VALUES OF GAUSS MAPS OF COMPLETE MINIMAL SURFACES IN $\mathbb{R}^m$ ON ANNULAR ENDS

LU JIN AND MIN RU

ABSTRACT. Let M be a complete minimal surface in $\mathbb{R}^m$ and let A be an annular end of M which is conformal to $\{z \mid 0 < 1/r < |z| < r\}$, where z is the conformal coordinate. Let G be the generalized Gauss map of M . We show that $G(A)$ must intersect every hyperplane in $\mathbb{P}^{m-1}(\mathbb{C})$, with the possible exception of $m(m+1)/2$ hyperplanes in general position.

1. INTRODUCTION

In 1988, H. Fujimoto (see [F1]) proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map G can omit at most 4 points, and the bound is sharp. In 1991, the second author (see [R]), based on an earlier result of Fujimoto, showed that if M is a complete non-flat minimal surface in $\mathbb{R}^m$, then its generalized Gauss map G cannot omit more than $m(m+1)/2$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$, located in general position. In the same year, Shu-Jung Kao (see [K]) extended Fujimoto's method and showed that the Gauss map of an end of a non-flat complete minimal surface in $\mathbb{R}^3$ that is conformally an annulus $\{z \mid 0 < 1/r < |z| < r\}$ must also assume every value, with at most 4 exceptions. This paper extends Shu-Jung Kao's result to minimal surfaces in $\mathbb{R}^m$. We prove the following result.

Main Theorem. *Let M be a complete non-flat minimal surface in $\mathbb{R}^m$ and let A be an annular end of M which is conformal to $\{z \mid 0 < 1/r < |z| < r\}$. Let G be the generalized Gauss map of M . Then $G(A)$ intersects with every hyperplane in $\mathbb{P}^{m-1}(\mathbb{C})$, with the possible exception of $m(m+1)/2$ hyperplanes in general position.*

2. THE MAIN LEMMA

(A) Associated curves and projective distance. Let f be a non-constant holomorphic map of $\Delta_R := \{z \in \mathbb{C}; |z| < R\}$ into $\mathbb{P}^n(\mathbb{C})$, where $0 < R \leq +\infty$. Assume that f is k -degenerate, i.e., $f(\Delta_R)$ is contained in a subspace of dimension $k \leq n$, but none of lower dimension. Without loss of generality, we can assume that $f(\Delta_R)$ is contained in $\mathbb{P}^k(\mathbb{C})$, so that $f : \Delta_R \rightarrow \mathbb{P}^k(\mathbb{C})$ is linearly non-degenerate. Take a reduced representation $F = (f_0, \dots, f_k)$ of f , where $F : \Delta_R \rightarrow \mathbb{C}^{k+1} \setminus \{0\}$ is

Received by the editors March 9, 2004 and, in revised form, January 14, 2005.

2000 *Mathematics Subject Classification.* Primary 53C42; Secondary 32H30.

The first author was supported by the National Natural Science Foundation of China (No. 10271029). The second author was supported in part by NSA under grant number H98230-05-1-0042.

©2006 American Mathematical Society
 Reverts to public domain 28 years from publication

a holomorphic map with $\mathbb{P}(F) = f$. Denote by $F^{(p)}$ the p -th derivative of F and define

$$F_p = F^{(0)} \wedge \cdots \wedge F^{(p)} : \Delta_R \rightarrow \bigwedge^{p+1} \mathbb{C}^{k+1}$$

for $0 \leq p \leq k$. Evidently $F_{k+1} \equiv 0$. Note that, for each $p = 0, 1, \dots, k$, the norm of F_{p+1} is

$$|F_{p+1}| = \left(\sum_{0 \leq j_0 < \cdots < j_p \leq k} |W(f_{j_0}, \dots, f_{j_p})|^2 \right)^{1/2},$$

where $W(f_{j_0}, \dots, f_{j_p})$ is the Wronskian of $f_{j_0}, \dots, f_{j_p}$. Let

