarrow H$ has a retraction in ${}^H_H\mathcal{YD}$, where H is regarded as an object in ${}^H_H\mathcal{YD}$ as in (14).

(3) There exists a left integral λ in H^* such that $\sum \lambda(h_{(1)}xSh_{(2)}) = \varepsilon(h)\lambda(x)$, $\forall h, x \in H$ and $\lambda(1_H) = 1_K$.

(4) The unit $u : K \rightarrow H$ has a section in \mathcal{YD}_H^H , where H is regarded as an object in \mathcal{YD}_H^H through the right analogues of the structures (14).

(5) There exists a right integral λ in H^* such that $\sum \lambda(Sh_{(1)}xh_{(2)}) = \varepsilon(h)\lambda(x)$, $\forall h, x \in H$ and $\lambda(1_H) = 1_K$.

(6) There exists an ad-invariant integral $\lambda \in H^*$.

Moreover, if these conditions are satisfied, an element $\lambda \in H^*$ satisfies (3) iff it satisfies (5) iff it is an ad-invariant integral. Such an element is unique.

Proof. (1) \Leftrightarrow (2) Through the quoted equivalence between $({}^H_H\mathfrak{M}_H^H, \square_H, H)$ and $({}^H_H\mathcal{YD}, \otimes, K)$, using the canonical isomorphisms $\eta_H^{-1} : (H \otimes H)^{coH} \rightarrow H$ (see (8)), where H is regarded as an object in ${}^H_H\mathcal{YD}$ as in (14) and $\eta_K^{-1} : H^{coH} \simeq K$, the morphism $\Delta : H \rightarrow H \otimes H$ corresponds to the unit $u : K \rightarrow H$.

(2) \Leftrightarrow (3) Since the structure of the left H -module of K is the one defined by ε , (2) \Leftrightarrow (3) follows by a direct computation.

(1) \Leftrightarrow (4) and (4) \Leftrightarrow (5) follow in an analogous way.

(6) \Rightarrow (3) is trivial.

(3), (5) \Rightarrow (6) Let $\lambda \in H^*$ as in (3) and $\gamma \in H^*$ as in (5). Then we have

$$\lambda = \gamma(1_H)\lambda = \gamma * \lambda = \gamma\lambda(1_H) = \gamma.$$

□

Remark 2.9. A complete treatment of the foregoing results regarding integrals can be found in [Ar1].

Definition 2.10. Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category.

An algebra (A, m, u) in \mathcal{M} is called *separable* if the multiplication $m : A \otimes A \rightarrow A$ has a section in the category of (A, A) -bimodules ${}_A\mathcal{M}_A$.

A coalgebra (C, Δ, ε) in \mathcal{M} is called *coseparable* if the comultiplication $\Delta : C \rightarrow C \otimes C$ has a retraction in ${}^C\mathcal{M}^C$.

Proposition 2.11. Let H be a Hopf algebra.

a) H is separable as an algebra in \mathfrak{M}^H if and only if H is semisimple.

b) H is separable as an algebra in ${}^H\mathfrak{M}^H$ if and only if there is an ad-coinvariant integral $t \in H$.

c) H is coseparable as a coalgebra in \mathfrak{M}_H if and only if H is cosemisimple.

d) H is coseparable as a coalgebra in ${}_H\mathfrak{M}_H$ if and only if there is an ad-invariant integral $\lambda \in H^*$.

Proof. To prove a) we remark that the category of (H, H) -bimodules in \mathfrak{M}^H is ${}_H\mathfrak{M}_H^H$. Then the conclusion follows by Lemma 2.3 and by Maschke's Theorem (see [Mo]).

To prove b) we remark that the category of (H, H) -bimodules in ${}^H\mathfrak{M}^H$ is ${}^H_H\mathfrak{M}_H^H$. Then the conclusion follows by Proposition 2.6.

To prove c) we remark that the category of (H, H) -bicomodules in \mathfrak{M}_H is ${}^H_H\mathfrak{M}_H^H$. Then the conclusion follows by Lemma 2.4 and by Dual Maschke's Theorem (see [Mo]).

To prove d) we remark that the category of (H, H) -bicomodules in ${}_H\mathfrak{M}_H$ is ${}^H_H\mathfrak{M}_H^H$. Then the conclusion follows by Proposition 2.8. □

In [AMS], we proved the following theorem.

Theorem 2.12. *Let (A, m, u) be a separable algebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ such that both the functors $A \otimes (-) : \mathcal{M} \rightarrow \mathcal{M}$ and $(-) \otimes A : \mathcal{M} \rightarrow \mathcal{M}$ are additive and right exact. Let $\pi : E \rightarrow A$ be an algebra homomorphism and let I denote the kernel of π . Assume that:*

- 1) π is an epimorphism;
- 2) there is $n \in \mathbb{N}$ such that $I^n = 0$;
- 3) for any $r = 1, \dots, n-1$, the canonical projection $p_r : E/I^{r+1} \rightarrow E/I^r$ splits in \mathcal{M} .

Then π has a section which is an algebra homomorphism.

Theorem 2.13. *Let H be a Hopf algebra. Let \mathcal{M} be either the monoidal category \mathfrak{M}^H or ${}^H\mathfrak{M}^H$. Suppose that $\pi : A \rightarrow H$ is a surjective morphism of algebras in \mathcal{M} such that $\text{Ker } \pi$ is the Jacobson radical J of A and J is nilpotent.*

- a) *Let $\mathcal{M} = \mathfrak{M}^H$. Assume that H is a semisimple Hopf algebra. Then $\pi : A \rightarrow A/J \simeq H$ has a section σ in \mathfrak{M}^H which is an algebra map.*
- b) *Let $\mathcal{M} = {}^H\mathfrak{M}^H$. Assume that H is an ad-coinvariant integral and that every canonical map $A/J^{n+1} \rightarrow A/J^n$ splits in ${}^H\mathfrak{M}^H$. Then $\pi : A \rightarrow H$ has a section σ in ${}^H\mathfrak{M}^H$ which is an algebra map.*

Proof. a) The Jacobson radical J of A is an H -subcomodule of A since π is a morphism of H -comodules. Hence, for every $n > 0$, J^n is also a subcomodule of A such that the canonical map $A/J^{n+1} \rightarrow A/J^n$ is H -colinear. Furthermore, J^n/J^{n+1} has a natural module structure over $A/J \simeq H$, and with respect to this structure J^n/J^{n+1} is an object in \mathfrak{M}_H^H . Hence J^n/J^{n+1} is a cofree right comodule (i.e. $J^n/J^{n+1} \simeq V \otimes H$). In particular J^n/J^{n+1} is an injective comodule. Thus the canonical map $A/J^{n+1} \rightarrow A/J^n$ has a section in \mathfrak{M}^H . By Proposition 2.11, we know that H is separable as an algebra in \mathfrak{M}^H so that we can apply Theorem 2.12.

b) We first remark that J^n is an (H, H) -subbicomodule of A and that the canonical maps $A/J^{n+1} \rightarrow A/J^n$ are morphisms of bicomodules. By Proposition 2.11, it results that H is separable in ${}^H\mathfrak{M}^H$, so we conclude by applying Theorem 2.12. \square

Corollary 2.14. *Let A be a Hopf algebra such that J , the Jacobson radical of A , is a nilpotent coideal in A . Let $H := A/J$, and let $\pi : A \rightarrow H$ be the canonical projection.*

- a) *If H is semisimple, then there is an algebra morphism in \mathfrak{M}^H that is a section of π .*
- b) *If H has an ad-coinvariant integral and every canonical map $A/J^{n+1} \rightarrow A/J^n$ splits in ${}^H\mathfrak{M}^H$, then there is an algebra morphism in ${}^H\mathfrak{M}^H$ that is a section of π .*
- c) *If H has an ad-coinvariant integral and any object in ${}^H_H\mathfrak{M}_H^H$ is injective as an (H, H) -bicomodule, then there is a section of π as in b).*

Proof. The first two assertions follow directly from the previous theorem, since we can regard A both as an algebra in \mathfrak{M}^H and as an algebra in ${}^H\mathfrak{M}^H$, π being a morphism of bialgebras.

Let us prove c). In view of b) it is enough to show that the canonical epimorphisms $A/J^{n+1} \rightarrow A/J^n$ split in ${}^H\mathfrak{M}^H$. Since A/J^n is an object in ${}^H\mathfrak{M}^H$

and the canonical epimorphism $A/J^{n+1} \rightarrow A/J^n$ is a morphism in ${}^H\mathfrak{M}^H$, it follows that $J^n/J^{n+1} \in {}^H\mathfrak{M}_H^H$, so it is an injective (H, H) -bicomodule. Therefore $A/J^{n+1} \rightarrow A/J^n$ has a section in ${}^H\mathfrak{M}^H$. \square

2.15. Let E be a coalgebra in a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. Let us recall (see [Mo, §5.2]) the definition of a wedge product of two subobjects X, Y of E in \mathcal{M} :

$$X \wedge_E Y := \text{Ker}[(\pi_X \otimes \pi_Y) \circ \Delta_E],$$

where $\pi_X : E \rightarrow E/X$ and $\pi_Y : E \rightarrow E/Y$ are the canonical quotient maps.

For the following theorem the reader is referred to [AMS].

Theorem 2.16. *Let (C, Δ, ε) be a coseparable coalgebra in an abelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ endowed with denumerable direct sums and such that both the functors $C \otimes (-) : \mathcal{M} \rightarrow \mathcal{M}$ and $(-) \otimes C : \mathcal{M} \rightarrow \mathcal{M}$ are additive and left exact. Let $\sigma : C \rightarrow E$ be coalgebra homomorphism. Assume that:*

- 1) σ is a monomorphism;
- 2) $\varinjlim C^{\wedge_E^n} = E$;
- 3) for any $r \in \mathbb{N}$ the canonical injection $i_r : C^{\wedge_E^r} \rightarrow C^{\wedge_E^{r+1}}$ cosplits in \mathcal{M} .

Then σ has a retraction which is a coalgebra homomorphism.

Theorem 2.17. *Let H be a Hopf algebra.*

a) *Let C be a coalgebra in \mathfrak{M}_H . If the coradical C_0 of C is H , then there is a coalgebra map $\pi_C : C \rightarrow H$ which is a morphism in \mathfrak{M}_H such that $\pi_C|_H = \text{Id}_H$.*

b) *Let C be a coalgebra in ${}_H\mathfrak{M}_H$. If $C_0 = H$, H has an ad -invariant integral and every C_n is a direct summand in C_{n+1} as an object in ${}_H\mathfrak{M}_H$, then there is a coalgebra map $\pi_C : C \rightarrow H$ which is a morphism in ${}_H\mathfrak{M}_H$ such that $\pi_C|_H = \text{Id}_H$.*

Proof. Let \mathcal{M} be one of the categories \mathfrak{M}_H or ${}_H\mathfrak{M}_H$. Let $\sigma : C_0 \rightarrow C$ be the canonical inclusion. Let us consider the coradical filtration $(C_n)_{n \in \mathbb{N}}$:

$$(15) \quad C_{n+1} = \{x \in C \mid \Delta(x) \in C \otimes C_n + C_0 \otimes C\} = (C_0)^{\wedge_C^{n+1}},$$

for every $n \geq 0$. Moreover we have $\varinjlim C_n = \bigcup C_n = C$.

a) Since $H = C_0$ is cosemisimple, by Proposition 2.11 it is coseparable in \mathfrak{M}_H , so that we can apply Theorem 2.16 in the case when $\mathcal{M} = \mathfrak{M}_H$.

In fact, by (15), C_{n+1}/C_n becomes a right $H = C_0$ -comodule with the structure induced by Δ . Hence C_{n+1}/C_n is an object in \mathfrak{M}_H^H , so it is free as a right H -module (by the fundamental theorem for Hopf modules). In conclusion the inclusion $C_n \subseteq C_{n+1}$ has a retraction in \mathfrak{M}_H .

b) By Proposition 2.11, H is coseparable in ${}_H\mathfrak{M}_H$ and moreover by assumption C_n is a direct summand of C_{n+1} as an object in ${}_H\mathfrak{M}_H$. Hence we can apply Theorem 2.16 in the case when $\mathcal{M} = {}_H\mathfrak{M}_H$. \square

Corollary 2.18. *Let C be a Hopf algebra such that C_0 , the coradical of C , is a Hopf subalgebra. Let $H := C_0$ and let $\sigma : H \rightarrow C$ be the canonical injection.*

a) *Since H is cosemisimple there is a coalgebra morphism in \mathfrak{M}_H that is a retraction of σ .*

b) *If H has an ad -invariant integral and every canonical map $C_n \rightarrow C_{n+1}$ cosplits in ${}_H\mathfrak{M}_H$, then there is a coalgebra morphism in ${}_H\mathfrak{M}_H$ that is a retraction of σ .*

c) *If H has an ad -invariant integral and any object in ${}_H\mathfrak{M}_H^H$ is projective as an (H, H) -bimodule, then there is a retraction of σ as in b).*

Proof. Since σ is a morphism of bialgebras, we can regard C both as a coalgebra in \mathfrak{M}_H and as a coalgebra in ${}^H\mathfrak{M}_H$ so that the first two assertions follow directly from the previous theorem.

Let us prove c). In view of b), it is enough to show that the canonical monomorphisms $C_n \rightarrow C_{n+1}$ split in ${}^H\mathfrak{M}_H$. Since C_n is an object in ${}^H\mathfrak{M}_H$ and the canonical monomorphism $C_n \rightarrow C_{n+1}$ is a morphism in ${}^H\mathfrak{M}_H$, it follows that $C_{n+1}/C_n \in {}^H_H\mathfrak{M}_H^H$, so it is a projective (H, H) -bimodule. Therefore $C_n \rightarrow C_{n+1}$ has a retraction in ${}^H\mathfrak{M}_H$. \square

Remarks 2.19. a) A. Masuoka informed us that the first statement of Theorem 2.17 follows easily from [Mas, Theorem 4.1].

b) Statement a) in Corollary 2.18 has already been proved by Masuoka; see [Mas, Theorem 3.1].

2.20. Let H be a Hopf algebra. By definition, an algebra A in ${}^H\mathfrak{M}^H$ is separable if and only if the multiplication $m : A \otimes A \rightarrow A$ has a section $\sigma : A \rightarrow A \otimes A$ which is a morphism of (A, A) -bimodules and (H, H) -comodules. Obviously, then A is separable as an algebra in \mathfrak{M}_K , but the converse does not hold in general. Nevertheless, if the forgetful functor $U : {}^H_A\mathfrak{M}_A^H \rightarrow {}_A\mathfrak{M}_A$ is separable, then A is separable as an algebra in $\mathcal{M} = {}^H\mathfrak{M}^H$. Before proving this result, let us recall the definition and basic properties of *separable functors*.

2.21. A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is called separable if, for all objects $C_1, C_2 \in \mathfrak{C}$, there is a map $\mathcal{P}_{C_1, C_2} : \text{Hom}_{\mathfrak{D}}(FC_1, FC_2) \rightarrow \text{Hom}_{\mathfrak{C}}(C_1, C_2)$ such that:

S1) For all morphisms $f \in \text{Hom}_{\mathfrak{C}}(C_1, C_2)$, $\mathcal{P}_{C_1, C_2}(F(f)) = f$.

S2) We have $\mathcal{P}_{C_1, C_2}(l) \circ f = g \circ \mathcal{P}_{C_1, C_2}(h)$ for every commutative diagram in \mathfrak{D} of type:

$$\begin{array}{ccc} F(C_1) & \xrightarrow{h} & F(C_2) \\ F(f) \downarrow & & \downarrow F(g) \\ F(C_3) & \xrightarrow{l} & F(C_4) \end{array}$$

Lemma 2.22. *Let $F : \mathfrak{C} \rightarrow \mathfrak{D}$ be a covariant separable functor and let $\alpha : X \rightarrow Y$ be a morphism in \mathfrak{C} . If $F(\alpha)$ has a section h (resp. a retraction l) in \mathfrak{D} , then α has a section (retraction) in \mathfrak{C} .*

Proof. It is sufficient to apply property S2) in the case when $g = \alpha, l = \text{Id}_{FY}, f = \text{Id}_Y$. Since $F(\alpha) \circ h = \text{Id}_{FY} \circ F(\text{Id}_Y)$, by S2) we get $\mathcal{P}_{Y, X}(\text{Id}_{FY}) \circ \text{Id}_Y = \alpha \circ \mathcal{P}_{Y, X}(h)$, so that, as $\text{Id}_{FY} = F(\text{Id}_Y)$, by S1) we conclude the proof. The dual case follows analogously applying properties S2) and S1) in the case when $f = \alpha, h = \text{Id}_{FX}, g = \text{Id}_X$. \square

We quote from [Raf] the so-called Rafael Theorem.

Theorem 2.23 (cf. [Raf]). *Let (T, U) be an adjunction, where $T : \mathfrak{C} \rightarrow \mathfrak{D}$ and $U : \mathfrak{D} \rightarrow \mathfrak{C}$. Then we have:*

1) *T is separable iff the unit $\eta : \text{Id}_{\mathfrak{C}} \rightarrow UT$ of the adjunction cosplits, i.e. there exists a natural transformation $\mu : UT \rightarrow \text{Id}_{\mathfrak{C}}$ such that $\mu \circ \eta = \text{Id}_{\text{Id}_{\mathfrak{C}}}$, the identity natural transformation on $\text{Id}_{\mathfrak{C}}$.*

2) *U is separable iff the counit $\epsilon : TU \rightarrow \text{Id}_{\mathfrak{D}}$ of the adjunction splits, i.e. there exists a natural transformation $\sigma : \text{Id}_{\mathfrak{D}} \rightarrow TU$ such that $\epsilon \circ \sigma = \text{Id}_{\text{Id}_{\mathfrak{D}}}$, the identity natural transformation on $\text{Id}_{\mathfrak{D}}$.*

2.24. The forgetful functor $U : {}^H_A\mathfrak{M}_A^H \rightarrow {}_A\mathfrak{M}_A$ has a right adjoint $T : {}_A\mathfrak{M}_A \rightarrow {}^H_A\mathfrak{M}_A^H$. On objects T is defined by $T(M) = H \otimes M \otimes H$, where $T(M)$ is a bicomodule via $\Delta_H \otimes M \otimes H$ and $H \otimes M \otimes \Delta_H$, and $T(M)$ is a bimodule with diagonal actions:

$$\begin{aligned} a(h \otimes m \otimes k) &= \sum a_{\langle -1 \rangle} h \otimes a_{\langle 0 \rangle} m \otimes a_{\langle 1 \rangle} k, \\ (h \otimes m \otimes k) a &= \sum h a_{\langle -1 \rangle} \otimes m a_{\langle 0 \rangle} \otimes k a_{\langle 1 \rangle}. \end{aligned}$$

Here we used the sigma notation: $(\rho_A^l \otimes H) \rho_A^r(a) = \sum a_{\langle -1 \rangle} \otimes a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle}$.

Let $\eta : \text{Id}_{{}^H_A\mathfrak{M}_A^H} \rightarrow TU$ be the unit of this adjunction. It is easy to see that, for any $M \in {}^H_A\mathfrak{M}_A^H$, we have $\eta_M : M \rightarrow H \otimes M \otimes H$, $\eta_M = (\rho_M^l \otimes H) \rho_M^r$.

Proposition 2.25. *Let H be a Hopf algebra. Let A be an H -bicomodule algebra and consider the forgetful functor $U : {}^H_A\mathfrak{M}_A^H \rightarrow {}_A\mathfrak{M}_A$. Assume there exists a left integral λ in H^* such that $\lambda(1_H) = 1$. Then, the morphism*

$$\mu_M : H \otimes M \otimes H \rightarrow M, \quad \mu_M(h \otimes m \otimes k) = \sum \lambda(Sh m_{\langle -1 \rangle}) m_{\langle 0 \rangle} \lambda(m_{\langle 1 \rangle} Sk)$$

is an (H, H) -bilinear morphism such that $\mu_M \circ \eta_M = \text{Id}_M$.

Moreover this gives rise to a functorial morphism $\mu : TU \rightarrow \text{Id}_{{}^H_A\mathfrak{M}_A^H}$.

