EQUILIBRIUM POINTS OF LOGARITHMIC POTENTIALS INDUCED BY POSITIVE CHARGE DISTRIBUTIONS. I. GENERALIZED DE BRUIJN-SPRINGER RELATIONS

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ABSTRACT. A notion of weighted multivariate majorization is defined as a preorder on sequences of vectors in Euclidean space induced by the Choquet ordering for atomic probability measures. We characterize this preorder both in terms of stochastic matrices and convex functions and use it to describe the distribution of equilibrium points of logarithmic potentials generated by discrete planar charge configurations. In the case of n positive charges we prove that the equilibrium points satisfy $\binom{n}{2}$ weighted majorization relations and are uniquely determined by $n - 1$ such relations. It is further shown that the Hausdorff geometry of the equilibrium points and the charged particles is controlled by the weighted standard deviation of the latter. By using finite-rank perturbations of compact normal Hilbert space operators we establish similar relations for infinite charge distributions. We also discuss a hierarchy of weighted de Bruijn-Springer relations and inertia laws, the existence of zeros of Borel series with positive l^1 -coefficients, and an operator version of the Clunie-Eremenko-Rossi conjecture.

1. INTRODUCTION

Let $\{z_i\}_{i \in \mathcal{N}}$ be a sequence of distinct points in a domain $\Omega \subseteq \mathbb{C}$, where $\mathcal{N} \subseteq \mathbb{N}$ with $|\mathcal{N}| \geq 2$. If at each z_i we place a particle with charge a_i – actually, an infinitely long thin rod perpendicular to the plane with uniform charge distribution – then up to complex conjugation and a constant factor the resulting electrostatic force is given by

$$(1.1) \quad f(z) = \sum_{i \in \mathcal{N}} \frac{a_i}{z - z_i}$$

and the logarithmic potential generated by these charged particles is

$$(1.2) \quad U(z) = \sum_{i \in \mathcal{N}} a_i \log \left| 1 - \frac{z}{z_i} \right|$$

(cf. [23, p. 10]). If $|\mathcal{N}| = \infty$, then the series in (1.1) and (1.2) converge absolutely for all $z \in \Omega \setminus \{z_i\}_{i \in \mathcal{N}}$ provided that certain growth conditions are satisfied (see §3.3 below). In this case f is meromorphic while U is subharmonic in Ω . Since $\nabla U(z) = \overline{f(z)}$ the critical points of the potential coincide with the equilibrium

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points of the field, that is, points upon which a free electron (or rod) once placed would remain. An important problem that goes back to Gauss is to describe the distribution and the relative geometry of the equilibrium points and charged particles. One of the main purposes of this paper is to study this problem by means of the *Hausdorff distance* and the notion of *weighted multivariate majorization* that we define in §2 as an extension of classical and matrix majorization. It is interesting to note that although our results are purely analytic or geometric in nature, they are obtained through a combination of the multilinear algebra and operator theoretic methods developed in [25, 26] and [25, 26, 28], respectively. The fact that such geometric properties could be established in this way was actually first noticed in *loc. cit.*, where long-standing conjectures on the geometry of polynomials were solved by applying this type of technique. Below we make extensive use of the key observation made in the aforementioned papers, namely that studying zeros and critical points of complex polynomials – or, more generally, poles and zeros of logarithmic potentials generated by finite positive charge configurations – is equivalent to studying spectra of normal matrices and their compressions.

The weighted multivariate generalization of vector and matrix majorization introduced in §2 is a natural preorder on finite sequences of vectors in Euclidean space induced by the Choquet ordering for nonnegative Radon measures (Definition 1). We characterize this new notion in terms of stochastic matrices and convex functions, and we recover in this way several related results on classical and multivariate majorization. In the process we use a Hahn-Banach type argument to give a new simple proof of a fundamental theorem of Sherman (Theorem 1).

In §3 we state most of our main results. These describe the distribution and the relative location of equilibrium points of logarithmic potentials for discrete bounded planar configurations of positive charges and are of three different types. First, we consider n positive charges and prove that the equilibrium points satisfy $\binom{n}{2}$ weighted multivariate majorization relations and are in fact uniquely determined by $n - 1$ such relations (Theorems 2–3). The higher de Bruijn-Springer relations thus obtained generalize some of the results of [25, 26] to arbitrary (not necessarily equal) positive charges. Second, we show that the relative Hausdorff geometry of the extended set of equilibrium points and the set of charged particles is controlled by the weighted standard deviation of the latter (Theorems 4–5). For equal charges the bounds given by these theorems are much better and also more natural than those predicted by the famous Sendov conjecture for complex polynomials and its converse; see, e.g., [3]. As we explain in the sequel [4] to this paper, these results also seem to suggest that the Hausdorff geometry is governed by natural dispersion measures associated with the given charge configuration. Finally, we establish weighted multivariate majorization relations for essentially any bounded discrete set of positively charged particles with finite total charge for which equilibrium points are known to exist (Theorems 6–7). We further conjecture that equilibrium points actually always exist under some mild assumptions (Conjectures 2–3 in §6).

The proofs of the aforementioned results are given in §4. As we already pointed out, for finite charge configurations these are based on the multilinear algebra and matrix theory techniques of [25, 26, 28]. In the process we are led to consider several natural related questions, such as a hierarchy of inertia laws for finite systems of planar charged particles that we propose and discuss in §5 (Conjecture 1). This would yield a whole new series of weighted generalized de Bruijn-Springer relations.

The techniques mentioned above actually extend to the case of infinite discrete distributions of charge, which allows us to make use of operator theoretic methods such as compressions of compact normal operators on separable Hilbert space in order to deal with this case. In this way we arrive at a great many related questions on the interplay between value distribution of meromorphic functions and variations of discrete spectra of compact normal Hilbert space operators under finite-rank perturbations. In §6–7 we give an operator theoretic approach to the well-known Clunie-Eremenko-Rossi conjecture [8] dealing with the existence of zeros of certain Borel series and we consider possible extensions of this conjecture as well as their potential implications (Conjectures 4–6).

2. WEIGHTED MULTIVARIATE MAJORIZATION

The concept of majorization was first studied by economists early in the twentieth century as a means for altering the unevenness of distribution of wealth or income. The majorization preorder on n -tuples of real numbers – also known as the strong spectral order, vector majorization or classical majorization – was used by Hardy, Littlewood and Pólya in their study of analytic inequalities [19] and by Schur in his work on the spectra of positive semidefinite Hermitian operators (cf. [12]). Classical majorization has since become an important tool in analysis, combinatorics and statistics [27] and has recently found remarkable applications to quantum mechanics and entanglement transformations in quantum computation and information theory (see [2] and references therein). The strong spectral order essentially quantifies the intuitive notion that the components of a real n -vector $\mathbf{x}$ are “less spread out” than the components of another such vector $\mathbf{y}$. Several matrix versions of this notion have been proposed and studied in various contexts (cf., e.g., [9] and [27]).