$$\mathbb{P} : \bigwedge^{p+1} \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P} \left(\bigwedge^{p+1} \mathbb{C}^{k+1} \right) \subset \mathbb{P}^{N_p}(\mathbb{C})$$

be the natural projection, where $N_p = \frac{(k+1)!}{(p+1)!(k-p)!} - 1$. Then F_p , $0 \leq p \leq k$, is projected down to a curve

$$f^p = \mathbb{P}(F_p) : \Delta_R \rightarrow \mathbb{P}^{N_p}(\mathbb{C})$$

which is called the p -th associated curve of f . Let $H = \{[x_0 : \cdots : x_k] \mid a_0 x_0 + \cdots + a_k x_k = 0\}$ be a hyperplane in $\mathbb{P}^k(\mathbb{C})$, where we assume that $\sum_{i=0}^k |a_i|^2 = 1$. Then $F(H) = a_0 f_0 + \cdots + a_k f_k$ measures the distance from $F(z)$ to H . In a similar way, we define the distance from $F_p(z)$ to H as

$$|F_p(H)|^2 = \sum_{0 \leq i_1 < \cdots < i_p \leq k} \left| \sum_{i_0 \neq i_1, \dots, i_p} a_{i_0} W(f_{i_0}, f_{i_1}, \dots, f_{i_p}) \right|^2.$$

Denote

$$(2.1) \quad \phi_0(H) = \frac{|F(H)|^2}{|F|^2}, \quad \phi_p(H) = \frac{|F_p(H)|^2}{|F_p|^2}.$$

Note that $0 \leq \phi_p(H) \leq \phi_{p+1}(H) \leq 1$ and $\phi_k(H) \equiv 1$.

(B) Nochka weights. Let $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C}) - \bigcup_{j=1}^q H_j$ be a non-constant holomorphic map, where $H_j, 1 \leq j \leq q$, are hyperplanes in general position. Assume that f is k -linearly non-degenerate, i.e., $f(\mathbb{C})$ is contained in a subspace of dimension $k \leq n$, but none of lower dimension. Without loss of generality, we assume that $f : \mathbb{C} \rightarrow \mathbb{P}^k(\mathbb{C})$. Then f is linearly non-degenerate. Furthermore, the hyperplanes $H_j \cap \mathbb{P}^k(\mathbb{C}), 1 \leq j \leq q$, are in n -subgeneral position. Without loss of generality, we still use H_j to denote $H_j \cap \mathbb{P}^k(\mathbb{C})$ in $\mathbb{P}^k(\mathbb{C})$. Recall that hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^k(\mathbb{C})$ are said to be in n -subgeneral position if at every point in $\mathbb{P}^k(\mathbb{C})$ there are at most n hyperplanes among them passing through this point. We have the following lemma (cf. [Ch] or [N]).

Lemma 2.1. *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in n -subgeneral position. Then there exist a function $\omega : \{1, \dots, q\} \rightarrow \mathbb{R}(0, 1]$ and a number $\theta > 0$ with the following properties:*

- (1) $0 < \omega(j)\theta \leq 1$ for all $j \in \{1, \dots, q\}$.
- (2) $q - 2n + k - 1 = \theta(\sum_{j=1}^q \omega(j) - k - 1)$,
- (3) $1 \leq (n+1)/(k+1) \leq \theta \leq (2n - k + 1)/(k+1)$.

We call ω the *Nochka weight* for the hyperplanes $\{H_j, 1 \leq j \leq q\}$.

(C) The Main Lemma.

