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EVERY REAL ELLIPSOID IN $\mathbb{C}^2$ ADMITS CR UMBILICAL POINTS

XIAOJUN HUANG AND SHANYU JI

To the memory of Professor S. S. Chern

ABSTRACT. We prove that every real ellipsoid $M \subset \mathbb{C}^2$ admits at least four umbilical points, which can be compared to the result of Webster that a generic real ellipsoid in $\mathbb{C}^n$ with $n \geq 3$ does not admit any umbilical point.

1. INTRODUCTION

By the Cartan-Chern-Moser theory [CM], the germ of a strongly pseudoconvex real analytic hypersurface $M \subset \mathbb{C}^n$ is determined, up to a local biholomorphic map, by a set of complete invariants which can be expressed by the curvatures of a connection or the coefficients in a normal form.

When $n \geq 3$, the fourth-order pseudoconformal curvature tensor $\mathcal{S}$ of Chern-Moser [CM] is of fundamental importance because it generates other invariants by differentiation. It is known that $\mathcal{S} \equiv 0$ if and only if M is locally biholomorphic to the sphere $\partial\mathbb{B}^n$. When $n = 2$, the fourth-order curvature tensor vanishes identically and its role is played by the Cartan six-order invariant curvature tensor $\mathcal{P}$ [Car]. Similarly, $\mathcal{P} \equiv 0$ if and only if M is locally biholomorphic to the 3-sphere $\partial\mathbb{B}^2$. In both cases, a point on M, at which the Chern-Moser tensor $\mathcal{S}$ (or the Cartan curvature tensor $\mathcal{P}$ for the case of $n = 2$) vanishes, is called a CR umbilical point, or briefly, an umbilical point ([CM]). CR umbilical points are biholomorphic differential invariants of M.

The study of CR umbilical points on a compact strongly pseudoconvex hypersurface M provides useful information for the holomorphic structure of its enclosed domain, as well as the intrinsic CR structure of M itself. However, different from the situation in the classical Differential Geometry, except in the trivial spherical case, where $\mathcal{S}$ or $\mathcal{P} \equiv 0$, computing umbilical points seems to be a very difficult problem. This is because the explicit formula for the fundamental Cartan-Chern-Moser curvature tensions is too complicated. Indeed, the situation is already non-trivial even in the simplest non-spherical case—where M is a real ellipsoid. Recently, based on his previous work on the complex dynamics property of real ellipsoids, Webster proved the following (see §3 for the definitions).

Theorem 1.1 (Webster [We2])**.** *A generic real ellipsoid in $\mathbb{C}^n$ with $n \geq 3$ does not admit any umbilical point.*

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Umbilical points on a certain class of real hypersurfaces of revolution were also studied by Webster [We3].

A natural question arising from [We2] is then to ask whether a generic real ellipsoid in $\mathbb{C}^2$ shares the same property as its analogy in higher dimensions. It is indeed this problem that motivated our present work, and we provide, in this paper, the following:

Theorem 1.2. *Every real ellipsoid $M \subset \mathbb{C}^2$ admits at least four umbilical points.*

Theorem 1.2 resembles the classical result for the umbilical points on the ellipsoids in $\mathbb{R}^3$ [Spv, p. 222]. A famous theorem of Hamburger [Ham] states that every compact real analytic convex surface in $\mathbb{R}^3$ admits at least two umbilical points. We do not know if there is a CR version of the Hamburger theorem. More precisely, it is an open question to us if every compact strongly convex hypersurface in $\mathbb{C}^2$ admits at least two CR umbilical points. Notice that only for $n = 2$, the fundamental curvature tension reduces to a function. It may not be surprising that it is more likely to find umbilical points on a hypersurface in $\mathbb{C}^2$ than to find umbilical points for a hypersurface in $\mathbb{C}^n$ $(n \geq 3)$.

The proof of Theorem 1.2 uses Chern's inhomogeneous coordinates for the projective G-structure bundle of the Segre family of a real analytic strongly pseudoconvex hypersurface [C], [CJ2], and a formula derived in Huang-Ji-Yau [HJY, Theorem 3.1] for the complexified Cartan fundamental curvature tension represented under these coordinates. The formula of [HJY] seems to fit particularly well with the computation here.

In the classical Differential Geometry [Spv], surfaces in $\mathbb{R}^3$ without umbilical points must be diffeomorphic to a torus. The boundary of a small thickening of the unit circle in $\mathbb{R}^2$ provides examples of closed surfaces without any umbilical point. However, this type of examples does not give compact CR manifolds without CR umbilical points. The following theorem gives a precise description for the set of umbilical points for the thickening of a closed real curve. It is not clear to us if there is any embeddable three-dimensional compact CR manifold which has no CR umbilical points.

Theorem 1.3. *Let $M_\epsilon \subset \mathbb{C}^2$ be the boundary of the ϵ-thickening of the unit circle $\{|z| = 1, w = 0\}$ in $\mathbb{C}^2$, defined by the equation $(1 - |z|)^2 + |w|^2 = \epsilon^2$, where ϵ is a sufficiently small positive number. Then the set of all umbilical points of M_ϵ forms a disjoint union of a closed real analytic curve and two two-dimensional totally real analytic tori.*

2. Umbilical points of real hypersurfaces in $\mathbb{C}^2$

In this section, we briefly review the Cartan-Chern-Moser theory (cf. [C], [HJ], [Hu]). We restrict ourselves to the case of $n = 2$. Let

$$M = \{(z, w) \in \mathbb{C}^2 : \ r(z, w, \overline{z}, \overline{w}) = 0\} \tag{2.1}$$

be a Levi non-degenerate smooth real analytic hypersurface with $(z_0, w_0) \in M$. Its complexification, called the *Segre family* of M, is then the complex three-fold

$$\mathcal{M} = \{(z, w, \zeta, \eta) \mid r(z, w, \zeta, \eta) = 0\} \subset \mathbb{C}^4.$$

Clearly $(z_0, w_0, \overline{z_0}, \overline{w_0}) \in \mathcal{M}$. Assume that

$$r_w(z_0, w_0, \overline{z_0}, \overline{w_0}) := \frac{\partial r}{\partial w}(z_0, w_0, \overline{z_0}, \overline{w_0}) \neq 0. \tag{2.2}$$

Define

(2.3)

$$S:\mathcal{M}\to\widetilde{\mathcal{M}}:=S(\mathcal{M})\subset\mathbb{C}^2\times\mathbb{P}^1,\quad (z,w,\zeta,\eta)\mapsto\left(z,w,\left[\frac{\partial r}{\partial z}:\frac{\partial r}{\partial w}\right](z,w,\zeta,\eta)\right).$$