Proof. Since, by Lemma 2.2, we have $\sum \lambda(Sh x_{(1)}) x_{(2)} = \sum h_{(1)} \lambda(Sh_{(2)} x)$, $\forall h, x \in H$, we obtain

$$\begin{aligned} \rho_M^l(\mu_M(h \otimes m \otimes k)) &= \sum \lambda(Sh m_{\langle -2 \rangle}) m_{\langle -1 \rangle} \otimes m_{\langle 0 \rangle} \lambda(m_{\langle 1 \rangle} Sk) \\ &= \sum h_1 \lambda(Sh_2 m_{\langle -1 \rangle}) \otimes m_{\langle 0 \rangle} \lambda(m_{\langle 1 \rangle} Sk) \\ &= \sum h_{(1)} \otimes \mu_M(h_{(2)} \otimes m \otimes k). \end{aligned}$$

Thus we have shown that μ_M is left H -colinear. Analogously it can be proved that μ_M is right H -colinear. It remains to show that μ_M is a retraction of η_M . In fact, we have

$$(\mu_M \eta_M)(m) = \sum \lambda(Sm_{\langle -2 \rangle} m_{\langle -1 \rangle}) m_{\langle 0 \rangle} \lambda(m_{\langle 1 \rangle} Sm_{\langle 2 \rangle}) = m.$$

It is easy to check that this gives rise to a functorial morphism $\mu : TU \rightarrow \text{Id}_{{}^H_A\mathfrak{M}_A^H}$. \square

Theorem 2.26. *Let H be a Hopf algebra. The following assertions are equivalent:*

- (1) *There exists a left integral λ in H^* such that $\lambda(1_H) = 1$.*
- (2) *The forgetful functor $U : {}^H_K\mathfrak{M}_K^H \rightarrow {}_K\mathfrak{M}_K$ is separable.*
- (3) *Any epimorphism (resp. monomorphism) in ${}^H\mathfrak{M}^H$ splits (cosplits) in ${}^H\mathfrak{M}^H$.*
- (4) *H is coseparable as a coalgebra in \mathfrak{M}_K .*
- (5) *H is coseparable as a coalgebra in \mathfrak{M}_H .*
- (6) *H is cosemisimple.*
- (7) *The unit $u : K \rightarrow H$ has a retraction in ${}^H\mathfrak{M}$.*

Proof. (1) \Rightarrow (2) By Proposition 2.25, the morphism

$$\mu_M : H \otimes M \otimes H \rightarrow M, \quad \mu_M(h \otimes m \otimes k) = \sum \lambda(Sh m_{\langle -1 \rangle}) m_{\langle 0 \rangle} \lambda(m_{\langle 1 \rangle} Sk)$$

is an (H, H) -bilinear morphism such that $\mu_M \circ \eta_M = \text{Id}_M$, and this gives rise to a functorial morphism $\mu : TU \rightarrow \text{Id}_{{}^H_A\mathfrak{M}_A^H}$. In view of Theorem 2.23, the functor U is separable.

(2) \Rightarrow (3) Let p be an epimorphism in ${}^H\mathfrak{M}^H$. Then $U(p)$ splits. By Lemma 2.22, we conclude the proof. Analogously any monomorphism in ${}^H\mathfrak{M}^H$ cosplits in ${}^H\mathfrak{M}^H$.

(3) \Rightarrow (4) The comultiplication $\Delta : H \rightarrow H \otimes H$ is a monomorphism in ${}^H\mathfrak{M}^H$.

(4) \Rightarrow (1) Let $\mu : H \otimes H \rightarrow H$ be an (H, H) -bilinear retraction of the comultiplication Δ , and set $\lambda_\mu := \varepsilon_H \mu(- \otimes 1_H) \in H^*$. Then λ_μ fulfills the conditions of (1).

(5) \Leftrightarrow (6) follows by Proposition 2.11.

(5) \Leftrightarrow (7) and (7) \Leftrightarrow (1) follow by Lemma 2.4. \square

Theorem 2.27. *Let H be a semisimple and cosemisimple Hopf algebra over a field K . Then:*

- 1) *there is an ad -invariant integral $\lambda \in H^*$;*
- 2) *there is an ad -coinvariant integral $t \in H$;*
- 3) *H is separable in ${}^H\mathfrak{M}^H$;*
- 4) *H is coseparable in ${}_H\mathfrak{M}_H$.*

Proof. First let us note that any semisimple Hopf algebra is finite dimensional (see [Mo]).

1) and 2) Since H is semisimple and cosemisimple, by [Ra2, Proposition 7] the Drinfeld double $D(H)$ is semisimple. By a result essentially due to Majid (see [Mo, Proposition 10.6.16]) and by [RT, Proposition 6], we get that the category ${}^H_H\mathcal{YD} \simeq {}_{D(H)}\mathfrak{M}$ is semisimple. Then the counit $\varepsilon : H \rightarrow K$ has a section in \mathcal{YD}_H^H so that, by Proposition 2.6, there is an ad -coinvariant integral. Analogously the unit $u : K \rightarrow H$ has a retraction in ${}^H_H\mathcal{YD}$ so that, by Proposition 2.8, there is an ad -invariant integral.

3) and 4) follow, in view of the foregoing, by Proposition 2.11. \square

Theorem 2.28. *Let A be a Hopf algebra such that J , the Jacobson radical of A , is a nilpotent coideal in A . Let $H := A/J$, and let $\pi : A \rightarrow H$ be the canonical projection. Assume that H is both semisimple and cosemisimple (e.g. H is semisimple over a field of characteristic 0). Then there is an algebra morphism in ${}^H\mathfrak{M}^H$ that is a section of π .*

Proof. In view of Theorem 2.27, if H is semisimple and cosemisimple, then it has an ad -coinvariant and an ad -invariant integral. Then, by Theorem 2.26, every bicomodule is injective. By c) of Corollary 2.14, we conclude the proof. \square

Theorem 2.29. *Let H be a Hopf algebra with an ad -invariant integral λ and let A be an H -bicomodule algebra. Then the forgetful functor $U : {}^H_A\mathfrak{M}_A^H \rightarrow {}_A\mathfrak{M}_A$ is separable.*

Proof. We use the notations of 2.24. Let $\eta : \text{Id}_{{}^H_A\mathfrak{M}_A^H} \rightarrow TU$ be the unit of the adjunction (U, T) . In view of Theorem 2.23, we have to construct a $\mu : TU \rightarrow \text{Id}_{{}^H_A\mathfrak{M}_A^H}$ such that $\mu \circ \eta = \text{Id}_{\text{Id}_{{}^H_A\mathfrak{M}_A^H}}$.

By Proposition 2.25, the morphism

$$\mu_M : H \otimes M \otimes H \rightarrow M, \quad \mu_M(h \otimes m \otimes k) = \sum \lambda(Sh m_{\langle -1 \rangle}) m_{\langle 0 \rangle} \lambda(m_{\langle 1 \rangle} Sk)$$

is a (H, H) -bilinear morphism such that $\mu_M \circ \eta_M = \text{Id}_M$. It is easy to check that this gives rise to a functorial morphism $\mu : TU \rightarrow \text{Id}_{{}^H_A\mathfrak{M}_A^H}$. In order to conclude

that μ_M is a morphism in ${}^H_A\mathfrak{M}_A^H$, it remains only to check that μ_M is a morphism of (A, A) -bimodules. We have

$$\begin{aligned}\mu_M(a(h \otimes m \otimes k)) &= \sum \lambda(ShSa_{(-2)} a_{(-1)} m_{(-1)}) a_{(0)} m_{(0)} \lambda(a_{(1)} m_{(1)} SkSa_{(2)}), \\ ad2) &= \sum \lambda(Shm_{(-1)}) a_{(0)} m_{(0)} \varepsilon(a_{(1)}) \lambda(m_{(1)} Sk) \\ &= a\mu_M(h \otimes m \otimes k).\end{aligned}$$

This relation proves that μ_M is left A -linear. Similarly, using the second equality of $ad2)$, one can show that μ_M is right A -linear. \square

Theorem 2.30. *Let H be a Hopf algebra over a field K and assume that H has an ad -invariant integral. An algebra A in the category ${}^H\mathfrak{M}^H$ is separable iff A is separable as an algebra in $(\mathfrak{M}_K, \otimes_K, K)$, i.e. as a usual algebra.*

Proof. It is enough to prove that if A is separable as an algebra in \mathfrak{M}_K , then it is separable as an algebra in ${}^H\mathfrak{M}^H$. If $m : A \otimes A \rightarrow A$ is the multiplication of the algebra A in the monoidal category ${}^H\mathfrak{M}^H$, then m also defines the multiplication of A as an algebra in \mathfrak{M}_K . By Theorem 2.29, the functor $U : {}^H_A\mathfrak{M}_A^H \rightarrow {}_A\mathfrak{M}_A$ is separable. Since $U(m) = m$ and m has a section in ${}_A\mathfrak{M}_A$ (A is separable in \mathfrak{M}_K), by Lemma 2.22 it follows that m has a section in ${}^H_A\mathfrak{M}_A^H$. \square

Corollary 2.31. *Let H be a semisimple and cosemisimple Hopf algebra over a field K . If A is an algebra in the category ${}^H\mathfrak{M}^H$, then A is separable as an algebra in ${}^H\mathfrak{M}^H$ iff A is separable as an algebra in $(\mathfrak{M}_K, \otimes_K, K)$.*

Proof. By Theorem 2.27, H has a non-zero ad -invariant integral. \square

2.32. The forgetful functor $U : {}^D_H\mathfrak{M}_H^D \rightarrow {}^D\mathfrak{M}^D$ has a left adjoint $T : {}^D\mathfrak{M}^D \rightarrow {}^D_H\mathfrak{M}_H^D$, $T(M) = H \otimes M \otimes H$, where $T(M)$ is a bimodule via $m_H \otimes M \otimes H$ and $H \otimes M \otimes m_H$, and $T(M)$ is a bicomodule with diagonal coactions:

$$\begin{aligned}\rho^l(h \otimes m \otimes k) &= \sum h_{(1)} m_{(-1)} k_{(1)} \otimes h_{(2)} \otimes m_{(0)} \otimes k_{(2)}, \\ \rho^r(h \otimes m \otimes k) &= \sum h_{(1)} \otimes m_{(0)} \otimes k_{(1)} \otimes h_{(2)} m_{(1)} k_{(2)}.\end{aligned}$$

Here we used the sigma notation: $(\rho_A^l \otimes A)\rho_A^r(a) = \sum a_{(-1)} \otimes a_{(0)} \otimes a_{(1)}$. Let $\epsilon : TU \rightarrow \text{Id}_{{}^D_H\mathfrak{M}_H^D}$ be the counit of this adjunction.

For any $M \in {}^D_H\mathfrak{M}_H^D$, we have $\epsilon_M : H \otimes M \otimes H \rightarrow M$, $\epsilon_M = (\mu_M^l \otimes H)\mu_M^r$.

Proposition 2.33. *Let H be a Hopf algebra. Let D be an H -bimodule coalgebra and consider the forgetful functor $U : {}^D_H\mathfrak{M}_H^D \rightarrow {}^D\mathfrak{M}^D$. Assume there exists a left integral t in H such that $\varepsilon(t) = 1$. Then, the morphism*

$$\sigma_M : M \rightarrow H \otimes M \otimes H, \quad \sigma_M(m) = \sum St_{(1)} \otimes t_{(2)} m \tilde{t}_{(1)} \otimes S\tilde{t}_{(2)}$$

is an (H, H) -bilinear morphism such that $\epsilon_M \circ \sigma_M = \text{Id}_M$.

Moreover this gives rise to a functorial morphism $\sigma : \text{Id}_{{}^D_H\mathfrak{M}_H^D} \rightarrow TU$.

Proof. Since, by Lemma 2.2, we have $\sum hSt_{(1)} \otimes t_{(2)} = \sum St_{(1)} \otimes t_{(2)}h, \forall h \in H$, we obtain

$$h\sigma_M(m) = \sum hSt_{(1)} \otimes t_{(2)} m \tilde{t}_{(1)} \otimes S\tilde{t}_{(2)} = \sum St_{(1)} \otimes t_{(2)} hm \tilde{t}_{(1)} \otimes S\tilde{t}_{(2)} = \sigma_M(hm).$$

Thus we have shown that σ_M is left H -linear. Analogously it can be proved that σ_M is right H -linear. It remains to show that σ_M is a section of ϵ_M . In fact, we have

$$(\epsilon_M \sigma_M)(m) = \sum St_{(1)} t_{(2)} m \tilde{t}_{(1)} \tilde{S} \tilde{t}_{(2)} = m.$$

It is easy to check that this gives rise to a functorial morphism $\sigma : \text{Id}_{D_H \mathfrak{M}_H^D} \rightarrow TU$. \square

Theorem 2.34. *Let H be a Hopf algebra. The following assertions are equivalent:*

- (1) *There exists a left integral t in H such that $\varepsilon(t) = 1$.*
- (2) *The forgetful functor $U : {}^K_H \mathfrak{M}_H^K \rightarrow {}^K \mathfrak{M}^K$ is separable.*
- (3) *Any epimorphism (resp. monomorphism) in ${}_H \mathfrak{M}_H$ splits (cosplits) in ${}_H \mathfrak{M}_H$.*
- (4) *H is separable as an algebra in \mathfrak{M}_K .*
- (5) *H is separable as an algebra in \mathfrak{M}^H .*
- (6) *H is semisimple.*
- (7) *The counit $\varepsilon : H \rightarrow K$ has a section in ${}_H \mathfrak{M}$.*

Proof. (1) \Rightarrow (2) By Proposition 2.25, the morphism

$$\sigma_M : M \rightarrow H \otimes M \otimes H, \quad \sigma_M(m) = \sum St_{(1)} \otimes t_{(2)} m \tilde{t}_{(1)} \otimes \tilde{S} \tilde{t}_{(2)}$$

is an (H, H) -bilinear morphism such that $\epsilon_M \circ \sigma_M = \text{Id}_M$, and this gives rise to a functorial morphism $\sigma : \text{Id}_{D_H \mathfrak{M}_H^D} \rightarrow TU$. In view of Theorem 2.23, the functor U is separable.

(2) \Rightarrow (3) Let p be an epimorphism in ${}_H \mathfrak{M}_H$. Then $U(p)$ splits. By Lemma 2.22, we conclude the proof. Analogously any monomorphism in ${}_H \mathfrak{M}_H$ cosplits in ${}_H \mathfrak{M}_H$.

(3) \Rightarrow (4) The multiplication $m : H \otimes H \rightarrow H$ is an epimorphism in ${}_H \mathfrak{M}_H$.

(4) \Rightarrow (1) Let $\sigma : H \rightarrow H \otimes H$ be an (H, H) -bilinear section of the multiplication m and set $t_\sigma := (H \otimes \varepsilon_H) \sigma(1_H) \in H$. Then t_σ fulfills the conditions of (1).

(5) \Leftrightarrow (6) follows by Proposition 2.11.

(5) \Leftrightarrow (7) and (7) \Leftrightarrow (1) follow by Lemma 2.3. \square

Theorem 2.35. *Let C be a Hopf algebra such that C_0 , the coradical of C , is a Hopf subalgebra. Let $H := C_0$ and let $\sigma : H \rightarrow C$ be the canonical injection. Assume that H is semisimple (e.g. H is finite dimensional over a field of characteristic 0). Then there is a coalgebra morphism π in ${}_H \mathfrak{M}_H$ that is a retraction of σ .*

Proof. In view of Theorem 2.27, if H is semisimple and cosemisimple, then it has an ad -coinvariant and an ad -invariant integral. Then, by Theorem 2.34, every bimodule is projective. By c) of Corollary 2.18, we conclude the proof. \square

Theorem 2.36. *Let H be a Hopf algebra with an ad -coinvariant integral t and let D be an (H, H) -bimodule coalgebra. Then the forgetful functor $U : {}^D_H \mathfrak{M}_H^D \rightarrow {}^D \mathfrak{M}^D$ is separable.*

Proof. We use the notations of 2.32. Let $\epsilon : TU \rightarrow \text{Id}_{D_H \mathfrak{M}_H^D}$ be the counit of the adjunction (T, U) . In view of Theorem 2.23, we have to construct a $\sigma : \text{Id}_{D_H \mathfrak{M}_H^D} \rightarrow TU$ such that $\epsilon \sigma = \text{Id}_{\text{Id}_{D_H \mathfrak{M}_H^D}}$. By Proposition 2.33, the morphism

$$\sigma_M : M \rightarrow H \otimes M \otimes H, \quad \sigma_M(m) = \sum St_{(1)} \otimes t_{(2)} m \tilde{t}_{(1)} \otimes \tilde{S} \tilde{t}_{(2)}$$

is a (H, H) -bilinear morphism such that $\epsilon_M \circ \sigma_M = \text{Id}_M$. It is easy to check that this gives rise to a functorial morphism $\sigma : \text{Id}_{D_H \mathfrak{M}_H^D} \rightarrow TU$. In order to conclude

that σ_M is a morphism in ${}^D_H\mathfrak{M}_H^D$, we have only to check that σ_M is a morphism of (D, D) -bicomodules. We have

$$\begin{aligned} \rho_{H \otimes M \otimes H}^l(\sigma_M(m)) &= \sum m_{\langle -1 \rangle} \tilde{t}_{(1)} S \tilde{t}_{(4)} \otimes St_{(1)} \otimes t_{(2)} m_0 \tilde{t}_{(2)} \otimes S \tilde{t}_{(3)}, \\ \text{cad}2) &= \sum m_{\langle -1 \rangle} \otimes St_{(1)} \otimes t_{(2)} m_0 \tilde{t}_{(1)} \otimes S \tilde{t}_{(2)} = (H \otimes \sigma_M) \rho_M^l(m). \end{aligned}$$

This relation proves that σ_M is left D -colinear. Similarly, using the second equality of $\text{cad}2$, one can show that σ_M is right D -colinear. \square

Theorem 2.37. *Let H be a Hopf algebra over a field K and assume that H has an ad -coinvariant integral. A coalgebra C in the category ${}_H\mathfrak{M}_H$ is coseparable iff C is coseparable as a coalgebra in $(\mathfrak{M}_K, \otimes_K, K)$, i.e. as a usual coalgebra.*

Proof. It is enough to prove that if C is coseparable as a coalgebra in \mathfrak{M}_K , then it is coseparable as a coalgebra in ${}_H^C\mathfrak{M}_H^C$. If $\Delta : C \rightarrow C \otimes C$ is the comultiplication of the coalgebra C in the monoidal category ${}_H^C\mathfrak{M}_H^C$, then Δ also defines the comultiplication of C as a coalgebra in \mathfrak{M}_K . Thus $U(\Delta) = \Delta$ has a retraction in ${}^C\mathfrak{M}^C$. Since, in view of Theorem 2.36, the functor $U : {}_H^C\mathfrak{M}_H^C \rightarrow {}^C\mathfrak{M}^C$ is coseparable, in view of Lemma 2.22, Δ has a retraction in ${}_H^C\mathfrak{M}_H^C$. Thus C is coseparable in ${}_H\mathfrak{M}_H$. \square

Corollary 2.38. *Let H be a semisimple and cosemisimple Hopf algebra over a field K . If C is a coalgebra in the category ${}_H\mathfrak{M}_H$, then C is coseparable as a coalgebra in ${}_H\mathfrak{M}_H$ iff C is coseparable as a coalgebra in $(\mathfrak{M}_K, \otimes_K, K)$.*

Proof. By Theorem 2.27, H has a non-zero ad -coinvariant integral. \square

Proposition 2.39. *Let H be a semisimple and cosemisimple Hopf algebra. Then we have:*

- 1) *If $\pi : A \rightarrow B$ is a surjective morphism of algebras in ${}^H\mathfrak{M}^H$ such that B is separable (as an algebra in \mathfrak{M}_K) and the kernel of π is nilpotent, then there is a section $\sigma : B \rightarrow A$ of π which is a morphism of algebras in ${}^H\mathfrak{M}^H$.*
- 2) *If $\sigma : C \rightarrow D$ is an injective morphism of coalgebras in ${}_H\mathfrak{M}_H$ such that C is coseparable (as a coalgebra in \mathfrak{M}_K) and the cokernel of σ is conilpotent, then there is a retraction $\pi : D \rightarrow C$ of σ which is a morphism of coalgebras in ${}_H\mathfrak{M}_H$.*

Proof. 1) By assumption H is semisimple and hence H is separable (see Theorem 2.34). Moreover by Corollary 2.31, B is separable as an algebra in the category ${}^H\mathfrak{M}^H$. Let n be a natural number such that $I^n = 0$, where $I = \text{Ker } \pi$. By Theorem 2.26, any epimorphism in the category ${}^H\mathfrak{M}^H$ splits in ${}^H\mathfrak{M}^H$. In particular, for every $r = 1, \dots, n-1$ the canonical morphism $\pi_r : A/I^{r+1} \rightarrow A/I^r$ has a section in the category ${}^H\mathfrak{M}^H$. We can now conclude by applying Theorem 2.12 to the algebra homomorphism $\pi : A \rightarrow B$.

2) By assumption H is cosemisimple, and hence H is coseparable (see Theorem 2.26). Moreover by Corollary 2.38, C is coseparable as a coalgebra in the category ${}_H\mathfrak{M}_H$. Let $(L, p) := \text{Coker}(\sigma)$. Then, for $n \geq 2$, we define L_n to be the coimage of $p^{\otimes n} \Delta_{n-1}$ where $\Delta_n : D \rightarrow D^{\otimes n+1}$ is the n -th iterated multiplication of D ($\Delta_1 := \Delta_D$), i.e. $L_n = D/\ker(p)^{\wedge n} = D/C^{\wedge n}$. Then (L, p) is called conilpotent if there is $n \geq 2$ such that $L_n = 0$ or equivalently $C^{\wedge n} = D$. So let n be such a natural number. By Theorem 2.34, any monomorphism in the category ${}_H\mathfrak{M}_H$ cosplits in ${}_H\mathfrak{M}_H$. In particular, for every $r = 1, \dots, n-1$ the canonical morphism $i_r : C^{\wedge r} \rightarrow C^{\wedge r+1}$ has a retraction in the category ${}_H\mathfrak{M}_H$. We can now conclude by applying Theorem 2.16 to the coalgebra homomorphism $\sigma : C \rightarrow D$. \square

Since any semisimple Hopf algebra H is separable, in the previous proposition we can choose $B = H$. Since any cosemisimple Hopf algebra H is coseparable, in the previous proposition we can choose $C = H$.

