THE G -STABLE PIECES OF THE WONDERFUL COMPACTIFICATION

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ABSTRACT. Let G be a connected, simple algebraic group over an algebraically closed field. There is a partition of the wonderful compactification $\bar{G}$ of G into finite many G -stable pieces, which was introduced by Lusztig. In this paper, we will investigate the closure of any G -stable piece in $\bar{G}$. We will show that the closure is a disjoint union of some G -stable pieces, which was first conjectured by Lusztig. We will also prove the existence of cellular decomposition if the closure contains finitely many G -orbits.

INTRODUCTION

An adjoint semi-simple group G has a “wonderful” compactification $\bar{G}$, introduced by De Concini and Procesi in [DP]. The variety $\bar{G}$ is a smooth variety with $G \times G$ action. Denote by G_{diag} the image of the diagonal embedding of G into $G \times G$. The G_{diag} -orbits of $\bar{G}$ were studied by Lusztig in [L4]. He introduced a partition of $\bar{G}$ into finitely many G -stable pieces. The G -orbits on each piece can be described explicitly. Based on the partition, he established the theory of “parabolic character sheaves” on $\bar{G}$.

The main results of this paper concern the closure of the G -stable pieces. The closure of each piece is a union of some other pieces, and if the closure contains finitely many G -orbits, then it admits a cellular decomposition. I believe that our results are necessary ingredients for establishing the (conjectural) Kazhdan-Lusztig theory on the “Parabolic Character Sheaves” on $\bar{G}$.

We now review the content of this paper in more detail.

In section 1, we recall the definition of G -stable pieces in [L4] and establish some basic results. The pieces are indexed by the pairs $\mathcal{I} = \{(J, w)\}$, where J is a subset of the simple roots and w is an element of the Weyl group W , which has minimal length in the coset wW_J . One interesting result is that any G -stable piece is the minimal G -stable subset that contains a particular $B \times B$ -orbit, where B is the Borel subgroup. The closure of any $B \times B$ -orbit in $\bar{G}$ was studied by Springer in [S]. Based on his result and the relations between G -stable pieces and $B \times B$ -orbits, we are able to investigate the closure of the G -stable pieces.

In section 2, we recall the definition of the “wonderful” compactification and introduce “compactification through the fibres”, a technique tool that will be used to prove the existence of cellular decomposition. In section 3, we describe a partial order on $\mathcal{I}$, which is the partial order that corresponds to the closure relation of the

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G -stable piece, as we will see in section 4. In section 4, we also discuss the closure of any G -stable piece that appears in [L3].

In section 5, we discuss the existence of cellular decomposition. Each piece does not have a cellular decomposition. However, a union of certain pieces has a cellular decomposition. (This is motivated by Springer in [S], in which he showed that a union of certain $B \times B$ -orbits is isomorphic to an affine space.) In fact, if the closure contains finitely many G -orbits, then it has a cellular decomposition.

The methods work for an arbitrary connected component of a disconnected algebraic group with identity component G . The results for that component is just a “twisted” version of the results for G itself.

1. THE G -STABLE PIECES

1.1. In the sequel G is a connected, semi-simple algebraic group of adjoint type over an algebraically closed field. Let B be a Borel subgroup of G , B^- be the opposite Borel subgroup and $T = B \cap B^-$. Let $(\alpha_i)_{i \in I}$ be the set of simple roots. For $i \in I$, we denote by s_i the corresponding simple reflection. For any element w in the Weyl group $W = N(T)/T$, we will choose a representative $\dot{w}$ in $N(T)$ in the same way as in [L1, 1.1].

For $J \subset I$, let $P_J \supset B$ be the standard parabolic subgroup defined by J and $P_J^- \supset B^-$ be the opposite of P_J . Set $L_J = P_J \cap P_J^-$. Then L_J is a Levi subgroup of P_J and P_J^- . We denote by Φ_J the set of roots that are a linear combination of $\{(a_j)_{j \in J}\}$. Let Z_J be the center of L_J and $G_J = L_J/Z_J$ be its adjoint group. We denote by π_{P_J} (resp. $\pi_{P_J^-}$) the projection of P_J (resp. P_J^-) onto G_J .

For any $J \subset I$, let $\mathcal{P}_J$ be the set of parabolic subgroups conjugate to P_J . We will write $\mathcal{B}$ for $\mathcal{P}_\emptyset$.

For any subset J of I , let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$ and W^J (resp. ${}^J W$) be the set of minimal length coset representatives of W/W_J (resp. $W_J \backslash W$). Let w_0^J be the unique element of maximal length in W_J . (We will simply write w_0^I as w_0 .) For $J, K \subset I$, we write ${}^J W^K$ for ${}^J W \cap W^K$.

For $w \in W$, we denote by $\text{supp}(w) \subset I$ the set of simple roots whose associated simple reflections occur in some (or equivalently, any) reduced decomposition of w .

For $J, K \subset I$, $P \in \mathcal{P}_J$, $Q \in \mathcal{P}_K$ and $u \in {}^J W^K$, we write $\text{pos}(P, Q) = u$ if there exists $g \in G$, such that ${}^g P = P_J$ and ${}^g Q = {}^u P_K$.

For any parabolic subgroup P , we denote by U_P its unipotent radical. We will simply write U for U_B and U^- for U_{B^-} . For $J \subset I$, set $U_J = U \cap L_J$ and $U_J^- = U^- \cap L_J$.

For any closed subgroup H of G , we denote by H_{diag} the image of the diagonal embedding of H in $G \times G$. For any subgroup H and $g \in G$, we write ${}^g H$ for gHg^{-1} . For any finite set A , we write $|A|$ for the cardinal of A .

1.2. Let $\hat{G}$ be a possibly disconnected reductive algebraic group over an algebraically closed field with identity component G . Let G^1 be a fixed connected component of $\hat{G}$. There exists an isomorphism $\delta : W \rightarrow W$ such that $\delta(I) = I$ and ${}^g P \in \mathcal{P}_{\delta(J)}$ for $g \in G^1$ and $P \in \mathcal{P}_J$. There also exists $g_0 \in G^1$ such that g_0 normalizes T and B . Moreover, g_0 can be chosen in such a way that ${}^{g_0} L_J = L_{\delta(J)}$ for $J \subset I$. We will fix such g_0 in the rest of this paper.

In particular, if $G^1 = G$, then $\delta = id$, where id is the identity map. In this case, we choose g_0 to be the unit element 1 of G .

1.3. We will follow the set-up of [L4].

Let $J, J' \subset I$ and $y \in {}^{J'}W^J$ be such that $\text{Ad}(y)\delta(J) = J'$. For $P \in \mathcal{P}_J$, $P' \in \mathcal{P}_{J'}$, define $A_y(P, P') = \{g \in G^1 \mid \text{pos}(P', {}^gP) = y\}$. Define

$$\tilde{Z}_{J,y,\delta} = \{(P, P', \gamma) \mid P \in \mathcal{P}_J, P' \in \mathcal{P}_{J'}, \gamma \in U_{P'} \backslash A_y(P, P') / U_P\}$$

with $G \times G$ action defined by $(g_1, g_2)(P, P', \gamma) = ({}^{g_2}P, {}^{g_1}P', g_1\gamma g_2^{-1})$.

By [L4, 8.9], $A_y(P, P')$ is a single P', P double coset. Thus $G \times G$ acts transitively on $\tilde{Z}_{J,y,\delta}$.

Let $z = (P, P', \gamma) \in \tilde{Z}_{J,y,\delta}$. Then there exists $g \in \gamma$ such that gP contains some Levi of $P \cap P'$. Now set $P_1 = g^{-1}({}^gP)^{(P'P)}g$, $P'_1 = P^{P'}$. Define

$$\alpha(P, P', \gamma) = (P_1, P'_1, U_{P'_1}gU_{P_1}).$$

By [L4, 8.11], the map α doesn't depend on the choice of g .

To $z = (P, P', \gamma) \in \tilde{Z}_{J,y,\delta}$, we associate a sequence $(J_k, J'_k, u_k, y_k, P_k, P'_k, \gamma_k)_{k \geq 0}$ with $J_k, J'_k \subset I$, $u_k \in W$, $y_k \in J'_k W^{\delta(J_k)}$, $\text{Ad}(y_k)\delta(J_k) = J'_k$, $P_k \in \mathcal{P}_{J_k}$, $P'_k \in \mathcal{P}_{J'_k}$, $\gamma_k = U_{P'_k}gU_{P_k}$ for some $g \in G$ that satisfies $\text{pos}(P'_k, {}^gP_k) = y_k$. The sequence is defined as follows:

$$P_0 = P, P'_0 = P', \gamma_0 = \gamma, J_0 = J, J'_0 = J', u_0 = \text{pos}(P'_0, P_0), y_0 = y.$$

Assume that $k \geq 1$, that $P_m, P'_m, \gamma_m, J_m, J'_m, u_m, y_m$ are already defined for $m < k$ and that $u_m = \text{pos}(P'_m, P_m)$, $P_m \in \mathcal{P}_{J_m}$, $P'_m \in \mathcal{P}_{J'_m}$ for $m < k$. Let

$$J_k = J_{k-1} \cap \delta^{-1}\text{Ad}(y_{k-1}^{-1}u_{k-1})J_{k-1}, J'_k = J_{k-1} \cap \text{Ad}(u_{k-1}^{-1}y_{k-1})\delta(J_{k-1}),$$

$$(P_k, P'_k, \gamma_k) = \alpha(P_{k-1}, P'_{k-1}, \gamma_{k-1}) \in \tilde{Z}_{J_k, y_k, \delta} \text{ (see [L4, 8.10])},$$

$$u_k = \text{pos}(P'_k, P_k), y_k = u_{k-1}^{-1}y_{k-1}, \gamma_k = U_{P'_k}g_{k-1}U_{P_k}.$$

It is known that the sequence is well defined. Moreover, for sufficiently large n , we have that $J_n = J'_n = J_{n+1} = J'_{n+1} = \dots = J_\infty$, $u_n = u_{n+1} = \dots = 1$, $y_n = y_{n+1} = \dots = y_\infty$, $P_n = P_{n+1} = \dots = P_\infty$, $P'_n = P'_{n+1} = \dots = P'_\infty$ and $\gamma_n = \gamma_{n+1} = \dots = \gamma_\infty$. Now we set $\beta(z) = y_\infty$. Then we have that $\beta(z) \in W^{\delta(J)}$. By [L4, 8.18] and [L3, 2.5], the sequence $(J_n, J'_n, u_n, y_n)_{n \geq 0}$ is uniquely determined by $\beta(z)$ and y .

For $w \in W^{\delta(J)}$, set

$$\tilde{Z}_{J,y,\delta}^w = \{z \in \tilde{Z}_{J,y,\delta} \mid \beta(z) = w\}.$$

Then $(\tilde{Z}_{J,y,\delta}^w)_{w \in W^{\delta(J)}}$ is a partition of $\tilde{Z}_{J,y,\delta}$ into locally closed G -stable subvarieties. We call $(\tilde{Z}_{J,y,\delta}^w)_{w \in W^{\delta(J)}}$ the G -stable pieces of $\tilde{Z}_{J,y,\delta}$. For $w \in W^{\delta(J)}$, let $(J_n, J'_n, u_n, y_n)_{n \geq 0}$ be the sequence determined by w and y . The restriction of the map α on $\tilde{Z}_{J,y,\delta}^w$ is a G -equivariant morphism from $\tilde{Z}_{J,y,\delta}^w$ onto $\tilde{Z}_{J_1, y_1, \delta}^w$. We also denote this morphism by α . It is known that α induces a bijection from the set of G -orbits on $\tilde{Z}_{J,y,\delta}^w$ to the set of G -orbits on $\tilde{Z}_{J_1, y_1, \delta}^w$.

We have a consequence $\tilde{Z}_{J,y,\delta}^w \xrightarrow{\alpha} \tilde{Z}_{J_1, y_1, \delta}^w \xrightarrow{\alpha} \tilde{Z}_{J_2, y_2, \delta}^w \xrightarrow{\alpha} \dots$. For sufficiently large n , $\vartheta = \alpha^n : \tilde{Z}_{J,y,\delta}^w \rightarrow \tilde{Z}_{J_\infty, w, \delta}^w$ is independent of the choice of n and is a G -equivariant morphism. Moreover, ϑ induces a bijection from the set of G -orbits on $\tilde{Z}_{J,y,\delta}^w$ to the set of G -orbits on $\tilde{Z}_{J_\infty, w, \delta}^w$.

In the rest of this section, we will fix J, y, δ, w and J_∞ . First, we will give an explicit description of J_∞ in terms of J, δ and w .

Lemma 1.4. *Keep the notion of section 1.3. Then*

$$J_\infty = \max\{K \subset J \mid \text{Ad}(w)\delta(K) = K\}.$$

Proof. Set $v = y_1 w^{-1}$. By [H, 2.2], $v \in W_J$. Now $J_1 = J \cap \delta^{-1}\text{Ad}(y_1^{-1})J$. Thus $\Phi_{\delta(J_1)} \subset \text{Ad}(y_1^{-1})\Phi_J = \text{Ad}(w^{-1})\text{Ad}(v^{-1})\Phi_J = \text{Ad}(w^{-1})\Phi_J$.