S is locally biholomorphic by the Levi non-degeneracy condition. (See Proposition 4.1 of [CJ2].) With the assumption in (2.2), we can regard (z,w,ζ) as a local non-homogeneous coordinates system for $\mathcal{M}$, and we can write $S(z,w,\zeta)=(z,w,-\frac{r_z}{r_w})$. Then we use (z,w,p) as a local coordinates system, called the Chern coordinates system, for $\widetilde{\mathcal{M}}$, where

$$p=-\frac{r_z}{r_w}. \tag{2.4}$$

Making use of the implicit function theorem, we can find a unique holomorphic function (in its argument) $h(z,\overline{z},\overline{w})$ near $(z_0,\overline{z_0},\overline{w_0})$ with $h(z_0,\overline{z_0},\overline{w_0})=w_0$ such that $w=h(z,\overline{z},\overline{w})$ solves the equation: $r(z,w,\overline{z},\overline{w})=0$. Then for $(z,w,\zeta,\eta)\in\mathcal{M}$, we have

$$p(z,w,\zeta)=\frac{\partial h(z,\zeta,\eta)}{\partial z}. \tag{2.5}$$

We have $dp=p_{11}dz+\hat{p}_1^1d\zeta+\hat{p}_1^2d\eta$, where $p_{11}=\frac{\partial^2h}{\partial z^2}$, $\hat{p}_1^1=\frac{\partial^2h}{\partial\zeta\partial z}$ and $\hat{p}_1^2=\frac{\partial^2h}{\partial\eta\partial z}$; and we have the identity

$$-pdz+dw-\hat{p}^1d\zeta-\hat{p}^2d\eta=0, \tag{2.6}$$

where $p=p_1=\frac{\partial h}{\partial z}$, $\hat{p}^1=\frac{\partial h}{\partial\zeta}$ and $\hat{p}^2=\frac{\partial h}{\partial\eta}$. Therefore, we obtain

$$dp|_{\mathcal{M}}=\left(p_{11}-\frac{\hat{p}_1^2p_1}{\hat{p}^2}\right)dz+\frac{\hat{p}_1^2}{\hat{p}^2}dw+\left(p_1^{,1}-\frac{p_1^{,2}p^{,1}}{p^{,2}}\right)d\zeta. \tag{2.7}$$

Hence, we have the following holomorphic coframe on $\widetilde{\mathcal{M}}$:

$$\begin{aligned}\theta &= dw-pdz=dw-p_1dz,\\ \theta^1 &= dz,\\ \theta_1 &= \frac{\hat{p}_1^2}{\hat{p}^2}\theta+\left(\hat{p}_1^1-\frac{\hat{p}_1^2\hat{p}^1}{\hat{p}^2}\right)d\zeta=dp-p_{11}dz.\end{aligned}$$

We emphasize again that p_{11} is a holomorphic function in (z,w,p) near (z_0,w_0,p_0) with $(z_0,w_0)\in M$ and $p_0=p(z_0,w_0,\overline{z_0})$; and p_{11} is given by the following formula:

$$p_{11}=\frac{\partial^2h}{\partial z^2}. \tag{2.8}$$

Define the holomorphic coframes over $\mathcal{M}$:

$$\omega=u\theta,\ \omega^1=u^1\theta+u_1^1\theta^1,\ \omega_1=v_1\theta+v_1^1\theta_1, \tag{2.9}$$

where u,u_1^1,u^1,v_1 are holomorphic functions with $u=iu_1^1v_1^1\neq0$.

Now, the fundamental Cartan-Chern-Moser theory [CM] gives the following:

Let $M=\{r=0\}\subset\mathbb{C}^2$, $(z_0,w_0)\in M$ such that (2.2) is satisfied and let $\widetilde{\mathcal{M}}$ be as in (2.3). Let $\widetilde{\pi}:\widetilde{\mathcal{Y}}\to\widetilde{\mathcal{M}}$ be the corresponding holomorphic projective structure bundle. Then besides the 3 holomorphic 1-forms in (2.9), there exist 5 more holomorphic 1-forms $\phi,\phi_1^1,\phi^1,\phi_1,\ \psi$, defined over $\widetilde{\mathcal{Y}}$, with holomorphic coordinates $z, w, p, u, u_1^1, u^1, v_1, t$, with $u, u_1^1 \neq 0$. These holomorphic 1-forms are $\mathbb{C}$-linearly independent, and satisfy the following structure equations:

$$
\begin{aligned}
d\omega &= i\omega^1 \wedge \omega_1 + \omega \wedge \phi, \\
d\omega^1 &= \omega^1 \wedge \phi_1^1 + \omega \wedge \phi^1, \\
d\omega_1 &= \phi_1^1 \wedge \omega_1 + \omega_1 \wedge \phi + \omega \wedge \phi_1, \\
d\phi &= i\omega^1 \wedge \phi_1 + i\phi^1 \wedge \omega_1 + \omega \wedge \psi, \\
d\phi_1^1 &= i\omega_1 \wedge \phi^1 - 2i\phi_1 \wedge \omega^1 - \tfrac{1}{2}\psi \wedge \omega, \\
d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi_1^1 - \tfrac{1}{2}\psi \wedge \omega^1 + L^{11}\omega \wedge \omega_1, \\
d\phi_1 &= \phi_1^1 \wedge \phi_1 - \tfrac{1}{2}\psi \wedge \omega_1 + P_{11}\omega \wedge \omega^1, \\
d\psi &= \phi \wedge \psi + 2i\phi^1 \wedge \phi_1 + H_1\omega \wedge \omega^1 + K^1\omega \wedge \omega_1.
\end{aligned}
\tag{2.10}
$$