Main Lemma. *Let $f = [f_0 : \cdots : f_k] : \Delta_R \rightarrow \mathbb{P}^k(\mathbb{C})$ be a non-degenerate holomorphic map, let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in n -subgeneral position, and let $\omega(j)$ be their Nochka weight. Let $F = (f_0, \dots, f_k) : \Delta_R \rightarrow \mathbb{C}^{k+1} \setminus \{0\}$ be a reduced representation of f . Assume that $q > 2n - k + 1$ and*

$$2q/N < \frac{\sum_{j=1}^q \omega(j) - (k+1)}{k(k+2)}.$$

Then there exists some positive constant C such that

$$\begin{aligned} & |F|^S \frac{\prod_{p=0}^{k-1} \prod_{j=1}^q |F_p(H_j)|^{4/N} |F_k|^{1+2q/N}}{\prod_{j=1}^q |F(H_j)|^{\omega_j}} \\ & \leq C \left(\frac{2R}{R^2 - |z|^2} \right)^{k(k+1)/2 + \sum_{p=0}^{k-1} (k-p)^2 2q/N}, \end{aligned}$$

where S is given by $\sum_{j=1}^q \omega(j) - (k+1) - (k^2 + 2k - 1)2q/N$.

For the proof, see [R].

3. PROOF OF THE MAIN THEOREM

Let M be a complete minimal surface in $\mathbb{R}^m$. Take an immersion $x : M \rightarrow \mathbb{R}^m$. Make M into a Riemann surface by decreeing that the 1-form $dx + \sqrt{-1}dy$ is of type $(1, 0)$, where (x, y) is any isothermal coordinates. The generalized Gauss map of x is defined to be

$$G : M \rightarrow \mathbb{P}^{m-1}(\mathbb{C}), \quad G(z) = \mathbb{P} \left(\frac{\partial x}{\partial z} \right).$$

If M is minimal, then G is a holomorphic map. Let A be an annular end of M , that is, $A = \{z \mid 0 < 1/r \leq |z| < r < \infty\}$, where z is the conformal coordinate. Assume that G is not constant on A ; then $G(A)$ is contained in a subspace of dimension k but none of lower dimension, with $0 < k \leq m - 1$. Without loss of generality, we assume that $G(A) \subset \mathbb{P}^k(\mathbb{C})$; we then can write $G = [g_0 : \cdots : g_k]$. Let $\tilde{G} = (g_0, \dots, g_k)$. Then the metric ds^2 on M induced from the standard metric on $\mathbb{R}^{m-1}$ is given by

$$ds^2 = 2|\tilde{G}|^2 |dz|^2.$$

Given hyperplanes $H_j, j = 1, \dots, q$, defined by

$$H_j = \{[z_0 : \cdots : z_k] \mid a_{j,0}z_0 + \cdots + a_{j,k}z_k = 0\},$$

then, for each pair (j, p) , there are $i_1, \dots, i_p$ such that

$$(3.1) \quad \psi_{j,p} = \sum_{l \neq i_1, \dots, i_p} a_{j,l} W(g_l, g_{i_1}, \dots, g_{i_p})$$

does not vanish identically. Every $\psi_{j,p}$ is holomorphic, so they have only isolated zeros.

Now assume the Main Theorem is not true, i.e., $G(A)$ does not intersect q hyperplanes in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ and $q > m(m+1)/2$.

Consider the numbers

$$\rho = \frac{1}{S} \left[\frac{k}{2}(k+1) + \frac{2q}{N} \sum_{p=0}^k (k-p)^2 \right],$$

$$\gamma = \left[\frac{k}{2}(k+1) + \frac{qk}{N}(k+1) + \frac{2q}{N} \sum_{p=0}^{k-1} p(p+1) \right],$$

and

$$\rho^* = \frac{1}{S - \gamma}.$$

Choose some N such that

$$\frac{\sum_{j=1}^q \omega(j) - (k+1) - k(k+1)/2}{k^2 + 2k - 1 + \sum_{p=0}^k (k-p)^2}$$

$$> 2q/N > \frac{\sum_{j=1}^q \omega(j) - (k+1) - k(k+1)/2}{2/q + (k^2 + 2k - 1) + k(k+1)/2 + \sum_{p=0}^{k-1} p(p+1)}$$

so that

$$(3.2) \quad 0 < \rho < 1 \text{ and } \frac{4\rho^*}{N} > 1.$$

Let

$$A' = \text{Int } A - \bigcup_{j,p} \{z; \psi_{j,p}(z) = 0\} \cup \{z; \psi_{j,p}(1/z) = 0\}.$$