3. SPLITTING MORPHISMS OF BIALGEBRAS

3.1. Let H be a Hopf algebra and let $(A, m, u, \Delta, \varepsilon)$ be a bialgebra.

By the results that we obtained in Corollary 2.14 and Theorem 2.28, we are led to investigate the following problem.

Problem 1. Characterize those bialgebras A with the property that there is a pair of K -linear maps,

$$\pi : A \rightarrow H \quad \text{and} \quad \sigma : H \rightarrow A,$$

such that π is a morphism of bialgebras and σ is an (H, H) -bilinear algebra section of π , i.e., $\pi\sigma = \text{Id}_H$.

Motivated by the results that we obtained in Corollary 2.18 and Theorem 2.35, we are also interested in studying the problem dual to Problem 1.

Problem 2. Characterize those bialgebras A with the property that there is a pair of K -linear maps,

$$\sigma : H \rightarrow A \quad \text{and} \quad \pi : A \rightarrow H,$$

such that σ is a morphism of bialgebras and π is an (H, H) -bilinear algebra retraction of σ , i.e. $\pi\sigma = \text{Id}_H$.

Our approach to Problem 1 is based on the observation that such a bialgebra can be viewed in a natural way as an object $A \in {}^H_H\mathfrak{M}_H^H$ such that A is an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}^H, \square_H, H)$. To explain this we will consider the following useful wider context.

Definition 3.2. Let R be an H -bicomodule algebra. Let A be an algebra in the category of vector spaces with multiplication $m : A \otimes A \rightarrow A$ and unit $u : K \rightarrow A$. Assume that A is an object in ${}^H_R\mathfrak{M}_R^H$. We say that (A, m, u) becomes an algebra in $({}^H_R\mathfrak{M}_R^H, \otimes_R, R)$ if m factorizes to a morphism

$$\overline{m} : A \otimes_R A \rightarrow A \text{ in } {}^H_R\mathfrak{M}_R^H$$

and u factorizes to a morphism

$$\overline{u} : R \rightarrow A \text{ in } {}^H_R\mathfrak{M}_R^H$$

such that $(A, \overline{m}, \overline{u})$ is an algebra in $({}^H_R\mathfrak{M}_R^H, \otimes_R, R)$.

Lemma 3.3. *With hypothesis and notations of the above definition, we have*

$$\overline{u}(r) = r \cdot 1_A = 1_A \cdot r.$$

Lemma 3.4. *Let R be an (H, H) -bicomodule algebra and let $(A, \overline{m}, \overline{u})$ be an algebra in the monoidal category $({}^H_R\mathfrak{M}_R^H, \otimes_R, R)$. Then A is in a natural way an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ which becomes an algebra in $({}^H_R\mathfrak{M}_R^H, \otimes_R, R)$.*

Proposition 3.5. *Let R be an (H, H) -bicomodule algebra. Let (A, m, u) be an algebra. The following assertions are equivalent:*

- (a) *A is an object in ${}^H_R\mathfrak{M}_R^H$ and (A, m, u) becomes an algebra in $({}^H_R\mathfrak{M}_R^H, \otimes_R, R)$.*
- (b) *A is an object in ${}^H_R\mathfrak{M}_R^H$, (A, m, u) is an (H, H) -bicomodule algebra and m factorizes to a morphism $\overline{m} : A \otimes_R A \rightarrow A$ in ${}^H_R\mathfrak{M}_R^H$.*

(c) (A, m, u) is an H -bicomodule algebra and there exists an algebra map $\sigma : R \rightarrow A$ which is a homomorphism in ${}^H\mathfrak{M}^H$. Moreover in (c) \Rightarrow (a) we have $\bar{u} = \sigma$, while in (a), (b) \Rightarrow (c) the map σ induces the (R, R) -bimodule structure of A .

Proof. (a) \Rightarrow (b) is trivial. (b) \Rightarrow (c) Let us set $a \cdot b = m(a \otimes b) = \bar{m}(a \otimes_R b)$. Then m is an R -balanced morphism of (R, R) -bimodules, i.e. for $a, b \in A$ and $r \in R$, we have

$$(16) \quad 1) \quad (ar) \cdot b = a \cdot (rb); \quad 2) \quad r(a \cdot b) = (ra) \cdot b; \quad 3) \quad (a \cdot b)r = a \cdot (br).$$

In particular the first relation gives us $1_A r = (1_A r) \cdot 1_A \stackrel{1)}{=} 1_A \cdot (r 1_A) = r 1_A$, for all $r \in R$. Let $\sigma : R \rightarrow A$ be defined by $\sigma(r) := r 1_A = 1_A r$. Let us prove that σ is an algebra map and (H, H) -bilinear. Since m is (R, R) -bilinear, we get

$$\sigma(rs) = (rs)1_A = r(s1_A) = r(1_A \cdot (s1_A)) \stackrel{2)}{=} (r1_A)(s1_A) = \sigma(r)\sigma(s).$$

Moreover, by right H -colinearity of the map defining the left R -module structure of A and right H -colinearity of u , we get

$$\begin{aligned} \sum \sigma(r)_{\langle 0 \rangle} \otimes \sigma(r)_{\langle 1 \rangle} &= \sum (r1_A)_{\langle 0 \rangle} \otimes (r1_A)_{\langle 1 \rangle} = \sum r_{\langle 0 \rangle} 1_A \otimes r_{\langle 1 \rangle} 1_H \\ &= \sum \sigma(r_{\langle 0 \rangle}) \otimes r_{\langle 1 \rangle} \end{aligned}$$

and analogously on the left.

(c) \Rightarrow (a) Clearly σ induces an R -bimodule structure over A . Let μ^l and μ^r be the maps defining the module structures. First of all, we have to prove that these structures make A an object in ${}^H_R\mathfrak{M}^H_R$, i.e. that they are (H, H) -bicomodule morphisms. We have

$$\begin{aligned} \rho_A^l(\mu^l(r \otimes a)) &= \rho^l(r \cdot a) = \rho^l(\sigma(r) \cdot a) = \sum (\sigma(r)a)_{\langle -1 \rangle} \otimes (\sigma(r)a)_{\langle 0 \rangle} \\ (*) &= \sum \sigma(r)_{\langle -1 \rangle} a_{\langle -1 \rangle} \otimes \sigma(r)_{\langle 0 \rangle} a_{\langle 0 \rangle} \\ (**) &= \sum r_{\langle -1 \rangle} a_{\langle -1 \rangle} \otimes \sigma(r_{\langle 0 \rangle}) a_{\langle 0 \rangle} \\ (***) &= \sum r_{\langle -1 \rangle} a_{\langle -1 \rangle} \otimes r_{\langle 0 \rangle} a_{\langle 0 \rangle} \\ &= (H \otimes \mu^l) \rho_{R \otimes A}^l(r \otimes a). \end{aligned}$$

Relation $(*)$ results by the fact that (A, m, u) is an H -bicomodule algebra, $(**)$ holds since σ is left H -colinear and $(***)$ follows by the definition of module structures. In a similar way one can prove that μ^l is right H -linear and that μ^r is a morphism of (H, H) -bicomodules. Since

$$(ar) \cdot b = (a \cdot \sigma(r)) \cdot b = a \cdot (\sigma(r) \cdot b) = a \cdot (rb),$$

the multiplication $m : A \otimes A \rightarrow A$ is R -balanced so that it factorizes to a map $\bar{m} : A \otimes_R A \rightarrow A$. The map \bar{m} is left R -linear as

$$r(a \cdot b) = \sigma(r) \cdot (a \cdot b) = (\sigma(r) \cdot a) \cdot b = (ra) \cdot b.$$

Analogously, one proves that \bar{m} is right R -linear. Obviously \bar{m} is (H, H) -bilinear since m is also (H, H) -bilinear. Since σ is an algebra morphism, we get that σ is a morphism of (R, R) -bimodules and that $\sigma \circ u_R = u$. Moreover, by assumption, σ is a morphism of (H, H) -bicomodules. Let $\bar{u} = \sigma$. We now prove that (A, \bar{m}, \bar{u}) is an algebra in $({}^H_R\mathfrak{M}^H_R \otimes_R, R)$. Clearly \bar{m} is associative. Moreover we have

$$\bar{m}(\bar{u} \otimes_R A)(r \otimes_R a) = \sigma(r)a = ra = l_A(r \otimes_R a),$$

where $l_A : R \otimes_R A \rightarrow A$ is the left unit constraint of the monoidal category $({}^H_R \mathfrak{M}_R^H, \otimes_R, R)$. Analogously one proves that $\overline{m}(\overline{u} \otimes_R A) = r_A$. \square

Our approach to Problem 2 of 3.1 is based on the observation that such a bialgebra can be viewed in a natural way as an object $A \in {}^H_H \mathfrak{M}_H^H$ such that A is a coalgebra in $({}^H_H \mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H \mathfrak{M}_H, \otimes_H, H)$. To explain this we will consider the following useful wider context.

Definition 3.6. Let H be a Hopf algebra and let D be an (H, H) -bimodule coalgebra. Let C be a coalgebra in the category of vector spaces with comultiplication Δ and counit ε . Assume that C is an object in ${}^D_H \mathfrak{M}_H^D$. We say that (C, Δ, ε) becomes a coalgebra in $({}^D_H \mathfrak{M}_H^D, \square_D, D)$ if Δ corestricts to a morphism

$$\overline{\Delta} : C \rightarrow C \square_D C \text{ in } {}^D_H \mathfrak{M}_H^D$$

and ε factorizes to a morphism

$$\overline{\varepsilon} : C \rightarrow D \text{ in } {}^D_H \mathfrak{M}_H^D$$

such that $(C, \overline{\Delta}, \overline{\varepsilon})$ is a coalgebra in $({}^D_H \mathfrak{M}_H^D, \square_D, D)$.

Lemma 3.7. *With hypothesis and notation of the above definition, we have*

$$\overline{\varepsilon}(a) = \sum \varepsilon(a_{\langle 0 \rangle}) a_{\langle 1 \rangle} = \sum \varepsilon(a_{\langle 0 \rangle}) a_{\langle -1 \rangle}.$$

Proof. Since $\overline{\varepsilon}$ is left D -colinear, we get $\sum c_{\langle -1 \rangle} \otimes \overline{\varepsilon}(c_{\langle 0 \rangle}) = \sum \overline{\varepsilon}(c)_{(1)} \otimes \overline{\varepsilon}(c)_{(2)}$. By applying $D \otimes \varepsilon_D$ on both sides, we obtain the first equality of the statement. The other one follows analogously. \square

Lemma 3.8. *Let D be an (H, H) -bimodule coalgebra and let $(C, \overline{\Delta}, \overline{\varepsilon})$ be a coalgebra in the monoidal category $({}^D_H \mathfrak{M}_H^D, \square_D, D)$. Then C is in a natural way a coalgebra in $({}_H \mathfrak{M}_H, \otimes, K)$ which becomes a coalgebra in $({}^D_H \mathfrak{M}_H^D, \square_D, D)$.*

Proposition 3.9. *Let D be an (H, H) -bimodule coalgebra. Let (C, Δ, ε) be a coalgebra. The following assertions are equivalent:*

- (a) C is an object in ${}^D_H \mathfrak{M}_H^D$ and (C, Δ, ε) becomes a coalgebra in $({}^D_H \mathfrak{M}_H^D, \square_D, D)$.
- (b) C is an object in ${}^D_H \mathfrak{M}_H^D$, (C, Δ, ε) is an (H, H) -bimodule coalgebra and Δ corestricts to a morphism $\overline{\Delta} : C \rightarrow C \square_D C$ in ${}^D_H \mathfrak{M}_H^D$.
- (c) (C, Δ, ε) is an (H, H) -bimodule coalgebra and there exists a coalgebra map $\pi : C \rightarrow D$ which is a homomorphism in ${}_H \mathfrak{M}_H$.

Moreover in (c) \Rightarrow (a) we have $\overline{\varepsilon} = \pi$, while in (a), (b) \Rightarrow (c) the map π induces the (D, D) -bicomodule structure of C .

Proof. Follows by duality from the proof of Proposition 3.5. \square

Proposition 3.10. *Let $\alpha : E \rightarrow L$ be a coalgebra map, where E and L are bialgebras. Then the following assertions are equivalent:*

- (1) E is an L -bicomodule algebra, i.e. an algebra in $({}^L \mathfrak{M}^L, \otimes, K)$, where the comodule structure of E is induced by α .
- (2) α is a bialgebra map.

Proof. (1) \Rightarrow (2) Since the multiplication of E is (L, L) -bilinear, we get

$$\begin{aligned} \sum \alpha(x_{(1)}) \alpha(y_{(1)}) \otimes x_{(2)} y_{(2)} &= \sum x_{\langle -1 \rangle} y_{\langle -1 \rangle} \otimes x_{\langle 0 \rangle} y_{\langle 0 \rangle} \\ &= \sum (xy)_{\langle -1 \rangle} \otimes (xy)_{\langle 0 \rangle} \\ &= \sum \alpha[(xy)_{(1)}] \otimes (xy)_{(2)}, \end{aligned}$$

so that, by applying $L \otimes \varepsilon_E$, we obtain $\alpha(x)\alpha(y) = \alpha(xy)$, for any $x, y \in E$. Also, $\rho^l u_E(k) = 1_L \otimes u_E(k)$ for any $k \in K$, so that by applying $\alpha \otimes \varepsilon_E$, we obtain $\alpha u_E = u_L$.

(2) \Rightarrow (1) Let us prove that the multiplication m_E is left L -colinear. Let $x, y \in E$. We have

$$\begin{aligned} \rho_E^l m_E(x \otimes y) &= \sum \alpha(x_{(1)}y_{(1)}) \otimes x_{(2)}y_{(2)} \\ &= \sum \alpha(x_{(1)})\alpha(y_{(1)}) \otimes x_{(2)}y_{(2)} \\ &= \sum x_{\langle -1 \rangle}y_{\langle -1 \rangle} \otimes x_{\langle 0 \rangle}y_{\langle 0 \rangle} = \sum (L \otimes m_E) \circ \rho_{E \otimes E}^l. \end{aligned}$$

Analogously one can check that m_E is right L -colinear. Since $\alpha(1_E) = 1_L$ we have $\rho^l u_E(k) = 1_L \otimes u_E(k)$ for any $k \in K$, i.e. the unit u_E is left L -colinear. Analogously one can check that u_E is right L -colinear. \square

Proposition 3.11. *Let $\alpha : E \rightarrow L$ be an algebra map, where E and L are bialgebras. Then the following assertions are equivalent:*

- (1) L is an E -bimodule coalgebra, i.e. a coalgebra in $({}_E\mathfrak{M}_E, \otimes, K)$, where the module structure of L is induced by α .
- (2) α is a bialgebra map.

Proof. Follows by duality from the proof of Proposition 3.10. \square

Theorem 3.12. *Let $(A, m, u, \Delta, \varepsilon_A)$ be a bialgebra and let H be a Hopf algebra. The following assertions are equivalent:*

- (a) A is an object in $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$, and A becomes an algebra in $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H\mathfrak{M}^H, \square_H, H)$.
- (b) There are a bialgebra map $\pi : A \rightarrow H$ and an (H, H) -bilinear algebra map $\sigma : H \rightarrow A$, where A is a (H, H) -bicomodule via π .

Furthermore, in this case, the counit ε_A is right H -linear if and only if $\pi\sigma = \text{Id}_H$.

Moreover, if (a) holds, we can choose π and σ such that

$$(17) \quad \pi(a) = \sum \varepsilon(a_{\langle 0 \rangle})a_{\langle 1 \rangle} = \sum \varepsilon(a_{\langle 0 \rangle})a_{\langle -1 \rangle} \quad \text{and} \quad \sigma(h) = h \cdot 1_A = 1_A \cdot h.$$

Proof. (a) \Rightarrow (b) Since A becomes a coalgebra in $({}^H\mathfrak{M}^H, \square_H, H)$, by Proposition 3.9 there exists a coalgebra map $\pi : A \rightarrow H$ which is a homomorphism in ${}_H\mathfrak{M}_H$ such that $(A, \overline{\Delta}, \pi = \overline{\varepsilon})$ is a coalgebra in $({}_H\mathfrak{M}_H, \square_H, H)$ and π induces the (H, H) -bicomodule structure of A . Note that by Lemma 3.7 and by Lemma 3.3, π and σ fulfill relations (17).

Since A becomes an algebra in $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$, by Proposition 3.5 it follows that A is an (H, H) -bicomodule algebra. Thus, by Proposition 3.10 we obtain that π is a bialgebra map.

Since A becomes an algebra in $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$, by Proposition 3.5 there exists an algebra map $\sigma : H \rightarrow A$ which is an homomorphism in ${}^H\mathfrak{M}^H$.

(b) \Rightarrow (a) By Proposition 3.10, applied to the bialgebra map $\alpha = \pi$, A is an algebra in $({}^H\mathfrak{M}^H, \otimes, K)$. By Proposition 3.5, applied in the case where “ R ” = H and using the fact that $\sigma : H \rightarrow A$ is an (H, H) -bilinear algebra map, A is an object in ${}_H^H\mathfrak{M}_H^H$ and (A, m, u) becomes an algebra in $({}_H^H\mathfrak{M}_H^H, \otimes_H, H)$. By Proposition 3.9 applied in the case “ H ” = K , “ C ” = A and “ D ” = H and using

the fact that “ π ” is a coalgebra map in \mathfrak{M}_K , we deduce that $(A, \Delta, \varepsilon_A)$ becomes a coalgebra in $({}^H\mathfrak{M}^H, \square_H, H)$.

We will now prove that if (a) or (b) holds true, the counit ε_A is right H -linear if and only if $\pi\sigma = \text{Id}_H$. Since σ is right H -colinear and the right coaction on A is induced by π , we have

$$\sum \sigma(h_{(1)}) \otimes h_{(2)} = \sum \sigma(h)_{(1)} \otimes \pi(\sigma(h)_{(2)}).$$

By applying $\varepsilon_A \otimes H$ to this equation we obtain

$$(18) \quad \sum \varepsilon_A(\sigma(h_{(1)})) h_{(2)} = (\pi\sigma)(h).$$

We point out that, by Proposition 3.5, σ induces the (H, H) -bimodule structure of A .

Assume that the counit ε_A is right H -linear. Then

$$\begin{aligned} \sum \varepsilon_A(\sigma(h_{(1)})) h_{(2)} &= \sum \varepsilon_A(1_A \sigma(h_{(1)})) h_{(2)} = \sum \varepsilon_A(1_A h_{(1)}) h_{(2)} \\ &= \sum \varepsilon_A(1_A) \varepsilon_H(h_{(1)}) h_{(2)} = h. \end{aligned}$$

Conversely, if $\pi\sigma = \text{Id}_H$, then, by applying ε_H to both sides of (18), we get $\varepsilon_A(\sigma(h)) = \varepsilon_H(h)$. Then

$$\varepsilon_A(ah) = \varepsilon_A(a\sigma(h)) = \varepsilon_A(a)\varepsilon_A(\sigma(h)) = \varepsilon_A(a)\varepsilon_H(h).$$

□

Theorem 3.13. *Let $(A, m, u, \Delta, \varepsilon_A)$ be a bialgebra and let H be a Hopf algebra. The following assertions are equivalent:*

- (a) *A is an object in $({}^H\mathfrak{M}_H^H, \otimes_H, H)$ and A becomes a coalgebra in $({}^H\mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H\mathfrak{M}_H, \otimes_H, H)$.*
- (b) *There are a bialgebra map $\sigma : H \rightarrow A$ and an (H, H) -bilinear coalgebra map $\pi : A \rightarrow H$, where A is an (H, H) -bimodule via σ .*

Furthermore, in this case, the unit u is right H -colinear if and only if $\pi\sigma = \text{Id}_H$.

Moreover, if (a) holds, we can choose π and σ so that (17) holds true.

Proof. Follows by duality from the proof of Theorem 3.12. □

Example 3.14. Let H be a Hopf algebra. By 1.9, we know that there is a monoidal category equivalence

$$({}_H\mathfrak{M}, \otimes, K) \xrightarrow{F} ({}_H\mathfrak{M}_H^H, \otimes_H, H) \xrightarrow{G} ({}_H\mathfrak{M}, \otimes, K).$$

Now let (R, m, u) be a left H -module algebra, i.e. an algebra in the monoidal category $({}_H\mathfrak{M}, \otimes, K)$. Then, by Proposition 1.5, $(F(R), m_{F(R)}, u_{F(R)})$ is an algebra in $({}_H\mathfrak{M}_H^H, \otimes_H, H)$. It is easy to check that by lifting the multiplication $m_{F(R)}$ to the usual tensor product $F(R) \otimes F(R)$, we obtain the so-called *smash product* $R \# H$ of R and H , i.e. the associative algebra defined on $R \otimes H$ by setting

$$(19) \quad (r \# h)(s \# k) = \sum r({}^{h_{(1)}}s) \# h_{(2)}k.$$

This algebra is unitary, with unit $1_R \# 1_H$. Here $r \# h := r \otimes h$.