Let $i \in J$. Assume that $\alpha_{\delta(i)} \in \text{Ad}(y_1^{-1})\Phi_J$. Then $\alpha_{\delta(i)} = \text{Ad}(y_1^{-1})\alpha = \text{Ad}(y^{-1})\text{Ad}(u_0)\alpha$ for some $\alpha \in \Phi_J$. Then $\alpha_{\text{Ad}(y)\delta(i)} = \text{Ad}(u_0)\alpha$. Note that $\alpha_{\text{Ad}(y)\delta(i)}$ is a simple root and $u_0 \in W^J$. Then $\alpha = \alpha_j$ for some $j \in J$. Hence $i = \delta^{-1}\text{Ad}(y_1^{-1})j$. Therefore, $i \in J \cap \delta^{-1}\text{Ad}(y_1^{-1})J = J_1$. So

$$J_1 = \max\{K \subset J \mid \Phi_{\delta(K)} \subset \text{Ad}(w^{-1})\Phi_J\}.$$

Set $J'_\infty = \max\{K \subset J \mid \text{Ad}(w)\delta(K) = K\}$. Then $J'_\infty \subset J$. Moreover, $\Phi_{\delta(J'_\infty)} = \text{Ad}(w^{-1})\Phi_{J'_\infty} \subset \text{Ad}(w^{-1})\Phi_J$. Thus $J'_\infty \subset J_1$. We can show by induction that $J'_\infty \subset J_n$ for all n . Thus $J'_\infty \subset J_\infty$. By the definition, $J_\infty = J_\infty \cap \delta^{-1}\text{Ad}(w^{-1})J_\infty$. Thus $\text{Ad}(w)\delta(J_\infty) = J_\infty$. So $J_\infty = J'_\infty$. The lemma is proved. $\square$

1.5. Now set $\tilde{h}_{J,y,\delta} = (P_J, \dot{y}^{-1}P_{J'}, U_{\dot{y}^{-1}P_{J'}}, g_0 U_{P_J}) \in Z_{J,y,\delta}$. For $w \in W^{\delta(J)}$ and $v \in W$, set $[\widetilde{J, w, v}]_{y,\delta} = (B \times B)(\dot{w}, \dot{v}) \cdot \tilde{h}_{J,y,\delta}$. Then we have the following result.

Lemma 1.6. *Keep the notion of section 1.3. Let $g \in P_{J_1}$. Set $z = (\dot{w}, g) \cdot \tilde{h}_{J,y,\delta}$ and $z' = (\dot{w}, g) \cdot \tilde{h}_{J_1,y_1,\delta}$. Then $\alpha(z) = z'$.*

Proof. Set $P = P_J$, $P' = \dot{w}\dot{y}^{-1}P_{J'}$, $g_1 = \dot{w}g_0g$ and $v = y_1 w^{-1}$. Then $v \in W_J$.

By the proof of [H, 2.3], $\dot{v}^{-1}L_{J'_1}$ is a Levi factor of $P \cap P'$ and $P^{P'} = \dot{v}^{-1}P_{J'_1}$, $(P')^P = \dot{w}\dot{y}^{-1}P_{\text{Ad}(y)\delta(J_1)}$. Moreover,

$$\dot{v}^{-1}L_{J'_1} = \dot{w}\dot{y}_1^{-1}L_{J'_1} = \dot{w}L_{\delta(J_1)} = \dot{w}g_0L_{J_1} \subset \dot{w}g_0P_J = \dot{w}g_0gP_J.$$

So ${}^{g_1}P$ contains some Levi of $P \cap P'$. We have that

$$\begin{aligned} g_1^{-1}({}^{g_1}P)^{(\dot{w}\dot{y}^{-1}P_{\text{Ad}(y_1)\delta(J_1)})}g_1 &= g^{-1}P^{(g_0^{-1}\dot{y}^{-1}P_{\text{Ad}(y)\delta(J_1)})} = g^{-1}P_{J \cap \delta^{-1}\text{Ad}(y^{-1})\text{Ad}(y)\delta(J_1)} \\ &= g^{-1}P_{J_1} = P_{J_1}. \end{aligned}$$

Thus $\alpha(z) = z'$. The lemma is proved. $\square$

Proposition 1.7. *We have that*

$$\tilde{Z}_{J,y,\delta}^w = G_{\text{diag}} \cdot [\widetilde{J, w, 1}]_{y,\delta} = G_{\text{diag}} \cdot (P_J, \dot{w}\dot{y}^{-1}P_{J'}, U_{\dot{w}\dot{y}^{-1}P_{J'}}, \dot{w}g_0(B \cap L_{J_\infty})U_{P_J}).$$

Proof. It is easy to see that $\tilde{Z}_{J_\infty,w,\delta}^w = G_{\text{diag}}(\dot{w}, L_{J_\infty}) \cdot \tilde{h}_{J_\infty,w,\delta}$. Thus for any $b \in B$, $\alpha^n((\dot{w}, b) \cdot \tilde{h}_{J,y,\delta}) \in \tilde{Z}_{J_\infty,w,\delta}^w$ for sufficiently large n . Therefore, $G_{\text{diag}}(\dot{w}, B) \cdot \tilde{h}_{J,y,\delta} \subset \tilde{Z}_{J,y,\delta}^w$.

Note that $\dot{w}g_0$ normalizes (L_{J_∞}) and $(L_{J_\infty}) \cap B$. Thus

$$\dot{w}g_0L_{J_\infty} = \{l\dot{w}g_0bl^{-1} \mid l \in L_{J_\infty}, b \in L_{J_\infty} \cap B\}.$$

Hence any element in $\tilde{Z}_{J_\infty,w,\delta}^w$ is G -conjugate to $(\dot{w}, l) \cdot \tilde{h}_{J_\infty,w,\delta}$ for some $l \in L_{J_\infty} \cap B$. Now let $z \in \tilde{Z}_{J,y,\delta}^w$. Then $\vartheta(z)$ is G -conjugate to $(\dot{w}, l) \cdot \tilde{h}_{J_\infty,w,\delta}$ for some $l \in L_{J_\infty} \cap B$. Set $z' = (\dot{w}, l) \cdot \tilde{h}_{J,y,\delta} \in \tilde{Z}_{J,y,\delta}^w$. Then $\vartheta(z')$ lies in the same G -orbit as $\vartheta(z)$. Since ϑ induces a bijection from the set of G -orbits on $\tilde{Z}_{J,y,\delta}^w$ to the set of G -orbits on $\tilde{Z}_{J_\infty,w,\delta}^w$, z is G -conjugate to z' and $\tilde{Z}_{J,y,\delta}^w = G_{\text{diag}}(\dot{w}, B \cap L_{J_\infty}) \cdot \tilde{h}_{J,y,\delta}$. The proposition is proved. $\square$

1.8. In [L4, 8.20], Lusztig showed that $\tilde{Z}_{J,y,\delta}^w$ is an iterated affine space bundle over a fibre bundle over $\mathcal{P}_{J_\infty}$ with fibres isomorphic to L_{J_∞} . In proposition 1.10, we will prove a similar (but more explicit) result, which will be used to establish the cellular decomposition. Before doing that, we include the following result (see [SL, page 26, lemma 4]) which we will use in the proof of proposition 1.10 and proposition 5.5.

Lemma 1.9. *Let H be a closed subgroup of G and $\Phi : X \rightarrow G/H$ be a G -equivariant morphism from the G -variety X to the homogeneous space G/H . Let $E \subset X$ be the fiber $\Phi^{-1}(H)$. Then E will be stabilized by H and the map $\Psi : G \times_H E \rightarrow X$ sending (g, e) to $g \cdot e$ defines an isomorphism of G -varieties.*

Proposition 1.10. *For $a \in W$, set $U_a = U \cap {}^a U^-$. Set*

$$\tilde{L}_{J,y,\delta}^w = (L_{J_\infty}, L_{J_\infty})(\dot{w}, 1) \cdot \tilde{h}_{J,y,\delta}.$$

Then we have the following results:

- (1) $\tilde{Z}_{J,y,\delta}^w$ is isomorphic to $G \times_{P_{J_\infty}} ((P_{J_\infty}) \cdot \tilde{L}_{J,y,\delta}^w)$.
- (2) $(P_{J_\infty}) \cdot \tilde{L}_{J,y,\delta}^w = (B \times B) \cdot \tilde{L}_{J,y,\delta}^w \cong (U \cap \dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-) \times \tilde{L}_{J,y,\delta}^w$, where $\tilde{L}_{J,y,\delta}^w$ is isomorphic to L_{J_∞} .
- (3) $G_{diag}(\dot{w}T, 1) \cdot \tilde{h}_{J,y,\delta}$ is dense in $\tilde{Z}_{J,y,\delta}^w$.

Proof. It is easy to see that $\tilde{L}_{J,y,\delta}^w = (\dot{w}, L_{J_\infty}) \cdot \tilde{h}_{J,y,\delta}$ is isomorphic to L_{J_∞} . By proposition 1.7, $\tilde{Z}_{J,y,\delta}^w = G_{diag} \cdot \tilde{L}_{J,y,\delta}^w$. Consider the G -equivariant map $p : \tilde{Z}_{J_\infty,w,\delta}^w \rightarrow \mathcal{P}_{J_\infty}$ defined by $p(P, P, \gamma) = P$ for $(P, P, \gamma) \in \tilde{Z}_{J_\infty,w,\delta}^w$. For $l \in L_{J_\infty}$ and $g \in G$, if $p \circ \vartheta((g, g)(\dot{w}, l) \cdot \tilde{h}_{J,y,\delta}) = P_{J_\infty}$, then $g \in P_{J_\infty}$. Thus $(P_{J_\infty})_{diag} \cdot \tilde{L}_{J,y,\delta}^w = (p \circ \vartheta)^{-1}(P_{J_\infty})$.

Assume that $(g, g)(\dot{w}, l_1) \cdot \tilde{h}_{J,y,\delta}^w = (\dot{w}, l_2) \cdot \tilde{h}_{J,y,\delta}^w$ for $g \in G$ and $l_1, l_2 \in L_{J_\infty}$. Then ${}^g P_{J_\infty} = p \circ \vartheta((g, g)(\dot{w}, l_1) \cdot \tilde{h}_{J,y,\delta}^w) = p \circ \vartheta((\dot{w}, l_2) \cdot \tilde{h}_{J,y,\delta}^w) = P_{J_\infty}$. So $g \in P_{J_\infty}$. Part (1) is proved.

We have that $(B \times B) \cdot \tilde{L}_{J,y,\delta}^w = (B)_{diag}(1, B) \cdot \tilde{L}_{J,y,\delta}^w$ and $p \circ \vartheta((1, B) \cdot \tilde{L}_{J,y,\delta}^w) = P_{J_\infty}$. Thus $(B \times B) \cdot \tilde{L}_{J,y,\delta}^w \subset (B)_{diag}(P_{J_\infty})_{diag} \tilde{L}_{J,y,\delta}^w = (P_{J_\infty})_{diag} \tilde{L}_{J,y,\delta}^w$. On the other hand, $(P_{J_\infty})_{diag} \tilde{L}_{J,y,\delta}^w \subset (P_{J_\infty}, P_{J_\infty}) \cdot \tilde{L}_{J,y,\delta}^w = (B \times B) \cdot \tilde{L}_{J,y,\delta}^w$. Hence $(P_{J_\infty}) \cdot \tilde{L}_{J,y,\delta}^w = (B \times B) \cdot \tilde{L}_{J,y,\delta}^w$.

Now consider $\pi : (U \cap \dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-) \times \tilde{L}_{J,y,\delta}^w \rightarrow (B \times B) \cdot \tilde{L}_{J,y,\delta}^w$ defined by $\pi(u, l) = (u, 1)l$ for $u \in U \cap \dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-$ and $l \in \tilde{L}_{J,y,\delta}^w$.

Note that $(1, BL_{J_\infty}) \cdot \tilde{h}_{J,y,\delta} = (1, U_{P_J} L_{J_\infty} U_J) \cdot \tilde{h}_{J,y,\delta} = (U_{\delta(J)}, L_{J_\infty}) \cdot \tilde{h}_{J,y,\delta}$. Since $w \in W^{\delta(J)}$, $B\dot{w}U_{\delta(J)} = B\dot{w} = U_{P_{J_\infty}} L_{J_\infty} \dot{w}$. Hence

$$(B\dot{w}, BL_{J_\infty}) \cdot \tilde{h}_{J,y,\delta} = (U_{P_{J_\infty}} \dot{w} L_{\delta(J_\infty)}, L_{J_\infty}) \cdot \tilde{h}_{J,y,\delta} = (U_{P_{J_\infty}} \dot{w}, L_{J_\infty}) \cdot \tilde{h}_{J,y,\delta}.$$

Since $wy^{-1} \in W^{J'}$ and $\text{Ad}(yw^{-1})J_\infty \subset J'$, then

$$\begin{aligned} U_{P_{J_\infty}} &= (U_{P_{J_\infty}} \cap \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-) (U_{P_{J_\infty}} \cap \dot{w} \dot{y}^{-1} U_{P_{J'}}) \\ &= (U \cap \dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-) (U_{P_{J_\infty}} \cap \dot{w} \dot{y}^{-1} U_{P_{J'}}). \end{aligned}$$

Therefore, $(B\dot{w}, BL_{J_\infty}) \cdot \tilde{h}_{J,y,\delta} = ((U \cap \dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-) \dot{w}, L_{J_\infty}) \cdot \tilde{h}_{J,y,\delta}$. So π is surjective.