In this section we generalize classical and matrix majorization by introducing a notion of weighted multivariate majorization that we shall use throughout both this paper and its sequel [4].

Notation 1. Let $k \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define

$$(2.1) \quad \begin{aligned} \mathcal{X}_n^k &= \{(\mathbf{x}_1, \dots, \mathbf{x}_n) \mid \mathbf{x}_i \in \mathbb{R}^k, 1 \leq i \leq n\}, \\ \mathcal{A}_n &= \left\{ (a_1, \dots, a_n) \mid a_i \in (0, 1), 1 \leq i \leq n, \sum_{i=1}^n a_i = 1 \right\}, \\ \mathbb{X}_n^k &= \mathcal{X}_n^k \times \mathcal{A}_n \end{aligned}$$

and set

$$(2.2) \quad \mathbb{X}^k = \bigcup_{n=1}^{\infty} \mathbb{X}_n^k.$$

Given $X = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathcal{X}_m^k$ we denote by $X^T = (\mathbf{x}_1, \dots, \mathbf{x}_m)^T$ its “transpose”, i.e., the ordered m -tuple consisting of the vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ written in column form. We identify X^T with the $m \times k$ real matrix whose i -th row consists of the coordinates of the vector $\mathbf{x}_i$ in the standard basis of $\mathbb{R}^k$. Let finally $\mathbb{M}_{m,n}^{rs}$ be the set of all row stochastic $m \times n$ matrices.

The notion of weighted multivariate majorization defined below (see Definition 1) is motivated by the following theorem of Sherman [31].

Theorem 1. Let $(X, \mathbf{a}) \in \mathbb{X}_m^k$ and $(Y, \mathbf{b}) \in \mathbb{X}_n^k$, where $X = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathcal{X}_m^k$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathcal{A}_m$, $Y = (\mathbf{y}_1, \dots, \mathbf{y}_n) \in \mathcal{X}_n^k$ and $\mathbf{b} = (b_1, \dots, b_n) \in \mathcal{A}_n$. The following conditions are equivalent:

(i) For any (continuous) convex function $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}$ one has

$$\sum_{i=1}^m a_i \Phi(\mathbf{x}_i) \leq \sum_{j=1}^n b_j \Phi(\mathbf{y}_j).$$

(ii) There exists a matrix $R \in \mathbb{M}_{m,n}^{rs}$ such that

$$\tilde{X}^T = R \tilde{Y}^T \text{ and } \mathbf{b} = \mathbf{a}R,$$

where $\tilde{X}^T$ is an $m \times k$ matrix and $\tilde{Y}^T$ is an $n \times k$ matrix obtained by some (and then any) ordering of the vectors in X^T and Y^T , respectively.

The special case when $m = n$ and all weights are equal in Theorem 1 was long assumed to be an open question and appears only as an implicit conjecture in [27, p. 433], which is the definite reference on majorization theory. Although essentially correct, Sherman's paper used rather complicated arguments and contained a series of misprints that required several subsequent errata, which may explain why the status of his proof seems to have remained unclear for such a long period. Below we present a short and simple proof of Theorem 1.

Before embarking on the actual proof let us note that condition (i) in Sherman's theorem may also be written in terms of the Choquet ordering for positive measures, a notion that we shall actually use in [4] (see Definition 9 in *loc. cit.*). Indeed, any pair $(X, \mathbf{a}) \in \mathbb{X}_m^k$ uniquely determines a discrete probability measure $\mu_{(X, \mathbf{a})}$ on $\mathbb{R}^k$ concentrating a mass a_i at $\mathbf{x}_i$, $1 \leq i \leq m$. Given two positive measures μ_1 and μ_2 on $\mathbb{R}^k$ one says that μ_2 dominates μ_1 in the Choquet ordering or that μ_2 is a dilation of μ_1 , denoted $\mu_1 \prec \mu_2$, if $\mu_1(\Phi) \leq \mu_2(\Phi)$ for any continuous convex function Φ on $\mathbb{R}^k$. Theorem 1 (i) may therefore be rewritten as $\mu_{(X, \mathbf{a})} \prec \mu_{(Y, \mathbf{b})}$. We refer to [5, 7, 13, 16, 29] for a detailed account of the Choquet ordering and various comparisons of nonnegative Radon measures and probability distributions in locally convex separable topological spaces. In particular, in the latter setting Sherman's theorem may in fact be viewed as a special case of the Cartier-Fell-Meyer theorem [7, 29, 5] and its subsequent extension due to Fischer-Holbrook [16]. The most general result of this type, namely a basic representation theorem for convex domination of measures, seems to be contained in [13].

Proof of Theorem 1. Since the implication (ii) $\Rightarrow$ (i) is an immediate consequence of Jensen's inequality we shall only focus on the converse statement. Assume that condition (i) in Theorem 1 holds, let $X = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathcal{X}_m^k$, $\mathbf{a} = (a_1, \dots, a_m) \in \mathcal{A}_m$, $Y = (\mathbf{y}_1, \dots, \mathbf{y}_n) \in \mathcal{X}_n^k$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathcal{A}_n$, and define the following set:

$$\mathcal{M}(X, Y) = \left\{ A = (a_{ij}) \in \mathbb{M}_{m,n}^{rs} \mid \mathbf{x}_i = \sum_{j=1}^n a_{ij} \mathbf{y}_j, 1 \leq i \leq m \right\}.$$

Lemma 1. $\mathcal{M}(X, Y)$ is a nonempty closed convex subset of $\mathbb{M}_{m,n}^{rs}$.