All of these forms $\omega, \omega^1, \omega_1, \phi, \phi_1^1, \phi^1, \phi_1$ and ψ, as well as all of the curvature functions L^{11}, P_{11}, H_1 and K^1, have been calculated explicitly in [HJY, Theorem 3.1]. In particular, we have

$$
\begin{aligned}
L^{11} &= -\frac{i(u_1^1)^2}{6u^3}\frac{\partial^4 p_{11}}{\partial p^4}, \\
P_{11} &= \frac{i}{u(u_1^1)^2}\Bigg[\frac{\partial^2 p_{11}}{\partial w^2} - \frac{1}{2}\frac{\partial p_{11}}{\partial w}\frac{\partial^2 p_{11}}{\partial p^2} + \frac{2}{3}\frac{\partial p_{11}}{\partial p}\frac{\partial^2 p_{11}}{\partial p\partial w} + \frac{p_{11}}{6}\frac{\partial^3 p_{11}}{\partial p^2\partial w} \\
&\quad - \frac{1}{6}\frac{\partial p_{11}}{\partial p}\left(\frac{\partial^3 p_{11}}{\partial p^2\partial z} + p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right) - \frac{2}{3}\left(\frac{\partial^3 p_{11}}{\partial z\partial w\partial p} + p_{11}\frac{\partial^3 p_{11}}{\partial p^2\partial w} + p\frac{\partial^3 p_{11}}{\partial p\partial w^2}\right) \\
&\quad + \frac{1}{6}\left(\frac{\partial^4 p_{11}}{\partial p^2\partial z^2} + p_{11}\frac{\partial^4 p_{11}}{\partial p^3\partial z} + p\frac{\partial^4 p_{11}}{\partial p^2\partial z\partial w}\right) + \frac{1}{6}\frac{\partial^3 p_{11}}{\partial p^3}\left(\frac{\partial p_{11}}{\partial z} + p\frac{\partial p_{11}}{\partial w}\right) \\
&\quad + \frac{p_{11}}{6}\left(\frac{\partial^4 p_{11}}{\partial z\partial p^3} + p_{11}\frac{\partial^4 p_{11}}{\partial p^4} + p\frac{\partial^4 p_{11}}{\partial p^3\partial w}\right) \\
&\quad + \frac{p}{6}\left(\frac{\partial^4 p_{11}}{\partial z\partial p^2\partial w} + p_{11}\frac{\partial^4 p_{11}}{\partial p^3\partial w} + p\frac{\partial^4 p_{11}}{\partial p^2\partial w^2}\right)\Bigg].
\end{aligned}
\tag{2.11}
$$

On the CR structure bundle $\hat{Y}$ over $\hat{M} = S(\{(z, w, \overline{z}, \overline{w}) : (z, w) \in M\})$, there are $\mathbb{R}$-linearly independent 1-forms $\omega, \omega^1, \overline{\omega^1}, \phi_1^1, \phi = \phi_1^1 + \overline{\phi_1^1}, \phi^1, \overline{\phi^1}, \psi$ satisfying the structure equations

$$
\begin{aligned}
d\omega &= i\omega^1 \wedge \overline{\omega^1} + \omega \wedge \phi, \\
d\omega^1 &= \omega^1 \wedge \phi_1^1 + \omega \wedge \phi^1, \\
d\phi_1^1 &= i\overline{\omega^1} \wedge \phi^1 - 2i\overline{\phi^1} \wedge \omega^1 - \tfrac{1}{2}\psi \wedge \omega, \\
d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi_1^1 - \tfrac{1}{2}\psi \wedge \omega^1 + \hat{L}^{11}\omega \wedge \overline{\omega^1}, \\
d\psi &= \phi \wedge \psi + 2i\phi^1 \wedge \overline{\phi_1} + (-\hat{H}_1\omega^1 - \overline{\hat{H}_1}\overline{\omega^1}) \wedge \omega.
\end{aligned}
\tag{2.12}
$$

It is known that the projective connection underlines the CR connection [C], [F]. Hence the structure equations (2.10), when restricted on $\hat{Y}$, reduce to (2.12). Consequently, $\hat{L}^{11} = L^{11}|_{\hat{Y}} = \overline{P_{11}|_{\hat{Y}}}$. $\hat{L}^{11}$, when pulled back to (Y, π, M), is the Cartan fundamental curvature function. Hence, $(z_0, w_0) \in M$ is an *umbilical point* if and only if $L^{11}|_{\hat{Y}} = 0$ along the fiber $\hat{\pi}^{-1}(z_0, w_0)$, where $\hat{\pi} : \hat{Y} \to \hat{M}$ is the natural projection. Notice that (z_0, w_0) is an umbilical point of M if and only if there is a biholomorphic change of coordinates under which (z_0, w_0) is mapped to the origin and $\hat{M}$ is defined by an equation of the form $\mathrm{Im}(w) = |z|^2 + o(|z|^6)$ (see [CM]).

From (2.11), we notice that L^{11} vanishes at a point in the fiber $\hat{\pi}^{-1}(S(z_0, w_0, \overline{z_0}, \overline{w_0}))$ if and only if L^{11} vanishes along the whole fiber $\hat{\pi}^{-1}(S(z_0, w_0, \overline{z_0}, \overline{w_0}))$. Since $u \neq 0, u_1^1 \neq 0$ in (2.11), we obtain

Theorem 2.1. *Let $M = \{r = 0\} \subset \mathbb{C}^2$. Let r and $(z_0, w_0) \in M$ be as in* (2.1)*. Assume that* (2.2) *is satisfied. Then $(z_0, w_0) \in M$ is an umbilical point if and only if*

$$\frac{\partial^4 p_{11}}{\partial p^4}(z_0, w_0, p_0) = 0$$

where $p_0 = -\frac{r_z}{r_w}(z_0, w_0, \overline{z_0}, \overline{w_0})$.

3. Umbilical points of ellipsoids in $\mathbb{C}^2$

Recall that a real ellipsoid $M \subset \mathbb{C}^n$ is the image of the unit sphere $\partial\mathbb{B}^n$ under a real-affine transformation of $\mathbb{R}^{2n} := \mathbb{C}^n$. It is known [We1] that after a holomorphic affine transformation, any real ellipsoid is given by an equation of the form $\sum_{j=1}^n (A_j x_j^2 + B_j y_j^2) = 1$ where $A_j \geq B_j > 0$ and $z_j = x_j + iy_j$. The complex structure of ellipsoids was first studied by Webster in his famous paper [We1]. He showed that when $n \geq 2$, two ellipsoids are biholomorphically equivalent if and only if the set of ratios $(A_j - B_j)/(A_j + B_j)$ is the same for the two. Hence any ellipsoid M can be made into the form

$$\sum_{j=1}^{n} \left((1 + a_j)x_j^2 + y_j^2 \right) = 1 \tag{3.1}$$

where $a_j \geq 0$. Notice that M is spherical if and only if $a_j = 0$ for all j. In particular, after a holomorphically linear change of coordinates, any ellipsoid M in $\mathbb{C}^2$ can be given by

$$(1 + a_1)x_1^2 + y_1^2 + (1 + a_2)x_2^2 + y_2^2 = 1, \quad a_1, a_2 \geq 0; \tag{3.2}$$

or equivalently,

$$a_1 z^2 + a_1 \overline{z}^2 + 2(2 + a_1)z\overline{z} + a_2 w^2 + a_2 \overline{w}^2 + 2(2 + a_2)w\overline{w} = 4. \tag{3.3}$$