Let $u \in U \cap \dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-$ and $l_1, l_2 \in L_{J_\infty}$. Assume that $(u, 1)(\dot{w}, l_1) \cdot \tilde{h}_{J,y,\delta} = (\dot{w}, l_2) \cdot \tilde{h}_{J,y,\delta}$. Note that the isotropy subgroup of $G \times G$ at the point $(\dot{w}, 1) \cdot \tilde{h}_{J,y,\delta}$ is $\{(U_{\dot{w}\dot{y}^{-1}P_{J'}} l', U_{P_J} g_0^{-1} \dot{w}^{-1} l' \dot{w} g_0) \mid l' \in \dot{w} L_{\delta(J)}\}$.

Thus $u \in U_{\dot{w}\dot{y}^{-1}P_{J'}} l'$ and $l_2^{-1} l_1 \in U_{P_J} g_0^{-1} \dot{w}^{-1} l' \dot{w} g_0$ for some $l' \in \dot{w} L_{\delta(J)}$. Then $l' \in L_{J_\infty}$ and $u = 1$. Thus π is injective.

In fact, we can show that the bijective morphism π is an isomorphism. The verification is omitted.

Part (3) can be proved in the same way as in [H, 2.7]. $\square$

1.11. For $P \in \mathcal{P}_J$, let H_P be the inverse image of the connected center of P/U_P under $P \rightarrow P/U_P$. We can regard H_P/U_P as a single torus Δ_J independent of P . Then Δ_J acts (freely) in the natural way on $\tilde{Z}_{J,y,\delta}$ and the action commutes with the action of G . Moreover, each piece $\tilde{Z}_{J,y,\delta}^w$ is Δ_J -stable.

Define

$$\begin{aligned} Z_{J,y,\delta} &= \{(P, P', \gamma) \mid P \in \mathcal{P}_J, P' \in \mathcal{P}_{J'}, \gamma \in H_{P'} \backslash A_y(P, P')/U_P\} \\ &= \{(P, P', \gamma) \mid P \in \mathcal{P}_J, P' \in \mathcal{P}_{J'}, \gamma \in U_{P'} \backslash A_y(P, P')/H_P\} \end{aligned}$$

with $G \times G$ action defined by $(g_1, g_2)(P, P', \gamma) = ({}^{g_2}P, {}^{g_1}P', g_1 \gamma g_2^{-1})$.

Then $Z_{J,y,\delta}$ and $\Delta_J \backslash \tilde{Z}_{J,y,\delta}$ can be identified in the natural way as varieties with G -action. Set $Z_{J,y,\delta}^w = \Delta_J \backslash \tilde{Z}_{J,y,\delta}^w$. Then

$$Z_{J,y,\delta} = \sqcup_{w \in W^{\delta(J)}} Z_{J,y,\delta}^w.$$

We call $(Z_{J,y,\delta}^w)_{w \in W^{\delta(J)}}$ the G -stable pieces of $Z_{J,y,\delta}$. Set

$$\begin{aligned} h_{J,y,\delta} &= (P_J, \dot{y}^{-1} P_{J'}, H_{\dot{y}^{-1}P_{J'}} g_0 U_{P_J}) \in Z_{J,y,\delta}, \\ L_{J,y,\delta}^w &= (L_{J_\infty}, L_{J_\infty})(\dot{w}, 1) \cdot h_{J,y,\delta}. \end{aligned}$$

For $w \in W^{\delta(J)}$ and $v \in W$, set $[J, w, v]_{y,\delta} = (B \times B)(\dot{w}, \dot{v}) \cdot h_{J,y,\delta}$. Then as a consequence of propositions 1.7 and 1.10, we have the following result.

Proposition 1.12. *For $w \in W^{\delta(J)}$, we have that*

- (1) $Z_{J,y,\delta}^w = G_{diag} \cdot [J, w, 1]_{y,\delta}$.
- (2) $Z_{J,y,\delta}^w$ is isomorphic to $G \times_{P_{J_\infty}} ((P_{J_\infty}) \cdot L_{J,y,\delta}^w)$.
- (3) $(P_{J_\infty}) \cdot L_{J,y,\delta}^w = (B \times B) \cdot L_{J,y,\delta}^w \cong (U \cap \dot{w}_0^{J_\infty} \dot{w} \dot{y}^{-1} \dot{w}_0^{J'} U^-) \times L_{J,y,\delta}^w$, where $L_{J,y,\delta}^w$ is isomorphic to $L_{J_\infty}/Z(L_J)$.
- (4) $G_{diag}(\dot{w}T, 1) \cdot h_{J,y,\delta}$ is dense in $Z_{J,y,\delta}^w$.

2. COMPACTIFICATION THROUGH THE FIBRES

2.1. For any connected, semi-simple algebraic group of adjoint type, De Concini and Procesi introduced its wonderful compactification $\bar{G}$ (see [DP]). It is an irreducible, projective smooth $G \times G$ -variety. The $G \times G$ -orbits Z_J of $\bar{G}$ are indexed by the subsets J of I . Moreover, $Z_J = (G \times G) \times_{P_J^- \times P_J} G_J$, where $P_J^- \times P_J$ acts on the right on $G \times G$ and on the left on G_J by $(q, p) \cdot z = \pi_{P_J^-}(q) z \pi_{P_J}(p)$. Denote by h_J the image of $(1, 1, 1)$ in Z_J . We will identify Z_J with $Z_{J, w_0 w_0^J, id}$ and h_J with $h_{J, w_0 w_0^J, id}$, where id is the identity map on I (see [H, 2.5]).

Let us consider the $B \times B$ -orbits on $\bar{G}$. For any $J \subset I$, $u \in W^J$ and $v \in W$, set $[J, u, v] = (B \times B)(\dot{u}, \dot{v}) \cdot h_J$. Then $\bar{G} = \bigsqcup_{J \subset I} \bigsqcup_{x \in W^J, w \in W} [J, x, w]$. The following result is due to Springer (see [S, 2.4]).

Theorem. *Let $x \in W^J$, $x' \in W^K$, $w, w' \in W$. Then $[K, x', w']$ is contained in the closure of $[J, x, w]$ if and only if $K \subset J$ and there exists $u \in W_K, v \in W_J \cap W^K$ with $xvu^{-1} \leq x'$, $w'u \leq wv$ and $l(wv) = l(w) + l(v)$. In particular, for $J \subset I$ and $w \in W^J$, the closure of $[J, w, 1]$ in $\bar{G}$ is $\sqcup_{K \subset J} \sqcup_{x \in W^K, u \in W_J, \text{ and } x \geq wu} [K, x, u]$.*

2.2. We have defined $Z_{J,y,\delta}$ in section 1.11. As we have seen, $Z_{J,y,\delta}$ is a locally trivial fibre bundle over $\mathcal{P}_J \times \mathcal{P}_{J'}$ with fibres isomorphic to $L_J/Z(L_J)$. Note that $L_J/Z(L_J)$ is a connected, semi-simple algebraic group of adjoint type. Thus we can define the wonderful compactification $\overline{L_J/Z(L_J)}$ of $L_J/Z(L_J)$. In this section, we will define $\overline{Z_{J,y,\delta}}$, which is a locally trivial fibre bundle over $\mathcal{P}_J \times \mathcal{P}_{J'}$ with fibres isomorphic to $\overline{L_J/Z(L_J)}$.

2.3. We keep the notation of section 1.3. Fix $g \in A_y(P, P')$. Then $A_y(P, P')g^{-1} = P'U_{gP}$ (see [L4, 8.9]). Set

$$L_{P,P',g} = {}^gP \cap P' / H_{gP \cap P'}.$$

Let $\Phi_g : H_{P'} \backslash A_y(P, P') / H_P \rightarrow L_{P,P',g}$ be the morphism defined by

$$H_{P'} \backslash A_y(P, P') / H_P \xrightarrow{\cdot g^{-1}} H_{P'} \backslash A_y(P, P') g^{-1} / H_{gP} \xleftarrow{i} L_{P,P',g},$$

where i is the obvious isomorphism.

The $P \times P'$ action on $H_{P'} \backslash A_y(P, P') / H_P$ induces a $P \times P'$ action on $L_{P,P',g}$. Now for $g, g' \in A_y(P, P')$, set $\Phi_{g,g'} = \Phi_{g'} \Phi_g^{-1} : L_{P,P',g} \xrightarrow{\cong} L_{P,P',g'}$. Then $\Phi_{g,g'}$ is compatible with the $P \times P'$ action. Moreover, $(L_{P,P',g}, \Phi_{g,g'})$ forms an inverse system and

$$H_{P'} \backslash A_y(P, P') / H_P = \varprojlim L_{P,P',g}.$$

Note that $L_{P,P',g}$ is a semi-simple group of adjoint type. Then we can define the De Concini-Procesi compactification $\overline{L_{P,P',g}}$ of $L_{P,P',g}$. The $P \times P'$ action on $L_{P,P',g}$ can be extended in the unique way to a $P \times P'$ action on $\overline{L_{P,P',g}}$. The isomorphism $\Phi_{g,g'} : L_{P,P',g} \xrightarrow{\cong} L_{P,P',g'}$ can be extended in the unique way to an isomorphism from $\overline{L_{P,P',g}}$ onto $\overline{L_{P,P',g'}}$. We will also denote this isomorphism by $\Phi_{g,g'}$. It is easy to see that this isomorphism is compatible with the $P \times P'$ action. Now $(\overline{L_{P,P',g}}, \Phi_{g,g'})$ forms an inverse system. Define

$$\overline{H_{P'} \backslash A_y(P, P') / H_P} = \varprojlim \overline{L_{P,P',g}}.$$

We also obtain a $P \times P'$ action on $\overline{H_{P'} \backslash A_y(P, P') / H_P}$. Thus we can identify $\overline{H_{P'} \backslash A_y(P, P') / H_P}$ with $\overline{L_{P,P',g}}$ as varieties with $P \times P'$ action.

Remark. $\overline{H_{P'} \backslash A_y(P, P') / H_P}$ is isomorphic to $\overline{L_{P,P',g}}$ as a variety. However, we are also concerned with the $P' \times P$ action. In this case, $\overline{H_{P'} \backslash A_y(P, P') / H_P}$ is regarded as $\overline{L_{P,P',g}}$ with a “twisted” $P' \times P$ action.

2.4. In this section, we will consider a special case, namely, $P = P' = G^0$. In this case, $A_y(P, P') = G^1$ and we will identify $H_G \backslash A_y(G, G)/H_G$ with G^1 .

Let $\mathcal{V}_G$ be the projective variety whose points are the $\dim(G)$ -dimensional Lie subalgebras of $\text{Lie}(G \times G)$. The $\hat{G} \times \hat{G}$ action on $\text{Lie}(G \times G)$ which is defined by $(g_1, g_2) \cdot (a, b) = (\text{Ad}(g_2)a, \text{Ad}(g_1)b)$ for $g_1, g_2 \in \hat{G}$ and $a, b \in \text{Lie}(G)$ induces a $\hat{G} \times \hat{G}$ action on $\mathcal{V}_G$. To each $g \in \hat{G}$, we associate a $\dim(G)$ -dimensional subspace $V_g = \{(a, \text{Ad}(g)a) \mid a \in \text{Lie}(G)\}$ of $\text{Lie}(G \times G)$. Then $V_{g_1 g g_2^{-1}} = (g_1, g_2) \cdot V_g$ for $g_1, g, g_2 \in \hat{G}$, and $g \mapsto V_g$ is an embedding $G^1 \subset \mathcal{V}_G$. We denote the image by $i(G^1)$.

If $G^1 = G$, then the closure of $i(G)$ in $\mathcal{V}_G$ is $\bar{G}$ (see [DP]). Note that $V_{gg_0} = (1, g_0^{-1})V_g$ for all $g \in G$. Thus $i(G^1) = (1, g_0^{-1})i(G)$. Hence the closure of $i(G^1)$ in $\mathcal{V}_G$ is $(1, g_0^{-1})\bar{G}$, which is just $\bar{G}^1$ defined above.

Remark. In [L4, 12.3], Lusztig defined the compactification of G^1 to be the closure of $i(G^1)$ in $\mathcal{V}_G$. As we have seen, our definition coincides with his definition.

2.5. In [L4, 12.3], Lusztig showed that

$$\bar{G}^1 = \sqcup_{J \subset I} Z_{J, w_0 w_0^{\delta(J)}, \delta} = \sqcup_{J \subset I} \sqcup_{w \in W^{\delta(J)}} Z_{J, w_0 w_0^{\delta(J)}, \delta}^w,$$

where the base point $h_{J, w_0 w_0^{\delta(J)}, \delta} = (P_J, P_{\delta(J)}^-, H_{P_{\delta(J)}^-} g_0 H_{P_J})$ is identified with the $\dim(G)$ -dimensional subalgebra $\{(lu, g_0 l g_0^{-1} u') \mid l \in L_J, u \in U_{P_J}, u' \in U_{P_{\delta(J)}^-}\}$ of $\text{Lie}(G \times G)$. We will simply write $h_{J, w_0 w_0^{\delta(J)}, \delta}$ as $h_{J, \delta}$, $[J, w, v]_{w_0 w_0^{\delta(J)}, \delta}$ as $[J, w, v]_{\delta}$ and $Z_{J, w_0 w_0^{\delta(J)}, \delta}^w$ as $Z_{J, \delta}^w$. We call $(Z_{J, \delta}^w)_{J \subset I, w \in W^{\delta(J)}}$ the G -stable pieces of $\bar{G}^1$. If $G^1 = G$, then $h_{J, id} = h_J$ and $[J, w, v]_{id} = [J, w, v]$.