By 1.11, the above equivalence induces a monoidal category equivalence

$$({}_H\mathcal{YD}, \otimes, K) \xrightarrow{F} ({}_H\mathfrak{M}_H^H, \otimes_H, H) \xrightarrow{G} ({}_H\mathcal{YD}, \otimes, K).$$

Thus, if we assume in addition that R is an algebra in ${}^H_H\mathcal{YD}$, then $F(R)$ is an algebra in the monoidal category $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ so that, by Lemma 3.4, $R\#H$ becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$, with respect to the structures

$$\begin{aligned} \rho_{R\#H}^l(r\#h) &= \sum r_{(-1)}h_{(1)} \otimes (r_{(0)}\#h_{(2)}), & \rho_{R\#H}^r(r\#h) &= \sum (r\#h_{(1)}) \otimes h_{(2)}, \\ k(r\#h) &= \sum {}^{k(1)}r\#k_{(2)}h, & (r\#h)k &= r\#hk. \end{aligned}$$

We will now prove that *any* algebra A that becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ is of this type, i.e. there is an algebra R in ${}^H_H\mathcal{YD}$ such that $A \simeq R\#H$.

Definition 3.15. Let H be a Hopf algebra and let $V \in {}^H_H\mathfrak{M}_H^H$. The space of right coinvariant elements of V will be called the *diagram* of V , and it will be denoted by R_V , or by R for short, if there is no danger of confusion.

Proposition 3.16. Let (A, m, u) be an algebra. Suppose that A is an object in ${}^H_H\mathfrak{M}_H^H$ such that A becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$. If $R = A^{Co(H)}$ is the diagram of A , then R is an algebra in ${}^H_H\mathcal{YD}$ with multiplication m_R , the restriction of m to $R \otimes R$, and unit $1_R = 1_A$.

Moreover, the canonical isomorphism $\epsilon_A : R\#H \rightarrow A$ is a morphism of algebras in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$.

Proof. Since A becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$, the multiplication m of A factors to a map $\overline{m} : A \otimes_H A \rightarrow A$ and the unit u of A factors to a map $\overline{u} : H \rightarrow A$. Moreover, by Proposition 3.5, $(A, \overline{m}, \overline{u})$ is an algebra in the monoidal category $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$. Therefore, by Proposition 1.5, $G(A) = R$, where G is the monoidal functor $({}^H_H\mathfrak{M}_H^H, \otimes_H, H) \xrightarrow{G} ({}^H_H\mathcal{YD}, \otimes, K)$ (see 1.11), is an algebra in the monoidal category $({}^H_H\mathcal{YD}, \otimes, K)$. The multiplication of R is exactly the one induced by the multiplication of A , and the unit is the same as A .

Now, by 1.11, the counit of the adjunction (F, G) , corresponding to the monoidal equivalence

$$({}^H_H\mathcal{YD}, \otimes, K) \xrightarrow{F} ({}^H_H\mathfrak{M}_H^H, \otimes_H, H) \xrightarrow{G} ({}^H_H\mathcal{YD}, \otimes, K),$$

is given by

$$\epsilon_M : M^{Co(H)} \otimes H \rightarrow M, \quad \epsilon_M(v \otimes h) = vh.$$

By Corollary 1.7, ϵ_A is an algebra isomorphism. \square

Example 3.17. Let H be a Hopf algebra. By 1.10, we know that there is a monoidal category equivalence

$$({}^H_H\mathfrak{M}, \otimes, K) \xrightarrow{F} ({}^H_H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H_H\mathfrak{M}, \otimes, K).$$

Let (D, Δ, ε) be a left H -comodule coalgebra, i.e. a coalgebra in the monoidal category $({}^H_H\mathfrak{M}, \otimes, K)$. Then, by Proposition 1.5, $(F(D), \Delta_{F(D)}, \varepsilon_{F(D)})$ is a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$. It is easy to check that embedding $D \square_H D$ inside $D \otimes D$, we obtain the so-called *smash coproduct* $D\#H$ of D and H , i.e. the coassociative and counitary coalgebra defined on $D \otimes H$ by setting

$$(20) \quad \Delta(d\#h) = \sum d_{(1)}\#(d_{(2)})_{(-1)}h_{(1)} \otimes (d_{(2)})_{(0)}\#h_{(2)},$$

$$(21) \quad \varepsilon(d\#h) = \varepsilon_D(d)\varepsilon_H(h).$$

By 1.12, the above equivalence induces a monoidal category equivalence

$$({}^H_H\mathcal{YD}, \otimes, K) \xrightarrow{F} ({}^H_H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H_H\mathcal{YD}, \otimes, K).$$

Thus, if we assume in addition that D is a coalgebra in ${}^H_H\mathcal{YD}$, then $F(D)$ is a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$ so that, by Lemma 3.8, $D\#H$ becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$, with respect to the structures

$$\begin{aligned} \rho_{R\#H}^l(d\#h) &= \sum d_{(-1)}h_{(1)} \otimes (d_{(0)}\#h_{(2)}), & \rho_{R\#H}^d(d\#h) &= \sum (d\#h_{(1)}) \otimes h_{(2)}, \\ k(d\#h) &= \sum {}^{k(1)}d\#k_{(2)}h, & (d\#h)k &= d\#hk. \end{aligned}$$

We will now prove that any coalgebra C that becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$ is of this type, i.e. there is a coalgebra D in ${}^H_H\mathcal{YD}$ such that $C \simeq D\#H$.

Proposition 3.18. *Let (C, Δ, ε) be a coalgebra. Suppose that C is an object in ${}^H_H\mathfrak{M}_H^H$ such that C becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$. Let $\bar{\varepsilon}$ be the counit of C as a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$. If $D = C^{Co(H)}$ is the diagram of C , then D is a coalgebra in ${}^H_H\mathcal{YD}$, where the comultiplication of D is given by*

$$\delta : D \rightarrow D \otimes D : d \mapsto \sum d_{(1)}S\bar{\varepsilon}d_{(2)} \otimes d_{(3)}$$

and the counit is induced by the counit of C .

Moreover, the canonical isomorphism $\epsilon_C : D\#H \rightarrow D$ is a morphism of coalgebras in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$.

Proof. Since C becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$, the comultiplication Δ of C corestricts to a map $\bar{\Delta} : C \rightarrow C\square_H C$ and the counit ε of C factorizes to a map $\bar{\varepsilon} : C \rightarrow H$ such that $\varepsilon = \varepsilon_H\bar{\varepsilon}$. Moreover, by Proposition 3.9, $(C, \bar{\Delta}, \bar{\varepsilon})$ is a coalgebra in the monoidal category $({}^H_H\mathfrak{M}_H^H, \square_H, H)$. Therefore, by Proposition 1.5, $G(C) = D$, where G is the monoidal functor $({}^H_H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H_H\mathcal{YD}, \otimes, K)$ (see 1.12), is a coalgebra in the monoidal category $({}^H_H\mathcal{YD}, \otimes, K)$. The comultiplication of D is

$$\delta : D \rightarrow D \otimes D : d \mapsto \sum d_{(1)}S\bar{\varepsilon}d_{(2)} \otimes d_{(3)},$$

and the counit is induced by the counit of C .

Now, by 1.12, the counit of the adjunction (F, G) , corresponding to the monoidal equivalence

$$({}^H_H\mathcal{YD}, \otimes, K) \xrightarrow{F} ({}^H_H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H_H\mathcal{YD}, \otimes, K),$$

is given by

$$\epsilon_M : M^{Co(H)} \otimes H \rightarrow M, \quad \epsilon_M(v \otimes h) = vh.$$

By Corollary 1.7, ϵ_A is a coalgebra isomorphism. \square

3.19. Suppose that H is a Hopf algebra. Let $V \in \mathfrak{M}_K$ and $W \in {}^H\mathfrak{M}$. It is well known that we have a functorial isomorphism

$$(22) \quad (V \otimes H)\square_H W \simeq V \otimes W,$$

which is given by $\sum_{i=1}^n v_i \square_H h_i \otimes w_i \mapsto \sum_{i=1}^n \varepsilon(h_i)v_i \otimes w_i$. The inverse of this map is $V \otimes \rho_W$.

Furthermore, the functor $F : \mathfrak{M}_K \rightarrow \mathfrak{M}^H$, $F(V) = V \otimes H$, has as a left adjoint the functor $G : \mathfrak{M}^H \rightarrow \mathfrak{M}_K$ that “forgets” the comodule structure. The maps that define this adjunction are

$$\begin{aligned} \alpha_{V,W} : \mathfrak{M}^H(V, W \otimes H) &\rightarrow \mathfrak{M}_K(V, W), & \alpha_{V,W}(f) &= (V \otimes \varepsilon)f, \\ \beta_{V,W} : \mathfrak{M}_K(V, W) &\rightarrow \mathfrak{M}^H, & \beta_{V,W}(g) &= (g \otimes H)\rho_V, \end{aligned}$$

where ρ_V defines the comodule structure on V .

Lemma 3.20. *Let V, W be two vector spaces and let Z be a left H -comodule. If we regard $Z \otimes H$ as a left H -comodule with diagonal coaction and a right comodule via $Z \otimes \Delta_H$, then there is a one-to-one correspondence between $\mathfrak{M}_K(V \otimes H, W \otimes Z)$ and $\mathfrak{M}^H(V \otimes H, (W \otimes H) \square_H (Z \otimes H))$. If $\gamma \in \mathfrak{M}_K(V \otimes H, W \otimes Z)$ and $\Gamma \in \mathfrak{M}^H(V \otimes H, (W \otimes H) \square_H (Z \otimes H))$ correspond to each other through this bijective map, then they are related by the following relations:*

$$(23) \quad \gamma = (W \otimes \varepsilon_H \otimes Z \otimes \varepsilon_H) \Gamma,$$

$$(24) \quad \Gamma(v \otimes h) = \sum \gamma^1(v \otimes h_{(1)}) \otimes \gamma^2(v \otimes h_{(1)})_{(-1)} h_{(2)} \otimes \gamma^2(v \otimes h_{(1)})_{(0)} \otimes h_{(3)},$$

where $\gamma(v \otimes h) = \sum \gamma^1(v \otimes h) \otimes \gamma^2(v \otimes h) \in W \otimes Z$ is a Sweedler-like notation for $\gamma(v \otimes h)$.

Proof. By (22) we have

$$(W \otimes H) \square_H (Z \otimes H) \simeq W \otimes Z \otimes H.$$

Hence

$$\mathfrak{M}^H(V \otimes H, (W \otimes H) \square_H (Z \otimes H)) \simeq \mathfrak{M}^H(V \otimes H, W \otimes Z \otimes H).$$

By composing this isomorphism with $\alpha_{V \otimes H, W \otimes Z}$, we obtain a bijective map

$$\mathfrak{M}^H(V \otimes H, (W \otimes H) \square_H (Z \otimes H)) \rightarrow \mathfrak{M}_K(V \otimes H, W \otimes Z).$$

Suppose now that γ and Γ correspond to each other through the above K -linear isomorphism. A straightforward but tedious computation shows us that γ and Γ verify (23) and (24). \square

3.21. Let $R \in {}^H\mathfrak{M}$ and let $\Delta_{R\#H} : (R\#H) \rightarrow (R\#H) \square_H (R\#H)$ be a right H -colinear map. By Lemma 3.20, if

$$(25) \quad \tilde{\delta} = (R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H) \Delta_{R\#H},$$

and for $r \in R, h \in H$ we write $\tilde{\delta}(r\#h) = \sum \tilde{\delta}^1(r\#h) \otimes \tilde{\delta}^2(r\#h) \in R \otimes R$, then

$$(26) \quad \Delta_{R\#H}(r\#h) = \sum \tilde{\delta}^1(r\#h_{(1)}) \# \tilde{\delta}^2(r\#h_{(1)})_{(-1)} h_{(2)} \otimes \tilde{\delta}^2(r\#h_{(1)})_{(0)} \# h_3.$$

Conversely if $\tilde{\delta} : R \otimes H \rightarrow R \otimes R$ is a linear map and $\Delta_{R\#H}$ is defined by (26), then $\Delta_{R\#H}$ is a right H -colinear map and $\text{Im}(\Delta_{R\#H}) \subseteq (R\#H) \square_H (R\#H)$.

3.22. Let H be a Hopf algebra and let A be a bialgebra with multiplication m , unit u , comultiplication Δ and counit ε_A .

In view of Theorem 3.12, Problem 1, as stated in 3.1, can be reformulated as follows: characterize all bialgebra A that are objects in ${}^H_H\mathfrak{M}_H^H$ such that A becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$, with the further requirement that ε_A is right H -linear.

By Proposition 3.16, the diagram (R, m, u) of A is an algebra in ${}^H_H\mathcal{YD}$, $R\#H$ is an algebra and the map $\epsilon_A : R\#H \rightarrow A$, $\epsilon_A(r \otimes h) = rh$ is an isomorphism of algebras in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$. Obviously, $R\#H$ is a bialgebra with comultiplication $\Delta_{R\#H}$ and counit $\varepsilon_{R\#H}$ given by

$$\Delta_{R\#H} := (\epsilon_A^{-1} \otimes \epsilon_A^{-1}) \Delta \epsilon_A \quad \text{and} \quad \varepsilon_{R\#H} := \varepsilon_A \epsilon_A.$$

Of course, with respect to this bialgebra structure, ϵ_A becomes an isomorphism of bialgebras.

Furthermore, since A becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$, the smash $R\#H$ has the same properties. In particular $\text{Im}(\Delta_{R\#H})$

$\subseteq (R\#H)\square_H(R\#H)$ and $\Delta_{R\#H}$ can be regarded as a right H -colinear map $\Delta_{R\#H} : R\#H \rightarrow (R\#H)\square_H(R\#H)$.

Hence, by 3.21, $\Delta_{R\#H}$ is uniquely determined by the K -linear map $\tilde{\delta} : R\#H \rightarrow R \otimes R$. In order to obtain the counit $\varepsilon_{R\#H}$, we consider the restriction of ε_A to R . For simplifying the notation, we shall denote it by ε .

Lemma 3.23. *The following assertions are equivalent:*

- (1) ε_A is right H -linear;
- (2) $\varepsilon_{R\#H}(r\#h) = \varepsilon(r)\varepsilon_H(h)$, for all $r \in R$ and $h \in H$;
- (3) $\varepsilon_A(1_A h) = \varepsilon_H(h)$, for all $h \in H$.

Proof. By definition, we have $\varepsilon_{R\#H}(r\#h) = \varepsilon_A(rh)$.

(1) \Rightarrow (2) By hypothesis we have $\varepsilon_A(rh) = \varepsilon(r)\varepsilon_H(h)$.

(2) \Rightarrow (3) By hypothesis we have $\varepsilon_{R\#H}(1_A\#h) = \varepsilon(1_A)\varepsilon_H(h) = \varepsilon_H(h)$.

(3) \Rightarrow (1) By relation 3) in (16) of Proposition 3.5, we have

$$\varepsilon_A(ah) = \varepsilon_A((a1_A)h) = \varepsilon_A(a(1_Ah)) = \varepsilon_A(a)\varepsilon(1_Ah).$$

□

All considerations above still hold if we work with an arbitrary algebra R in ${}^H_H\mathcal{YD}$. To be more precise we reformulate our problem of characterizing algebras A as above in the following way.

Problem 3.24. Let R be an algebra in ${}^H_H\mathcal{YD}$. Suppose that $\tilde{\delta} : R\#H \rightarrow R \otimes R$, $\varepsilon : R \rightarrow K$ are K -linear maps. Let $\Delta_{R\#H}$ be defined by (26) and let $\varepsilon_{R\#H} := \varepsilon \otimes \varepsilon_H$. Find a necessary and sufficient condition such that $(R\#H, \Delta_{R\#H}, \varepsilon_{R\#H})$ is a bialgebra that becomes a coalgebra in $({}^H\mathfrak{M}^H, \square_H, H)$.

Note that $R\#H$ always becomes an algebra in $({}^H\mathfrak{M}^H, \otimes_H, H)$, as claimed in Example 3.14. Of course, by solving the above problem we also get an answer to our initial question of finding all bialgebras A , where A is an H -Hopf bimodule, that become an algebra in $({}^H\mathfrak{M}^H, \otimes_H, H)$ and a coalgebra in $({}^H\mathfrak{M}^H, \square_H, H)$ such that ε_A is right H -linear. It is enough to take R to be the diagram of A and $\tilde{\delta}, \varepsilon$ as in 3.22. Therefore, we fix the following **notation**:

- R is an algebra in ${}^H_H\mathcal{YD}$;
- $\tilde{\delta} : R\#H \rightarrow R \otimes R$ and $\varepsilon : R \rightarrow K$ are K -linear maps;
- $\Delta_{R\#H}$ is defined by (26);
- $\varepsilon_{R\#H} := \varepsilon \otimes \varepsilon_H$;
- $\rho_{R\#H} : R\#H \rightarrow H \otimes R\#H : r\#h \mapsto \sum r_{\langle -1 \rangle} h_{(1)} \otimes (r_{\langle 0 \rangle} \# h_{(2)})$ denotes the map that defines the left coaction on $R\#H$ (see Example 3.14).

3.25. To simplify the computation we shall sometimes use the method of representing morphisms in a braided category by diagrams. For details, the reader is referred to [Ka, Chapter XIV.1]. Here we shall only mention that the morphisms are represented by arrows oriented downwards.

We shall apply this method in the category ${}^H_H\mathcal{YD}$ of Yetter–Drinfeld modules. Recall that, for every $V, W \in {}^H_H\mathcal{YD}$ the braiding is given by

$$(27) \quad c_{V,W} : V \otimes W \rightarrow W \otimes V, \quad c_{V,W}(v \otimes w) = \sum v_{\langle -1 \rangle} w \otimes v_{\langle 0 \rangle}.$$

Two examples of diagrams in this category can be found in Figure 1. Note that in both pictures the crossings represent $c_{R,H}$.

$$\rho_{R\#H} = \text{diagram} \quad \Delta_{R\#H} = \text{diagram}$$

FIGURE 1. Definitions of $\rho_{R\#H}$ and $\Delta_{R\#H}$.

$$\text{diagram} = \text{diagram} \quad \text{diagram} = \text{diagram}$$

FIGURE 2. Properties of ε_H and Δ_H .

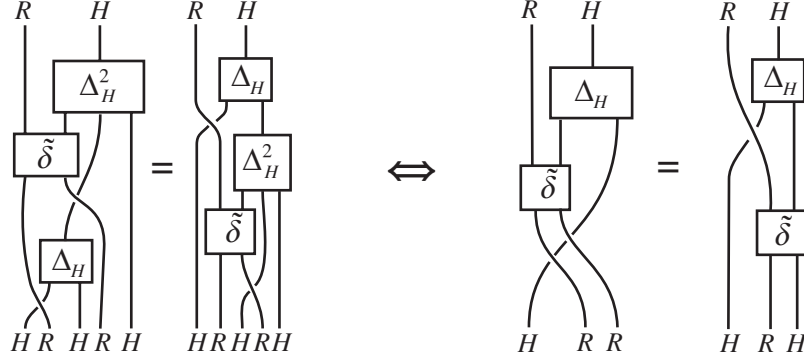
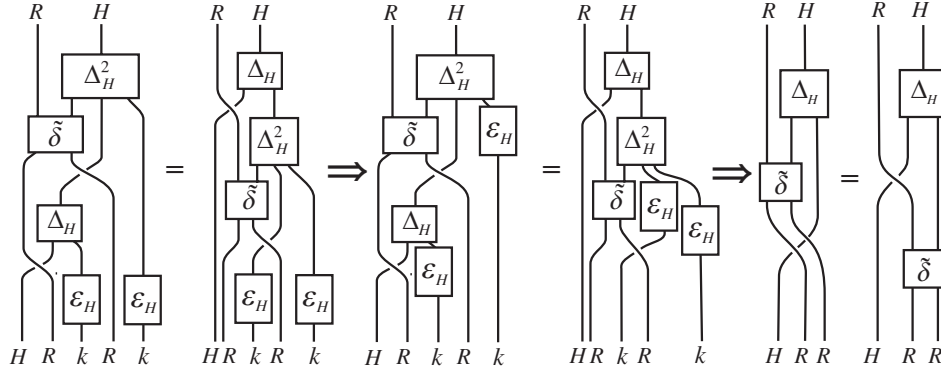
Lemma 3.26. *Let H be a Hopf algebra. Then:*

- a) $(\varepsilon_H \otimes R)c_{R,H} = c_{R,K}(R \otimes \varepsilon_H)$.
- b) $(\Delta_H \otimes R)c_{R,H} = (H \otimes c_{R,H})(c_{R,H} \otimes H)(R \otimes \Delta_H)$.

Proof. Straightforward. □

Remark 3.27. The equations from the previous lemma admit the representations from Figure 2. Note that in both equalities the right-hand side is obtained from the left-hand side by pulling ε_H , respectively Δ_H , under the crossing. This is a general property that works for arbitrary diagrams related to braided categories: a morphism can be moved along the string, and it can be pulled under or over crossings.