We notice from (3.2) that M can be parameterized by three real parameters $\alpha, \beta \in [0, 2\pi]$, $c \in [0, 1]$ through the following equation:

$$z = \frac{c}{\sqrt{1 + a_1}} cos\ \alpha + i\ c\ sin\ \alpha, \quad w = \frac{\sqrt{1 - c^2}}{\sqrt{1 + a_2}} cos\ \beta + i\sqrt{1 - c^2} sin\ \beta. \tag{3.4}$$

In fact, for any $c \in [0, 1]$, consider $w = x_2 + iy_2$ with $(1 + a_2)x_2^2 + y_2^2 = 1 - c^2$. Then $w = \frac{\sqrt{1-c^2}}{\sqrt{1+a_2}} cos\ \beta + i\sqrt{1 - c^2} sin\ \beta$ for $\beta \in [0, 2\pi]$. Since $(1 + a_1)x_1^2 + y_1^2 = c^2$, the formula for $z = x_1 + iy_1 = \frac{c}{\sqrt{1+a_1}} cos\ \alpha + i\ c\ sin\ \alpha$ follows.

Complexifying (3.3), we obtain the Segre family $\mathcal{M} \subset \mathbb{C}^2 \times \mathbb{C}^2$ of M, defined by the equation

$$a_1 z^2 + a_1 \zeta^2 + 2(2 + a_1)z\zeta + a_2 w^2 + a_2 \eta^2 + 2(2 + a_2)w\eta = 4. \tag{3.5}$$

Choose the defining function of M to be $r := a_1 z^2 + a_1\overline{z}^2 + 2(2+a_1)z\overline{z} + a_2 w^2 + a_2\overline{w}^2 + 2(2+a_2)w\overline{w} - 4$. Then a point (z, w) satisfies (2.2) if and only if $a_2 w + (2+a_2)\overline{w} \neq 0$. By (3.4), this is equivalent to the condition that $c \neq 1$, or equivalently, $w \neq 0$. We assume

$$c \neq 1, \quad \text{i.e.,} \quad w \neq 0. \tag{3.6}$$

Then making use of the implicit function theorem, we have a unique function $w = h(z, \overline{z}, \overline{w})$, which solves the the equation $r = 0$ near the point under study. Applying $\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial z^2}$ to (3.5), we get $a_1 z + (2+a_1)\zeta + a_2 w \frac{\partial h}{\partial z} + (2+a_2)\eta \frac{\partial h}{\partial z} = 0$ and $a_1 + a_2\left(\frac{\partial h}{\partial z}\right)^2 + a_2 w \frac{\partial^2 h}{\partial z^2} + (2+a_2)\eta \frac{\partial^2 h}{\partial z^2} = 0$. Since $p = \frac{\partial h}{\partial z}$ and $p_{11} = \frac{\partial^2 h}{\partial z^2}$ on $\mathcal{M}$, we obtain

$$a_1 z + (2+a_1)\zeta + a_2 wp + (2+a_2)\eta p = 0 \quad \text{and} \tag{3.7}$$

$$a_1 + a_2 p^2 + a_2 w p_{11} + (2+a_2)\eta p_{11} = 0. \tag{3.8}$$

At the point $(z, w, \overline{z}, \overline{w}) \in \mathcal{M}$, we then have

$$p = -\frac{a_1 z + (2+a_1)\overline{z}}{a_2 w + (2+a_2)\overline{w}}. \tag{3.9}$$

Now, we can use (3.5) (3.6) (3.7) and (3.8) to cancel out ξ, η as follows:

Multiplying $(2+a_1)^2$ to the equation (3.5) and making use of the equality: $(2+a_1)\zeta = -a_1 z - a_2 wp - (2+a_2)\eta p$ from (3.7), we have

$$\begin{aligned}
&(2+a_1)^2 a_1 z^2 + a_1\Big[a_1 z + a_2 wp + (2+a_2)\eta p\Big]^2 \\
&+2(2+a_1)^2 z\Big(- a_1 z - a_2 wp - (2+a_2)\eta p\Big) \\
&+a_2(2+a_1)^2 w^2 + a_2(2+a_1)^2\eta^2 \\
&+2(2+a_1)^2(2+a_2)w\eta = 4(2+a_1)^2.
\end{aligned} \tag{3.10}$$

Multiplying (3.10) by $(2+a_2)^2 p_{11}^2$ and making use of (3.8): $(2+a_2)\eta p_{11} = -a_1 - a_2 p^2 - a_2 w p_{11}$, we obtain the following:

$$\begin{aligned}
&a_1(2+a_1)^2(2+a_2)^2 z^2 p_{11}^2 + a_1(2+a_2)^2\Big(a_1 z p_{11} - a_1 p - a_2 p^3\Big)^2 \\
&-2(2+a_1)^2(2+a_2)^2 p_{11} z(a_1 z p_{11} - a_1 p - a_2 p^3) \\
&+a_2(2+a_1)^2(2+a_2)^2 p_{11}^2 w^2 + a_2(2+a_1)^2\Big(a_1 + a_2 p^2 + a_2 w p_{11}\Big)^2 \\
&-2(2+a_1)^2(2+a_2)^2 w p_{11}(a_1 + a_2 p^2 + a_2 w p_{11}) = 4(2+a_1)^2(2+a_2)^2 p_{11}^2.
\end{aligned} \tag{3.11}$$

Write (3.11) as

$$\widetilde{A} p_{11}^2 + 2\widetilde{B} p_{11} + \widetilde{C} = 0, \ \text{where} \tag{3.12}$$

$$\widetilde{A} = -4a_1(1+a_1)(2+a_2)^2 z^2 - 4a_2(1+a_2)(2+a_1)^2 w^2 - 4(2+a_1)^2(2+a_2)^2, \tag{3.13}$$

$$\widetilde{B} = 4(a_1 + a_2 p^2)\Big[(1+a_1)(2+a_2)^2 zp - (2+a_1)^2(1+a_2)w\Big], \tag{3.14}$$

$$\widetilde{C} = (a_1 + a_2 p^2)^2\Big[a_1(2+a_2)^2 p^2 + a_2(2+a_1)^2\Big]. \tag{3.15}$$