Note that h_J corresponds to the $\dim(G)$ -dimensional subalgebra $\{(lu, lu') \mid l \in L_J, u \in U_{P_J}, u' \in U_{P_J^-}\}$ of $\text{Lie}(G \times G)$. Thus $h_{J, \delta} = (1, g_0^{-1})h_{\delta(J)}$. Hence

$$\begin{aligned} [J, w, v]_{\delta} &= (B \times B)(\dot{w}, \dot{v}) \cdot h_{J, \delta} = (B \times B)(\dot{w}, \dot{v})(1, g_0^{-1}) \cdot h_J \\ &= (1, g_0^{-1})(B \times B)(\dot{w}, \dot{\delta(v)}) \cdot h_{\delta(J)} = (1, g_0^{-1})[\delta(J), w, \delta(v)]. \end{aligned}$$

Thus we have the following result.

Proposition. *Let $J \subset I$ and $w \in W^{\delta(J)}$. Then the closure of $[J, w, 1]_{\delta}$ in $\bar{G}^1$ is $\sqcup_{K \subset J} \sqcup_{x \in W^{\delta(K)}, u \in W_J, \text{ and } x \geq w\delta(u)} [K, x, u]_{\delta}$.*

2.6. Define

$$\underline{Z_{J, y, \delta}} = \{(P, P', \gamma) \mid P \in \mathcal{P}_J, P' \in \mathcal{P}_{J'}, \gamma \in \overline{H_{P'} \backslash A_y(P, P')/H_P}\}$$

with $G \times G$ action defined by $(g_1, g_2)(P, P', \gamma) = (g_2 P, g_1 P', g_1 \gamma g_2^{-1})$.

Set $P = P_J$ and $P' = \dot{y}^{-1} P_{J'}$. Then $\overline{A_y(P, P')}$ can be identified with $\overline{L_{P, P', g_0} g_0}$ as varieties with $P' \times P$ action. Moreover, we have a canonical isomorphism between $\overline{L_{P, P', g_0}}$ and $\overline{L_{\delta(J)}}$. For $K \subset J$, I will identify $h_{\delta(K)} g_0$ with the corresponding element in $\overline{A_y(P, P')}$.

Then the $G \times G$ -orbits in $\underline{Z_{J, y, \delta}}$ are in one-to-one correspondence with the subsets of J , i.e.,

$$\underline{Z_{J, y, \delta}} = \sqcup_{K \subset J} (G \times G) \cdot (P, P', h_{\delta(K)} g_0).$$

Set $y_K = yw_0^{\delta(J)}w_0^{\delta(K)}$. Note that $U_{P_J}(L_J \cap U_{P_K}) = U_{P_K}$ and

$$\begin{aligned} U_{P'}(\dot{y}^{-1}L_{J'} \cap U_{P_{\delta(K)}^-}) &= (\dot{y}_K^{-1}U_{\dot{y}_K P'})\dot{y}_K^{-1}(\dot{y}_K \dot{y}^{-1}L_{J'} \cap \dot{y}_K U_{P_{\delta(K)}^-}) \\ &= \dot{y}_K^{-1}(U_{P_{J'}}(L_{J'} \cap U_{P_{\text{Ad}(y_K)\delta(K)}})) = \dot{y}_K^{-1}U_{P_{\text{Ad}(y_K)\delta(K)}}. \end{aligned}$$

The isotropic subgroup of $G \times G$ at $(P, P', h_{\delta(K)}g_0)$ is $\{(l_1 u_1, g_0^{-1}l_2 g_0 u_2) \mid l_1, l_2 \in L_{\delta(K)}, l_1 l_2^{-1} \in Z(L_{\delta(K)}), u_1 \in U_{\dot{y}_K^{-1}P_{\text{Ad}(y_K)\delta(K)}}, u_2 \in U_{P_K}\}$. Now set $Q = P_K, Q' = \dot{y}_K^{-1}P_{\text{Ad}(y_K)\delta(K)}$ and $\gamma = H_{Q'}g_0H_Q$. Then $\text{pos}(Q', {}^{g_0}Q) = y_K$ and $(Q, Q', \gamma) \in Z_{K, y_K, \delta}$. The isotropic subgroup of $G \times G$ at $(P, P', h_{\delta(K)}g_0)$ is the same as the isotropic subgroup of $G \times G$ at $(Q, Q', \gamma) \in Z_{K, y_K, \delta}$. Thus we can identify $(P, P', h_{\delta(K)}g_0)$ with (Q, Q', γ) and $(G \times G) \cdot (P, P', h_{\delta(K)}g_0)$ with $Z_{K, y_K, \delta}$ as varieties with $G \times G$ action. In other words,

$$\underline{Z_{J, y, \delta}} = \sqcup_{K \subset J} Z_{K, yw_0^{\delta(J)}w_0^{\delta(K)}, \delta}.$$

3. PARTIAL ORDER ON $\mathcal{I}_\delta$

In this section, we will only consider subvarieties of G and for any subvariety X of G , we denote by $\bar{X}$ the closure of X in G .

3.1. Let $y, w \in W$. Then $y \leq w$ if and only if for any reduced expression $w = s_1 s_2 \cdots s_q$, there exists a subsequence $i_1 < i_2 < \cdots < i_r$ of $1, 2, \dots, q$ such that $y = s_{i_1} s_{i_2} \cdots s_{i_r}$ (see [L2, 2.4]).

The following assertions follow from the above property:

- (1) If $l(wu) = l(w) + l(u)$, then for any $w_1 \leq w$ and $u_1 \leq u$, $w_1 u_1 \leq wu$.
- (2) Let $u, v \in W$ and $i \in I$. Assume that $s_i v < v$; then $u \leq v \Leftrightarrow su \leq v$.
- (3) Let $u, v \in W$ and $i \in I$. Assume that $u < s_i u$; then $u \leq v \Leftrightarrow u \leq s_i v$.

The assertion (1) follows directly from the above property. The proofs of assertions (2) and (3) can be found in [L2, 2.5].

3.2. It is known that $G = \sqcup_{w \in W} B\dot{w}B$ and for $w, w' \in W$, $B\dot{w}B \subset \overline{B\dot{w}'B}$ if and only if $w \leq w'$. Moreover,

$$\overline{B\dot{s}_i B\dot{w}B} = \begin{cases} \overline{B\dot{w}B}, & \text{if } s_i w < w; \\ \overline{B\dot{s}_i \dot{w}B}, & \text{if } s_i w > w. \end{cases}$$

Similarly, $G = \sqcup_{w \in W} B\dot{w}B^-$ and for $w, w' \in W$, $B\dot{w}B^- \subset \overline{B\dot{w}'B^-}$ if and only if $w \geq w'$. Moreover,

$$\overline{B\dot{s}_i B\dot{w}B^-} = \begin{cases} \overline{B\dot{w}B^-}, & \text{if } s_i w > w; \\ \overline{B\dot{s}_i \dot{w}B^-}, & \text{if } s_i w < w. \end{cases}$$

Lemma 3.3. *Let $u, w \in W$. Then*

(1) *The subset $\{vw \mid v \leq u\}$ of W contains a unique minimal element y . Moreover, $l(y) = l(w) - l(yw^{-1})$ and $\overline{B\dot{u}B\dot{w}B^-} = \overline{B\dot{y}B^-}$.*

(2) *The subset $\{vw \mid v \leq u\}$ of W contains a unique maximal element y' . Moreover, $l(y') = l(w) + l(y'w^{-1})$ and $\overline{B\dot{u}B\dot{w}B} = \overline{B\dot{y}'B}$.*

Proof. We will only prove part (1). Part (2) can be proved in the same way.

For any $v \leq u$, $B\dot{v} \subset \overline{B\dot{u}B}$. Thus $\overline{B\dot{v}B\dot{w}B^-} \subset \overline{B\dot{u}B\dot{w}B^-} \subset \overline{B\dot{u}B\dot{w}B^-}$. On the other hand, $\overline{B\dot{u}B\dot{w}B^-}$ is an irreducible, closed, $B \times B^-$ -stable subvariety of G . Thus there exists $y \in W$, such that $\overline{B\dot{u}B\dot{w}B^-} = \overline{B\dot{y}B^-}$. Since $B\dot{v}B\dot{w}B^- \subset \overline{B\dot{y}B^-}$,

we have that $vw \geq y$. Now it suffices to prove that $y = vw$ for some $v \leq u$ with $l(vw) = l(w) - l(v)$.

We argue by induction on $l(u)$. If $l(u) = 0$, then $u = 1$ and the statement is clear. Assume now that $l(u) > 0$. Then there exists $i \in I$, such that $s_i u < u$. We denote $s_i u$ by u' . Now

$$\overline{B\dot{u}B\dot{w}B^-} = \overline{B\dot{s}_i B\dot{u}' B\dot{w}B^-} = \overline{B\dot{s}_i \overline{B\dot{u}' B\dot{w}B^-}}.$$

By induction hypothesis, there exists $v' \leq u'$, such that $l(v'w) = l(w) - l(v')$ and $\overline{B\dot{u}' B\dot{w}B^-} = \overline{B\dot{v}' \dot{w}B^-}$. Thus

$$\overline{B\dot{s}_i \overline{B\dot{u}' B\dot{w}B^-}} = \overline{B\dot{s}_i \overline{B\dot{v}' \dot{w}B^-}} = \overline{B\dot{s}_i B\dot{v}' \dot{w}B^-} = \begin{cases} \overline{B\dot{v}' \dot{w}B^-}, & \text{if } s_i v' w > v' w; \\ \overline{B\dot{s}_i \dot{v}' \dot{w}B^-}, & \text{if } s_i v' w < v' w. \end{cases}$$

Note that $s_i u < u$ and $v' \leq s_i u < u$. Thus $s_i v' \leq u$. Moreover, if $s_i v' w < v' w$, then $l(s_i v' w) = l(v' w) - 1 = l(w) - l(v') - 1$. Thus we have that $l(s_i v') = l(v) + 1$ and $l(s_i v' w) = l(w) - l(s_i v')$. Therefore, the statement holds for u . $\square$

Corollary 3.4. *Let $u, w, w' \in W$ with $w' \leq w$. Then*

- (1) *There exists $v \leq u$, such that $vw' \leq uw$.*
- (2) *There exists $v' \leq u$, such that $uw' \leq v'w$.*

Proof. Let $v \leq u$ be the element of W such that vw' is the unique minimal element in $\{v'w' \mid v' \leq u\}$. Then $\overline{B\dot{u}B\dot{w}'B^-} = \overline{B\dot{v} \dot{w}'B^-}$. Since $w' \leq w$, we have that $B\dot{w}B^- \subset \overline{B\dot{w}'B^-}$. Thus

$$B\dot{u} \dot{w} B^- \subset B\dot{u} B\dot{w} B^- \subset \overline{B\dot{u} \overline{B\dot{w}'B^-}} \subset \overline{B\dot{u} B\dot{w}'B^-} = \overline{B\dot{v} \dot{w}'B^-}.$$

So $uw \geq vw'$. Thus part (1) is proved. Part (2) can be proved in the same way. $\square$

3.5. We will recall some known results about W^J .

- (1) If $w \in W^J$ and $i \in I$, then there are three possibilities:
 - (a) $s_i w > w$ and $s_i w \in W^J$;
 - (b) $s_i w > w$ and $s_i w = ws_j$ for some $j \in J$;
 - (c) $s_i w < w$ in which case $s_i w \in W^J$.
- (2) If $w \in W^J$, $v \in W_J$ and $K \subset J$; then $v \in W^K$ if and only if $vw \in W^K$.
- (3) If $w \in {}^{J'}W^J$ and $u \in W_{J'}$, then $uw \in W^J$ if and only if $u \in W^K$, where $K = J' \cap \text{Ad}(w)J$.

Lemma 3.6. *Let $w \in {}^{J'}W^J$, $u \in W_{J'}$ and $K = J' \cap \text{Ad}(w)J$; then $uw = vwu'$ for some $v \in W_{J'} \cap W^K$ and $u' \in W_{\text{Ad}(w^{-1})K}$.*

Proof. We argue by induction on $l(u)$. If $u = 1$, then the statement is clear. Now assume that $u = s_i u_1$ for some $i \in J'$ and $l(u_1) < l(u)$. Then by induction hypothesis, $u_1 w = v_1 w u'_1$ for some $v_1 \in W_{J'} \cap W^K$ and $u'_1 \in W_{\text{Ad}(w^{-1})K}$.

If $s_i v_1 w \in W^J$, then the statement holds for u . Now assume that $s_i v_1 w \notin W^J$. Then $s_i v_1 w > v_1 w$. Hence $s_i v_1 > v_1$. Moreover, $s_i v_1 \notin W^K$. Thus $s_i v_1 = v_1 s_k$ for some $k \in K$. Note that $s_k w = ws_l$ for some $l \in \text{Ad}(w^{-1})K$. Thus the statement holds for u . The lemma is proved. $\square$

3.7. Let $J \subset I$ and $w, w' \in W$ with $l(w) = l(w')$. We say that w' can be obtained from w via a (J, δ) -cyclic shift if $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression and either (1) $i_1 \in J$ and $w' = s_{i_1} w s_{\delta(i_1)}$ or (2) $i_n \in \delta(J)$ and $w' = s_{\delta^{-1}(i_n)} w s_{i_n}$. We say that w and w' are equivalent in J if there exists a finite sequence of elements $w = w_0, w_1, \dots, w_m = w'$ such that w_{k+1} can be obtained from w_k via a (J, δ) -cyclic shift. (We then write $w \sim_{J, \delta} w'$.)