Recall that we are seeking conditions such that $(R\#H, \Delta_{R\#H}, \varepsilon_{R\#H})$ becomes a coalgebra in the monoidal category $({}^H\mathfrak{M}^H, \square_H, H)$ so that we need $\Delta_{R\#H}$ to be an (H, H) -bilinear map. Note that the left H -comodule structure of $(R\#H) \otimes (R\#H)$ is given by $\rho_{R\#H} \otimes R\#H$ and the right one by $R\#H \otimes \rho_{R\#H}^r$. By 3.22, we already know that $\Delta_{R\#H}$ is right H -colinear. The following result deals with left H -colinearity which is expressed by relation (28).

FIGURE 3. Representation of (28) \iff (29).FIGURE 4. The proof of (28) \implies (29).

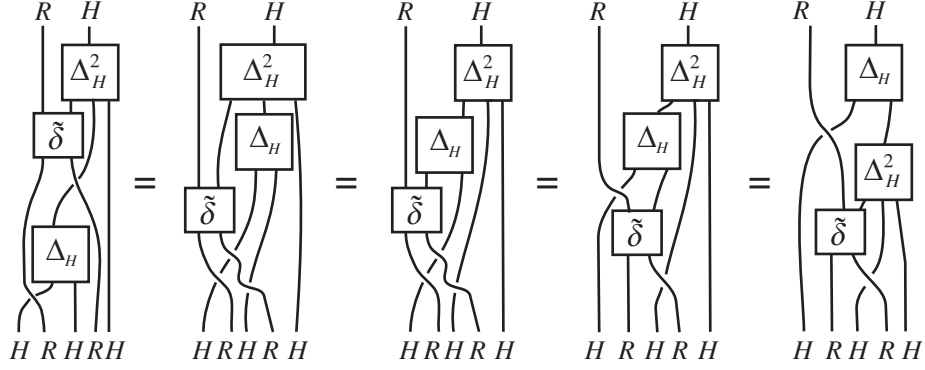
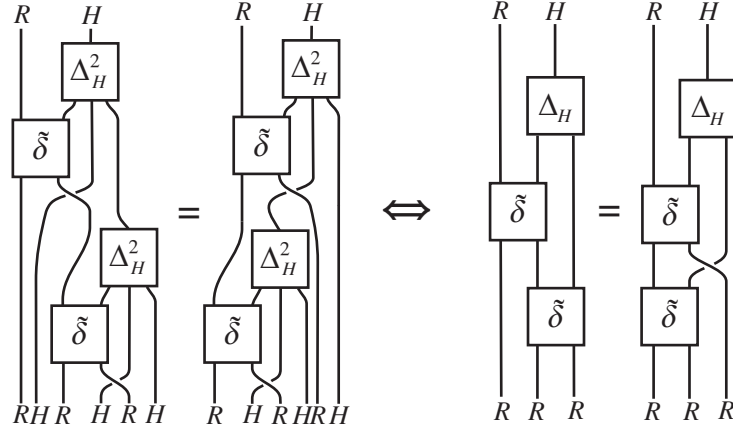
Lemma 3.28. *The following two relations are equivalent:*

$$(28) \quad [\rho_{R\#H} \otimes (R\#H)]\Delta_{R\#H} = (H \otimes \Delta_{R\#H})\rho_{R\#H},$$

$$(29) \quad (H \otimes \tilde{\delta})\rho_{R\#H} = (c_{R,H} \otimes R)(R \otimes c_{R,H})(\tilde{\delta} \otimes H)(R \otimes \Delta_H).$$

Proof. Note that the equivalence that we have to prove can be represented as in Figure 3. We prove that (28) \implies (29) in Figure 4. The first equality there was obtained by composing both sides of (28) with $H \otimes R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H$. The second equation holds because ε_H and Δ_H can be pulled under the string in a crossing; see Remark 3.27. We conclude the proof of this implication by using the fact that ε_H is the counit of H .

The other implication is proved in Figure 5. By Remark 3.27 we can drag Δ_H under the braiding, so we get the first equality. Since the comultiplication in H is coassociative, we have the second and last relations. The third one follows since, by assumption, (29) holds. \square

FIGURE 5. The proof of (29) \implies (28).FIGURE 6. Representation of (30) \iff (31).

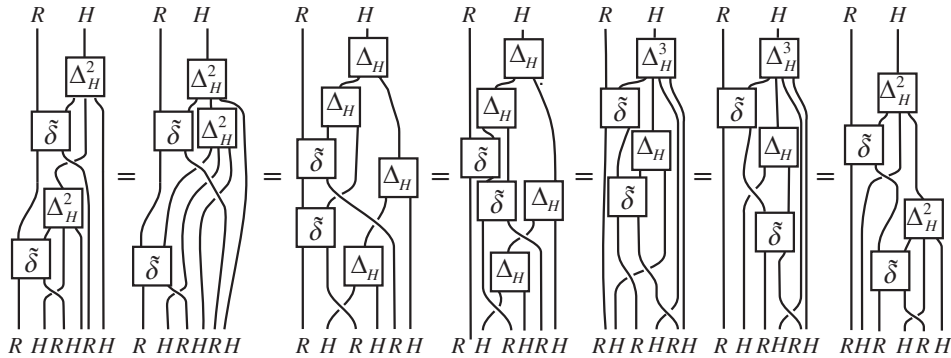
Lemma 3.29. Assume that $\Delta_{R\#H}$ is left H -colinear (i.e. satisfies (28)). Then the following two relations are equivalent:

$$(30) \quad [\Delta_{R\#H} \otimes (R\#H)]\Delta_{R\#H} = [(R\#H) \otimes \Delta_{R\#H}]\Delta_{R\#H},$$

$$(31) \quad (\tilde{\delta} \otimes R)(R \otimes c_{R,H})(\tilde{\delta} \otimes H)(R \otimes \Delta_H) = (R \otimes \tilde{\delta})(\tilde{\delta} \otimes H)(R \otimes \Delta_H).$$

Proof. The diagrammatic representation of the equivalence is given in Figure 6. It is easy to see that (30) implies (31). Indeed it is enough to add $(R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H)$ on the bottom of the diagram representing (30), then drag ε_H under the crossings and use the fact that ε_H is a counit. The other implication is proved in Figure 7. \square

3.30. Let R and S be two algebras in the braided category ${}^H_H\mathcal{YD}$. We can define a new algebra structure on $R \otimes S$ by using the braiding (27) and not the usual flip

FIGURE 7. The proof of (31) \implies (30).

morphism. The multiplication in this case is defined by the formula

$$(32) \quad (r \otimes s)(t \otimes v) = \sum r(s_{(-1)}t) \otimes s_{(0)}v.$$

Let us remark that, for any algebra R in ${}^H_H\mathcal{YD}$, the smash product $R \# H$ is a particular case of this construction. Just take $S = H$ with the left adjoint action and usual left H -comodule structure. Another example that we are interested in is $R \otimes R$, where R is the diagram of a bialgebra A as in 3.22. For such an algebra R in ${}^H_H\mathcal{YD}$ we shall always use this algebra structure on $R \otimes R$.

Lemma 3.31. *Let $\tilde{\delta} : R \otimes H \rightarrow R \otimes R$ be a K -linear map. Then the following two relations are equivalent:*

$$(33) \quad \Delta_{R \# H}((r \# h)(s \# k)) = \Delta_{R \# H}(r \# h) \Delta_{R \# H}(s \# k),$$

$$(34) \quad \tilde{\delta}((r \# h)(s \# k)) = \sum \tilde{\delta}(r \# h_{(1)})^{h_{(2)}} \tilde{\delta}(s \# k),$$

where, for every $h \in H$ and $r, t \in R$, we have ${}^h(r \otimes t) = \sum {}^{h_{(1)}}r \otimes {}^{h_{(2)}}t$.

Proof. Let $r \# h$ and $s \# k \in R \# H$. Thus we have

$$\begin{aligned} \Delta(r \# h) &= \sum \tilde{\delta}^1(r \# h_{(1)}) \# \tilde{\delta}^2(r \# h_{(1)})_{(-1)} h_{(2)} \otimes \tilde{\delta}^2(r \# h_{(1)})_{(0)} \# h_{(3)}, \\ \Delta(s \# k) &= \sum \tilde{\delta}^1(s \# k_{(1)}) \# \tilde{\delta}^2(s \# k_{(1)})_{(-1)} k_{(2)} \otimes \tilde{\delta}^2(s \# k_{(1)})_{(0)} \# k_{(3)}, \\ \Delta_{R \# H}((r \# h)(s \# k)) &= \sum \tilde{\delta}^1(r^{h_{(1)}} s \# h_{(2)} k_{(1)}) \# \tilde{\delta}^2(r^{h_{(1)}} s \# h_{(2)} k_{(1)})_{(-1)} h_{(3)} k_{(2)} \\ &\quad \otimes \tilde{\delta}^2(r^{h_{(1)}} s \# h_{(2)} k_{(1)})_{(0)} \# h_{(4)} k_{(3)}. \end{aligned}$$

By substituting in (33) the elements involving $\Delta_{R \# H}$ with the right-hand sides of the above three relations, and then by applying $R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H$, it results in

$$(35) \quad \begin{aligned} &\tilde{\delta}((r \# h)(s \# k)) \\ &= \sum \tilde{\delta}^1(r \# h_{(1)}) \tilde{\delta}^2(r \# h_{(1)})_{(-1)} h_{(2)} \tilde{\delta}^1(s \# k) \otimes \tilde{\delta}^2(r \# h_{(1)})_{(0)} {}^{h_{(3)}} \tilde{\delta}^2(s \# k) \end{aligned}$$

Since in $R \otimes R$ the multiplication is defined by (32), it follows that the right-hand sides of (34) and (35) are equal, so the equality (34) holds.

Conversely, if (34) holds true, then we have (35). We can replace the left-hand side of this relation by $\sum \tilde{\delta}^1(r^{h_{(1)}} s \# h_{(2)} k) \otimes \tilde{\delta}^2(r^{h_{(1)}} s \# h_{(2)} k)$. A very long

computation, using this equivalent form of (35), ends the proof of the proposition. \square

3.32. Let $\tilde{\delta} : R \otimes H \rightarrow R \otimes R$ be a K -linear map. For every $r \in R$ and $h \in H$ we introduce the notation

$$(36) \quad \delta(r) = \tilde{\delta}(r \# 1), \quad \omega(h) = \tilde{\delta}(1 \# h).$$

Then $\delta : R \rightarrow R \otimes R$ and $\omega : H \rightarrow R \otimes R$ are K -linear maps. Recall that $R \otimes R$ is an algebra in ${}^H_H\mathcal{YD}$ with the multiplication defined in 3.30. For example we can compute the product $\delta(r)\omega(h)$ in $R \otimes R$. Now, using the notation above, we can give a new interpretation of (33).

Lemma 3.33. *Let $\tilde{\delta} : R \otimes H \rightarrow R \otimes R$ be a K -linear map. Then $\Delta_{R \# H}$ is a morphism of algebras iff $\delta(1_R) = 1_R \otimes 1_R$, $\omega(1_H) = 1_R \otimes 1_R$ and $\delta, \tilde{\delta}$ and ω satisfy the following four relations:*

$$(37) \quad \tilde{\delta}(r \# h) = \delta(r)\omega(h),$$

$$(38) \quad \delta(rs) = \delta(r)\delta(s),$$

$$(39) \quad \omega(hk) = \sum \omega(h_{(1)})^{h_{(2)}}\omega(k),$$

$$(40) \quad \sum \delta(h_{(1)}r)\omega(h_{(2)}) = \sum \omega(h_{(1)})^{h_{(2)}}\delta(r).$$

Proof. By Lemma 3.31, the map $\Delta_{R \# H}$ is multiplicative if and only if (34) holds, i.e.

$$\tilde{\delta}((r \# h)(s \# k)) = \sum \tilde{\delta}(r \# h_{(1)})^{h_{(2)}}\tilde{\delta}(s \# k).$$

Now assume that (34) holds. Then setting $h = 1_H = k$ we obtain (38), while for $r = 1_R = s$ we obtain (39). Also for $h = 1_H$ and $s = 1_R$ we get (37) and for $r = 1_R$ and $k = 1_H$ we get (40), by means of (37). Conversely assume that (38), (40), (37) and (39) hold true. Then by (37), (38) and (39) we have

$$\tilde{\delta}((r \# h)(s \# k)) = \sum \delta(r)^{h_{(1)}}\omega(h_{(2)}k) = \sum \delta(r)\delta(h_{(1)}s)\omega(h_{(2)})^{h_{(3)}}\omega(k).$$

So, by (40) and by the fact that $R \otimes R$ is an algebra in ${}^H_H\mathcal{YD}$ (hence an H -module algebra), we get

$$\tilde{\delta}((r \# h)(s \# k)) = \sum \delta(r)\omega(h_{(1)})^{h_{(2)}}\delta(s)^{h_{(3)}}\omega(k) = \sum \delta(r)\omega(h_{(1)})^{h_{(2)}}[\delta(s)\omega(k)].$$

Now we can prove (34) by using (37) once again. Obviously $\Delta_{R \# H}$ is a morphism of unitary rings if and only if $\delta(1_R) = 1_R \otimes 1_R$ and $\omega(1_H) = 1_R \otimes 1_R$. \square

Remark 3.34. By (37) we can recover $\tilde{\delta}$ from $\delta : R \rightarrow R \otimes R$ and $\omega : H \rightarrow R \otimes R$. Equation (38) says that δ is multiplicative with the algebra structure on $R \otimes R$ introduced in 3.30. We have already noted that $R \otimes R$ is a left H -module algebra. For an arbitrary left H -module algebra A , Sweedler, in [Sw2], defined a non-commutative 1-cocycle with a coefficient in A to be a K -linear map $\theta : H \rightarrow A$ such that

$$\theta(hk) = \sum \theta(h_{(1)})^{h_{(2)}}\theta(k).$$

Hence (39) means that ω is a 1-cocycle with coefficients in A .

Lemma 3.35. *Assume that $\Delta_{R \# H}$ is multiplicative. Then (29) holds iff δ and ω are left H -colinear (where H is a left H -comodule with left adjoint coaction).*

Proof. Assume that (29) holds and let $r \in R$, $h \in H$. By evaluating (29) at $r \# 1$ we get

$$\rho_{R \otimes R}^l(\delta(r)) = \sum r_{\langle -1 \rangle} \otimes \delta(r_{\langle 0 \rangle}),$$

so δ is H -colinear. Similarly, for $1 \# h$ we have

$$\begin{aligned} & \sum h_{(1)} \otimes \omega^1(h_{(2)}) \otimes \omega^2(h_{(2)}) \\ &= \sum \omega^1(h_{(1)})_{\langle -1 \rangle} \omega^2(h_{(1)})_{\langle -1 \rangle} h_{(2)} \otimes \omega^1(h_{(1)})_{\langle 0 \rangle} \otimes \omega^2(h_{(1)})_{\langle 0 \rangle}, \end{aligned}$$

i.e. we get

$$(41) \quad \sum h_{(1)} \otimes \omega(h_{(2)}) = \sum \omega(h_{(1)})_{\langle -1 \rangle} h_{(2)} \otimes \omega(h_{(1)})_{\langle 0 \rangle}.$$

On the other hand,

$$\begin{aligned} \sum \omega(h)_{\langle -1 \rangle} \otimes \omega(h)_{\langle 0 \rangle} &= \sum \omega(h_{(1)})_{\langle -1 \rangle} h_{(2)} S(h_{(3)}) \otimes \omega(h_{(1)})_{\langle 0 \rangle} \\ &= \sum h_{(1)} S(h_{(3)}) \otimes \omega(h_{(2)}), \end{aligned}$$

where the last equality holds in view of (41). Hence ω is left H -colinear.

Conversely, assume that δ and ω are left H -colinear. Relation (29), that we have to prove, is equivalent to $A_l(r, s) = A_r(r, s)$, where

$$(42) \quad A_l(r, s) = \sum \tilde{\delta}^1(r \# h_{(1)})_{\langle -1 \rangle} \tilde{\delta}^2(r \# h_{(1)})_{\langle -1 \rangle} h_{(2)} \otimes \tilde{\delta}^1(r \# h_{(1)})_{\langle 0 \rangle} \tilde{\delta}^2(r \# h_{(1)})_{\langle 0 \rangle},$$

$$(43) \quad A_r(r, s) = \sum r_{\langle -1 \rangle} h_{(1)} \otimes \tilde{\delta}(r_{\langle 0 \rangle} \# h_{(2)}).$$

Then, since $\Delta_{R \# H}$ is multiplicative, by (37) we have

$$\begin{aligned} A_l(r, s) &= \sum \tilde{\delta}(r \otimes h_{(1)})_{\langle -1 \rangle} h_{(2)} \otimes \tilde{\delta}(r \otimes h_{(1)})_{\langle 0 \rangle} \\ &= \sum \delta(r)_{\langle -1 \rangle} \omega(h_{(1)})_{\langle -1 \rangle} h_{(2)} \otimes \delta(r)_{\langle 0 \rangle} \omega(h_{(1)})_{\langle 0 \rangle}. \end{aligned}$$

Since δ and ω are left colinear it results in

$$\begin{aligned} & \sum \delta(r)_{\langle -1 \rangle} \omega(h_{(1)})_{\langle -1 \rangle} h_{(2)} \otimes \delta(r)_{\langle 0 \rangle} \omega(h_{(1)})_{\langle 0 \rangle} \\ &= \sum r_{\langle -1 \rangle} h_{(1)} \otimes \delta(r_{\langle 0 \rangle}) \omega(h_{(2)}) = A_r(r, s), \end{aligned}$$

so $A_l(r, s) = A_r(r, s)$, and the lemma is proved. \square

3.36. To simplify the notation, for every $r \in R$, let $\delta(r) := \sum r^{(1)} \otimes r^{(2)}$. This is a kind of sigma notation that we shall use for δ .

Lemma 3.37. *Assume that $\Delta_{R \# H}$ is a morphism of algebras such that δ is left H -colinear. Then (31) holds iff the following two relations hold true for any $r \in R$ and $h \in H$:*

$$(44) \quad \sum r^{(1)} \otimes \delta(r^{(2)}) = \sum \delta(r^{(1)}) \omega(r_{\langle -1 \rangle}^{(2)}) \otimes r_{\langle 0 \rangle}^{(2)},$$

$$\begin{aligned} (45) \quad & \sum \omega^1(h_{(1)}) \otimes \delta(\omega^2(h_{(1)})) \omega(h_{(2)}) \\ &= \sum \delta(\omega^1(h_{(1)})) \omega(\omega^2(h_{(1)})_{\langle -1 \rangle} h_{(2)}) \otimes \omega^2(h_{(1)})_{\langle 0 \rangle}. \end{aligned}$$

Proof. Since $\Delta_{R\#H}$ is multiplicative it is straightforward to prove that (31) holds iff, for every $r \in R$ and $h \in H$, we have $B_l(r, h) = B_r(r, h)$, where

$$(46) \quad B_l(r, h) = \sum r^{(1)} \left(r^{(2)}_{\langle -1 \rangle} \omega^1(h_{(1)}) \right) \otimes \delta \left(r^{(2)}_{\langle 0 \rangle} \omega^2(h_{(1)}) \right) \omega(h_{(2)}),$$

(47)

$$B_r(r, h) = \sum \delta \left(r^{(1)} r^{(2)}_{\langle -2 \rangle} \omega^1(h_{(1)}) \right) \omega \left(r^{(2)}_{\langle -1 \rangle} \omega^2(h_{(1)})_{\langle -1 \rangle} h_{(2)} \right) \otimes r^{(2)}_{\langle 0 \rangle} \omega^2(h_{(1)})_{\langle 0 \rangle}.$$

Since $\Delta_{R\#H}$ is a morphism of algebras we have $\delta(1_R) = 1_R \otimes 1_R$ and $\omega(1_H) = 1_R \otimes 1_R$. Hence one can easily see that (44) and (45) are equivalent to $B_l(r, 1) = B_r(r, 1)$ and $B_l(1, h) = B_r(1, h)$, respectively. In particular, (31) implies (44) and (45). In order to prove the converse, let us denote by $C_l(h)$ and $C_r(h)$ the left- and right-hand sides of (45). Since δ is left H -colinear, and by using (38), it results in

$$B_l(r, h) = \sum \left(r^{(1)} \otimes \delta(r^{(2)}) \right) C_l(h),$$

where the product is performed in $R \otimes R \otimes R$, which is an algebra with multiplication given by

$$(r \otimes s \otimes t)(r' \otimes s' \otimes t') = \sum r^{s_{\langle -1 \rangle} t_{\langle -2 \rangle}} r' \otimes s_{\langle 0 \rangle}^{t_{\langle -1 \rangle}} s' \otimes t_{\langle 0 \rangle} t'.$$

Similarly, by (37) and (34), it follows that

$$B_r(r, h) = \sum \left(\delta(r^{(1)}) \omega(r^{(2)}_{\langle -1 \rangle}) \otimes r^{(2)}_{\langle 0 \rangle} \right) C_r(h).$$

By multiplying (44) and (45) side by side in $R \otimes R \otimes R$, we deduce that $B_l(r, h) = B_r(r, h)$. \square

3.38. We are seeking conditions such that $(R\#H, \Delta_{R\#H}, \varepsilon_{R\#H})$ becomes a coalgebra in the monoidal category $({}^H\mathfrak{M}^H, \square_H, H)$. Note that, in this case, $\varepsilon_{R\#H} = \varepsilon_H \circ \bar{\varepsilon}_{R\#H}$, where, by Lemma 3.7,

$$(48) \quad \bar{\varepsilon}_{R\#H}(r\#h) = \sum \varepsilon_{R\#H}[(r\#h)_{\langle 0 \rangle}](r\#h)_{\langle 1 \rangle} = \sum \varepsilon_{R\#H}(r\#h_{(1)})h_{(2)} = \varepsilon(r)h.$$

Note that $\bar{\varepsilon}_{R\#H}$ is a map in the category ${}^H\mathfrak{M}^H$.