Proof. It is clear that $\mathcal{M}(X, Y)$ is closed and convex. To show that $\mathcal{M}(X, Y) \neq \emptyset$ it is enough to prove that if Theorem 1 (i) holds, then $\mathbf{x}_i \in \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ for $1 \leq i \leq m$, where $\text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ is the convex hull of the vectors $\mathbf{y}_1, \dots, \mathbf{y}_n$. Supposing that $\mathbf{x}_l \notin \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ for some $l \in \{1, \dots, m\}$ let d denote the

Euclidean distance in $\mathbb{R}^k$ and consider the function $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}$ given by $\Phi(\mathbf{u}) = d(\mathbf{u}, \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n))$. Note that $\Phi(\mathbf{x}_i) > 0$, $\Phi(\mathbf{y}_j) = 0$ for $1 \leq j \leq n$, and Φ is a convex function since $\text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ is a nonempty convex subset of $\mathbb{R}^k$. It follows that $\sum_{i=1}^m a_i \Phi(\mathbf{x}_i) > \sum_{j=1}^n b_j \Phi(\mathbf{y}_j)$, which contradicts condition (i) of Theorem 1. Thus $\mathbf{x}_i \in \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ for $1 \leq i \leq m$, as required. $\square$

Given $M = (m_{ij}) \in \mathcal{M}(X, Y)$ we define a vector $\mathbf{s}(M) = (s_1(M), \dots, s_n(M)) \in \mathbb{R}^n$ by $s_j(M) = \sum_{i=1}^m a_i m_{ij}$, $1 \leq j \leq n$. Set

$$\mathcal{L}(X, Y) = \{\mathbf{s}(M) \mid M \in \mathcal{M}(X, Y)\}.$$

Note that $\alpha \mathbf{s}(M) + (1 - \alpha) \mathbf{s}(N) = \mathbf{s}(\alpha M + (1 - \alpha)N)$ for any $M, N \in \mathcal{M}(X, Y)$ and $\alpha \in [0, 1]$, so that $\mathcal{L}(X, Y)$ is a convex subset of $\mathbb{R}^n$. Moreover, since $\mathcal{M}(X, Y)$ is closed the same must be true for $\mathcal{L}(X, Y)$.

Lemma 2. *With the above notations one has $\mathbf{b} \in \mathcal{L}(X, Y)$. In other words, there exists a matrix $M_0 \in \mathcal{M}(X, Y)$ such that $\mathbf{s}(M_0) = \mathbf{b}$.*

Proof. Assume that $\mathbf{b} \notin \mathcal{L}(X, Y)$. Since $\mathcal{L}(X, Y)$ is a closed convex set it follows from the Hahn-Banach theorem that there exist $c \in \mathbb{R}$ and $(r_1, \dots, r_n) \in \mathbb{R}^n$ such that

$$(2.3) \quad \sum_{j=1}^n r_j s_j < c < \sum_{j=1}^n r_j b_j \text{ for } (s_1, \dots, s_n) \in \mathcal{L}(X, Y).$$

Given an arbitrary vector $\mathbf{y} \in \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ we define the simplex

$$W(\mathbf{y}) = \left\{ (\lambda_1, \dots, \lambda_n) \in [0, 1]^n \mid \mathbf{y} = \sum_{j=1}^n \lambda_j \mathbf{y}_j, \sum_{j=1}^n \lambda_j = 1 \right\}.$$

Note that $W(\mathbf{y})$ is compact for any $\mathbf{y} \in \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ since it is complete and totally bounded. We may therefore define a function $G : \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n) \rightarrow \mathbb{R}$ by setting

$$G(\mathbf{y}) = \max \left\{ \sum_{j=1}^n r_j \lambda_j \mid (\lambda_1, \dots, \lambda_n) \in W(\mathbf{y}) \right\} \text{ for } \mathbf{y} \in \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n).$$

Let $\lambda_j(\mathbf{y}) \in [0, 1]$, $1 \leq j \leq n$, be such that $G(\mathbf{y}) = \sum_{j=1}^n r_j \lambda_j(\mathbf{y})$. Since $\lambda_j(\mathbf{y}_i) = \delta_{ij}$ one has $G(\mathbf{y}_j) \geq r_j$ for $1 \leq j \leq n$ and thus

$$(2.4) \quad \sum_{j=1}^n b_j G(\mathbf{y}_j) \geq \sum_{j=1}^n b_j r_j > c.$$

On the other hand the left inequality in (2.3) implies that for any matrix $M = (m_{ij}) \in \mathcal{M}(X, Y)$ one has

$$(2.5) \quad c > \sum_{j=1}^n r_j s_j(M) = \sum_{i=1}^m \sum_{j=1}^n a_i r_j m_{ij}.$$

Let us now consider the extension $\tilde{G}$ of the function G to $\mathbb{R}^k$ given by $\tilde{G}(\mathbf{y}) = G(\mathbf{y})$ if $\mathbf{y} \in \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ and $\tilde{G}(\mathbf{y}) = 0$ for $\mathbf{y} \in \mathbb{R}^k \setminus \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$. It is not difficult to see that $\tilde{G}$ is a concave function on $\mathbb{R}^k$. By Lemma 1 one has $\mathbf{x}_i \in \text{conv}(\mathbf{y}_1, \dots, \mathbf{y}_n)$, $1 \leq i \leq m$, so that the numbers $\lambda_j(\mathbf{x}_i)$ are well defined for

$1 \leq i \leq m$, $1 \leq j \leq n$. Note that $(\lambda_j(\mathbf{x}_i)) \in \mathcal{M}(X, Y)$ and that by (2.5) and (2.4) the following holds:

$$\sum_{i=1}^m a_i \tilde{G}(\mathbf{x}_i) = \sum_{i=1}^m \sum_{j=1}^n a_i r_j \lambda_j(\mathbf{x}_i) < \sum_{j=1}^n b_j r_j \leq \sum_{j=1}^n b_j \tilde{G}(\mathbf{y}_j).$$

This however contradicts condition (i) of Theorem 1 and thus we are done. It follows that $\mathbf{b} \in \mathcal{L}(X, Y)$, which completes the proof of the lemma. $\square$

The fact that (ii) $\Rightarrow$ (i) in Theorem 1 is equivalent to Lemma 2. $\square$

Remark 1. The authors are grateful to the referee for pointing out the similarity between the Hahn-Banach type argument used above and the one given in the original proof of the Cartier-Fell-Meyer theorem [7].

Definition 1. The pair $(X, \mathbf{a}) \in \mathbb{X}_m^k$ is said to be *weightily majorized* by the pair $(Y, \mathbf{b}) \in \mathbb{X}_n^k$, denoted $(X, \mathbf{a}) \prec (Y, \mathbf{b})$, if the conditions of Theorem 1 are satisfied.

Remark 2. If $(X, \mathbf{a}) \prec (Y, \mathbf{b})$, then the $\mathbf{a}$ -barycenter of X must coincide with the $\mathbf{b}$ -barycenter of Y , that is, $\sum_{i=1}^m a_i \mathbf{x}_i = \sum_{j=1}^n b_j \mathbf{y}_j$.