Proposition 3.8. *Let $(J, w) \in \mathcal{I}_\delta$ and $w' \in W$. The following conditions on w' are equivalent:*

- (1) $w' \geq u^{-1} w \delta(u)$ for some $u \in W_J$.
- (2) $w' \geq u^{-1} w \delta(v)$ for some $u \leq v \in W_J$.
- (3) $w' \geq w_1$ for some $w_1 \sim_{J, \delta} w$.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. The implication (3) $\Rightarrow$ (1) follows from the definition. We now prove the implication (2) $\Rightarrow$ (3) by induction on $|J|$. Assume that the implication holds for all $J' \subset I$ with $|J'| < |J|$. Then we prove that the implication holds for J by induction on $l(v)$.

Set $w = xy$ with $x \in W_J$ and $y \in {}^J W^{\delta(J)}$. Set $K = J \cap \delta^{-1} \text{Ad}(y^{-1})J$, $v = v_1 v_2$ with $v_1 \in W_K, v_2 \in {}^K W$ and $u = u_1 u_2$ with $u_1 \leq v_1, u_2 \leq v_2$ and $l(u) = l(u_1) + l(u_2)$. There are two cases.

Case 1. $u_2 = v_2 = 1$.

In this case, $u, v \in W_K$ and $w \in W^{\delta(K)}$. If $|K| < |J|$, then by induction hypothesis, $u^{-1} w \delta(v) \geq w_1$ for some $w_1 \sim_{K, \delta} w$. If $K = J$, then since $w = xy \in W^{\delta(J)}$, we have that $x = 1$. Thus $u^{-1} w \delta(v) \geq w$. The implication is proved in this case.

Case 2. $v_2 \neq 1$.

In this case, $l(v_1) < l(v)$. By induction hypothesis, there exists $w_1 \sim_{J, \delta} w$, such that $w_1 \leq u_1^{-1} w \delta(v_1)$. Let $u_3 \leq u_2$ be the element in W such that $u_3^{-1} w_1$ is the unique minimal element in $\{(u')^{-1} w_1 \mid u' \leq u_2\}$. Then $l(u_3^{-1} w_1) = l(w_1) - l(u_3)$ and $u_3^{-1} w_1 \leq u_2^{-1} u_1^{-1} w \delta(v_1) = u^{-1} w \delta(v_1)$. By lemma 3.6, $u^{-1} w = ab$ for some $a \in W^{\delta(J)}$ and $b \in W_{\delta(K)}$. Thus $l(u^{-1} w \delta(v_1 v_2)) = l(ab \delta(v_1 v_2)) = l(a) + l(b \delta(v_1 v_2)) = l(a) + l(b \delta(v_1)) + l(\delta(v_2)) = l(ab \delta(v_1)) + l(\delta(v_2)) = l(u^{-1} w \delta(v_1)) + l(\delta(v_2))$. By section 3.1, $u_3^{-1} w_1 \delta(u_3) \leq u^{-1} w \delta(v)$.

Now assume that $u_3 = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $u_3^{-1} w_1 = s_{j_1} s_{j_2} \cdots s_{j_l}$ are reduced expressions. For $m = 1, 2, \dots, k+1$, set

$$x_m = (s_{i_m} s_{i_{m+1}} \cdots s_{i_k}) (s_{j_1} s_{j_2} \cdots s_{j_l}) (s_{\delta(i_1)} s_{\delta(i_2)} \cdots s_{\delta(i_{m-1})}).$$

Then $l(x_m) \leq k + l = l(w_1)$ for all m . On the other hand, for any m , there exists $y_m \in W_J$, such that $x_m = y_m^{-1} w \delta(y_m)$. Note that $w \in W^J$; we have that $l(y_m^{-1} w \delta(y_m)) \geq l(w \delta(y_m)) - l(y_m^{-1}) = l(w) = l(w_1)$ for all $y_m \in W_J$. Thus $l(x_m) = l(w_1)$ and $x_m \sim_{J, \delta} w_1$ for all m . In particular, $u_3^{-1} w_1 \delta(u_3) = x_{k+1} \sim_{J, \delta} w_1$. The implication is proved in this case. $\square$

Remark. We see from the proof that $u^{-1} w \delta(v) \geq x^{-1} w \delta(x)$ for some $x \leq u$. This result will be used in the proof of lemma 5.2.

3.9. Let $(J, w) \in \mathcal{I}_\delta$ and $w' \in W$; we say that $w' \geq_{J, \delta} w$ if w' satisfies the equivalent conditions of proposition 3.8 (1)-(3). It is easy to see that $x \geq w \Rightarrow x \geq_{J, \delta} w \Rightarrow l(x) \geq l(w)$.

Now for $(J_1, w_1), (J_2, w_2) \in \mathcal{I}_\delta$, we say that $(J_1, w_1) \leq_\delta (J_2, w_2)$ if $J_1 \subset J_2$ and $w_1 \geq_{J, \delta} w_2$. In the end of this section, we will show that $\leq$ is a partial order on $\mathcal{I}_\delta$. (The definition of partial order can be found in section 3.12).

Lemma 3.10. *Let $J \subset I$, $w \in W^J$, $u \in W$ with $l(uw) = l(u) + l(w)$. Assume that $uw = xv$ with $x \in W^J$ and $v \in W_J$. Then for any $v' \leq v$, there exists $u' \leq u$, such that $u'w = xv'$.*

Proof. We argue by induction on $l(u)$. If $l(u) = 0$, then $u = 1$ and the statement is clear. Assume now that $l(u) > 0$. Then there exists $i \in I$, such that $s_i u < u$. We denote $s_i u$ by u_1 . Let $u_1 w = x_1 v_1$ with $x_1 \in W^J$ and $v_1 \in W_J$. Then $s_i x_1 > x_1$.

If $s_i x_1 \in W^J$, then the lemma holds by induction hypothesis. If $s_i x_1 \notin W^J$, then there exists $j \in J$, such that $s_i x_1 = x_1 s_j$. In this case, $s_j v_1 > v_1$. Let $v' \leq s_j v_1$. If $v' \leq v_1$, then the lemma holds by induction hypothesis. If $v' \not\leq v_1$, then $v' = s_j v'_1$ for some $v'_1 \leq v_1$. By induction hypothesis, there exists $u'_1 \leq u_1$, such that $u'_1 w = x_1 v'_1$. Thus $s_i u'_1 w = x_1 s_j v'_1$. The lemma holds in this case. $\square$

Lemma 3.11. *Fix $J \subset I$ and $w \in W^{\delta(J)}$. For any $K \subset J$, $w' \in W^{\delta(K)}$ with $w' \geq_{J, \delta} w$, there exists $x \in W^{\delta(K)}$, $u \in W_J$ and $u_1 \in W_K$, such that $x \geq w\delta(u)$ and $w' = u_1^{-1} u^{-1} x \delta(u_1)$.*

Proof. Since $w' \geq_{J, \delta} w$, there exists $v_1 \in W_J$, such that $w' \geq v_1^{-1} w \delta(v_1)$. By corollary 3.4, there exists $v' \leq v_1$, such that $v' w' \geq w \delta(v_1) \geq w \delta(v')$. Let v be a minimal element in the set $\{v \in W_J \mid v w' \geq w \delta(v)\}$. Then $l(v w') = l(v) + l(w')$. Now assume that $v w' = x \delta(v')$ for some $x \in W^{\delta(K)}$ and $v' \in W_K$. Then there exists $v'_1 \leq v'$, such that $x \geq w \delta(v) \delta(v'_1)^{-1}$. By lemma 3.10, $x \delta(v'_1) = v_2 w'$ for some $v_2 \leq v$. Since $l(x \delta(v'_1)) = l(x) + l(v'_1)$, $v_2 w' = x \delta(v'_1) \geq w \delta(v) \geq w \delta(v_2)$. Therefore, $v_2 = v$ and $v'_1 = v'$. So $x \geq w \delta(v) \delta(v')^{-1}$. Now set $u = v(v')^{-1}$ and $u_1 = v'$. Then $w' = v^{-1} x \delta(v') = u_1^{-1} u^{-1} x \delta(u_1)$. $\square$

3.12. A relation $\leq$ is a partial order on a set S if it has:

1. Reflexivity: $a \leq a$ for all $a \in S$.
2. Antisymmetry: $a \leq b$ and $b \leq a$ implies $a = b$.
3. Transitivity: $a \leq b$ and $b \leq c$ implies $a \leq c$.

Proposition 3.13. *The relation $\leq_\delta$ on the set $\mathcal{I}_\delta$ is a partial order.*

Proof. Reflexivity is clear from the definition.

For $(J_1, w_1), (J_2, w_2) \in \mathcal{I}_\delta$ with $(J_1, w_1) \leq_\delta (J_2, w_2)$ and $(J_2, w_2) \leq_\delta (J_1, w_1)$, we have that $J_1 = J_2$ and $l(w_1) = l(w_2)$. Since $w_1 \geq w'_2$ for some $w'_2 \sim_{J, \delta} w_2$ and $l(w_1) = l(w_2) = l(w'_2)$, $w_1 = w'_2 \in W^{\delta(J_2)}$. Hence $w_1 = w'_2 = w_2$. Therefore $(J_1, w_1) = (J_2, w_2)$. Antisymmetry is proved.

Let $(J_1, w_1), (J_2, w_2)$ and $(J_3, w_3) \in \mathcal{I}_\delta$. Assume that $(J_1, w_1) \leq_\delta (J_2, w_2)$ and $(J_2, w_2) \leq_\delta (J_3, w_3)$. Then $J_1 \subset J_2 \subset J_3$. Moreover, there exists $x \in W^{\delta(J_2)}$, $u \in W_{J_3}$ and $u_1 \in W_{J_2}$, such that $x \geq w_3 \delta(u)$ and $w_2 = u_1^{-1} u^{-1} x \delta(u_1)$. Since $w_1 \geq_{J, \delta} w_2$, there exists $u_2 \in W_{J_2}$, such that $w_1 \geq u_2^{-1} u^{-1} x \delta(u_2)$. Note that $l(x \delta(u_2)) = l(x) + l(u_2)$ and $x \geq w_3 \delta(u)$. Thus $x \delta(u_2) \geq w_3 \delta(u u_2)$. By corollary 3.4, there exists $v \leq u u_2$, such that $w_1 \geq v^{-1} w_3 \delta(u u_2)$. By section 3.7, $w_1 \geq_{J_3, \delta} w_3$. Transitivity is proved. $\square$

4. THE CLOSURE OF ANY G -STABLE PIECE

4.1. We have that

$$G^1 = \sqcup_{w \in W} B\dot{w}U^- \dot{w}_0^{\delta(J)} g_0 = \sqcup_{w \in W} B\dot{w}\dot{w}_0^{\delta(J)} U_{P_{\delta(J)}}^- U_{\delta(J)} g_0.$$

Moreover, $B\dot{w}U^- = \sqcup_{b \in U_J^- \cap \dot{w}^{-1}U^-} B\dot{w}U_{P_J^-} b$. Thus

$$\begin{aligned} B\dot{w}U_{P_J^-} U_J &= B\dot{w}\dot{w}_0^J U^- \dot{w}_0^J = \sqcup_{b \in U_J^- \cap (\dot{w}\dot{w}_0^J)^{-1}U^-} B\dot{w}\dot{w}_0^J U_{P_J^-} b\dot{w}_0^J \\ &= \sqcup_{b \in U_J \cap \dot{w}^{-1}U^-} B\dot{w}U_{P_J^-} b. \end{aligned}$$

Note that if $w = w'u$ with $w' \in W^J$ and $u \in W_J$, then

$$U_J \cap \dot{w}^{-1}U^- = \dot{u}^{-1}(\dot{u}U_J \cap (\dot{w}')^{-1}U^-) = \dot{u}^{-1}(\dot{u}U_J \cap U_J^-) = U_J \cap \dot{u}^{-1}U_J^-.$$

Lemma 4.2. *Let $(J, w) \in \mathcal{I}_\delta$. For any $u \in W$ and $b \in B$, there exists $v \leq u$, such that $\dot{u}b\dot{w} \in B\dot{v}\dot{w}U_{P_{\delta(J)}}^- U_{\delta(J)}$.*

Proof. We will prove the statement by induction on $l(u)$.

If $u = 1$, then the statement holds. If $u = s_i u_1$ with $l(u_1) = l(u) - 1$, then by induction hypothesis, there exists $v_1 \leq u_1$, such that $\dot{u}_1 b\dot{w} \in b'\dot{v}_1 \dot{w} U_{P_{\delta(J)}}^- U_{\delta(J)}$ for some $b' \in B$. Write $b' = b_1 b_2$, where $b_1 \in U_{P_{\{i\}}}$ and $b_2 \in U_{\{i\}}$. Then $\dot{s}_i b_1 \dot{s}_i^{-1} \dot{s}_i b_2 \dot{v}_1 \dot{w}$ with $\dot{s}_i b_1 \dot{s}_i^{-1} \in B$.