Lemma 3.39. *Let R be an algebra in ${}^H_H\mathcal{YD}$ and let $\varepsilon : R \rightarrow K$ be a K -linear map. The map $\varepsilon_{R\#H} : R\#H \rightarrow K$, $\varepsilon_{R\#H}(r \otimes h) := \varepsilon(r)\varepsilon_H(h)$ is an algebra map and $\bar{\varepsilon}_{R\#H} : R\#H \rightarrow H$, defined as in (48), is a left H -colinear map if and only if ε is an algebra map in ${}^H_H\mathcal{YD}$.*

Proof. The map $\bar{\varepsilon}_{R\#H}$ is left H -colinear if and only if

$$(49) \quad \sum \varepsilon(r)h_{(1)} \otimes h_{(2)} = \sum r_{\langle -1 \rangle} h_{(1)} \varepsilon(r_{\langle 0 \rangle}) \otimes h_{(2)}, \forall r \in R, h \in H.$$

Note that relation (49) holds true if and only if

$$(50) \quad \varepsilon(r)h = \sum r_{\langle -1 \rangle} h \varepsilon(r_{\langle 0 \rangle}), \forall r \in R, h \in H,$$

i.e. if and only if ε is left H -colinear. In fact from (50), (49) follows easily. Conversely, by applying $H \otimes \varepsilon_H$ to (49), we get (50). Now let us prove that $\varepsilon_{R\#H}$ is an algebra map if and only if ε is an H -linear algebra map.

“ \Rightarrow ” Assume that the map $\varepsilon_{R\#H} : R\#H \rightarrow K, \varepsilon_{R\#H}(r \otimes h) := \varepsilon(r)\varepsilon_H(h)$ is an algebra map. By the definition of the multiplication in $R\#H$ and the definition of $\varepsilon_{R\#H}$ we get

$$(51) \quad \begin{aligned} \sum \varepsilon(r^{h_{(1)}}s)\varepsilon_H(h_{(2)}v) &= \varepsilon_{R\#H}((r\#h)(s\#v)) = \varepsilon_{R\#H}(r\#h)\varepsilon_{R\#H}(s\#v) \\ &= \varepsilon(r)\varepsilon_H(h)\varepsilon(s)\varepsilon_H(v). \end{aligned}$$

Then $\varepsilon_{R\#H}(1_R \otimes 1_H) = 1$, i.e. $\varepsilon(r) = 1$. Thus, from (51), we have

$$\varepsilon(hs) = \sum \varepsilon(1_R^{h_{(1)}}s)\varepsilon_H(h_{(2)}1_H) = \varepsilon(1_R)\varepsilon_H(h)\varepsilon(s)\varepsilon_H(1_H) = \varepsilon_H(h)\varepsilon(s)$$

and

$$\varepsilon(rs) = \sum \varepsilon(rs)\varepsilon_H(1_H1_H) = \varepsilon(r)\varepsilon_H(1_H)\varepsilon(s)\varepsilon_H(1_H) = \varepsilon(r)\varepsilon(s),$$

i.e. ε is an H -linear algebra map.

“ \Leftarrow ” Now assume that ε is an H -linear algebra map. Thus

$$\begin{aligned} \varepsilon_{R\#H}((r\#h)(s\#k)) &= \sum \varepsilon_{R\#H}(r^{h_{(1)}}s\#h_{(2)}k) \\ &= \sum \varepsilon(r^{h_{(1)}}s)\varepsilon_H(h_{(2)}k) \\ &= \sum \varepsilon(r)\varepsilon_H(h_{(1)})\varepsilon(s)\varepsilon_H(h_{(2)})\varepsilon_H(k) \\ &= \varepsilon_{R\#H}((r\#h)(s\#k)). \end{aligned}$$

□

Lemma 3.40. Assume that ε is an algebra map in ${}^H_H\mathcal{YD}$. Then $\varepsilon_{R\#H} : R\#H \rightarrow K$ is a counit for $\Delta_{R\#H}$ if and only if, for every $r \in R$ and $h \in H$, we have

$$(52) \quad \sum \varepsilon(\tilde{\delta}^1(r \otimes h))\tilde{\delta}^2(r \otimes h) = \varepsilon_H(h)r = \sum \tilde{\delta}^1(r \otimes h)\varepsilon(\tilde{\delta}^2(r \otimes h)).$$

Proof. Assume that $\varepsilon_{R\#H}$ is a counit for $\Delta_{R\#H}$. Then, by the definition of $\Delta_{R\#H}$ (see (26)), it results in

$$r \otimes h = \sum \tilde{\delta}^1(r \otimes h_{(1)}) \otimes \tilde{\delta}^2(r \otimes h_{(1)})_{(-1)}h_{(2)}\varepsilon(\tilde{\delta}^2(r \otimes h_{(1)})_{(0)})\varepsilon_H(h_{(3)}).$$

By applying $R \otimes \varepsilon_H$ to this relation, we get the second equality of (52). The other one can be proved similarly.

Conversely assume that the equality (52) holds. Since ε is left H -colinear, we have

$$\begin{aligned} (R\#H \otimes \varepsilon_{R\#H})\Delta_{R\#H} &= \sum \tilde{\delta}^1(r \otimes h_{(1)})\varepsilon(\tilde{\delta}^2(r \otimes h_{(1)})) \otimes h_{(2)} \\ &= \sum r\varepsilon_H(h_{(1)}) \otimes h_{(2)} = r \otimes h. \end{aligned}$$

We can prove the second relation analogously. □

Lemma 3.41. Assume that $\Delta_{R\#H}$ is multiplicative and that $\varepsilon : R \rightarrow K$ is an algebra map in ${}^H_H\mathcal{YD}$. Then (52) holds if and only if

$$(53) \quad (\varepsilon \otimes R)\delta = (R \otimes \varepsilon)\delta = \text{Id}_R,$$

$$(54) \quad (\varepsilon \otimes R)\omega = (R \otimes \varepsilon)\omega = \varepsilon_H 1_R.$$

Proof. First let us observe that $\varepsilon \otimes R : R \otimes R \rightarrow R$ and $R \otimes \varepsilon : R \otimes R \rightarrow R$ are algebra maps (recall that $R \otimes R$ is an algebra with multiplication $(m_R \otimes m_R \otimes R)(R \otimes c_{R,R})$, where c is the braiding in ${}^H_H\mathcal{YD}$). Clearly (52) holds if and only if

$$(\varepsilon \otimes R)\tilde{\delta}(r\#h) = \varepsilon_H(h)r = (R \otimes \varepsilon)\tilde{\delta}(r\#h), \forall r \in R, \forall h \in H.$$

Assume now that (53) and (54) hold. Then

$$(\varepsilon \otimes R)\tilde{\delta}(r\#h) = (\varepsilon \otimes R)\delta(r) \cdot (\varepsilon \otimes R)\omega(h) = \varepsilon_H(h)r.$$

Analogously we can deduce the second equality of (52). The other implication is trivial. \square

To state the main results of this part more easily, in the next definition we collect together all the required properties of δ , ω and ε .

Definition 3.42. Let H be a Hopf algebra and let R be an algebra in $({}^H_H\mathcal{YD}, \otimes, K)$. Assume that $\varepsilon : R \rightarrow K$, $\delta : R \rightarrow R \otimes R$ and $\omega : H \rightarrow R \otimes R$ are K -linear maps. The quadruple $(R, \varepsilon, \delta, \omega)$ will be called a *Yetter–Drinfeld quadruple* if and only if, for all $r, s \in R$ and $h, k \in H$, the following relations are satisfied:

$$(55) \quad \varepsilon(hr) = \varepsilon(r)\varepsilon_H(h) \quad \text{and} \quad \sum r_{\langle -1 \rangle} \varepsilon(r_{\langle 0 \rangle}) = \varepsilon(r)1_H;$$

$$(56) \quad \varepsilon(rs) = \varepsilon(r)\varepsilon(s) \quad \text{and} \quad \varepsilon(1_R) = 1;$$

$$(57) \quad \rho_{R \otimes R}(\delta(r)) = \sum r_{\langle -1 \rangle} \otimes \delta(r_{\langle 0 \rangle});$$

$$(58) \quad \rho_{R \otimes R}(\omega(h)) = \sum h_{(1)} S(h_{(3)}) \otimes \omega(h_{(2)});$$

$$(59) \quad \delta(rs) = \delta(r)\delta(s) \quad \text{and} \quad \delta(1_R) = 1_R \otimes 1_R;$$

$$(60) \quad \omega(hk) = \sum \omega(h_{(1)}) (h_{(2)} \omega(k)) \quad \text{and} \quad \omega(1_H) = 1_R \otimes 1_R;$$

$$(61) \quad \sum \delta(h_{(1)}r) \omega(h_{(2)}) = \sum \omega(h_{(1)}) h_{(2)} \delta(r);$$

$$(62) \quad \sum r^{(1)} \otimes \delta(r^{(2)}) = \sum \delta(r^{(1)}) \omega(r_{\langle -1 \rangle}^{(2)}) \otimes r_{\langle 0 \rangle}^{(2)};$$

$$(63) \quad \begin{aligned} \sum \omega^1(h_{(1)}) \otimes \delta(\omega^2(h_{(1)})) \omega(h_{(2)}) \\ = \sum \delta(\omega^1(h_{(1)})) \omega(\omega^2(h_{(1)})_{\langle -1 \rangle} h_{(2)}) \otimes \omega^2(h_{(1)})_{\langle 0 \rangle}; \end{aligned}$$

$$(64) \quad (\varepsilon \otimes R)\delta = (R \otimes \varepsilon)\delta = \text{Id}_R;$$

$$(65) \quad (\varepsilon \otimes R)\omega = (R \otimes \varepsilon)\omega = \varepsilon_H 1_R.$$

Remark 3.43. Note that these relations can be interpreted as follows:

- (55) – ε is a morphism in ${}^H_H\mathcal{YD}$;
- (56) – ε is a morphism of algebras;
- (57) – δ is left H -colinear;
- (58) – ω is left H -colinear, where H is a comodule with the adjoint coaction;
- (59) – δ is a morphism of algebras where on $R \otimes R$ we consider the algebra structure that uses the braiding c ;
- (60) – ω is a normalized cocycle;
- (61) – ω measures how far δ is to a morphism of left H -modules (if ω is trivial, i.e. for every $h \in H$ we have $\omega(h) = \varepsilon(h)1_R \otimes 1_R$, then δ is left H -linear); we shall say that δ is a *twisted morphism* of left H -modules;

(62) – it was derived from the fact that $\Delta_{R\#H}$ is coassociative, so we shall say that δ is ω -coassociative (when ω is trivial then (62) is equivalent to the fact that δ is coassociative);

(63) – is the only property that has not an equivalent in the theory of bialgebras; we shall just say that δ and ω are compatible;

(64) – δ is a counitary map with respect to ε ;

(65) – ω is a counitary map with respect to ε .

Since ε satisfies the last two relations, we shall call it the *counit* of the Yetter–Drinfeld quadruple R . By analogy δ will be called the *comultiplication* of R . Finally, we shall say that ω is the *cocycle* of R .

3.44. To every Yetter–Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$, we associate the K -linear maps $\Delta_{R\#H} : R\#H \rightarrow (R\#H) \otimes (R\#H)$ and $\varepsilon_{R\#H} : R\#H \rightarrow K$, which are defined by

(66)

$$\Delta_{R\#H}(r \otimes h) = \sum \tilde{\delta}^1(r \otimes h_{(1)}) \otimes \tilde{\delta}^2(r \otimes h_{(1)})_{\langle -1 \rangle} h_{(2)} \otimes \tilde{\delta}^2(r \otimes h_{(1)})_{\langle 0 \rangle} \otimes h_{(3)},$$

(67)

$$\varepsilon_{R\#H}(r\#h) = \varepsilon(r)\varepsilon_H(h),$$

where $\tilde{\delta}(r\#h) := \delta(r)\omega(h)$, and we use the notation

$$\tilde{\delta}(r\#h) = \sum \tilde{\delta}^1(r \otimes h) \otimes \tilde{\delta}^2(r \otimes h).$$

Theorem 3.45. *Let R be an algebra in ${}^H_H\mathcal{YD}$. If $\varepsilon : R \rightarrow K$, $\delta : R \rightarrow R \otimes R$ and $\omega : H \rightarrow R \otimes R$ are linear maps, then the following assertions are equivalent:*

(a) $(R, \varepsilon, \delta, \omega)$ is a Yetter–Drinfeld quadruple.

(b) The smash product algebra $R\#H$ is a bialgebra with comultiplication $\Delta_{R\#H}$ and counit $\varepsilon_{R\#H}$ defined by (66) and (67) such that $R\#H$ becomes an algebra in $({}^H_H\mathfrak{M}^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}^H, \square_H, H)$.

Proof. (a) \Rightarrow (b) In view of the definition of $\Delta_{R\#H}$, we have that (37) holds. Then, since also (59), (60) and (61) hold, by Lemma 3.33 it results that $\Delta_{R\#H}$ is a unitary algebra morphism.

Since $\Delta_{R\#H}$ is multiplicative, we can apply Lemma 3.28 and Lemma 3.35 to deduce that $\Delta_{R\#H}$ is left H -colinear by using relations (57) and (58), i.e. that δ and ω are left H -colinear. On the other hand, by 3.21, we get that $\Delta_{R\#H}$ is right colinear, so $\Delta_{R\#H}$ is a morphism of (H, H) -bicomodules. Also, by 3.21, it follows that the image of $\Delta_{R\#H}$ is included in $(R\#H)\square_H(R\#H)$.

Since $\Delta_{R\#H}$ is multiplicative and left H -colinear and since δ is also left H -colinear, by (62) and (63) it results that $\Delta_{R\#H}$ is coassociative (use Lemma 3.37 and Lemma 3.29).

To prove that $\varepsilon_{R\#H}$ is a morphism of algebras we use Lemma 3.39, (55) and (56). Finally, in view of (64) and (65), Lemma 3.41 and Lemma 3.40 imply that $\varepsilon_{R\#H}$ is a counit for $\Delta_{R\#H}$. All these properties together mean that $R\#H$ is a bialgebra that, in view of (b) \Rightarrow (a) of Proposition 3.9, becomes a coalgebra in $({}^H_H\mathfrak{M}^H, \square_H, H)$. We conclude by remarking that $R\#H$ always becomes an algebra in $({}^H_H\mathfrak{M}^H, \otimes_H, H)$; see Example 3.14.

(b) \Rightarrow (a) Since $\Delta_{R\#H}$ is morphism of algebras, by Lemma 3.33 it follows that (59), (60) and (61) hold true. As $\Delta_{R\#H}$ is left H -colinear and multiplicative, by Lemma 3.28 and Lemma 3.35, δ and ω are H -colinear, so that (57) and (58) hold.

Since we have already proved that δ is left H -colinear, we can apply Lemma 3.37 and Lemma 3.29 to deduce (62) and (63) from the fact that $\Delta_{R\#H}$ is coassociative.

Since $(R\#H, \Delta_{R\#H}, \varepsilon_{R\#H})$ becomes a coalgebra in $({}^H\mathfrak{M}^H, \square_H, H)$ and $\varepsilon_{R\#H}$ is an algebra map, in view of 3.38 we can apply Lemma 3.39 to get that ε is an algebra map in ${}^H_H\mathcal{YD}$, so that (55) and (56) hold true.

Since $\varepsilon_{R\#H}$ is a counit for $\Delta_{R\#H}$ and since $\Delta_{R\#H}$ is multiplicative, by Lemma 3.40 and Lemma 3.41, we conclude that (64) and (65) hold. Thus $(R, \varepsilon, \delta, \omega)$ is a Yetter–Drinfeld quadruple. \square

Definition 3.46. Let $(R, \varepsilon, \delta, \omega)$ be a Yetter–Drinfeld quadruple. The smash product algebra $R\#H$, endowed with the bialgebra structure described in Theorem 3.45, will be called the *bosonization* of $(R, \varepsilon, \delta, \omega)$ and will be denoted by $R\#_b H$.

Proposition 3.47. Let R be an H -bicomodule algebra. Let $\phi : (A, m_A, u_A) \rightarrow (B, m_B, v_B)$ be an isomorphism of algebras in the category of vector spaces. If $A \in {}^H_R\mathfrak{M}^H_R$, then B can be endowed, via ϕ , with obvious Hopf bicomodule structures and $\phi : A \rightarrow B$ is an isomorphism in ${}^H_R\mathfrak{M}^H_R$. Moreover, if A becomes an algebra in $({}^H_R\mathfrak{M}^H_R, \otimes_R, R)$, then (B, m_B, u_B) also becomes an algebra in $({}^H_R\mathfrak{M}^H_R, \otimes_R, R)$ such that $\phi : (A, \overline{m}_A, \overline{u}_A) \rightarrow (B, \overline{m}_B, \overline{u}_B)$ is an algebra isomorphism in the category $({}^H_R\mathfrak{M}^H_R, \otimes_R, R)$.

Proof. Obvious. \square

Proposition 3.48. Let D be an H -bimodule coalgebra. Let $\phi : (A, \Delta_A, \varepsilon_A) \rightarrow (B, \Delta_B, \varepsilon_B)$ be an isomorphism of coalgebras in the category of vector spaces. If $A \in {}^D_H\mathfrak{M}^D_H$, then B can be endowed, via ϕ , with obvious Hopf bicomodule structures and $\phi : A \rightarrow B$ is an isomorphism in ${}^D_H\mathfrak{M}^D_H$. Moreover, if A becomes a coalgebra in $({}^D_H\mathfrak{M}^D_H, \square_D, D)$, then $(B, \Delta_B, \varepsilon_B)$ also becomes a coalgebra in $({}^D_H\mathfrak{M}^D_H, \square_D, D)$ such that $\phi : (A, \overline{\Delta}_A, \overline{\varepsilon}_A) \rightarrow (B, \overline{\Delta}_B, \overline{\varepsilon}_B)$ is a coalgebra isomorphism in the category $({}^D_H\mathfrak{M}^D_H, \square_D, D)$.

Proof. Obvious. \square

We have been informed that the dual form of the equivalence (b) \Leftrightarrow (c) below, as stated in Theorem 3.64, has already been proved by P. Schauenburg (see 6.1 and Theorem 5.1 in [Sch2]). Nevertheless, for sake of completeness, we decided to keep our proof.

Theorem 3.49. Let A be a bialgebra and let H be a Hopf algebra. The following assertions are equivalent:

- (a) A is an object in ${}^H_H\mathfrak{M}^H_H$, the counit $\varepsilon_A : A \rightarrow K$ is right H -linear and A becomes an algebra in $({}^H_H\mathfrak{M}^H_H, \otimes_H, H)$ and a coalgebra in $({}^H\mathfrak{M}^H, \square_H, H)$.
- (b) There is an algebra R in ${}^H_H\mathcal{YD}$, and there are maps $\varepsilon_R : R \rightarrow k, \delta : R \rightarrow R \otimes R$ and $\omega : H \rightarrow R \otimes R$ such that $(R, \varepsilon_R, \delta, \omega)$ is a Yetter–Drinfeld quadruple and A is isomorphic as a bialgebra to the bosonization $R\#_b H$ of this Yetter–Drinfeld quadruple.
- (c) There are a bialgebra map $\pi : A \rightarrow H$ and an (H, H) -bilinear algebra map $\sigma : H \rightarrow A$ such that $\pi\sigma = \text{Id}_H$.