It is clear from Definition 1 that the weighted majorization relation is a preorder on $\mathbb{X}^k$. Indeed, by Theorem 1 this relation is both reflexive and transitive. Moreover, if $m = n$ and $a_i = b_j = \frac{1}{n}$, $1 \leq i, j \leq n$, then Theorem 1 and Birkhoff's theorem [27, Theorem A.2] imply that the weighted multivariate majorization relation induces a partial ordering on the set of unordered n -tuples of vectors in $\mathbb{R}^k$. We shall refer to this partial ordering as the *ordinary multivariate majorization* relation, for which Sherman's theorem takes a particularly simple form:

Corollary 1. If $X = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{X}_n^k$ and $Y = (\mathbf{y}_1, \dots, \mathbf{y}_n) \in \mathcal{X}_n^k$, then the following conditions are equivalent:

- (i) The inequality $\sum_{i=1}^n \Phi(\mathbf{x}_i) \leq \sum_{j=1}^n \Phi(\mathbf{y}_j)$ holds for any convex function $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}$.
- (ii) There exists a doubly stochastic $n \times n$ matrix S such that $\tilde{X}^T = S \tilde{Y}^T$, where $\tilde{X}^T$ and $\tilde{Y}^T$ are $n \times k$ matrices obtained by some ordering of the vectors in X^T and Y^T , respectively.

Remark 3. For $k = 1$ Corollary 1 amounts to a well-known description of classical (vector) majorization due to Schur and to Hardy-Littlewood-Pólya [19, 27].

Remark 4. In the sequel we shall often consider weighted pairs of the form $(X, \mathbf{a})$ and $(Y, \mathbf{b})$, where X and Y are unordered tuples of complex numbers. In this case we define the weighted majorization relation $(X, \mathbf{a}) \prec (Y, \mathbf{b})$, when appropriate, by identifying $\mathbb{C}$ with $\mathbb{R}^2$ in Theorem 1 and Definition 1.

Another interesting consequence of Sherman's theorem is as follows:

Corollary 2. Assume that $m < n \leq k$ and let $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathcal{X}_m^k$, $(\mathbf{y}_1, \dots, \mathbf{y}_n) \in \mathcal{X}_n^k$, $(a_1, \dots, a_m) \in \mathcal{A}_m$ and $(b_1, \dots, b_n) \in \mathcal{A}_n$. For $m+1 \leq i \leq n$ let $(c_{i1}, \dots, c_{in}) \in \mathcal{A}_n$ be such that $b'_j := b_j - \sum_{i=m+1}^n a_i c_{ij} \neq 0$, $1 \leq j \leq n$, and set $\mathbf{x}_i = \sum_{j=1}^n c_{ij} \mathbf{y}_j$. If $\mathbf{y}_1, \dots, \mathbf{y}_n$ are linearly independent, then the following conditions are equivalent:

- (i) $\sum_{i=1}^m a_i \Phi(\mathbf{x}_i) \leq \sum_{j=1}^n b_j \Phi(\mathbf{y}_j)$ for all convex functions $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}$.
- (ii) $\sum_{i=1}^m a_i \Phi(\mathbf{x}_i) \leq \sum_{j=1}^n b'_j \Phi(\mathbf{y}_j)$ for all convex functions $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}$.

If either of these conditions is fulfilled, then $b'_j > 0$ for $1 \leq j \leq n$.

Proof. The implication (ii) $\Rightarrow$ (i) being trivial let us show that (i) $\Rightarrow$ (ii). Set $X = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathcal{X}_n^k$, $X' = (\mathbf{x}_1, \dots, \mathbf{x}_m)$, $Y = (\mathbf{y}_1, \dots, \mathbf{y}_n)$, $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$. Then condition (i) is equivalent to $(X, \mathbf{a}) \prec (Y, \mathbf{b})$. Note that $\sum_{j=1}^n b'_j = \sum_{i=1}^m a_i =: \alpha \in (0, 1)$ and let $\mathbf{a}' = (\alpha^{-1}a_1, \dots, \alpha^{-1}a_m) \in \mathcal{A}_m$ and $\mathbf{b}' = (\alpha^{-1}b'_1, \dots, \alpha^{-1}b'_n)$. Using Theorem 1 and the fact that $\mathbf{y}_1, \dots, \mathbf{y}_n$ are linearly independent one can easily show that $\mathbf{b}' \in \mathcal{A}_n$ and $(X', \mathbf{a}') \prec (Y, \mathbf{b}')$, which is equivalent to condition (ii). $\square$

Remark 5. For an arbitrarily given convex function $\Phi : \mathbb{R}^k \rightarrow \mathbb{R}$ the inequality in Corollary 2 (i) is *a priori* weaker than the one in (ii). As a special case let us consider three complex numbers z_1, z_2, z_3 with $z_2 \bar{z}_3 \notin \mathbb{R}$ and $\alpha, \beta, \gamma \in (0, 1)$ such that $\gamma \neq (1 - \alpha)\beta$ and $1 - \gamma \neq (1 - \alpha)(1 - \beta)$. Then Corollary 2 implies that if

$$\alpha\Phi(z_1) + (1 - \alpha)\Phi(\beta z_2 + (1 - \beta)z_3) \leq \gamma\Phi(z_2) + (1 - \gamma)\Phi(z_3)$$

for all convex functions $\Phi : \mathbb{C} \rightarrow \mathbb{R}$, then $\gamma > (1 - \alpha)\beta$, $1 - \gamma > (1 - \alpha)(1 - \beta)$, and the inequality

$$\alpha\Phi(z_1) + (1 - \alpha)(\beta\Phi(z_2) + (1 - \beta)\Phi(z_3)) \leq \gamma\Phi(z_2) + (1 - \gamma)\Phi(z_3)$$

holds for any convex function $\Phi : \mathbb{C} \rightarrow \mathbb{R}$.

Remark 6. Theorem 1 generalizes several known results on vector and matrix majorization, many of which seem in fact to be regularly rediscovered in various contexts; see, e.g., [9] and references therein. Two new interesting extensions of classical and multivariate majorization have recently been introduced and studied in [26].

3. GEOMETRY OF EQUILIBRIUM POINTS: STATEMENT OF MAIN RESULTS

In this section we formulate our main results. These describe the distribution of equilibrium points of logarithmic potentials for discrete configurations of positive charges (Theorems 2–3 and Theorems 6–7) and the Hausdorff geometry of the extended set of equilibrium points for finite charge configurations (Theorems 4–5).