Assume that $\widetilde{A} \neq 0$ at the point $(z, w) \in M$ with $w \neq 0$. We can then solve p_{11} from (3.12):

$$p_{11} = \frac{-\widetilde{B} \pm \widetilde{H}}{\widetilde{A}} \tag{3.16}$$

where

$$\begin{aligned}\widetilde{H}^2 = \widetilde{B}^2 - \widetilde{A}\widetilde{C} = 4(a_1 + a_2p^2)^2 \\ \Big\{4\Big[(1+a_1)(2+a_2)^2zp - (1+a_2)(2+a_1)^2w\Big]^2 \\ +\Big[a_1(2+a_2)^2p^2 + a_2(2+a_1)^2\Big]\cdot\Big[a_1(1+a_1)(2+a_2)^2z^2 \\ +a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big]\Big\}.\end{aligned} \tag{3.17}$$

After taking out the common factor $2(a_1 + a_2p^2)$, (3.16) can be simplified as

$$p_{11} = \frac{-\hat{B} \pm \hat{H}}{\widetilde{A}} \cdot 2(a_1 + a_2p^2) \tag{3.18}$$

where $2(a_1 + a_2p^2)\hat{B} = \widetilde{B}$, and
(3.19)

$$\begin{aligned}\hat{H}^2 = 4\Big[(1+a_1)(2+a_2)^2zp - (1+a_2)(2+a_1)^2w\Big]^2 \\ +\Big[a_1(2+a_2)^2p^2 + a_2(2+a_1)^2\Big] \\ \cdot\Big[a_1(1+a_1)(2+a_2)^2z^2 + a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big].\end{aligned}$$

Write

$$\hat{H}^2 = Ap^2 + Bp + C, \text{ where} \tag{3.20}$$

$$\begin{aligned}A = 4(1+a_1)^2(2+a_2)^4z^2 + a_1(2+a_2)^2\Big[a_1(1+a_1)(2+a_2)^2z^2 \\ +a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big],\end{aligned} \tag{3.21}$$

$$B = -8(1+a_1)(1+a_2)(2+a_1)^2(2+a_2)^2zw, \tag{3.22}$$

$$\begin{aligned}C = 4(1+a_2)^2(2+a_1)^4w^2 \ + a_2(2+a_1)^2\Big[a_1(1+a_1)(2+a_2)^2z^2 \\ +a_2(1+a_2)(2+a_1)^2w^2 + (2+a_1)^2(2+a_2)^2\Big].\end{aligned} \tag{3.23}$$

Assume that $\hat{H}^2 = Ap^2 + Bp + C \neq 0$ at the point $(z, w) \in M$ with p being given as before. Notice that $\widetilde{A}$ is independent of p and that the degree of $\hat{B}$ in p is 1. From the formula of p_{11} in (3.18), it follows that at $(z, w, \overline{z}, \overline{w})$,

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \Leftrightarrow \frac{\partial^4}{\partial p^4}\Big((a_1 + a_2p^2)\hat{H}\Big) = 0. \tag{3.24}$$

Assume that $\hat{H}(z^*, w^*, p^*) = 0$ with $(z^*, w^*) \in M$ and $p^* = p(z^*, w^*, \overline{z^*})$, where

$$w^*,\ A(z^*, w^*), \widetilde{A}(z^*, w^*) \neq 0.$$

Since $p_{11}(z, w, p)$ is a holomorphic function for $(z, w, p) \approx (z^*, w^*, p^*)$, we easily see from (3.18) that $J(z, w, p) := \hat{H} \cdot (a_1 + a_2p^2)$ is also holomorphic for $(z, w, p) \approx (z^*, w^*, p^*)$. In particular, $J(z^*, w^*, p)$ is holomorphic in p for $p \approx p^*$. Now, suppose that $2A(z^*, w^*)p^* + B(z^*, w^*) \neq 0$. Then $\hat{H} = \pm(p - p^*)^{1/2}h^*$ with $h^* \neq 0$ holomorphic for $p \approx p^*$, by (3.20). This clearly contradicts the fact that $J(z^*, w^*, p)$ is holomorphic in p for $p \approx p^*$. Hence, we conclude that $\hat{H}(z^*, w^*, p^*) = 0$ can only occur at the point where

$$2A(z^*, w^*)p^* + B(z^*, w^*) = 0. \tag{3.25}$$

Next, we have

$$\frac{\partial^4}{\partial p^4}\Big((a_1 + a_2p^2)\hat{H}\Big) = 12a_2\frac{\partial^2 \hat{H}}{\partial p^2} + 8a_2p\frac{\partial^3 \hat{H}}{\partial p^3} + (a_1 + a_2p^2)\frac{\partial^4 \hat{H}}{\partial p^4}. \tag{3.26}$$

Since $\hat{H}^2 = Ap^2 + Bp + C$, we get $2\hat{H}\frac{\partial \hat{H}}{\partial p} = 2Ap + B$. We continue to differentiate it to get $\left(\frac{\partial \hat{H}}{\partial p}\right)^2 + \hat{H}\frac{\partial^2 \hat{H}}{\partial p^2} = A$. Hence

$$\begin{aligned} \frac{\partial^2 \hat{H}}{\partial p^2} &= \frac{A - (\frac{\partial \hat{H}}{\partial p})^2}{\hat{H}} = \frac{4A\hat{H}^2 - (2Ap + B)^2}{4\hat{H}^3} \\ &= \frac{4A(Ap^2 + Bp + C) - (4A^2p^2 + 4ABp + B^2)}{4\hat{H}^3} = \frac{4AC - B^2}{4\hat{H}^3}. \end{aligned} \tag{3.27}$$

Continuing differentiation on $\left(\frac{\partial \hat{H}}{\partial p}\right)^2 + \hat{H}\frac{\partial^2 \hat{H}}{\partial p^2} = A$, we obtain $3\frac{\partial \hat{H}}{\partial p}\frac{\partial^2 \hat{H}}{\partial p^2} + \hat{H}\frac{\partial^3 \hat{H}}{\partial p^3} = 0$ and thus

$$\begin{aligned} \frac{\partial^3 \hat{H}}{\partial p^3} &= -\frac{3}{\hat{H}} \cdot \frac{\partial \hat{H}}{\partial p} \cdot \frac{\partial^2 \hat{H}}{\partial p^2} \\ &= -\frac{3}{\hat{H}} \cdot \frac{2Ap + B}{2\hat{H}} \cdot \frac{4AC - B^2}{4\hat{H}^3} = -\frac{3}{8\hat{H}^5}(2Ap + B)(4AC - B^2). \end{aligned} \tag{3.28}$$