Moreover, if (c) holds, we can choose the Yetter–Drinfeld quadruple $(R, \varepsilon_R, \delta, \omega)$, where

$$R = A^{Co(H)}, \quad \varepsilon_R = \varepsilon_{A|H},$$

$$\delta(r) = r_{(1)}\sigma S\pi(r_{(2)}) \otimes r_{(3)}, \quad \omega(h) = \sigma(h)_{(1)}\sigma S\pi[\sigma(h)_{(2)}] \otimes \sigma(h)_{(3)}\sigma S\pi[\sigma(h)_{(4)}].$$

Proof. (a) \Rightarrow (b) By 3.22 the canonical map $\epsilon_A : R\#H \rightarrow A$ in ${}^H_H\mathfrak{M}_H^H$ is an isomorphism of bialgebras, where the coalgebra structure on $R\#H$ is defined by $\Delta_{R\#H} := (\epsilon_A^{-1} \otimes \epsilon_A^{-1})\Delta\epsilon_A$ and $\varepsilon_{R\#H} := \varepsilon_A\epsilon_A$. Clearly, by Proposition 3.47 and Proposition 3.48, $R\#H$ becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$, since A does. Let ε be the restriction of ε_A to R . As explained in 3.21, if

$$\tilde{\delta} = (R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H)\Delta_{R\#H},$$

and for $r \in R$, $h \in H$ we write $\tilde{\delta}(r\#h) = \sum \tilde{\delta}^1(r\#h) \otimes \tilde{\delta}^2(r\#h) \in R \otimes R$, then, as $\Delta_{R\#H}$ is right H -colinear, we have

$$\Delta_{R\#H}(r\#h) = \sum \tilde{\delta}^1(r\#h_{(1)})\#\tilde{\delta}^2(r\#h_{(1)})_{\langle -1 \rangle}h_{(2)} \otimes \tilde{\delta}^2(r\#h_{(1)})_{\langle 0 \rangle}\#h_3.$$

Let us define the K -linear maps δ and ω as in (36). Since $\Delta_{R\#H}$ is a morphism of algebras, by Lemma 3.33 it follows that $\tilde{\delta}(r\#h) := \delta(r)\omega(h)$. Thus we can apply Theorem 3.45 to conclude that $(R, \varepsilon, \delta, \omega)$ is a Yetter–Drinfeld quadruple. Note that the bosonization of this Yetter–Drinfeld quadruple is the bialgebra $R\#H$ constructed above.

(b) \Rightarrow (a) By Proposition 3.47 and Proposition 3.48, A is an object in ${}^H_H\mathfrak{M}_H^H$ and A becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$.

Since $\varepsilon_{R\#H}$ is defined by (67), it is right H -linear, so that the map $\varepsilon_A : A \rightarrow K$ is right H -linear, too.

(a) \Leftrightarrow (c) follows by Theorem 3.12.

The last statement follows by direct computation, using the canonical isomorphism $\epsilon_A : R\#H \rightarrow A$ in ${}^H_H\mathfrak{M}_H^H$, which turns out to be $\epsilon_A(r\#h) = r\sigma(h)$, the inverse being defined by $\epsilon_A^{-1}(a) = a_{(1)}\sigma S\pi(a_{(2)}) \otimes \pi(a_{(3)})$. \square

Remark 3.50. Let $(R, \varepsilon, \delta, \omega)$ be a Yetter–Drinfeld quadruple such that ω is trivial. Recall that this means that

$$\omega(h) = \varepsilon_H(h)1_R \otimes 1_R, \text{ for all } h \in H.$$

Then it is easy to check that relations (55)–(65) are equivalent to the fact that (R, δ, ε) is a bialgebra in $({}^H_H\mathcal{YD}, \otimes, K)$. Conversely, starting with a bialgebra (R, δ, ε) in the monoidal category $({}^H_H\mathcal{YD}, \otimes, K)$, we can consider the Yetter–Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$, where ω is the trivial cocycle. Furthermore, the bosonization of this Yetter–Drinfeld quadruple is the usual bosonization of the bialgebra R , i.e. as an algebra is the smash product $R\#H$ and as a coalgebra is the smash coproduct. Recall that the comultiplication and counit of the smash coproduct are respectively defined by

$$\begin{aligned} \Delta_{R\#H}(r\#h) &= \sum r^{(1)} \otimes r^{(2)}_{\langle -1 \rangle}h_{(1)} \otimes r^{(2)}_{\langle 0 \rangle} \otimes h_{(2)}, \\ \varepsilon_{R\#H}(r\#h) &= \varepsilon(r)\varepsilon(h), \end{aligned}$$

where, by notation, $\delta(r) = \sum r^{(1)} \otimes r^{(2)}$.

Corollary 3.51 (D. Radford). *Let H be a Hopf algebra and let A be a bialgebra. Then the following statements are equivalent:*

- (a) *A is an object in ${}^H_H\mathfrak{M}_H^H$, A becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$.*
- (b) *The diagram R of A is a bialgebra in $({}^H_H\mathcal{YD}, \otimes, K)$ such that A is isomorphic as a bialgebra to the usual bosonization of R by H .*
- (c) *There are two bialgebra morphisms $\pi : A \rightarrow H$, $\sigma : H \rightarrow A$ such that $\pi\sigma = \text{Id}_H$.*

Proof. (a) \Rightarrow (c) Note that the map $\varepsilon_A : A \rightarrow K$ is right H -linear, so that, by Theorem 3.12 and by Theorem 3.13, we conclude the proof.

(c) \Rightarrow (b) We apply Theorem 3.49. Let $(R, \varepsilon_R, \delta, \omega)$ be the the Yetter-Drinfeld quadruple that corresponds to π and σ , i.e.

$$R = A^{Co(H)}, \quad \varepsilon_R = \varepsilon_{A|R},$$

$$\delta(r) = r_{(1)}\sigma S\pi(r_{(2)}) \otimes r_{(3)}, \quad \omega(h) = \sigma(h)_{(1)}\sigma S\pi[\sigma(h)_{(2)}] \otimes \sigma(h)_{(3)}\sigma S\pi[\sigma(h)_{(4)}].$$

Since σ is a coalgebra map, then ω is trivial.

(b) \Rightarrow (a) follows by Proposition 3.47 and Proposition 3.48. In fact, as explained in Example 3.14 and in Example 3.17, the usual bosonization $R\#H$ of R by H is an object in ${}^H_H\mathfrak{M}_H^H$ that becomes an algebra in $({}^H_H\mathfrak{M}_H^H, \otimes_H, H)$ and a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$. \square

Lemma 3.52. *Let A be a bialgebra over a field K and let I be a nilpotent ideal and coideal of A . If the quotient bialgebra A/I has an antipode, then A is a Hopf algebra.*

Proof. Let us point out that an element x in a ring R is invertible if it is invertible modulo a nil ideal L of R . We apply this to the ring $R = \text{Hom}_K(A, A)$ endowed with the convolution product, to the nil ideal $L = \text{Hom}_K(A, I)$ and to $x = \text{Id}_A$. The quotient R/L is isomorphic to the algebra $\text{Hom}_K(A, A/I)$ and, through this identification, the class of Id_A corresponds to the canonical projection $p : A \rightarrow A/I$. We conclude by remarking that the inverse of p in $\text{Hom}_K(A, A/I)$ is $p \circ S$, where S is the antipode of A/I . \square

Theorem 3.53. *Let A be a bialgebra over a field K . If the Jacobson radical J of A is a nilpotent coideal such that $H := A/J$ is a Hopf algebra which has an ad-coinvariant integral and that every canonical map $A/J^{n+1} \rightarrow A/J^n$ splits in ${}^H_H\mathfrak{M}_H^H$, then A is isomorphic as a bialgebra to the bosonization $R\#_b H$ of a certain Yetter-Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$. In fact A and $R\#_b H$ are isomorphic Hopf algebras.*

Proof. By Theorem 2.13 there is an (H, H) -bilinear algebra section $\sigma : H \rightarrow A$ of the canonical projection $\pi : A \rightarrow H$. We conclude by applying Theorem 3.49 and Lemma 3.52. \square

Theorem 3.54. *Let A be a bialgebra over a field K . If the Jacobson radical J of A is a nilpotent coideal such that $H := A/J$ is a Hopf algebra which is both semisimple and cosemisimple (e.g. when H is semisimple over a field of characteristic 0), then A is isomorphic as a bialgebra to the bosonization of a certain Yetter-Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$. In fact A and $R\#_b H$ are isomorphic Hopf algebras.*

Proof. Apply Theorem 2.28, Theorem 3.49 and Lemma 3.52. \square

3.55. We now go back to Problem 2, as stated in 3.1, i.e. to investigate those bialgebras A with the property that there is a pair of K -linear maps

$$\sigma : H \rightarrow A \quad \text{and} \quad \pi : A \rightarrow H$$

such that σ is a morphism of bialgebras and π is an (H, H) -bilinear algebra retraction of σ , i.e. $\pi\sigma = \text{Id}_H$. To this aim, we proceed as follows.

3.56. Let $R \in {}_H\mathfrak{M}$ and let $m_{R\#H} : (R\#H) \otimes_H (R\#H) \rightarrow (R\#H)$ be a right H -linear map. In analogy with 3.21, if

$$(68) \quad \tilde{m} = m_{R\#H}(R \otimes u_H \otimes R \otimes u_H),$$

and for $r \in R$, $h \in H$ we write $\tilde{m}(r \otimes s) = \sum \tilde{m}^0(r \otimes s) \otimes \tilde{m}^1(r \otimes s) \in R\#H$, then

$$(69) \quad m_{R\#H}[(r\#h) \otimes_H (s\#l)] = \sum \tilde{m}^0(r \otimes^{h_{(1)}} s) \otimes \tilde{m}^1(r \otimes^{h_{(1)}} s) h_{(2)}l.$$

Conversely if $\tilde{m} : R \otimes R \rightarrow R \otimes H$ is a linear map and $m_{R\#H}$ is defined by (69), then $m_{R\#H}$ is a right H -linear map.

Futhermore, if (R, δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$, for every $r \in R$ and $h \in H$ we introduce the notation

$$(70) \quad m = (R \otimes \varepsilon_H)\tilde{m}, \quad \xi = (\varepsilon \otimes H)\tilde{m}.$$

Then $m : R \otimes R \rightarrow R$ and $\xi : R \otimes R \rightarrow H$ are K -linear maps.

3.57. Let H be a Hopf algebra and let A be a bialgebra with multiplication m , unit u_A , comultiplication Δ and counit ε_A .

In view of Theorem 3.13, Problem 2 can be reformulated as follows: to characterize all bialgebras A that are objects in ${}^H_H\mathfrak{M}_H^H$ such that A becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H\mathfrak{M}_H, \otimes_H, H)$, with the further requirement that u_A is right H -colinear.

By Proposition 3.18, the diagram (R, δ, ε) of A is a coalgebra in ${}^H_H\mathcal{YD}$, the smash coproduct $R\#H$ is a coalgebra and the map $\epsilon_A : R\#H \rightarrow A$, $\epsilon_A(r \otimes h) = rh$ is an isomorphism of coalgebras in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$. Obviously, $R\#H$ is a bialgebra with multiplication $m_{R\#H}$ and unit $u_{R\#H}$ given by

$$m_{R\#H} := \epsilon_A^{-1} m(\epsilon_A \otimes \epsilon_A) \quad \text{and} \quad u_{R\#H} := \epsilon_A^{-1} u_A.$$

Of course, with respect to this bialgebra structure, ϵ_A becomes an isomorphism of bialgebras.

Furthermore, since A becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H\mathfrak{M}_H, \otimes_H, H)$, the smash $R\#H$ has the same properties. In particular $m_{R\#H}$ factorizes to a morphism of right H -modules $\overline{m}_{R\#H} : (R\#H) \otimes_H (R\#H) \rightarrow R\#H$.

Hence, by 3.56, $m_{R\#H}$ is uniquely determined by a K -linear map $\tilde{m} : R \otimes R \rightarrow R\#H$. In order to obtain the unit $u_{R\#H}$, we consider the corestriction of u_A to R . For simplifying the notation, we shall denote it by u .

3.58. Let R and S be two coalgebras in the braided category ${}^H_H\mathcal{YD}$. We can define a new coalgebra structure on $R \otimes S$ by using the braiding (27) and not the usual flip morphism. The comultiplication in this case is defined by the formula

$$(71) \quad \delta_{R \otimes S}(r \otimes s) = \sum r^{(1)} \otimes r^{(2)}_{(-1)} s^{(1)} \otimes r^{(2)}_{(0)} \otimes s^{(2)}.$$

Let us remark that, for any coalgebra R in ${}^H_H\mathcal{YD}$, the smash coproduct $R\#H$ is a particular case of this construction. Just take $S = H$ with the left adjoint coaction and the usual left H -module structure. Another example that we are interested in

is $R \otimes R$, where R is the diagram of a bialgebra A as in 3.57. For such a coalgebra R in ${}^H_H\mathcal{YD}$ we shall always use this coalgebra structure on $R \otimes R$.

Definition 3.59. Let H be a Hopf algebra and let (R, δ, ε) be a coalgebra in the category $({}^H_H\mathcal{YD}, \otimes, K)$. Set $\delta(r) = \sum r^{(1)} \otimes r^{(2)}$. Assume that $u : K \rightarrow R$, $m : R \otimes R \rightarrow R$, and $\xi : R \otimes R \rightarrow H$ are K -linear maps. The quadruple (R, u, m, ξ) will be called a *dual Yetter–Drinfeld quadruple* if and only if, for all $r, s, t \in R$ and $h \in H$, the following relations are satisfied:

$$(72) \quad hu(1) = \varepsilon_H(h)u(1) \quad \text{and} \quad \rho_R u(1) = 1_H \otimes u(1);$$

$$(73) \quad \delta u(1) = u(1) \otimes u(1) \quad \text{and} \quad \varepsilon u(1) = 1_K;$$

$$(74) \quad {}^h m(r \otimes s) = \sum m({}^{h_{(1)}} r \otimes {}^{h_{(2)}} s);$$

$$(75) \quad \sum \xi({}^{h_{(1)}} r \otimes {}^{h_{(2)}} s) = \sum h_{(1)} \xi(r \otimes s) S h_{(2)};$$

$$(76) \quad \delta m = (m \otimes m) \delta_{R \otimes R} \quad \text{and} \quad \varepsilon m = m_K(\varepsilon \otimes \varepsilon);$$

$$(77) \quad \Delta_H \xi = (m_H \otimes H)(\xi \otimes H \otimes \xi)(R \otimes R \otimes \rho_{R \otimes R}) \delta_{R \otimes R} \quad \text{and} \quad \varepsilon_H \xi = m_K(\varepsilon \otimes \varepsilon);$$

$$(78) \quad c_{R,H}(m \otimes \xi) \delta_{R \otimes R} = (m_H \otimes R)(\xi \otimes H \otimes m)(R \otimes R \otimes \rho_{R \otimes R}) \delta_{R \otimes R};$$

$$(79) \quad m(R \otimes m) = m(m \otimes R)(R \otimes R \otimes \mu_R)(R \otimes R \otimes \xi \otimes R)(\delta_{R \otimes R} \otimes R);$$

$$(80) \quad \begin{aligned} m_H(\xi \otimes H)(R \otimes m \otimes \xi)(R \otimes \delta_{R \otimes R}) \\ = m_H(\xi \otimes H)(R \otimes c_{H,R})(m \otimes \xi \otimes R)(\delta_{R \otimes R} \otimes R); \end{aligned}$$

$$(81) \quad m(R \otimes u) = \text{Id}_R = m(u \otimes R);$$

$$(82) \quad \xi(R \otimes u) = \xi(u \otimes R) = \varepsilon 1_H.$$

Remark 3.60. Note that these relations can be interpreted as follows:

(72) – u is a morphism in ${}^H_H\mathcal{YD}$;

(73) – u is a coalgebra map;

(74) – m is left H -linear;

(75) – ξ is left H -linear, where H is a module with the adjoint action;

(76) – m is a morphism of coalgebras, where on $R \otimes R$ we consider the coalgebra structure that uses the braiding c ;

(77) – ξ is a normalized cocycle; more generally, if C is a left H -comodule coalgebra, then a map $\psi : C \rightarrow H$ is called a non-commutative 1 cocycle if

$$\Delta_H(\psi(c)) = \sum \psi(c_{(1)}) (c_{(2)})_{(-1)} \otimes \psi((c_{(2)})_{(0)});$$

(78) – ξ measures how far m is to be a morphism of left H -comodules (if ξ is trivial, i.e. for every $r, s \in R$ we have $\xi(r \otimes s) = \varepsilon(r)\varepsilon(s)$, then m is left H -colinear); we shall say that m is a *twisted morphism* of left H -comodules; we shall use the notation $m(r \otimes s) = rs$, so (78) can be rewritten as follows:

$$\begin{aligned} \sum (r^{(1)} r_{(-1)}^{(2)} s^{(1)})_{(-1)} \xi(r_{(0)}^{(2)} \otimes s^{(2)}) \otimes (r^{(1)} r_{(-1)}^{(2)} s^{(1)})_{(0)} \\ = \sum \xi(r^{(1)} \otimes r_{(-1)}^{(2)} s^{(1)}) r_{(-1)}^{(2)} s_{(-1)}^{(2)} \otimes r_{(0)}^{(2)} s_{(0)}^{(2)}; \end{aligned}$$

(79) – when ξ is trivial then (79) is equivalent to the fact that m is associative; so, in general, we shall say that m is ξ -associative; here μ_R denotes the H -action on R ;

(80) – we shall just say that m and ξ are *compatible*; it is equivalent to

$$\begin{aligned} & \sum \xi(r \otimes s^{(1)} s_{(-1)}^{(2)} t^{(1)}) \xi(s_{(0)}^{(2)} \otimes t^{(2)}) \\ &= \sum \xi(r^{(1)} r_{(-1)}^{(2)} s^{(1)} \otimes \xi(r_{(0)}^{(2)} \otimes s^{(2)})_{(1)} t) \xi(r_{(0)}^{(2)} \otimes s^{(2)})_{(2)}; \end{aligned}$$

(81) – m is a *unitary map* with respect to u ;

(82) – ξ is a *unitary map* with respect to u .

Since u satisfies the last two relations, we shall call it the *unit* of the dual Yetter–Drinfeld quadruple R . By analogy m will be called the *multiplication* of R . Finally, we shall say that ξ is the *cocycle* of R .

3.61. To every dual Yetter–Drinfeld quadruple (R, u, m, ξ) , we associate the K –linear maps $m : R \# H \otimes (R \# H) \rightarrow (R \# H)$ and $u_{R \# H} : K \rightarrow R \# H$, which are defined by

$$(83) \quad m_{R \# H}[(r \# h) \otimes_H (s \# k)] = \sum \tilde{m}^0(r \otimes^{h(1)} s) \otimes \tilde{m}^1(r \otimes^{h(1)} s) h_{(2)} k,$$

$$(84) \quad u_{R \# H}(1) = u(1) \# 1_H,$$

where $\tilde{m}(r \otimes s) = (m \otimes \xi) \delta_{R \otimes R}(r \otimes s) = \sum m(r^{(1)} \otimes r_{(-1)}^{(2)} s^{(1)}) \otimes \xi(r_{(0)}^{(2)} \otimes s^{(2)})$ and we use the notation $\tilde{m}(r \otimes s) = \sum \tilde{m}^0(r \otimes s) \otimes \tilde{m}^1(r \otimes s)$.

Theorem 3.62. *Let (R, δ, ε) be a coalgebra in ${}^H_H \mathcal{YD}$. If $u : K \rightarrow R$, $m : R \otimes R \rightarrow R$ and $\xi : R \otimes R \rightarrow H$ are linear maps, then the following assertions are equivalent:*

(a) (R, u, m, ξ) is a dual Yetter–Drinfeld quadruple.

(b) *The smash coproduct coalgebra $R \# H$ is a bialgebra with multiplication $m_{R \# H}$ and unit $u_{R \# H}$ defined by (83) and (84) such that $R \# H$ becomes a coalgebra in $({}^H_H \mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H \mathfrak{M}_H, \otimes_H, H)$.*

Proof. Dual to Theorem 3.49. □

Definition 3.63. Let (R, u, m, ξ) be a dual Yetter–Drinfeld quadruple. The smash product coalgebra $R \# H$, endowed with the bialgebra structure described in Theorem 3.62, will be called the *bosonization* of (R, u, m, ξ) and will be denoted by $R \# {}^b H$.

As we already remarked before Theorem 3.49, the equivalence (b) \Leftrightarrow (c) below has already been proved by P. Schauenburg (see 6.1 and Theorem 5.1 in [Sch2]).

Theorem 3.64. *Let A be a bialgebra and let H be a Hopf algebra. The following assertions are equivalent:*

(a) *A is an object in ${}^H_H \mathfrak{M}_H^H$, the unit $u : K \rightarrow A$ is right H –colinear and A becomes a coalgebra in $({}^H_H \mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H \mathfrak{M}_H, \otimes_H, H)$.*

(b) *There is a coalgebra R in ${}^H_H \mathcal{YD}$ and there are maps $u : K \rightarrow R$, $m : R \otimes R \rightarrow R$ and $\xi : R \otimes R \rightarrow H$ such that (R, u, m, ξ) is a dual Yetter–Drinfeld quadruple and A is isomorphic, as a bialgebra, to the bosonization $R \# {}^b H$ of (R, u, m, ξ) .*

(c) *There are a bialgebra map $\sigma : H \rightarrow A$ and an (H, H) –bilinear coalgebra map $\pi : A \rightarrow H$ such that $\pi \sigma = \text{Id}_H$.*

Moreover, if (c) holds, we can choose the Yetter–Drinfeld quadruple (R, u, m, ξ) , where

$$\begin{aligned} R &= A^{Co(H)}, & u &= u_A^{|R|}, \\ m(r \otimes s) &= \sum r_{(1)} s_{(1)} \sigma S \pi(r_{(2)} s_{(2)}), & \xi(r \otimes s) &= \pi(rs). \end{aligned}$$

Proof. (a) \Rightarrow (b) By 3.57 the canonical map $\epsilon_A : R\#H \rightarrow A$ in ${}^H_H\mathfrak{M}_H^H$ is an isomorphism of bialgebras, where the algebra structure on $R\#H$ is defined by $m_{R\#H} := \epsilon_A^{-1}m(\epsilon_A \otimes \epsilon_A)$ and $u_{R\#H} := \epsilon_A^{-1}u_A$. Clearly, by Proposition 3.47 and Proposition 3.48, $R\#H$ becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H\mathfrak{M}_H, \otimes_H, H)$, since A does. Let u be the corestriction of u_A to R .