3.1. Weighted majorization relations for finite charge configurations. Let z_i , $1 \leq i \leq n$, be distinct points in the complex plane carrying positive charges a_i , $1 \leq i \leq n$, respectively, where $n \geq 2$. Throughout this section we assume wlog that the a_i are normalized so that $\sum_{i=1}^n a_i = 1$. The logarithmic potential generated by these charged particles is given by

$$(3.1) \quad U(z) = \sum_{i=1}^n a_i \log \left| 1 - \frac{z}{z_i} \right|,$$

and the resulting electrostatic force is

$$(3.2) \quad f(z) = \sum_{i=1}^n \frac{a_i}{z - z_i}.$$

The class of functions of the form (3.2) is essentially the same as Sz-Nagy's class of generalized derivatives [32]. As explained in §1, the critical points of $U(z)$ coincide with the equilibrium points of the electrostatic field, that is, the zeros of $f(z)$. By an argument reminiscent of the Gauss-Lucas theorem one can see that all equilibrium points lie in $\text{conv}(z_1, \dots, z_n)$; cf., e.g., Proposition 3.1 in [26]. Remarkable generalizations of this simple fact were recently obtained in *loc. cit.*, where it was

shown that these points actually satisfy a whole sequence of majorization relations. Theorems 2–3 below further extend the aforementioned results.

Notation 2. Let $w_1, \dots, w_{n-1}$ be the critical points of $U(z)$ counted with multiplicity and recall the set $\mathcal{A}_m$ defined in (2.1). For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$ let

$$(3.3) \quad \begin{aligned} W(\lambda, \mu) &= (\lambda w_1 + \mu, \dots, \lambda w_{n-1} + \mu) \in \mathbb{C}^{n-1}, \\ Z(\lambda, \mu) &= (\lambda z_1 + \mu, \dots, \lambda z_n + \mu) \in \mathbb{C}^n, \\ \mathbf{a} &= \left(\frac{1}{n-1}, \dots, \frac{1}{n-1} \right) \in \mathcal{A}_{n-1}, \quad \mathbf{b} = \left(\frac{1-a_1}{n-1}, \dots, \frac{1-a_n}{n-1} \right) \in \mathcal{A}_n. \end{aligned}$$

Theorem 2. *There exists a matrix $R \in \mathbb{M}_{n-1,n}^{rs}$ such that $W(\lambda, \mu)^T = RZ(\lambda, \mu)^T$ and $\mathbf{b} = \mathbf{a}R$ for any $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$, so that $(W(\lambda, \mu), \mathbf{a}) \prec (Z(\lambda, \mu), \mathbf{b})$. Equivalently, the inequality*

$$(3.4) \quad \sum_{j=1}^{n-1} \Phi(\lambda w_j + \mu) \leq \sum_{i=1}^n (1-a_i) \Phi(\lambda z_i + \mu)$$

is satisfied by any (continuous) convex function $\Phi : \mathbb{C} \rightarrow \mathbb{R}$.

Remark 7. The special case of (3.4) when $\lambda = 1$, $\mu = 0$ and $a_i = \frac{1}{n}$, $1 \leq i \leq n$, is already a considerable improvement of the Gauss-Lucas theorem and was originally conjectured by de Bruijn and Springer [6]. This conjecture was recently proved by S. Malamud [25, 26] and R. Pereira [28] independently of each other.

Remark 8. Theorem 2 implies in particular that the following ordinary multivariate majorization relation holds in $\mathbb{R}^2$:

$$(w_1, \dots, w_{n-1}, \zeta) \prec (z_1, \dots, z_{n-1}, z_n),$$

where $\zeta = \sum_{i=1}^n a_i z_i$ is the $(a_1, \dots, a_n)$ -barycenter of $Z(1, 0)$ (cf. Remarks 2 and 4). This relation was previously obtained in [26, Proposition 4.3] and [28, Theorem 5.4]. Note though that if $n \geq 3$, then even in the case of equal charges the above relation is in general weaker than the corresponding one given by Theorem 2 since the z_i are linearly dependent (compare with Corollary 2).

As we shall now see, (3.4) is actually but one relation in a sequence of $\binom{n}{2}$ weighted multivariate majorization relations satisfied by the zeros and poles of $f(z)$. Moreover, the equilibrium points of $U(z)$ are in fact uniquely determined by $n-1$ such relations. To formulate these results we need some new notation.

Notation 3. Given $k \in \mathbb{N}$ and $1 \leq m \leq k$ let $\Pi_{k,m}$ denote the m -th elementary symmetric function on k symbols. For $\lambda \in \mathbb{C} \setminus \{0\}$, $\mu \in \mathbb{C}$, $k \in \{1, \dots, n-1\}$, and

$1 \leq m \leq k$ set

$$\begin{aligned}
 (3.5) \quad W_m^{[k]}(\lambda, \mu) &= \left(\omega_m[r_1, \dots, r_k](\lambda, \mu) \right)_{1 \leq r_1 < \dots < r_k \leq n-1} \in \mathbb{C}^{\binom{n-1}{k}}, \\
 \text{where } \omega_m[r_1, \dots, r_k](\lambda, \mu) &= \Pi_{k,m}(\lambda w_{r_1} + \mu, \dots, \lambda w_{r_k} + \mu), \\
 Z_m^{[k]}(\lambda, \mu) &= \left(\zeta_m[s_1, \dots, s_k](\lambda, \mu) \right)_{1 \leq s_1 < \dots < s_k \leq n} \in \mathbb{C}^{\binom{n}{k}}, \\
 \text{where } \zeta_m[s_1, \dots, s_k](\lambda, \mu) &= \Pi_{k,m}(\lambda z_{s_1} + \mu, \dots, \lambda z_{s_k} + \mu), \\
 \mathbf{a}^{[k]} &= \left(\binom{n-1}{k}^{-1}, \dots, \binom{n-1}{k}^{-1} \right) \in \mathcal{A}_{\binom{n-1}{k}}, \\
 \mathbf{b}^{[k]} &= \left(\binom{n-1}{k}^{-1} \left(1 - \sum_{i=1}^k a_{s_i} \right) \right)_{1 \leq s_1 < \dots < s_k \leq n} \in \mathcal{A}_{\binom{n}{k}}.
 \end{aligned}$$

Below we shall always assume that the components of $W_m^{[k]}(\lambda, \mu)$, $Z_m^{[k]}(\lambda, \mu)$ and $\mathbf{b}^{[k]}$ are arranged increasingly with respect to the lexicographic order on the set of all indices consisting of k -tuples of increasing numbers in $\{1, \dots, n\}$. In particular, for any $1 \leq m \leq \binom{n}{k}$ the m -th coordinates of $Z_m^{[k]}(\lambda, \mu)$ and $\mathbf{b}^{[k]}$ are indexed by the same k -tuple $(s_1, \dots, s_k)$ satisfying $1 \leq s_1 < \dots < s_k \leq n$. Note also that with this ordering (3.3) and (3.5) imply that $\mathbf{a}^{[1]} = \mathbf{a}$, $\mathbf{b}^{[1]} = \mathbf{b}$, $W_1^{[1]}(\lambda, \mu) = W(\lambda, \mu)$ and $Z_1^{[1]}(\lambda, \mu) = Z(\lambda, \mu)$.