If $(\dot{v}_1 \dot{w})^{-1} b_2 \dot{v}_1 \dot{w} \in U_{P_{\delta(J)}}^- U_{\delta(J)}$, then $\dot{s}_i b_2 \dot{v}_1 \dot{w} \in \dot{s}_i \dot{v}_1 \dot{w} U_{P_{\delta(J)}}^- U_{\delta(J)}$. Otherwise, $b_2 \neq 1$ and $(\dot{v}_1 \dot{w})^{-1} U_{\{i\}}^- \dot{v}_1 \dot{w} \subset U_{P_{\delta(J)}}^- U_{\delta(J)}$. Note that $\dot{s}_i b_2 \in BU_{\{i\}}^-$. Thus $\dot{s}_i b_2 \dot{v}_1 \dot{w} \in B\dot{v}_1 \dot{w} U_{P_{\delta(J)}}^- U_{\delta(J)}$. The statement holds in both cases. $\square$

4.3. Let $z \in (G, 1) \cdot h_{J, \delta}$. Then z can be written as $z = (b\dot{w}\dot{u}b', 1) \cdot h_{J, \delta}$ with $b \in B$, $w \in W^{\delta(J)}$, $u \in W_{\delta(J)}$ and $b' \in U_{\delta(J)} \cap \dot{u}^{-1}U_{\delta(J)}^-$. Moreover, w, u, b' are uniquely determined by z .

Set $J_0 = J$. To $z \in (G, 1) \cdot h_{J, \delta}$, we associate a sequence $(J_i, w_i, v_i, v'_i, c_i, z_i)_{i \geq 1}$ with $J_i \subset J$, $w_i \in W^{\delta(J)}$, $v_i \in W_{J_{i-1}} \cap J_i W$, $v'_i \in W_{J_i}$, $c_i \in U_{\delta(J_{i-1})} \cap \dot{\delta}(v_i^{-1})U_{\delta(J_{i-1})}^-$ and $z_i \in (B\dot{w}_i \dot{\delta}(v'_i)U_{\delta(J)} \dot{\delta}(v_i)c_i, 1) \cdot h_{J, \delta}$ and in the same G_{diag} -orbit as z . The sequence is defined as follows.

Assume that $z \in (B\dot{w}\dot{\delta}(u)U_{\delta(J)}, 1) \cdot h_{J, \delta}$ with $w \in W^{\delta(J)}$ and $u \in W_{\delta(J)}$. Then set $z_1 = z$, $J_1 = J$, $w_1 = w$, $v_1 = 1$, $v'_1 = u$ and $c_1 = 1$.

Assume that $k \geq 1$, that $J_k, w_k, v_k, v'_k, c_k, z_k$ are already defined and that $J_k \subset J_{k-1}$, $w_k \in W^{\delta(J)}$, $W_{J_{k-1}} w_k \subset W^{\delta(J)} W_{\delta(J_k)}$, $v_k \in W_{J_{k-1}} \cap J_k W$, $v'_k \in W_{J_k}$, $c_k \in U_{\delta(J_{k-1})} \cap \dot{\delta}(v_k^{-1})U_{\delta(J_{k-1})}^-$ and $z_k \in (B\dot{w}_k \dot{\delta}(v'_k)U_{\delta(J)} \dot{\delta}(v_k)c_k, 1) \cdot h_{J, \delta}$. Set $z_{k+1} = (g_0^{-1} \dot{\delta}(v_k)c_k g_0, g_0^{-1} \dot{\delta}(v_k)c_k g_0) z_k$. Then $z_{k+1} \in (G, 1) \cdot h_{J, \delta}$. Moreover, by lemma 4.2, there exists $x_k \leq v_k$, such that $z_{k+1} \in (B\dot{x}_k w_k \dot{\delta}(v'_k)U_{\delta(J)}, 1) \cdot h_{J, \delta}$.

Let y_{k+1} be the unique element of the minimal length in $W_{J_k} x_k w_k \dot{\delta}(v'_k) W_{\delta(J)}$. Set $J_{k+1} = J_k \cap \delta^{-1} \text{Ad}(y_{k+1}^{-1}) J_k$. Since $W_{J_{k-1}} w_k \subset W^{\delta(J)} W_{\delta(J_k)}$, then we have that $x_k w_k \dot{\delta}(v'_k) = w_{k+1} \dot{\delta}(v'_{k+1} v_{k+1})$ for some $w_{k+1} \in W^{\delta(J)}$, $v'_{k+1} \in W_{J_{k+1}}$ and $v_{k+1} \in W_{J_k} \cap J_{k+1} W$. Note that $W_{J_k} w_{k+1} \subset W^{\delta(J)} W_{\delta(J) \cap \text{Ad}(y_{k+1}^{-1}) J_k}$. On the other hand, $W_{J_k} w_{k+1} \subset W_{J_{k-1}} w_k W_{\delta(J_k)} \subset W^{\delta(J)} W_{\delta(J_k)}$. Thus $W_{J_k} w_{k+1} \subset (W^{\delta(J)} W_{\delta(J) \cap \text{Ad}(y_{k+1}^{-1}) J_k}) \cap (W^{\delta(J)} W_{\delta(J_k)}) = W^{\delta(J)} W_{\delta(J_{k+1})}$.

Moreover $z_{k+1} \in (B\dot{w}_{k+1}\dot{\delta}(v'_{k+1})U_{\delta(J)}\dot{\delta}(v_{k+1})c_{k+1}, 1) \cdot h_{J,\delta}$ for a unique $c_{k+1} \in U_{J_k} \cap \dot{\delta}(v_{k+1}^{-1})U_{J_k}^-$.

This completes the inductive definition. Moreover, for sufficient large n , we have that $J_n = J_{n+1} = \dots$, $w_n = w_{n+1} = \dots$, $v'_n = v'_{n+1} = \dots$ and $v_n = v_{n+1} = \dots = 1$.

4.4. Let $K \subset J$, $y \in {}^K W^{\delta(K)}$ and $K = \text{Ad}(y)\delta(K)$. Then for any $u \in W_K$, we have that

$$(B\dot{y}\dot{\delta}(u)U_{\delta(K)}, 1) \cdot h_{J,\delta} \subset G_{\text{diag}}(\dot{y}L_{\delta(K)}, B) \cdot h_{J,\delta} = G_{\text{diag}}(\dot{y}L_{\delta(K)}, U_{P_K}) \cdot h_{J,\delta}.$$

For any $l \in L_K$, there exists $l' \in L_K$, such that $l'\dot{y}g_0l(l')^{-1} \in \dot{y}g_0(L_K \cap B)$. Thus $(L_K)_{\text{diag}}(\dot{y}(L_{\delta(K)} \cap B), U_{P_K}) \cdot h_{J,\delta} = (\dot{y}L_{\delta(K)}, U_{P_K}) \cdot h_{J,\delta}$. Hence

$$(B\dot{y}\dot{\delta}(u)U_{\delta(K)}, 1) \cdot h_{J,\delta} \subset G_{\text{diag}}(\dot{y}(L_{\delta(K)} \cap B), U_{P_K}) \cdot h_{J,\delta} = G_{\text{diag}}(\dot{y}, B) \cdot h_{J,\delta} = Z_{J,\delta}^y.$$

Now for any $z \in (G, 1) \cdot h_{J,\delta}$, let $(z_i, J_i, w_i, v_i, v'_i, c_i)_{i \geq 1}$ be the sequence associated to z . Assume that $J_n = J_{n+1} = \dots$, $w_n = w_{n+1} = \dots$, $v'_n = v'_{n+1} = \dots$ and $v_n = v_{n+1} = \dots = 1$. Then we have showed that $z_n \in Z_{J,\delta}^{w_n}$. Thus $z \in Z_{J,\delta}^w$.

Note that for any $z \in Z_{J,\delta}$, z is in the same G -orbit as an element of the form $(G, 1) \cdot h_{J,\delta}$. Therefore, given $z \in Z_{J,\delta}$, our procedure determines the G -stable piece $Z_{J,\delta}^w$ that contains z .

Now we are able to describe the closure of $Z_{J,\delta}^w$. In theorem 4.5, we will only consider subvarieties of $\bar{G}^1$ and for any subvariety X of $\bar{G}^1$, we denote by $\bar{X}$ the closure of X in $\bar{G}^1$.

Theorem 4.5. *For any $(J, w) \in \mathcal{I}_\delta$, we have that*

$$\overline{Z_{J,\delta}^w} = \sqcup_{(K,w') \leq_\delta (J,w)} Z_{K,\delta}^{w'}.$$

Proof. Define $\pi' : G \times \overline{[I, 1, 1]_\delta} \rightarrow \bar{G}^1$ by $\pi(g, z) = (g, g) \cdot z$. The morphism is invariant under the B -action defined by $b(g, z) = (gb^{-1}, \pi'(b, z))$. Denote by $G \times_B \overline{[I, 1, 1]_\delta}$ the quotient; we obtain a morphism $\pi : G \times_B \overline{[I, 1, 1]_\delta} \rightarrow \bar{G}^1$. Because G/B is projective, π is proper and hence surjective.

Note that $\overline{[J, w, 1]_\delta} = \sqcup_{K \subset J} \sqcup_{x \in W^{\delta(K)}, u \in W_J, \text{ and } x \geq w\delta(u)} [K, x, u]_\delta$. Since $Z_{J,\delta}^w = \pi(G \times_B [J, w, 1]_\delta)$, we have that

$$\overline{Z_{J,\delta}^w} = \sqcup_{K \subset J} \sqcup_{x \in W^{\delta(K)}, u \in W_J, \text{ and } x \geq w\delta(u)} G_{\text{diag}} \cdot [K, x, u]_\delta.$$

For any $z \in [K, x, u]_\delta$ with $x \in W^{\delta(K)}$, $u \in W_J$ and $x \geq w\delta(u)$, we have that $z \in (B\dot{x}, B\dot{u}) \cdot h_{K,\delta} = G_{\text{diag}}(\dot{u}^{-1}B\dot{x}, 1) \cdot h_{K,\delta} \subset \sqcup_{v \leq u^{-1}} G_{\text{diag}}(B\dot{v}\dot{x}U_{\delta(K)}, 1) \cdot h_{K,\delta}$.

Fix $v \leq u^{-1}$ and $z' \in (B\dot{v}\dot{x}U_{\delta(K)}, 1) \cdot h_{K,\delta}$. Let $(z_i, J_i, w_i, v_i, v'_i, c_i)_{i \geq 1}$ be the sequence associated to z' . Then for any i , there exists $x_i \leq v_i$, such that $x_i w_i \delta(v'_i) = w_{i+1} \delta(v'_{i+1} v_{i+1})$. Assume that $J_n = J_{n+1} = \dots$, $w_n = w_{n+1} = \dots$, $v'_n = v'_{n+1} = \dots$ and $v_n = v_{n+1} = \dots = 1$. Set $x_\infty = x_n x_{n-1} \dots x_2$ and $v_\infty = v'_n (v_n v_{n-1} \dots v_2)$. Note that $x_1 = v_1 = 1$. Then $x_\infty v x = x_\infty w_1 \delta(v'_1) = w_n \delta(v_\infty)$. Since $v'_n \in W_{J_{n+1}}$ and $v_i \in W_{J_i} \cap {}^{J_{i+1}} W$, we have that $l(v_\infty) = l(v'_n) + l(v_n) + l(v_{n-1}) + \dots + l(v_2)$. Thus $x_\infty \leq v_\infty$. By section 4.4, $z' \in Z_{K, x_\infty v x \delta(v_\infty^{-1})}$. Note that $v^{-1} \leq u$ and $l(wu) = l(w) + l(u)$. Thus $w\delta(v^{-1}) \leq w\delta(u) \leq x$. Similarly, $w\delta(v^{-1} x_\infty^{-1}) \leq x\delta(v_\infty^{-1})$. By corollary 3.4, there exist $v' \leq v^{-1} x_\infty^{-1}$, such that $(v')^{-1} w\delta(v^{-1} x_\infty^{-1}) \leq x_\infty v x \delta(v_\infty^{-1})$. Thus by proposition 3.8, $x_\infty v x \delta(v_\infty^{-1}) \geq_{J,\delta} w$.

For any $K \subset J$ and $w' \in W^{\delta(K)}$ with $w' \geq_{J,\delta} w$, there exists $x \in W^{\delta(K)}$, $u \in W_J$ and $u_1 \in W_K$, such that $x \geq w\delta(u)$ and $w' = u_1^{-1} u^{-1} x \delta(u_1)$. Since $[K, x, u]_\delta \subset$

$\overline{[J, w, 1]_\delta}$. We have that $(\dot{x}T, \dot{u}) \cdot h_{K, \delta} \subset \overline{Z_{J, \delta}^w}$. Therefore $(\dot{u}^{-1}\dot{x}T, 1) \cdot h_{K, \delta} \subset \overline{Z_{J, \delta}^w}$. Note that $u^{-1}x = u_1w'\delta(u_1)^{-1}$. Then $(\dot{u}_1\dot{w}'\dot{\delta}(u_1)^{-1}T, 1) \cdot h_{K, \delta} \subset \overline{Z_{J, \delta}^w}$. Thus

$$(\dot{u}_1^{-1}, \dot{u}_1^{-1})(\dot{u}_1\dot{w}'\dot{\delta}(u_1)^{-1}T, 1) \cdot h_{K, \delta} = (\dot{w}'T, 1) \cdot h_{K, \delta} \subset \overline{Z_{J, \delta}^w}.$$

By proposition 1.12, $Z_{K, \delta}^{w'} \subset \overline{Z_{J, \delta}^w}$. The theorem is proved. $\square$

Our method also works in another situation.