As explained in 3.56, if

$$\tilde{m} = m_{R\#H}(R \otimes u_H \otimes R \otimes u_H),$$

and for $r \in R$, $h \in H$ we write $\tilde{m}(r \otimes s) = \sum \tilde{m}^0(r \otimes s) \otimes \tilde{m}^1(r \otimes s) \in R\#H$, then, as $m_{R\#H}$ is right H -linear, we have

$$m_{R\#H}[(r\#h) \otimes_H (s\#l)] = \sum \tilde{m}^0(r \otimes^{h_{(1)}} s) \otimes \tilde{m}^1(r \otimes^{h_{(1)}} s) h_{(2)}l.$$

Let us define the K -linear maps m and ξ as in (70). Since $m_{R\#H}$ is a morphism of coalgebras, analogous to Lemma 3.33, it follows that $\tilde{m} = (m \otimes \xi)\delta_{R \otimes R}$. Thus we can apply Theorem 3.62 to conclude that (R, u, m, ξ) is a dual Yetter–Drinfeld quadruple. Note that the bosonization of this dual Yetter–Drinfeld quadruple is the bialgebra $R\#H$ constructed above.

(b) \Rightarrow (a) By Proposition 3.47 and Proposition 3.48, A is an object in ${}^H_H\mathfrak{M}_H^H$ and A becomes a coalgebra in $({}^H_H\mathfrak{M}_H^H, \square_H, H)$ and an algebra in $({}_H\mathfrak{M}_H, \otimes_H, H)$. Since $u_{R\#H}$ is defined by (84), it is right H -colinear, so that the map $u_A : K \rightarrow A$ is right H -colinear, too.

(a) \Leftrightarrow (c) follows by Theorem 3.13.

The last statement follows by direct computation, using the canonical isomorphism $\epsilon_A : R\#H \rightarrow A$ in ${}^H_H\mathfrak{M}_H^H$, which turns out to be $\epsilon_A(r\#h) = r\sigma(h)$, the inverse being defined by $\epsilon_A^{-1}(a) = \sum a_{(1)}\sigma S\pi(a_{(2)}) \otimes \pi(a_{(3)})$. \square

Remark 3.65. Let (R, u, m, ξ) be a dual Yetter–Drinfeld quadruple such that ξ is trivial. Recall that this means that

$$\xi(r \otimes s) = \varepsilon(r)\varepsilon(s)1_H, \text{ for all } r, s \in R.$$

Then it is easy to check that relations (72)–(82) are equivalent to the fact that (R, m, u) is a bialgebra in $({}^H_H\mathcal{YD}, \otimes, K)$. Conversely, starting with a bialgebra (R, m, u) in the monoidal category $({}^H_H\mathcal{YD}, \otimes, K)$, we can consider the dual Yetter–Drinfeld quadruple (R, u, m, ξ) , where ξ is the trivial cocycle. Furthermore, the bosonization of this dual Yetter–Drinfeld quadruple is the usual bosonization of the bialgebra R , i.e. as an algebra is the smash product $R\#H$ and as a coalgebra is the smash coproduct. Recall that the multiplication and the unit of the smash product are respectively defined by

$$\begin{aligned} m_{R\#H}[(r\#h) \otimes (s\#k)] &= \sum r^{h_{(1)}}s \otimes h_{(2)}k, \\ u_{R\#H}(1) &= u(1) \otimes 1_H. \end{aligned}$$

Theorem 3.66. *Let A be bialgebra over a field K . Suppose that the coradical H of A is a semisimple sub-bialgebra of A with antipode. Then A is isomorphic as a bialgebra to the bosonization $R\#^b H$ of a certain dual Yetter–Drinfeld quadruple (R, u, m, ξ) . In fact A and $R\#^b H$ are isomorphic Hopf algebras.*

Proof. In view of a famous Takeuchi’s result (see [Mo, Lemma 5.2.10]), A is a Hopf algebra. Let $\sigma : H \rightarrow A$ be the canonical injection. By Theorem 2.35, there is a coalgebra morphism $\pi : A \rightarrow H$ in ${}_H\mathfrak{M}_H$ such that $\pi\sigma = \text{Id}_H$. In view of

Theorem 3.64, there exists a dual Yetter–Drinfeld quadruple (R, u, m, ξ) such that A is isomorphic as a bialgebra to the bosonization of this dual Yetter–Drinfeld quadruple. \square

Example 3.67. Let p be an odd prime and let K be an infinite field containing a primitive p -th root of the unit λ . Let C be a cyclic group of order p^2 with generator c . For every $a \in K, a \neq 0$, let $A := H(a)$ be the Hopf algebra constructed by Beattie, Dăscălescu and Grünenfelder in [BDG]. A has dimension p^4 , with basis $\{c^i x_1^j x_2^r \mid 0 \leq i \leq p^2 - 1, 0 \leq j, r \leq p - 1\}$, where c, x_1, x_2 are subject to

$$\begin{aligned} c^{p^2} &= 1, x_1^p = c^p - 1, x_2^p = c^p - 1, \\ x_1 c &= \lambda^{-1} c x_1, x_2 c = \lambda c x_2, x_2 x_1 = \lambda x_1 x_2 + a(c^2 - 1), \\ \Delta(c) &= c \otimes c, \Delta(x_1) = c \otimes x_1 + x_1 \otimes 1, \Delta(x_2) = c \otimes x_2 + x_2 \otimes 1. \end{aligned}$$

A is a pointed Hopf algebra with coradical $H := KC$. Let $\sigma : H \rightarrow A$ be the canonical injection and let $\pi : A \rightarrow H$ be the obvious projection. It is straightforward to show that A, H, π and σ fulfill the requirements of Theorem 3.64(c). Let

$$R = A^{coH} = \left\{ b \in A \mid \sum b_{(1)} \otimes \pi(b_{(2)}) = b \otimes 1 \right\}.$$

We have that R is the K -subspace of A spanned by the products $x_1^j x_2^r$, where $0 \leq j, r \leq p - 1$. In view of Theorem 3.64, one gets a dual Yetter–Drinfeld quadruple (R, u, m, ξ) such that A is isomorphic as a bialgebra to the bosonization $R \#^b H$ of R by H . Moreover $\xi(r \otimes s) = \pi(rs)$. We point out that ξ is not trivial. In fact we have

$$\xi(x_2 \otimes x_1) = \pi(x_2 x_1) = \pi[\lambda x_1 x_2 + a(c^2 - 1)] = a(c^2 - 1).$$

Clearly, the dual Hopf algebra A^* fulfills the requirements of Theorem 3.49 with respect to H^*, σ^* and π^* . Let $\iota : R \rightarrow A$ be the canonical injection. Then we have that the restriction Λ of ι^* to $(A^*)^{coH^*}$,

$$\Lambda : (A^*)^{coH^*} \rightarrow R^*,$$

is an isomorphism. Let $\alpha : R^* \otimes R^* \rightarrow (R \otimes R)^*$ be the usual isomorphism. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^* & \xrightarrow{\omega} & (A^*)^{coH^*} \otimes (A^*)^{coH^*} \\ \xi^* \downarrow & & \downarrow \Lambda \otimes \Lambda \\ (R \otimes R)^* & \xleftarrow{\alpha} & R^* \otimes R^* \end{array}$$

In fact we have

$$\begin{aligned} [(\alpha(\Lambda \otimes \Lambda)\omega)(\chi)](r \otimes s) &= (\varepsilon_R \# \chi) m_{R \# H}(r \# 1 \otimes s \# 1) \\ &= \sum \varepsilon_R \left[m \left(r^{(1)} \otimes r_{(-1)}^{(2)} s^{(1)} \right) \right] \chi \left[\xi \left(r_{(0)}^{(2)} \otimes s^{(2)} \right) \right] \\ &= \sum \varepsilon_R \left(r^{(1)} \right) \varepsilon_R \left(r_{(-1)}^{(2)} s^{(1)} \right) \chi \left[\xi \left(r_{(0)}^{(2)} \otimes s^{(2)} \right) \right] \\ &= \sum \varepsilon_R \left(r^{(1)} \right) \varepsilon_H \left(r_{(-1)}^{(2)} \right) \varepsilon_R \left(s^{(1)} \right) \chi \left[\xi \left(r_{(0)}^{(2)} \otimes s^{(2)} \right) \right] \\ &= \chi[\xi(r \otimes s)] = [\xi^*(\chi)](r \otimes s). \end{aligned}$$

It follows that we can identify the Yetter–Drinfeld quadruple $((A^*)^{coH^*}, \varepsilon, \delta, \omega)$ with the Yetter–Drinfeld quadruple $(R^*, (u_R)^*, m^*, \xi^*)$, where $(u_R)^* : R^* \rightarrow K$ is the evaluation at $1 \in R$. In particular we observe that we get a non-trivial bosonization since ω is not trivial.

The remaining part of this section is devoted to the proof of Theorem 3.71.

Proposition 3.68. *Let H be a cosemisimple Hopf algebra. Suppose that C is a coalgebra in \mathfrak{M}_H such that the coradical C_0 of C is H . Then C is an object in ${}^H\mathfrak{M}_H^H$ such that R , the space of right coinvariant elements of C , is an H –comodule coalgebra and C is isomorphic as a coalgebra, via a morphism in ${}^H\mathfrak{M}_H^H$, with the smash coproduct coalgebra $R\#H$ of R by H .*

Moreover there is a right H –linear coalgebra morphism $\pi_R : C \rightarrow R$ such that $\pi_R|_R = \text{Id}_R$, where R is regarded as a right module with trivial action.

Proof. Let H be a cosemisimple Hopf algebra. Suppose that (C, Δ, ε) is a coalgebra in $(\mathfrak{M}_H, \otimes, K)$ such that the coradical of C is H . Then, by Theorem 2.17, there is a coalgebra map $\pi_C : C \rightarrow H$ which is right H –linear and $\pi_C(h) = h$, for any $h \in H$. Since π_C is a morphism of coalgebras in $(\mathfrak{M}_H, \otimes, K)$, then C is an object in ${}^H\mathfrak{M}_H^H$ and Δ corestricts to a morphism

$$\overline{\Delta} : C \rightarrow C \square_H C \text{ in } {}^H\mathfrak{M}_H^H$$

such that $(C, \overline{\Delta}, \pi_C)$ is a coalgebra in $({}^H\mathfrak{M}_H^H, \square_H, H)$.

Let G be the monoidal functor $({}^H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H\mathfrak{M}, \otimes, K)$ (see 1.10). Then, by Proposition 1.5, $G(C) = R$ is a coalgebra in the monoidal category $({}^H\mathfrak{M}, \otimes, K)$ and the comultiplication of R is

$$(85) \quad \delta : R \rightarrow R \otimes R : r \mapsto \sum r_{(1)} S\pi_C(r_{(2)}) \otimes r_{(3)}$$

while the counit is induced by the counit of C .

Now, by 1.10, the counit of the adjunction (F, G) , corresponding to the monoidal equivalence

$$({}^H\mathfrak{M}, \otimes, K) \xrightarrow{F} ({}^H\mathfrak{M}_H^H, \square_H, H) \xrightarrow{G} ({}^H\mathfrak{M}, \otimes, K),$$

is given by

$$\epsilon_M : M^{Co(H)} \otimes H \rightarrow M, \quad \epsilon_M(v \otimes h) = vh.$$

By Corollary 1.7, ϵ_C is a coalgebra isomorphism in ${}^H\mathfrak{M}_H^H$. Note that the coalgebra structure of $FG(C) = C^{Co(H)} \otimes H = R \otimes H$ is exactly the one defining the smash coproduct of R by H (see Example 3.17). It is easy to check that the map $\pi : R\#H \rightarrow R$, given by $\pi(r\#h) = \varepsilon_H(h)r$, is a morphism of coalgebras, it is right H –linear and $\pi|_R = \text{Id}_R$. As the canonical map $\epsilon_C : R\#H \rightarrow C$ is an isomorphism of coalgebras in ${}^H\mathfrak{M}_H^H$ and $\epsilon_C(r\#1) = r$, for every $r \in R$ we get that $\pi_R := \pi\epsilon_C^{-1}$ has the same properties. \square

Lemma 3.69. *Let C be a coalgebra. Suppose that there is a group–like element $c_0 \in C$ such that $C_0 = Kc_0$, i.e. C is connected. Let $(C'_n)_{n \in \mathbb{N}}$ be a coalgebra filtration in C such that $C'_0 = C_0$. Then, for every $c \in C'_n$, we have*

$$(86) \quad \Delta(c) - c \otimes c_0 - c_0 \otimes c \in C'_{n-1} \otimes C'_{n-1}.$$

In particular, if $c \in C'_1$, then $\Delta(c) = c \otimes c_0 + c_0 \otimes c - \varepsilon(c)c_0 \otimes c_0$.

Proof. Since $(C'_n)_{n \in \mathbb{N}}$ is a coalgebra filtration, we have $\Delta(C'_n) \subseteq \sum_{i+j=n} C'_i \otimes C'_j$. Hence there are $c', c'' \in C'_n$ and $x \in C'_{n-1} \otimes C'_{n-1}$ such that

$$(87) \quad \Delta(c) = c' \otimes c_0 + c_0 \otimes c'' + x.$$

By applying $\varepsilon \otimes C$ and $C \otimes \varepsilon$ to this relation, we deduce that

$$\begin{aligned} c &= \varepsilon(c')c_0 + c'' + x_1 = c'' + y_1, \\ c &= \varepsilon(c'')c_0 + c' + x_2 = c' + y_2, \end{aligned}$$

where $x_1 = (\varepsilon \otimes C)(x)$, $x_2 = (C \otimes \varepsilon)(x)$ are in C'_{n-1} , since $x \in C'_{n-1} \otimes C'_{n-1}$. Then $y_1 = \varepsilon(c')c_0 + x_1 \in C'_{n-1}$ and $y_2 = \varepsilon(c'')c_0 + x_2 \in C'_{n-1}$.

We conclude the first part of the lemma by substituting c' and c'' in (87). Now, if $c \in C'_1$, then $\Delta(c) = c \otimes c_0 + c_0 \otimes c + \alpha c_0 \otimes c_0$, for a certain α in K . By applying $\varepsilon \otimes \varepsilon$ we deduce that $\alpha = -\varepsilon(c)$. \square

3.70. Let H be a cosemisimple Hopf algebra. We shall denote by \widehat{H} the set of isomorphism classes of simple left H -comodules. It is well known that, for every $\tau \in \widehat{H}$, there is a simple subcoalgebra $C(\tau)$ of H such that $\rho_V(V) \subseteq C(\tau) \otimes V$, where (V, ρ_V) is an arbitrary comodule in τ . Moreover, we have $H = \bigoplus_{\tau \in \widehat{H}} C(\tau)$.

Theorem 3.71. *Let H be a cosemisimple Hopf algebra. Suppose that (C, Δ, ε) is a coalgebra in \mathfrak{M}_H such that the coradical C_0 of C is H . Let $(C_n)_{n \in \mathbb{N}}$ be the coradical filtration of C .*

a) *For every natural number n , we have $C_n \simeq R_n \# H$ (an isomorphism in ${}^H\mathfrak{M}_H^H$). In particular C_n is freely generated as an H -module by elements $r \in C$ satisfying the relation*

$$(88) \quad \Delta(r) = \sum r_{\langle -1 \rangle} \otimes r_{\langle 0 \rangle} + r \otimes 1_H + C_{n-1} \otimes C_{n-1}.$$

b) C_1 *verifies the following equation:*

$$(89) \quad C_1 = C_0 + \sum_{\tau \in \widehat{H}} (C(\tau) \wedge K1_H) H.$$

Proof. a) By Proposition 3.68, C_n is the smash coproduct coalgebra $R'_n \# H$. By the construction of R'_n we have $R'_n = R \cap C_n$. Since C_n is isomorphic in ${}^H\mathfrak{M}_H^H$ to $R'_n \# H$, it results that C_n is free as a right H -module.

Note that $(R'_n)_{n \in \mathbb{N}}$ is not *a priori* a coalgebra filtration in R , since R is not a subcoalgebra of C (its comultiplication is δ ; see (85) for its definition). Let $\delta(r) = \sum r^{(1)} \otimes r^{(2)}$.

Let us prove that $(R'_n)_{n \in \mathbb{N}}$ is indeed a coalgebra filtration. Let π_R be the coalgebra morphism from Proposition 3.68. Then $R'_n = \pi_R(C_n)$, so $(R'_n)_{n \in \mathbb{N}}$ is a coalgebra filtration of R , as π_R is surjective. By [Mo, Corollary 5.3.5], the coradical of R is included in $\pi_R(H) = K1_H$, hence R is connected and $R'_0 = R_0$. By Lemma 3.69 applied to the filtration $(R'_n)_{n \in \mathbb{N}}$ we deduce that

$$\delta(r) \in r \otimes 1_H + 1_H \otimes r + R'_n \otimes R'_n,$$

for any $r \in R'_{n+1}$. By induction it results that $R'_n \subseteq R_n$, for every n . On the other hand, for $r \in R_{n+1}$ we have

$$\delta(r) \in r \otimes 1_H + 1_H \otimes r + R_n \otimes R_n.$$

Since C is isomorphic to the smash coproduct coalgebra via ϵ_C , we get $\Delta(r) = (\epsilon_C \otimes \epsilon_C)\Delta_{R\#H}(r\#1_H)$, so that

$$(90) \quad \Delta(r) = \sum r^{(1)}r^{(2)}_{\langle -1 \rangle} \otimes r^{(2)}_{\langle 0 \rangle} \in \sum r_{\langle -1 \rangle} \otimes r_{\langle 0 \rangle} + r \otimes 1_H + R_n H \otimes R_n H.$$

If we assume, by induction, that $R_n = R'_n$, then $\Delta(r) \in C \otimes C_n + H \otimes C$, that is, $r \in C_{n+1}$. Thus $r \in C_{n+1} \cap R = R'_{n+1}$. In conclusion, the filtrations $(R'_n)_{n \in \mathbb{N}}$ and $(R_n)_{n \in \mathbb{N}}$ are equal, and $C_n \simeq R_n \# H$. Note also that, by (90), every element in R_n satisfies (88), and hence a) is proved.

b) By the proof of the first part, it follows that every R_n is a subobject in ${}^H\mathfrak{M}_H^H$ of R . Let us decompose R_1 as a direct sum of left H -comodules

$$(91) \quad R_1 = K1_H \oplus R'_1 = K1_H \oplus \left(\bigoplus_{i=1}^n V_i \right),$$

where each V_i is simple. Let τ_i be the isomorphism class of V_i . Take $i \in \{1, \dots, n\}$ and $r \in V_i$. As in the proof of (90), by using the second equality in Lemma 3.69, one can show that

$$\Delta(r) = \sum r_{\langle -1 \rangle} \otimes r_{\langle 0 \rangle} + r \otimes 1_H - \varepsilon(r)1_H \otimes 1_H = \sum r_{\langle -1 \rangle} \otimes r_{\langle 0 \rangle} + (r - \varepsilon(r)1_H) \otimes 1_H.$$

Hence $\Delta(r) \in C(\tau_i) \otimes C + C \otimes K1_H$ which proves that $r \in C(\tau_i) \wedge K1_H$. Thus, in view of the decomposition (91), we have proved the inclusion “ \subseteq ” of (89), as C is generated as a right H -module by R . The other inclusion is trivial since, for $\tau \in \widehat{H}$ and $c \in C(\tau) \wedge K1_H$, we have

$$\Delta(c) \in C(\tau) \otimes C + C \otimes K1_H \subseteq H \otimes C + C \otimes H.$$

Thus $c \in H \wedge H = C_1$, so we deduce $(C(\tau) \wedge K1_H)H \subseteq C_1$, as C_1 is a right submodule of C . \square

Remark 3.72. Let A be a Hopf algebra such that A_0 , the coradical of A , is a subalgebra. In [AS5, Lemma 4.2] it is shown that (89) holds true for $C := grA$. In [CDMM, Remark 3.2] it is pointed out that the proof of (89), given in [AS5] for grA , also works in the case $C := A$, since A is a cosmash by [Mas, Theorem 3.1].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FERRARA, VIA MACHIAVELLI 35, FERRARA, I-44100, ITALY

E-mail address: `alessandro.ardizzoni@unife.it`

URL: `http://web.unife.it/utenti/alessandro.ardizzoni`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FERRARA, VIA MACHIAVELLI 35, I-44100, FERRARA, ITALY

E-mail address: `men@dns.unife.it`

URL: `http://web.unife.it/utenti/claudia.menini`

FACULTY OF MATHEMATICS, UNIVERSITY OF BUCHAREST, STRADA ACADEMIEI 14, BUCHAREST, RO-70109, ROMANIA

E-mail address: `dstefan@al.math.unibuc.ro`