Theorem 3.11 in [26] describes all possible sets of equilibrium points for logarithmic potentials generated by a fixed configuration of particles that are allowed to carry arbitrary positive charges. A natural related inverse problem is to describe geometrically the equilibrium points of a logarithmic potential associated to a given set of particles with prescribed positive charges. The following theorem solves this problem and provides a characterization of the zeros of the logarithmic potential $U(z)$ defined in (3.1) by means of $n-1$ weighted majorization relations.

Theorem 3. *The following conditions are equivalent:*

(i) *The zeros of $f(z)$ counted with multiplicity are $w_1, \dots, w_{n-1}$, that is,*

$$(3.6) \quad \sum_{i=1}^n a_i \prod_{\substack{j=1 \\ j \neq i}}^n (z - z_j) = \prod_{k=1}^{n-1} (z - w_k), \quad z \in \mathbb{C}.$$

(ii) *There exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$ such that*

$$\left(W_k^{[k]}(\lambda, \mu), \mathbf{a}^{[k]} \right) \prec \left(Z_k^{[k]}(\lambda, \mu), \mathbf{b}^{[k]} \right), \quad 1 \leq k \leq n-1.$$

(iii) *There exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$ such that for any $k \in \{1, \dots, n-1\}$ one can find an $\binom{n-1}{k} \times \binom{n}{k}$ row stochastic matrix $R_k(\lambda, \mu)$ satisfying*

$$W_k^{[k]}(\lambda, \mu)^T = R_k(\lambda, \mu) Z_k^{[k]}(\lambda, \mu)^T \text{ and } \mathbf{b}^{[k]} = \mathbf{a}^{[k]} R_k(\lambda, \mu).$$

(iv) *There exist $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$ such that for any $k \in \{1, \dots, n-1\}$ the inequality*

$$\sum_{1 \leq r_1 < \dots < r_k \leq n-1} \Phi\left(\omega_k[r_1, \dots, r_k](\lambda, \mu)\right) \\ \leq \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(1 - \sum_{i=1}^k a_{s_i}\right) \Phi\left(\zeta_k[s_1, \dots, s_k](\lambda, \mu)\right)$$

holds for all (continuous) convex functions $\Phi : \mathbb{C} \rightarrow \mathbb{R}$.

(v) *Either of conditions (ii)–(iv) is true for $\lambda = 1$ and $\mu = 0$.*

If the above conditions are fulfilled, then for any $k \in \{1, \dots, n-1\}$ there exists an $\binom{n-1}{k} \times \binom{n}{k}$ row stochastic matrix R_k such that

$$(3.7) \quad W_k^{[k]}(\lambda, \mu)^T = R_k Z_k^{[k]}(\lambda, \mu)^T \text{ and } \mathbf{b}^{[k]} = \mathbf{a}^{[k]} R_k$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$.

Remark 9. Theorem 3 extends [26, Theorem 4.7] to the case of arbitrary (not necessarily equal) charges. If $a_i = \frac{1}{n}$, $1 \leq i \leq n$, then Theorem 3 reduces to the characterization of the zeros and critical points of degree n complex polynomials by means of $n-1$ majorization relations which was given in [26, Theorem 4.10].

From Theorem 3 we deduce a number of corollaries.

Corollary 3. *For any $k \in \{1, \dots, n-1\}$ there exists an $\binom{n-1}{k} \times \binom{n}{k}$ row stochastic matrix R_k such that*

$$(3.8) \quad W_m^{[k]}(\lambda, \mu)^T = R_k Z_m^{[k]}(\lambda, \mu)^T \text{ and } \mathbf{b}^{[k]} = \mathbf{a}^{[k]} R_k$$

whenever $1 \leq m \leq k$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. In particular, for all such parameters k, m, λ, μ and any convex function $\Phi : \mathbb{C} \rightarrow \mathbb{R}$ the following inequality holds:

$$(3.9) \quad \sum_{1 \leq r_1 < \dots < r_k \leq n-1} \Phi\left(\omega_m[r_1, \dots, r_k](\lambda, \mu)\right) \\ \leq \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(1 - \sum_{i=1}^k a_{s_i}\right) \Phi\left(\zeta_m[s_1, \dots, s_k](\lambda, \mu)\right).$$

Remark 10. Note that (3.8) is *a priori* stronger than (3.9) since it asserts that there exist matrices R_k of the desired type that simultaneously realize the weighted majorization relations between the pairs $(W_m^{[k]}(\lambda, \mu), \mathbf{a}^{[k]})$ and $(Z_m^{[k]}(\lambda, \mu), \mathbf{b}^{[k]})$ for all admissible parameters m, λ and μ .

Setting $m = 1$ in Corollary 3 yields the following generalization of Theorem 2, which corresponds to the case when $k = 1$ in (3.10) below.

Corollary 4. *For any $k \in \{1, \dots, n-1\}$ there exists an $\binom{n-1}{k} \times \binom{n}{k}$ row stochastic matrix R_k such that*

$$W_1^{[k]}(\lambda, \mu)^T = R_k Z_1^{[k]}(\lambda, \mu)^T \text{ and } \mathbf{b}^{[k]} = \mathbf{a}^{[k]} R_k$$

for any $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. Thus

$$\left(W_1^{[k]}(\lambda, \mu), \mathbf{a}^{[k]}\right) \prec \left(Z_1^{[k]}(\lambda, \mu), \mathbf{b}^{[k]}\right), \quad 1 \leq k \leq n-1,$$

or equivalently, for all convex functions $\Phi : \mathbb{C} \rightarrow \mathbb{R}$ and $k \in \{1, \dots, n-1\}$ one has

$$(3.10) \quad \sum_{1 \leq r_1 < \dots < r_k \leq n-1} \Phi(\omega_1[r_1, \dots, r_k](\lambda, \mu)) \leq \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(1 - \sum_{i=1}^k a_{s_i}\right) \Phi(\zeta_1[s_1, \dots, s_k](\lambda, \mu)).$$

Remark 11. The condition $\lambda \neq 0$ may actually be omitted from the above inequalities. However, if $\lambda = 0$ the vectors defined in (3.5) are independent of the z_i and w_j , and the aforementioned inequalities become equalities. Thus the corresponding weighted majorization relations are trivially satisfied in this case.