On A' , define a new metric

$$d\tilde{s}^2 = \left(\frac{\prod_{j=1}^q |\tilde{G}(H_j)|^{\omega(j)}}{|\tilde{G}_k|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |\psi_{j,p}|^{4/N}} \right)^{2\rho^*} |dz|^2.$$

Since $\tilde{G}(H_j)$, $\tilde{G}_k$ and $\psi_{j,p}$ are all holomorphic, the metric $d\tilde{s}^2$ is flat on A' . We now prove the following claim.

Claim. $d\tilde{s}^2$ is complete on the set $\{z \mid |z| = r\} \cup_{j,p} \{z \mid \psi_{j,p}(z) = 0 \text{ or } \psi_{j,p}(1/z) = 0\}$, i.e., set $\{z \mid |z| = r\} \cup_{j,p} \{z \mid \psi_{j,p}(z) = 0 \text{ or } \psi_{j,p}(1/z) = 0\}$ is at infinite distance from any interior point.

In fact, if either $\tilde{G}_k(z_0) = 0$ or $\psi_{j,p}(z_0) = 0$ for some positive p and j , then

$$|d\tilde{s}| \sim \frac{c}{|z - z_0|^{\delta_0}} |dz|$$

around z_0 where $c > 0$, $\delta_0 = 4\rho^*/N > 1$. Therefore, $d\tilde{s}^2$ is complete on

$$\bigcup_{j,p} \{z \mid \psi_{j,p}(z) = 0 \text{ or } \psi_{j,p}(1/z) = 0\}.$$

Now assume that $d\tilde{s}^2$ is not complete on the set $\{z \mid |z| = r\}$. There exists $\gamma : [0, 1) \rightarrow A'$, where $\gamma(1) \in \{z \mid |z| = r\}$ so that $|\gamma| < \infty$. Furthermore, we may also assume that $\text{dist}(\gamma(0), \{z \mid |z| = 1/r\}) > 2|\gamma|$. Consider a small disk Δ with center at $\gamma(0)$. Since $d\tilde{s}$ is flat, Δ is isometric to an ordinary disk in the plane. Let $B : \{|w| < \eta\} \rightarrow \Delta$ be the isometry. Extend B , as a local isometry into A' , to the largest disk $\{|w| < R\}$. Then $R \leq |\gamma|$. The reason B cannot be extended to a larger disk must be that the image goes to the outside boundary $\{z \mid |z| = r\}$ of

A' . More precisely, there exists a point w_0 with $|w_0| = R$ such that if we consider the curve

$$L_{w_0} : w = tw_0, \quad 0 \leq t < 1,$$

then the image Γ of L_{w_0} by B is a divergent curve on A . Let $Z = \tilde{G} \circ B : \{|w| < R\} \rightarrow \mathbb{C}^{k+1}$. Then the metric on $\{|w| < R\}$ induced from $ds^2 = 2|\tilde{G}|^2|dz|^2$ through B is given by

$$(3.3) \quad B^*ds^2 = 2|\tilde{G}(B(w))|^2 \left| \frac{dz}{dw} \right|^2 |dw|^2.$$

We now calculate $\left| \frac{dz}{dw} \right|$. We know that

$$|dw| = |d\tilde{s}| = \left(\frac{\prod_{j=1}^q |\tilde{G}(H_j)|^{\omega(j)}}{|\tilde{G}_k|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |\psi_{j,p}|^{4/N}} \right)^{\rho^*} |dz|.$$