Again from the equation $3\frac{\partial \hat{H}}{\partial p}\frac{\partial^2 \hat{H}}{\partial p^2} + \hat{H}\frac{\partial^3 \hat{H}}{\partial p^3} = 0$, we get by differentiation

$$3\Big(\frac{\partial^2 \hat{H}}{\partial p^2}\Big)^2 + 4\frac{\partial \hat{H}}{\partial p}\frac{\partial^3 \hat{H}}{\partial p^3} + \hat{H}\frac{\partial^4 \hat{H}}{\partial p^4} = 0, \text{ and thus}$$

$$\begin{aligned} \frac{\partial^4 \hat{H}}{\partial p^4} &= \frac{1}{\hat{H}}\Big[-3\Big(\frac{\partial^2 \hat{H}}{\partial p^2}\Big)^2 - 4\frac{\partial \hat{H}}{\partial p}\frac{\partial^3 \hat{H}}{\partial p^3}\Big] \\ &= \frac{3(4AC - B^2)}{16\hat{H}^7}\Big(B^2 - 4AC + 4(2Ap + B)^2\Big). \end{aligned} \tag{3.29}$$

By Theorem 2.1, (3.24), (3.26), (3.27), (3.28) and (3.29), $(z, w) \in M$ is an umbilical point if and only if

$$\begin{aligned} &\frac{a_2(4AC - B^2)}{\hat{H}^3} - \frac{a_2p(2Ap + B)(4AC - B^2)}{\hat{H}^5} \\ &+\Big(a_1 + a_2p^2\Big)\frac{(4AC - B^2)[B^2 - 4AC + 4(2Ap + B)^2]}{16\hat{H}^7} = 0, \end{aligned}$$

which amounts to saying that either $4AC - B^2 = 0$ or

$$a_2\hat{H}^4 - a_2p(2Ap + B)\hat{H}^2 + \frac{1}{16}[a_1 + a_2p^2][B^2 - 4AC + 4(2Ap + B)^2] = 0. \tag{3.30}$$

Since $\hat{H}^2 = Ap^2 + Bp + C$, it follows from (3.30) that

$$4a_2(Bp + 2C)^2 + 4a_1(2Ap + B)^2 + (a_1 + a_2p)(B^2 - 4AC) = 0.$$

Hence, we have proved the following criterion on umbilical points.

Theorem 3.1. *Let $M \subset \mathbb{C}^2$ be as in* (3.2)*. Let $(z,w) \in M$ be such that $w \neq 0$, $\widetilde{A}(z,w) \neq 0$ and $\hat{H}(z,w,p(z,w,\overline{z})) = Ap^2+Bp+C \neq 0$. Then (z,w) is an umbilical point if and only if either $4AC - B^2 = 0$ or*

$$4a_2(Bp+2C)^2 + 4a_1(2Ap+B)^2 + (a_1+a_2p)(B^2-4AC) = 0 \tag{3.31}$$

at (z,w,p). Here p is as in (3.9)*; A, B and C are as in* (3.21)*,* (3.22) *and* (3.23)*.*

4. Proof of Theorem 1.2

Lemma 4.1. *Let M be as in* (3.2)*. Assume that $a_1 > 0$. If $16a_1+16a_1a_2+3a_1a_2^2-4a_2^2 > 0$, then M is umbilical at $(\frac{c}{\sqrt{1+a_1}}, i\sqrt{1-c^2}) \in M$ for a certain $c \in (0,1)$.*

Proof. From (3.4) and (3.9), Consider the curve $\Gamma \subset M$ with the parameter $c \in [0,1]$, defined by:

$$z(c) = \frac{c}{\sqrt{1+a_1}}, \tag{4.1}$$

$$w(c) = i\sqrt{1-c^2}, \quad 0 \leq c < 1. \tag{4.2}$$

Then along Γ, from (3.9), we have

$$p(c) = -\frac{a_1z+(2+a_1)\overline{z}}{aw+(2+a)\overline{w}} = -\frac{i(\sqrt{1+a_1})c}{\sqrt{1-c^2}}. \tag{4.3}$$

By (3.21),(3.22) and (3.23), we have

$$\begin{aligned} A(c) = {} & 4(1+a_1)(2+a_2)^4c^2 + a_1(2+a_2)^2\Big[a_1(2+a_2)^2c^2 \\ & -a_2(1+a_2)(2+a_1)^2(1-c^2) + (2+a_1)^2(2+a_2)^2\Big], \end{aligned} \tag{4.4}$$

$$B(c) = -8(1+a_1)(1+a_2)(2+a_1)^2(2+a_2)^2ic\frac{\sqrt{1-c^2}}{\sqrt{1+a_1}}, \tag{4.5}$$

$$\begin{aligned} C(c) = {} & -4(1+a_2)^2(2+a_1)^4(1-c^2) \ + a_2(2+a_1)^2\Big[a_1(2+a_2)^2c^2 \\ & -a_2(1+a_2)(2+a_1)^2(1-c^2) + (2+a_1)^2(2+a_2)^2\Big]. \end{aligned} \tag{4.6}$$

By Theorem 3.1, it is enough to show that there is a certain $c \in (0,1)$ such that at the point $(z(c), w(c), p(c)) \in \widetilde{\mathcal{M}}$

$$\widetilde{A} \neq 0, \tag{4.7}$$

$$Ap^2 + Bp + C \neq 0, \quad \text{and} \tag{4.8}$$

$$4a_2(Bp+2C)^2 + 4a_1(2Ap+B)^2 + (a_1+a_2p^2)(B^2-4AC) = 0. \tag{4.9}$$

We first prove that (4.7) holds for any point in Γ. By (3.13), $\widetilde{A} = 0$ at $(z(c), w(c)) \in \Gamma$ if and only if

$$-4a_1(2+a_2)^2c^2 + 4a_2(1+a_2)(2+a_1)^2(1-c^2) - 4(2+a_1)^2(2+a_2)^2 = 0,$$

namely,

$$-4a_1(2+a_2)^2c^2 - 4(2+a_1)^2[4+3a_2+c^2a_2+c^2a_2^2] = 0.$$

But this is a contradiction because the left-hand side is strictly negative for any $c \in [0,1]$.

We also notice that $A > 0$ along Γ, too.