Taking $\lambda = 1$, $\mu = 0$ and $\Phi(z) = |z|^\alpha$ with $\alpha \geq 1$ in Corollary 3 we obtain the following moment inequalities for the charged particles and the equilibrium points.

Corollary 5. For any $k \in \{1, \dots, n-1\}$, $1 \leq m \leq k$, and $\alpha \geq 1$ one has

$$\sum_{1 \leq r_1 < \dots < r_k \leq n-1} |\Pi_{k,m}(w_{r_1}, \dots, w_{r_k})|^\alpha \leq \sum_{1 \leq s_1 < \dots < s_k \leq n} |\Pi_{k,m}(z_{s_1}, \dots, z_{s_k})|^\alpha,$$

where $\Pi_{k,m}$ denotes as before the m -th elementary symmetric function on k symbols.

In particular, by setting $a_i = \frac{1}{n}$, $1 \leq i \leq n$, and $m = 1$ in Corollary 5 we get

$$(3.11) \quad \binom{n-1}{k}^{-1} \sum_{1 \leq r_1 < \dots < r_k \leq n-1} \left| \sum_{i=1}^k w_{r_i} \right|^\alpha \leq \binom{n}{k}^{-1} \sum_{1 \leq s_1 < \dots < s_k \leq n} \left| \sum_{i=1}^k z_{s_i} \right|^\alpha$$

for all $\alpha \geq 1$ and $k \in \{1, \dots, n-1\}$. Note that for $k = 2$ and $\alpha \geq 1$ the inequalities in (3.11) are quite similar to the inequalities

$$\binom{n-1}{2}^{-1} \sum_{1 \leq r_1 < r_2 \leq n-1} |w_{r_1} - w_{r_2}|^\alpha \leq \binom{n}{2}^{-1} \sum_{1 \leq s_1 < s_2 \leq n} |z_{s_1} - z_{s_2}|^\alpha$$

that were conjectured by Rahman and Schmeisser in [30, Remark 2.3.9]. In §5 we conjecture a whole hierarchy of weighted de Bruijn-Springer relations and inertia laws; see Conjecture 1 below. The latter actually contains as special cases all types of moment inequalities like those that we just described.

3.2. Relative Hausdorff geometry. The metric function between (closed) subsets of Euclidean space introduced by Hausdorff makes use of the following notions.

Definition 2. The *relative* (alternatively, *directed* or *oriented*) *Hausdorff distance* from a subset Ω_1 of $\mathbb{C}$ to another such subset Ω_2 is given by

$$h(\Omega_1, \Omega_2) = \sup_{w \in \Omega_1} \text{dist}(w, \Omega_2) = \sup_{w \in \Omega_1} \inf_{z \in \Omega_2} |w - z|.$$

Other frequent names and notations are the *forward Hausdorff distance* from Ω_1 to Ω_2 , $h_f(\Omega_1, \Omega_2) := h(\Omega_1, \Omega_2)$, and the *backward Hausdorff distance* from Ω_1 to Ω_2 , $h_b(\Omega_1, \Omega_2) := h(\Omega_2, \Omega_1)$. The (*symmetrized*) *Hausdorff distance* between Ω_1 and Ω_2 is then given by

$$H(\Omega_1, \Omega_2) = \max(h(\Omega_1, \Omega_2), h(\Omega_2, \Omega_1)).$$

Let $\mathcal{Z} = \{z_1, \dots, z_n\}$ be a set of charged particles and $\mathcal{W} = \{w_1, \dots, w_{n-1}\}$ be the corresponding set of equilibrium points of the logarithmic potential defined in (3.1). Theorems 2–3 yield a series of relations between the sets $\mathcal{W}$ and $\mathcal{Z}$ when all their elements are considered simultaneously. These relations go much beyond the obvious inclusion $\mathcal{W} \subset \text{conv}(\mathcal{Z})$ and suggest that no equilibrium point can actually lie too far away from the particles. Our next result gives a precise meaning to this intuitive fact and shows that the relative Hausdorff distance $h(\mathcal{W}, \mathcal{Z})$ is dominated by the **a-weighted standard deviation** of the particles. The latter is given by the square root of the **a-weighted variance** of $\mathbf{z} = (z_1, \dots, z_n)$, which we define as

$$(3.12) \quad \sigma_2(\mathbf{z}; \mathbf{a})^2 = \inf_{\alpha \in \mathbb{C}} \sum_{i=1}^n a_i |z_i - \alpha|^2 = \sum_{i=1}^n a_i |z_i - \zeta|^2,$$

where $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{A}_n$ and ζ stands as before for the **a-weighted barycenter** $\sum_{i=1}^n a_i z_i$ (cf. Remark 8). We shall actually prove a stronger statement involving the *extended set of equilibrium points* of the given system of positive charges, i.e., the set

$$(3.13) \quad \mathcal{W}_e = \mathcal{W} \cup \{\zeta\} = \{w_1, \dots, w_{n-1}, \zeta\}.$$

Theorem 4. *In the above notation one has $h(\mathcal{W}, \mathcal{Z}) \leq h(\mathcal{W}_e, \mathcal{Z}) \leq \sigma_2(\mathbf{z}; \mathbf{a})$.*

Remark 12. The bound given in Theorem 4 is sharp, as one can see for instance by placing n equal charges at the vertices of a regular n -gon.

A natural question in this context is whether there is any analog of Theorem 4 for the relative Hausdorff distance from $\mathcal{Z}$ to $\mathcal{W}_e$. The following theorem provides an affirmative answer to this question when all particles are assumed to lie on the same line and shows that in this case the Hausdorff geometry of the set of charged particles and the extended set of equilibrium points is controlled by the weighted standard deviation of the particles.

Theorem 5. *If $z_1, \dots, z_n$ are collinear, then $H(\mathcal{Z}, \mathcal{W}_e) \leq \sigma_2(\mathbf{z}; \mathbf{a})$.*

Remark 13. Note that Theorem 5 is sharp and that a similar result cannot hold if $\mathcal{W}_e$ is replaced by $\mathcal{W}$. To see this one may simply consider two distinct points in the plane carrying unequal positive charges.

3.3. Infinite discrete distributions of charge. As we point out in §6, several difficulties occur when trying to extend the results of §3.1 to logarithmic potentials generated by an infinite discrete set of positive charges. To begin with, in this case the fundamental question dealing with the *existence* of equilibrium points is already quite delicate and, as a matter of fact, still an open problem (cf. [8, 14, 24]).