Proposition 4.6. *The closure of $Z_{J, 1, \delta}^w$ in $Z_{J, 1, \delta}$ is $\sqcup_{w' \in W^{\delta(J)}, w \geq_J w'} Z_{J, 1, \delta}^{w'}$.*

Proof. In the proof, we will only consider subvarieties of $Z_{J, 1, \delta}$, and for any subvariety X of $Z_{J, 1, \delta}$, we denote by $\bar{X}$ its closure in $Z_{J, 1, \delta}$.

Note that the morphism $\pi : Z_{J, 1, \delta} \rightarrow \mathcal{P}_J$, defined by $\pi(P, Q, \gamma) = P$ for $(P, Q, \gamma) \in Z_{J, 1, \delta}$ is a locally trivial fibration with isomorphic fibers. Moreover, $i : \pi^{-1}(P_J) \rightarrow G^1/H_{P_J}$, defined by $i(P, Q, \gamma) = \gamma$ for $(P, Q, \gamma) \in \pi^{-1}(P_J)$, is an isomorphism. Now $[J, w, 1]_{1, \delta} \subset \pi^{-1}(P_J)$ and $i([J, w, 1]_{1, \delta}) = B\dot{w}Bg_0/H_{P_J}$. Thus $\overline{[J, w, 1]_{1, \delta}} = \sqcup_{w' \leq w} [J, w', 1]_{1, \delta}$. For any $w' \in W^J$ with $w \geq_{J, \delta} w'$, there exists $u \in W_J$, such that $w \geq u^{-1}w'\delta(u)$. Thus $(\dot{u}^{-1}\dot{w}'\dot{\delta}(u)T, 1) \cdot h_{J, 1, \delta} \subset \overline{[J, w, 1]_{1, \delta}}$. Hence $G_{diag}(\dot{u}^{-1}\dot{w}'\dot{\delta}(u)T, 1) \cdot h_{J, 1, \delta} = G_{diag}(\dot{w}'T, 1) \cdot h_{J, 1, \delta} \subset \overline{Z_{J, 1, \delta}^w}$. So $Z_{J, 1, \delta}^{w'} \subset \overline{Z_{J, 1, \delta}^w}$.

On the other hand, for any $z \in [J, w', 1]_{1, \delta}$, by the similar argument as we did in sections 4.3 and 4.4, there exists $u \leq v \in W_J$, such that $uw'\delta(v^{-1}) \in W^{\delta(J)}$ and $z \in Z_{J, 1, \delta}^{uw'v^{-1}}$. If moreover, $w' \leq w$, then $w \geq u^{-1}(uw'\delta(v^{-1}))\delta(v)$. Thus $w \geq_{J, \delta} uw'v^{-1}$. Therefore $z \in \sqcup_{w' \in W^{\delta(J)}, w \geq_{J, \delta} w'} Z_{J, 1, \delta}^{w'}$. The proposition is proved. $\square$

5. THE CELLULAR DECOMPOSITION

5.1. A finite partition of a variety X into subsets is said to be an α -partition if the subsets in the partition can be indexed $X_1, X_2, \dots, X_n$ in such a way that $X_1 \cup X_2 \cup \dots \cup X_i$ is closed in X for $i = 1, 2, \dots, n$. We say that a variety has a cellular decomposition if it admits an α -partition into subvarieties which are affine spaces. It is easy to see that if a variety X admits an α -partition into subvarieties and each subvariety has a cellular decomposition, then X has a cellular decomposition.

Lemma 5.2. *Let $(J, w) \in \mathcal{I}_\delta$, $K \subset J$ and $w' \in W$ with $\text{Ad}(w')\delta(K) = K$. If $w'v \geq_{J, \delta} w$ for some $v \in W_{\delta(K)}$, then $w' \geq_{J, \delta} w$.*

Proof. Fix w' and (J, w) . It suffices to prove the following statement:

Let $u \in W_J$ and $v \in W_{\delta(K)}$. If $w'v \geq u^{-1}w\delta(u)$, then $w' \geq_{J, \delta} w$.

We argue by induction on $l(u)$. Assume that the statement holds for all $u' < u$. Then I will prove that the statement holds for u by induction on $l(v)$. If $l(v) = 0$, then $v = 1$ and the statement holds in this case. Now assume that $l(v) > 0$.

Set $u = u_1u_2$ with $u_1 \in W^K$ and $u_2 \in W_K$. If $u_2 = 1$, then $u \in W^K$ and $w\delta(u) \in W^{\delta(K)}$. By corollary 3.4, there exists $u' \leq u$, such that $u'w'v \geq w\delta(u)$. Assume that $v = v's_k$ for $v' < v$ and $k \in \delta(K)$. Then $w\delta(u) < w\delta(u)s_k$. By section 3.1, $w\delta(u) \leq u'w'v'$. By corollary 3.4, there exists $u'_1 \leq u' \leq u$, such that $w'v' \geq (u'_1)^{-1}w\delta(u)$. By the remark of proposition 3.8, $w'v' \geq (u'_2)^{-1}w\delta(u'_2)$ for some $u'_2 \leq u'_1$. Thus by induction hypothesis, $w' \geq_{J, \delta} w$.

If $u_2 \neq 1$, then $l(u_1) < l(u)$. By corollary 3.4, there exists $u_3 \leq u_2$ and $u_4 \leq u_2^{-1}$, such that $u_3w'v\delta(u_4) \geq u_1^{-1}w\delta(u_1)$. Note that $u_3w'vu_4 = w'((w')^{-1}u_3w')v\delta(u_4) \in w'W_{\delta(K)}$. By induction hypothesis on $l(u_1)$, $w' \geq_{J, \delta} w$. $\square$

5.3. Let $J \subset I$. For $w \in W$, set

$$\begin{aligned} I_1(J, w, \delta) &= \max\{K \subset J \mid w \in W^{\delta(K)}\}, \\ I_2(J, w, \delta) &= \max\{K \subset J \mid \text{Ad}(w)\Phi_{\delta(K)} = \Phi_K\}. \end{aligned}$$

Now let $(J, w) \in \mathcal{I}_\delta$. Set

$$W_\delta(J, w) = \{u \in W \mid u \geq_{J, \delta} w, I_2(J, u, \delta) \subset I_1(J, u, \delta)\}.$$

For any $u \in W_\delta(J, w)$, set

$$\begin{aligned} X_u^{(J, w, \delta)} &= \sqcup_{K \subset I_1(J, u, \delta)} \sqcup_{v \in W_{\delta(I_2(J, u, \delta))} \cap W^{\delta(K)}} Z_{K, \delta}^{uv} \\ &= \sqcup_{v \in W_{\delta(I_2(J, u, \delta))}} \sqcup_{K \subset I_1(J, uv, \delta)} Z_{K, \delta}^{uv}. \end{aligned}$$

For $w' \geq_{J, \delta} w$, we have that $w' = uv$ for some $u \in W^{\delta(I_2(J, w'))}$ and $v \in W_{\delta(I_2(J, w'))}$. Then $I_2(J, u, \delta) = I_2(J, w', \delta) \subset I_1(J, u, \delta)$. By lemma 5.2, $u \geq_{J, \delta} w$. Thus $u \in W_\delta(J, w)$ and $\sqcup_{K \subset I_1(J, w', \delta)} Z_{K, \delta}^{w'} \subset X_u^{(J, w, \delta)}$.

For $u_1, u_2 \in W(J, w)$ and $v_1 \in W_{\delta(I_2(J, u_1))}, v_2 \in W_{\delta(I_2(J, u_2))}$ with $u_1 v_1 = u_2 v_2$, we have that $I_2(J, u_1, \delta) = I_2(J, u_1 v_1, \delta) = I_2(J, u_2 v_2, \delta) = I_2(J, u_2, \delta)$. Note that $u_1, u_2 \in W^{\delta(I_2(J, u_1))}$. Thus $u_1 = u_2$ and $v_1 = v_2$.

Therefore $\overline{Z_{J, \delta}^w} = \sqcup_{u \in W_\delta(J, w)} X_u^{(J, w, \delta)}$.

Lemma 5.4. Let $(J, w) \in \mathcal{I}_\delta$. Set $I_2 = I_2(J, w, \delta)$. For $K \subset J$, we have that

$$\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (L_{I_2})_{diag}(\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K, \delta} = (L_{I_2}, L_{I_2})(\dot{w}, 1) \cdot h_{K, \delta}.$$

Proof. At first, we will prove the case when $K \subset I_2$. In this case, set $g_1 = g_0 \dot{w}$. Then $g_1 L_{\delta(I_2)} g_1^{-1} = L_{\delta(I_2)}$ and $g_1 (L_{\delta(I_2)} \cap B) g_1^{-1} = L_{\delta(I_2)} \cap B$. Now consider $\overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})} g_1$ (a variety that is isomorphic to $\overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})}$, but with “twisted” $L_{\delta(I_2)} \times L_{\delta(I_2)}$ action; see section 2.3). We have that

$$\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (L_{\delta(I_2)})_{diag}(\dot{v}, B \cap L_{\delta(I_2)}) \cdot (h_{\delta(K)} g_1) = (L_{\delta(I_2)}, L_{\delta(I_2)}) \cdot (h_{\delta(K)} g_1).$$

(In the case when $g_1^n \in L_{\delta(I_2)}$ for some $n \in \mathbf{N}$, $L_{\delta(I_2)} g_1$ is a connected component of the group generated by $L_{\delta(I_2)}$ and g_1 . In this case, the left hand side is the union of some $L_{\delta(I_2)}$ -stable pieces and the equality follows from [L4, 12, 3]. The general case can be shown in the same way.)

Therefore

$$\begin{aligned} &\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, g_0) (L_{I_2})_{diag}(\dot{w}, g_0^{-1})(\dot{v}, B \cap L_{\delta(I_2)}) \cdot h_{\delta(K)} \\ &= \sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (1, g_1) (L_{\delta(I_2)})_{diag}(1, g_1^{-1})(\dot{v}, B \cap L_{\delta(I_2)}) \cdot h_{\delta(K)} \\ &= (L_{\delta(I_2)}, L_{\delta(I_2)})(\dot{w}, 1) \cdot h_{\delta(K)}. \end{aligned}$$

Note that $h_{K, \delta} = h_{\delta(K)} g_0$. Then $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1) (L_{I_2})_{diag}(\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K, \delta} = (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K, \delta}$. Hence $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (L_{I_2})_{diag}(\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K, \delta} = (L_{I_2}, L_{I_2}) \cdot h_{K, \delta}$.

In the general case, Consider $\pi : (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K, \delta} \rightarrow \overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})} g_0$ defined by $\pi((l_1, l_2) h_{K, \delta}) = (l_1, l_2) \cdot (h_{\delta(K) \cap \delta(I_2)} g_0)$ for $l_1 \in L_{\delta(I_2)}, l_2 \in L_{I_2}$. Here $h_{\delta(K) \cap \delta(I_2)}$ on the right side is the base point in $\overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})}$ that corresponds to $\delta(K) \cap \delta(I_2)$. It is easy to see that the morphism is well-defined. Now define the T -action on $(L_{\delta(I_2)}, L_{I_2}) \cdot h_{K, \delta}$ by $t \cdot ((l_1, l_2) h_{K, \delta}) = (tl_1, l_2) h_{K, \delta}$ for $t \in T$ and $l_1 \in L_{\delta(I_2)}, l_2 \in$

L_{I_2} . Then T acts transitively on $\pi^{-1}(a)$ for any $a \in (L_{\delta(I_2)}, L_{I_2}) \cdot (h_{\delta(K) \cap \delta(I_2)} g_0)$. Now

$$\begin{aligned} & \sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} \pi((\dot{w}^{-1}, 1)(L_{I_2})_{diag} \cdot (\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K, \delta}) \\ &= \sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1)(L_{I_2})_{diag} (\dot{w}\dot{v}, B \cap L_{I_2}) \cdot (h_{\delta(K) \cap \delta(I_2)} g_0) \\ &= (L_{\delta(I_2)}, L_{I_2}) \cdot (h_{\delta(K) \cap \delta(I_2)} g_0). \end{aligned}$$

Moreover $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1)(L_{I_2})_{diag} \cdot (\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K, \delta}$ is stable under T -action. Thus $\sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (\dot{w}^{-1}, 1)(L_{I_2})_{diag} \cdot (\dot{w}\dot{v}, B \cap L_{I_2}) \cdot h_{K, \delta} = (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K, \delta}$. The lemma is proved. $\square$

Proposition 5.5. *Let $(J, w) \in \mathcal{I}_\delta$ and $u \in W_\delta(J, w)$. Set $I_1 = I_1(J, u, \delta)$, $I_2 = I_2(J, u, \delta)$ and $L_u^{(J, w, \delta)} = \sqcup_{K \subset I_1} (L_{I_2}, L_{I_2})(\dot{u}, 1) \cdot h_{K, \delta}$. Then we have that*

- (1) $L_u^{(J, w, \delta)}$ is a fibre bundle over $\overline{L_{I_2}/Z(L_{I_2})}$ with fibres isomorphic to an affine space of dimension $|I_1| - |I_2|$.
- (2) $X_u^{(J, w, \delta)} = G_{diag} \cdot L_u^{(J, w, \delta)}$ is isomorphic to $G \times_{P_{I_2}} ((P_{I_2})_{diag} \cdot L_u^{(J, w, \delta)})$.
- (3) $(P_{I_2})_{diag} \cdot L_u^{(J, w, \delta)} = (B \times B) \cdot L_u^{(J, w, \delta)} \cong (U \cap \dot{w}_0^{I_2} \dot{u} \dot{w}_0 U^-) \times L_u^{(J, w, \delta)}$.