Next, after being restricted to Γ, (4.9) can be written as

$$\begin{aligned}&\Big[4a_2B^2+16a_1A^2+a_2(B^2-4AC)\Big]p^2+(16a_2BC+16a_1AB)p\\&+\Big[16a_2C^2+4a_1B^2+a_1(B^2-4AC)\Big]=0.\end{aligned}\tag{4.10}$$

In order to solve the equation (4.9), by (4.3) and (4.10), it is enough to show that there exists a point $c\in(0,1)$ such that $K(c)=0$, where

$$\begin{aligned}K(c)\quad&:=\Big[4a_2B^2+16a_1A^2+a_2(B^2-4AC)\Big]\Big(a_1z+(2+a_1)\overline{z}\Big)^2\\&-(16a_2BC+16a_1AB)\Big(a_1z+(2+a_1)\overline{z}\Big)\Big(a_2w+(2+a_2)\overline{w}\Big)\\&+\Big[16a_2C^2+4a_1B^2+a_1(B^2-4AC)\Big]\Big(a_2w+(2+a_2)\overline{w}\Big)^2.\end{aligned}\tag{4.11}$$

By (4.11) (4.4) (4.5) and (4.6), $K(c)$ is a real-valued function defined on $[0,1]$. When $c=0$, we have $z=0$, $w=i$ and

$$\begin{aligned}A&=a_1(2+a_2)^2\Big[-a_2(1+a_2)(2+a_1)^2+(2+a_1)^2(2+a_2)^2\Big]\\&=a_1(2+a_1)^2(2+a_2)^2(4+3a_2),\quad B=0,\\C&=-4(1+a_2)^2(2+a_1)^4+a_2(2+a_1)^2\Big[-a_2(1+a_2)(2+a_1)^2\\&+(2+a_1)^2(2+a_2)^2\Big]=-(2+a_1)^4(2+a_2)^2.\end{aligned}$$

Hence

$$K(0)=-16C(4a_2C-a_1A)<0,\tag{4.12}$$

by noticing that $C<0$ and $A>0$.

When $c=1$, we have $z=\frac{1}{\sqrt{1+a_1}}$, $w=0$ and

$$\begin{aligned}A&=4(1+a_1)(2+a_2)^4+a_1(2+a_2)^2\Big[a_1(2+a_2)^2+(2+a_1)^2(2+a_2)^2\Big]\\&=(2+a_2)^4(1+a_1)(2+a_1)^2,\quad B=0,\\C&=a_2(2+a_1)^2\Big[a_1(2+a_2)^2+(2+a_1)^2(2+a_2)^2\Big]\\&=a_2(2+a_1)^2(2+a_2)^2(1+a_1)(4+a_1).\end{aligned}$$

Hence

$$\begin{aligned}K(1)\quad&=4A(4a_1A-a_2C)4(1+a_1)\\&=d^*[4a_1(2+a_2)^2-a_2^2(4+a_1)].\end{aligned}\tag{4.13}$$

Here $d^*>0$. Hence, when

$$4a_1(2+a_2)^2-a_2^2(4+a_1)=16a_1+16a_1a_2+3a_1a_2^2-4a_2^2>0,$$

$K(0)<0$ and $K(1)>0$. Thus, $K(c)=0$ for a certain $c\in(0,1)$. Namely, we showed that (4.9) holds for a certain c.

It remains to prove that (4.8) cannot hold for the above $c\in(0,1)$. Suppose that $\hat{H}(c)^2=0$. Since $\widetilde{A}(c)>0$ and $A(c)>0$, w conclude by (3.25), that $2Ap+B=0$.

Making use of (4.3), (4.4) and (4.5), we thus have

$$-8(1+a_1)(1+a_2)(2+a_1)^2(2+a_2)^2\frac{ic\sqrt{1-c^2}}{\sqrt{1+a_1}} = \frac{2i(\sqrt{1+a_1})c}{\sqrt{1-c^2}}\cdot A(c). \tag{4.14}$$

This is a contradiction, for after dividing the fact i, the left-hand side of (4.14) is negative, while its right-hand side is strictly positive. The proof of Lemma 4.1 is complete. □

Proof of Theorem 1.2. If M is spherical, then every point is an umbilical point. We assume that M is not spherical. Then $a_1 + a_2 > 0$. We notice that $(1+a_1)x_1^2 + y_1^2 + (1+a_2)x_2^2 + y_2^2 = 1$ is holomorphically equivalent to the ellipsoid defined by $(1+a_2)x_1^2 + y_1^2 + (1+a_1)x_2^2 + y_2^2 = 1$ through the map $(z,w) \to (w,z)$. Hence, we need only to prove Theorem 1.2 for the case when $a_1 \geq a_2$. Then the assumption in Lemma 4.1 holds automatically and thus we have an umbilical point of the form $(\frac{c}{\sqrt{1+a_1}}, i\sqrt{1-c^2})$ $(c \in (0,1))$. Notice that M has automorphisms sending (z,w) to $(\pm z, \pm w)$. We easily conclude that M possesses at least four umbilical points. □

5. Proof of Theorem 1.3

The ϵ-thickening Ω_ϵ of the unit circle $\{|z| = 1, w = 0\}$ is defined to be the set of points whose distance to the circle is less than ϵ. It is straightforward to verify that the boundary M_ϵ of Ω_ϵ is defined by the following equation, which is strictly plurisubharmonic when $0 < \epsilon < 1/4$:

$$|z|^2 - 2|z| + 1 + |w|^2 = \epsilon^2. \tag{5.1}$$

Here and in what follows, we assume $0 < \epsilon << 1$. Also, since Ω_ϵ is a Reinhardt domain, we need only to study the points $(z,w) \in M_\epsilon$ with $z = x_1 \geq 0$ and $w = x_2 \geq 0$. Also, we assume that $x_2 > 0$. Notice that when $\epsilon << 1$, $x_2 \approx 1$.