Instead of z , we change to the coordinate w for the right-hand side of the above expression. Precisely speaking, we let $Z(w) = \tilde{G}(B(w))$ and let $\hat{\psi}_{j,p}(w)$ be defined from Z in the same way the $\psi_{j,p}$ was defined from $\tilde{G}$ (see (3.1)). We note that $Z \wedge Z' \cdots \wedge Z^{(p)} = (\tilde{G} \wedge \cdots \wedge \tilde{G}^{(p)}) \left(\frac{dz}{dw} \right)^{p(p+1)/2}$, hence we have

$$\begin{aligned} \left| \frac{dw}{dz} \right| &= \left(\frac{\prod_{j=1}^q |\tilde{G}(H_j)|^{\omega(j)}}{\prod_{p=0}^{k-1} \prod_{j=1}^q |\psi_{j,p}|^{4/N} |\tilde{G}_k|^{1+2q/N}} \right)^{\rho^*} \\ &= \left(\frac{\prod_{j=1}^q |Z(H_j)|^{\omega(j)}}{|Z_k|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |\hat{\psi}_{j,p}|^{4/N}} \right)^{\rho^*} \frac{1}{\left| \frac{dw}{dz} \right|^{\rho^* \gamma}}. \end{aligned}$$

Hence,

$$\begin{aligned} \left| \frac{dz}{dw} \right| &= \left(\frac{|Z_k|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |\hat{\psi}_{j,p}|^{4/N}}{\prod_{j=1}^q |Z(H_j)|^{\omega(j)}} \right)^{\frac{\rho^*}{1+\rho^* \gamma}} \\ &= \left(\frac{|Z_k|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |\hat{\psi}_{j,p}|^{4/N}}{\prod_{j=1}^q |Z(H_j)|^{\omega(j)}} \right)^{\frac{1}{S}}. \end{aligned}$$

By the definition of $\hat{\psi}_{j,p}(w)$, we have $|\hat{\psi}_{j,p}(w)| \leq |Z_p(H_j)|$, therefore

$$(3.4) \quad \left| \frac{dz}{dw} \right| \leq \left(\frac{|Z_k|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |Z_p(H_j)|^{4/N}}{\prod_{j=1}^q |Z(H_j)|^{\omega(j)}} \right)^{\frac{1}{S}}.$$

Combining (3.3) and (3.4), we have

$$B^*ds = 2|Z| \left(\frac{|Z_k|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |Z_p(H_j)|^{4/N}}{\prod_{j=1}^q |Z(H_j)|^{\omega(j)}} \right)^{\frac{1}{S}} |dw|.$$

Using the Main Lemma, we obtain

$$B^*ds^2 \leq C \left(\frac{2R}{R^2 - |w|^2} \right)^\rho |dw|,$$

where C is a positive constant. Hence, we deduce from $0 < \rho < 1$ that

$$|\Gamma| \leq \int_{(o, w_0)} ds = \int_{L_{w_0}} B^* ds \leq C \int_0^R \left(\frac{2R}{R^2 - |w|^2} \right)^\rho |dw| < +\infty,$$

which is impossible, since Γ is also a divergent curve on M and M is complete.

Define $ds'^2 = \lambda(z)|dz|^2$ on A' , where

$$\begin{aligned} \lambda(z) = & \left(\frac{\prod_{j=1}^q |\tilde{G}(z)(H_j)|^{\omega(j)}}{|\tilde{G}_k(z)|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |\psi_{j,p}(z)|^{4/N}} \right)^{2\rho^*} \\ & \times \left(\frac{\prod_{j=1}^q |\tilde{G}(1/z)(H_j)|^{\omega(j)}}{|\tilde{G}_k(1/z)|^{1+2q/N} \prod_{p=0}^{k-1} \prod_{j=1}^q |\psi_{j,p}(1/z)|^{4/N}} \right)^{2\rho^*}. \end{aligned}$$

Then ds' is complete and flat on A' . Let $u(z) = \log \lambda(z)$. Then $u(z)$ is a harmonic function on A' . Let D be the universal covering surface of A' . In a neighborhood of any point of D , we may introduce an analytic function $f(z)$ whose real part is $u(z)$, and the mapping

$$w(z) = \int e^{f(z)} dz$$

satisfies

$$(3.5) \quad \left| \frac{dw}{dz} \right| = \left| e^{f(z)} \right| = e^{u(z)} = \lambda.$$