Proof. For part (1), note that $L_u^{(J, u, \delta)} = \sqcup_{K \subset I_1} (\dot{u}, 1)(L_{\delta(I_2)}, L_{I_2}) \cdot h_{K, \delta} = (\dot{u} L_{\delta(I_2)}, L_{I_2}) \cdot (\sqcup_{K \subset I_1} (T, 1) h_{K, \delta})$ is a variety. Consider the morphism

$$\pi' : \sqcup_{K \subset I_1} (L_{\delta(I_2)}, L_{I_2}) \cdot h_{K, \delta} \rightarrow \overline{L_{\delta(I_2)}/Z(L_{\delta(I_2)})} g_0$$

defined by $\pi'((l_1, l_2) h_{K, \delta}) = (l_1, l_2) \cdot (h_{\delta(K) \cap \delta(I_2)} g_0)$ for $l_1 \in L_{\delta(I_2)}$, $l_2 \in L_{I_2}$. It is easy to see that π' is well defined and is a locally trivial fibration with fibers isomorphic to an affine space of dimension $|I_1| - |I_2|$.

Let $v \in W_{\delta(I_2)}$. For $K \subset J$, if $\text{Ad}(uv)\delta(K) = K$, then

$$\text{Ad}(u)\Phi_{\delta(K)} = \text{Ad}(uv^{-1}u^{-1})\Phi_K.$$

Since $uv^{-1}u^{-1} \in W_{I_2}$, we have that $\text{Ad}(u)\Phi_{\delta(K)} \subset \Phi_{K \cup I_2}$. Thus $\text{Ad}(u)\Phi_{\delta(K \cup I_2)} \subset \Phi_{K \cup I_2}$. By the maximal property of I_2 , $K \cup I_2 \subset I_2$. Thus $I_2(J, uv, \delta) \subset I_2$. Therefore,

$$\begin{aligned} G_{diag} \cdot L_u^{(J, w, \delta)} &= G_{diag}(\sqcup_{K \subset I_1} (L_{I_2}, L_{I_2})(\dot{u}, 1) \cdot h_{K, \delta}) \\ &= G_{diag}(\sqcup_{K \subset I_1} \sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} (L_{I_2})_{diag} (\dot{u}\dot{v}, B \cap L_{I_2(J, u, \delta)}) \cdot h_{K, \delta}) \\ &= \sqcup_{K \subset I_1} \sqcup_{v \in W_{\delta(I_2)} \cap W^{\delta(K)}} G_{diag}(\dot{u}\dot{v}, B \cap L_{I_2}) \cdot h_{K, \delta} \\ &= \sqcup_{K \subset I_1} \sqcup_{v \in W_{\delta(I_2)}} Z_{J, \delta}^{uv} = X_u^{(J, w, \delta)}. \end{aligned}$$

Assume that $(g, g)a = b$ for some $g \in G$ and $a, b \in L_u^{(J, u, \delta)}$. Then a, b are in the same G orbit. Note that any element in $L_u^{(J, u, \delta)}$ is conjugate by L_{I_2} to an element of the form $(\dot{u}\dot{v}, l) h_{K, \delta}$ with $v \in W_{\delta(I_2)}$, $K \subset I_1(J, uv, \delta)$ and $l \in L_{I_2} \cap B$. Moreover, $(\dot{u}\dot{v}, L_{I_2} \cap B) \cdot h_{K, \delta} \subset Z_{K, \delta}^{uv}$. Thus if $v_1 \neq v_2$ or $K_1 \neq K_2$, then for any $l, l' \in L_{I_2} \cap B$, $(\dot{u}\dot{v}_1, l) \cdot h_{K_1, \delta}$ and $(\dot{u}\dot{v}_2, l') \cdot h_{K_2, \delta}$ are not in the same G orbit. Thus $(g, g)(\dot{u}\dot{v}, l_1) \cdot h_{K, \delta} = (\dot{u}\dot{v}, l_2) \cdot h_{K, \delta}$ for some $v \in W_{\delta(I_2)}$, $K \subset I_1(J, uv, \delta)$ and $l_1, l_2 \in L_{I_2} \cap B$. By proposition 1.12, $g \in P_{I_2(K, uv, \delta)}$. Since $I_2(K, uv, \delta) \subset I_2(J, uv, \delta) \subset I_2$, we have that $g \in P_{I_2}$. By lemma 1.9, $X_u^{(J, w, \delta)} \cong G \times_{P_{I_2}} ((P_{I_2})_{diag} \cdot L_u^{(J, w, \delta)})$. Part (2) is proved.

For part (3), it is easy to see that $(P_{I_2(J,u,\delta)})_{diag} \cdot L_u^{(J,w,\delta)} \subset (B \times B) \cdot L_u^{(J,w,\delta)}$. On the other hand,

$$\begin{aligned} (B \times B) \cdot L_u^{(J,w,\delta)} &= (U_{P_{I_2}}, U_{P_{I_2}})(L_{I_2})_{diag} (\sqcup_{v \in W_{\delta(I_2)}} \sqcup_{K \subset I_2(J,uv,\delta)} (\dot{u}v, B) \cdot h_{K,\delta}) \\ &= (L_{I_2})_{diag} (U_{P_{I_2}}, U_{P_{I_2}})(\sqcup_{v \in W_{\delta(I_2)}} \sqcup_{K \subset I_2(J,uv,\delta)} (\dot{u}v, B) \cdot h_{K,\delta}). \end{aligned}$$

By proposition 1.12,

$$\begin{aligned} (U_{P_{I_2}}, U_{P_{I_2}})(\dot{u}v, B) \cdot h_{K,\delta} &= (B \times B)(\dot{u}v, 1) \cdot h_{K,\delta} \\ &\subset (P_{I_2(K,uv,\delta)})_{diag} \cdot (L_{I_2(K,uv,\delta)}, L_{I_2(K,uv,\delta)})(\dot{u}v, 1) h_{K,\delta}. \end{aligned}$$

We have showed that $I_2(K, uv, \delta) \subset I_2$. Hence

$$(U_{P_{I_2}}, U_{P_{I_2}})(\dot{u}v, B) \cdot h_{K,\delta} \subset (P_{I_2(J,u,\delta)})_{diag} \cdot L_u^{(J,w,\delta)}.$$

Therefore, $(P_{I_2})_{diag} \cdot L_u^{(J,w,\delta)} = (B \times B) \cdot L_u^{(J,w,\delta)}$.

Consider the morphism $\pi : (U \cap \dot{w}_0^{I_2} \dot{u} \dot{w}_0 U^-) \times L_u^{(J,w,\delta)} \rightarrow (B \times B) \cdot L_u^{(J,w,\delta)}$ defined by $\pi(b, l) = (b, 1) \cdot l$ for $b \in U \cap \dot{w}_0^{I_2} \dot{u} \dot{w}_0 U^-$ and $l \in L_u^{(J,w,\delta)}$. By the similar argument as we did in proposition 1.10, we can show that π is an isomorphism. $\square$

Corollary 5.6. *We keep the notation of proposition 5.5. If moreover, $I_2 = \emptyset$, then $X_u^{(J,w,\delta)}$ admits a cellular decomposition.*

Proof. If $I_2 = \emptyset$, then $L_u^{(J,w,\delta)}$ is an affine space. Thus $X_u^{(J,w,\delta)}$ is isomorphic to $G \times BC$, where $C = (B \times B) \cdot L_u^{(J,w,\delta)}$. By part (3) of proposition 5.5, C is an affine space. It is easy to see that B acts linearly on C . Therefore $X_u^{(J,w,\delta)}$ is a vector bundle over $\mathcal{B}$. Note that $\mathcal{B}$ admits a cellular decomposition. By a well-known result (see [Q], [Su] or [VS]), $X_u^{(J,w,\delta)}$ admits a cellular decomposition. $\square$

5.7. For $w_1, w_2 \in W_{\delta}(J, w)$, we say $w_2 \leq' w_1$ if there exists $w_1 = x_0, x_1, \dots, x_n = w_2$, $v_i \in \delta(I_2(J, x_{i+1}, \delta))$ for all i , such that $x_{i+1} v_i \geq_{I_1(J, x_i, \delta), \delta} x_i$.

By theorem 4.5, $\overline{X_{u_1}^{(J,w,\delta)}} \cap X_{u_2}^{(J,w,\delta)} = \emptyset$ if $u_2 \not\leq' u_1$. Hence if $\leq'$ is a partial order on $W_{\delta}(J, w)$, then $\overline{Z_{J,\delta}^w} = \sqcup_{u \in W_{\delta}(J,w)} X_u^{(J,w,\delta)}$ is an α -partition. We will show that $\leq'$ is a partial order if $\overline{Z_{J,\delta}^w}$ contains finitely many G -orbits.

Lemma 5.8. *Let $J \subset I$, $u \in W$, $w \in W^J$ and $v \in W_J$. Assume that $uwv = w'v'$ for some $w' \in W^J$ and $v' \in W_J$. If $l(uwv) = l(wv) - l(u)$, then $w' \leq w$. If moreover, $w' = w$, then $\text{Ad}(w^{-1})\text{supp}(u) \subset J$.*

Proof. If $u = s_i$ for some $i \in J$ and $l(s_j wv) = l(wv) - 1$, then either $s_i w < w$ and $s_i w \in W^J$ or $s_i w = w s_j$ for some $j \in J$. It is easy to check that the statement holds in both cases.

The general case can be proved by induction on $l(u)$. $\square$

Lemma 5.9. *If $w_1, w_2 \in W_{\delta}(J, w)$ with $w_1 \leq' w_2$ and $w_2 \leq' w_1$ and $I_2(J, w_1, \delta) = I_2(J, w_2, \delta) = \emptyset$, then $w_1 = w_2$.*

Proof. We will prove the case: if $w_1 \geq_{I_1(J, w_2, \delta), \delta} w_2$, $w_2 \geq_{I_1(J, w_1, \delta), \delta} w_1$ and $I_2(J, w_1, \delta) = \emptyset$, then $w_1 = w_2$. The general case can be proved in the same way.

We argue by induction on $|J|$. Since $l(w_1) \geq l(w_2)$ and $l(w_2) \geq l(w_1)$, we have that $l(w_1) = l(w_2)$. Thus $w_1 = u_2^{-1}w_2\delta(u_2)$ and $w_2 = u_1^{-1}w_1\delta(u_1)$ for some $u_1 \in W_{I_1(J, w_1, \delta)}$ and $u_2 \in W_{I_1(J, w_2, \delta)}$. By induction hypothesis, it suffices to prove the case when $J = \text{supp}(u_1) \cup \text{supp}(u_2)$.

We have that $w_1 = w'_1\delta(v_1)$ and $w_2 = w'_2\delta(v_2)$ for some $w'_1, w'_2 \in W^{\delta(J)}$ and $v_1, v_2 \in W_J$. Note that $w'_1\delta(v_1) = u_2^{-1}w'_2\delta(v_2u_2)$ and $l(u_2^{-1}w'_2\delta(v_2u_2)) = l(w'_2\delta(v_2u_2)) - l(u_2)$. By lemma 5.8, $w'_1 \leq w'_2$. Similarly $w'_2 \leq w'_1$. Therefore $w'_1 = w'_2$. By 5.8, $\text{Ad}(w'_2)^{-1}\text{supp}(u_2) \subset \delta(J)$ and $\text{Ad}(w'_1)^{-1}\text{supp}(u_1) \subset \delta(J)$. Therefore $\text{Ad}(w'_1)^{-1}J \subset \delta(J)$. Hence $\text{Ad}(w_1)^{-1}\Phi_J = \Phi_{\delta(J)}$. Since $I_2(J, w_1, \delta) = \emptyset$, we have that $J = \emptyset$. Therefore $w_1 \geq w_2$ and $w_2 \geq w_1$. Thus $w_1 = w_2$. The case is proved. $\square$

As a summary, we have the following result.

Theorem 5.10. *If $\overline{Z_{J, \delta}^w}$ contains only finitely many G -orbits, then it has a cellular decomposition.*

Proof. If $\overline{Z_{J, \delta}^w}$ contains only finitely many G -orbits, then $I_2(J, u, \delta) = \emptyset$ for $u \geq_{J, \delta} w$. In this case, $W_\delta(J, w) = \{u \in W \mid u \geq_{J, \delta} w\}$. We have that

$$\overline{Z_{J, \delta}^w} = \sqcup_{u \in W_\delta(J, \delta)} X_u^{(J, w, \delta)}.$$

By section 5.7 and lemma 5.9, the partition is an α -partition. Thus by corollary 5.6, $\overline{Z_{J, \delta}^w}$ has a cellular decomposition. $\square$

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