The complexification of (5.1) is given by

$$r := z\zeta - 2(z\zeta)^{1/2} + 1 + w\eta - \epsilon^2 = 0. \tag{5.2}$$

As in §3, we have

$$r_z = \zeta - (z\zeta)^{-1/2}\zeta + p\eta = 0, \text{ and} \tag{5.3}$$

$$r_{zz} = \frac{1}{2}(z\zeta)^{-3/2}\zeta^2 + p_{11}\eta = 0. \tag{5.4}$$

From (5.3), we have

$$z\zeta - (z\zeta)^{1/2} + pz\eta = 0. \tag{5.5}$$

Subtracting (5.2) from (5.5), we obtain

$$(z\zeta)^{1/2} = 1 - \epsilon^2 + (w - pz)\eta. \tag{5.6}$$

Returning to (5.4) and making use of (5.6), we get

$$1 - \epsilon^2 + (w - pz)\eta + 2\eta z^2 p_{11} = 0. \tag{5.7}$$

Here, we remark that near the point under study, $\eta \approx x_2 \neq 0$. Hence $\frac{1-\epsilon^2}{\eta} + (w - pz) + 2z^2 p_{11} = 0$ and

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \iff \frac{\partial^4}{\partial p^4}\left(\frac{1}{\eta}\right) = 0. \tag{5.8}$$

Set $X = \frac{1}{\eta}$. Substituting (5.6) into (5.2), we get

$$\Big[(1-\epsilon)^2 + (w-pz)\eta\Big]^2 - 2\Big[(1-\epsilon^2) + (w-pz)\eta\Big] + 1 + w\eta - \epsilon^2 = 0, \text{ or}$$

$$(w-pz)^2\eta^2 + \Big[2(1-\epsilon^2)(w-pz) - 2(w-pz) + w\Big]\eta$$
$$+ (1-\epsilon^2)^2 - 2(1-\epsilon^2) + (1-\epsilon^2) = 0,$$
$$-\epsilon^2(1-\epsilon^2)X^2 + \Big[-2\epsilon^2(w-pz) + w\Big]X + (w-pz)^2 = 0.$$

Hence

$$X = \frac{-(-2\epsilon^2(w-pz)+w) \pm H}{-2\epsilon^2(1-\epsilon^2)} \tag{5.9}$$

where

$$H^2 := (2\epsilon^2(w-pz)-w)^2 + 4\epsilon^2(1-\epsilon^2)(w-pz)^2.$$

Hence

$$\frac{\partial^4 p_{11}}{\partial p^4} = 0 \iff \frac{\partial^4 H}{\partial p^4} = 0. \tag{5.10}$$

Write $H^2 = Ap^2 + Bp + C$ where

$$\begin{aligned} A &= 4\epsilon^2 z^2 + 4\epsilon^2(1-\epsilon^2)z^2 = 4\epsilon^2 z^2, \\ B &= -4\epsilon^2 z(2\epsilon^2 w) - 8\epsilon^2(1-\epsilon^2)wz = -4\epsilon^2 wz, \\ C &= \epsilon^2 w^2. \end{aligned} \tag{5.11}$$

By (3.29), we conclude that $\frac{\partial^4 p_{11}}{\partial p^4} = 0$ if and only if

$$\text{either } 4AC - B^2 = 0 \text{ or } B^2 - 4A + 4(2Ap+B)^2 = 0. \tag{5.12}$$

Since $4AC - B^2 = 4\epsilon^2(zw)^2(1-4\epsilon^2) \neq 0$, the first equality in (5.12) never occurs. The second equality in (5.12) is equivalent to $4AC - B^2 = 4(2Ap+B)^2$, namely,

$$2\epsilon zw\sqrt{1-4\epsilon^2} = \pm 2(2Ap+B). \tag{5.13}$$

At the point in M with $z = x_1 > 0$ and $w = x_2 > 0$, by (5.3), we find $x_1 - 1 + px_2 = 0$, or

$$p = \frac{1-x_1}{x_2}. \tag{5.14}$$

Hence we get from (5.11)

$$A = 4\epsilon^2 x_1^2, \; B = -4\epsilon^2 x_1 x_2 \text{ and } C = \epsilon^2 x_2^2. \tag{5.15}$$

Then (5.13) is equivalent to

$$2\epsilon x_1 x_2\sqrt{1-4\epsilon^2} = \pm 2\left(8\epsilon^2 x_1^2 \cdot \frac{1-x_1}{x_2} - 4\epsilon^2 x_1 x_2\right). \tag{5.16}$$

Since $x_1 \approx 1$, we get from (5.16):

$$x_2^2\sqrt{1-4\epsilon^2} = \pm\left(8\epsilon(x_1 - x_1^2) - 4\epsilon x_2^2\right). \tag{5.17}$$

Recall $x_2^2 = \epsilon^2 - (1-x_1)^2$. Let $T = 1 - x_1$. Then $x_1 - x_1^2 = T - T^2$ and $x_2^2 = \epsilon^2 - T^2$. Hence (5.17) is equivalent to

$$(\epsilon^2 - T^2)\sqrt{1-4\epsilon^2} = \pm 4\epsilon(2T - T^2 - \epsilon^2), \tag{5.18}$$

or

$$f(T) := (\sqrt{1-4\epsilon^2} \mp 4\epsilon)T^2 \pm 8\epsilon T + (-\epsilon^2\sqrt{1-4\epsilon^2} \mp \epsilon^2) = 0.$$

Notice that $-\epsilon < T < \epsilon$. From the fact that

$$f'(T) = 2(\sqrt{1-4\epsilon^2} \mp 4\epsilon)T \pm 8\epsilon = 0 \iff |T| \approx 4\epsilon$$

for $\epsilon << 1$, we conclude that the real-valued function $f(T)$ is monotonic for $T \in (-\epsilon, \epsilon)$. We further compute

$$f(-\epsilon) = (\sqrt{1-4\epsilon^2} \mp 4\epsilon)\epsilon^2 \mp 8\epsilon^2 + (-\epsilon^2\sqrt{1-4\epsilon^2} \mp 4\epsilon^3) \approx \mp 8\epsilon^2$$

and

$$f(\epsilon) = (\sqrt{1-4\epsilon^2} \mp 4\epsilon)\epsilon^2 \pm 8\epsilon^2 + (-\epsilon^2\sqrt{1-4\epsilon^2} \mp 4\epsilon^3) \approx \pm 8\epsilon^2$$

for $\epsilon << 1$. Then we see that (5.12) has two solutions in $(-\epsilon, \epsilon)$. A little more effort actually shows that these two solutions are different. Therefore, by Theorem 2.1, we conclude that M admits two distinct umbilical points with $z = x_1 > 0, w = x_2 > 0$. One can similarly verify that points in M with $w = 0$ are umbilical points. The statement of Theorem 1.3 thus follows from the Reinhardt property of Ω_ϵ. $\square$

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Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903
E-mail address: huangx@math.rutgers.edu

Department of Mathematics, University of Houston, Houston, Texas 77204
E-mail address: shanyuji@math.uh.edu