Thus the length of any curve on D with respect to the metric ds' is equal to the length of its image in the w -plane. By the simple connectivity of D , there exists a global map of D into the w -plane which satisfies (3.4), and by the completeness of D , this map must be a one-to-one map of D onto the entire w -plane. Thus D is conformally equivalent to the plane, which is impossible by H. Cartan's theorem which says any holomorphic map $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ omitting $2n+1$ hyperplanes in general position must be constant. This proves the Main Theorem. $\square$

ACKNOWLEDGMENTS

The second author wishes to thank the Department of Mathematics, Fudan University, People's Republic of China for their kind hospitality during which part of the work on this paper took place.

REFERENCES

- [A] Ahlfors, L.: The theory of meromorphic curves. Acta. Soc. Sci. Fenn. Nova Ser. A **3**(4), 171-183 (1941). MR0004309 (2:357b)
- [Ca] Cartan, H.: Sur les zeros des combinaisons lineaires de p fonctions holomorphes donnees. Mathematica (Cluj) **7**, 80-103 (1933).
- [Ch] Chen, W.: Cartan conjecture: Defect relation for meromorphic maps from parabolic manifold to projective space, Thesis, University of Notre Dame, 1987.
- [CO] Chern, S.S. and Osserman, R.: Complete minimal surface in Euclidean n -space, J. Analyse Math., **19**, 15-34 (1967). MR0226514 (37:2103)
- [CG] Cowen, M.J. and Griffiths, P.A.: Holomorphic curves and metric of negative curvature, J. Analyse Math., **29**, 93-153 (1976). MR0508156 (58:22677)
- [F1] Fujimoto, H.: On the number of exceptional values of the Gauss map of minimal surfaces, J. Math. Soc. Japan, **49**, 235-385 (1988). MR0930599 (89b:53013)
- [F2] Fujimoto, H.: Modified defect relations for the Gauss map of minimal surfaces, II, J. Differential Geometry, **31**, 365-385 (1990). MR1037406 (91a:53011)

- [H] Huber, A.: On subharmonic functions and differential geometry in the large, *Comm. Math. Helv.*, **32**, 13–72 (1957). MR0094452 (20:970)
- [K] Kao, S.J.: On values of Gauss maps of complete minimal surfaces on annular ends, *Math. Ann.*, 315–318 (1991). MR1129370 (92k:53018)
- [N] Nochka, E.I.: On the theory of meromorphic functions, *Soviet. Math. Dokl.*, **27(2)**, 377–381 (1983). MR0701289 (85i:32038)
- [O] Osserman, R.: A survey of minimal surfaces, 2nd ed., Dover, New York, 1986. MR0852409 (87j:53012)
- [OR] Osserman, R. and Ru, M.: An estimate for the Gauss curvature on minimal surfaces in $\mathbb{R}^m$ whose Gauss map omits a set of hyperplanes, *J. Diff. Geometry*, **46**, 578–593 (1997). MR1484891 (98k:53014)
- [R] Ru, M.: On the Gauss map of minimal surfaces immersed in $\mathbb{R}^n$, *J. Diff. Geometry*, **34**, 411–423 (1991). MR1131437 (92k:53117)
- [Sha] Shabat, B.V.: Distribution of values of holomorphic mappings. *Translations of Mathematical Monographs*, Vol. **61**, American Math. Soc., 1985. MR0807367 (86k:32023)
- [St] Stoll, W.: The Ahlfors-Weyl theory of meromorphic maps on parabolic manifolds, *Lecture Notes in Math.*, Vol. **981**, Springer, Berlin, 101–219 (1981). MR0699135 (85c:32045)

DEPARTMENT OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI, PEOPLE'S REPUBLIC OF CHINA
E-mail address: jinluk@online.sh.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77204
E-mail address: minru@math.uh.edu