Nevertheless, it turns out that in many situations one can obtain new information on the geometry of the electrostatic field through natural infinite-dimensional versions of Theorem 2. As we shall now explain, this is for instance the case for collinear positive charges and, more generally, for essentially any bounded discrete set of positively charged particles with finite total charge for which equilibrium points are known to exist. Let $\{a_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers and let $\{z_i\}_{i \in \mathbb{N}}$ be a sequence of distinct complex numbers satisfying the conditions

$$(3.14) \quad a_i > 0 \text{ and } |z_i| < \rho \text{ for } i \in \mathbb{N}, \quad \sum_{i=1}^{\infty} a_i = 1, \quad \lim_{i \rightarrow \infty} z_i = \rho,$$

where ρ is a fixed positive number. Using the same electrostatic interpretation as before – that is, placing a charge a_i at z_i for $i \in \mathbb{N}$ – we get the resulting force

$$(3.15) \quad f(z) = \sum_{i=1}^{\infty} \frac{a_i}{z - z_i}$$

and an associated logarithmic potential given by

$$U(z) = \sum_{i=1}^{\infty} a_i \log \left| 1 - \frac{z}{z_i} \right|.$$

Set $\mathbb{D}(\rho) = \{z \in \mathbb{C} : |z| < \rho\}$ and $\mathbb{D}_f(\rho) = \mathbb{D}(\rho) \setminus \{z_i\}_{i \in \mathbb{N}}$. By (3.14) one has

$$\sum_{i=1}^{\infty} \frac{a_i}{|z_i|} < \infty,$$

so that the series in the right-hand side of (3.15) converges absolutely for all $z \in \mathbb{D}_f(\rho)$ and thus f is a meromorphic function in $\mathbb{D}(\rho)$. One can see in a similar fashion that U is subharmonic in $\mathbb{D}(\rho)$. We have the following analog of Theorem 2.

Theorem 6. *If $\{z_i\}_{i \in \mathbb{N}}$ is a real sequence satisfying (3.14), then the function f defined by (3.15) has an infinite discrete set of real zeros $\{w_j\}_{j \in \mathbb{N}}$ and the sequences $\{z_i\}_{i \in \mathbb{N}}$ and $\{w_j\}_{j \in \mathbb{N}}$ interlace on the real axis. Moreover, there exists an infinite matrix $S = (s_{ij})_{i,j=1}^{\infty}$ such that for all $(i, j) \in \mathbb{N}^2$ one has*

$$s_{ij} \geq 0, \quad \sum_{k=1}^{\infty} s_{ik} = 1, \quad \sum_{l=1}^{\infty} s_{lj} = 1 - a_j, \quad w_i = \sum_{k=1}^{\infty} s_{ik} z_k.$$

In particular, for any $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$ the inequality

$$\sum_{j=1}^{\infty} \Phi(\lambda w_j + \mu) \leq \sum_{i=1}^{\infty} (1 - a_i) \Phi(\lambda z_i + \mu)$$

holds for any convex function $\Phi : \mathbb{C} \rightarrow \mathbb{R}$.

Our last main result is a natural analog of Theorem 6 for all bounded discrete configurations of positive charges for which equilibrium points are known to exist.

Theorem 7. *Let $\{z_i\}_{i \in \mathbb{N}}$ be a sequence of distinct complex numbers satisfying (3.14) and let f be the meromorphic function in $\mathbb{D}(\rho)$ defined by (3.15). If f has a non-empty (discrete) set of zeros $\{w_j\}_{j \in \mathbb{N}}$ then the inequality*

$$\sum_{j=1}^{\infty} \Phi(\lambda w_j + \mu) \leq \sum_{i=1}^{\infty} (1 - a_i) \Phi(\lambda z_i + \mu)$$

holds for all $\lambda \in \mathbb{C} \setminus \{0\}$, $\mu \in \mathbb{C}$ and any nonnegative convex function $\Phi : \mathbb{C} \rightarrow \mathbb{R}$.

Remark 14. The fact that inequality (4.18) in §4.3 may be strict explains why the inequality in Theorem 7 holds in general only for *nonnegative* convex functions. The connection between majorization and such inequalities for nonnegative convex functions was first found in [16].

Remark 15. Theorem 7 is reminiscent of Kato's bound for the variation of discrete spectra of Hilbert space selfadjoint operators under compact perturbations [21].

4. PROOF OF MAIN RESULTS

4.1. Proof of Theorems 2–3. We shall use various methods from multilinear algebra and matrix analysis. These require a few notations and preliminary results, and we refer to [1] and [27] for background material.

Notation 4. Given $m \in \mathbb{N}$ and $1 \leq k \leq m$ let

$$Q_{k,m} = \{\mathbf{i} = (i_1, \dots, i_k) \mid 1 \leq i_1 < \dots < i_k \leq m\}.$$

If $B = (b_{ij})$ is an $m \times m$ matrix and $\mathbf{i}, \mathbf{j} \in Q_{k,m}$ we denote by

$$B(\mathbf{i}, \mathbf{j}) = B \begin{pmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{pmatrix}$$

the $k \times k$ submatrix of B lying in rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$. Below we shall always assume that the elements of $Q_{k,m}$ are arranged in lexicographic order, i.e., if $\mathbf{i}$ and $\mathbf{j}$ are distinct k -tuples in $Q_{k,m}$, then $\mathbf{i} \geq \mathbf{j}$ if the first non-zero term in the sequence $i_1 - j_1, \dots, i_k - j_k$ is positive.

We fix an n -dimensional complex Hilbert space $\mathcal{H}$ with unitarily invariant scalar product $\langle \cdot, \cdot \rangle$ and identity operator $I = I_{\mathcal{H}} = I_n \in L(\mathcal{H})$, where $L(\mathcal{H})$ is the set of all linear operators on $\mathcal{H}$. Let $A \in L(\mathcal{H})$ be a normal operator with spectrum $\Sigma(A) = \{z_1, \dots, z_n\}$ and choose an orthonormal basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ of $\mathcal{H}$ such that $A\mathbf{e}_i = z_i\mathbf{e}_i$ for $1 \leq i \leq n$. Define a unit vector

$$(4.1) \quad \mathbf{v}_n = \sum_{i=1}^n \sqrt{a_i} \mathbf{e}_i$$

and let P be the orthogonal projection on the subspace $\mathcal{K} := \mathbf{v}_n^\perp$ of $\mathcal{H}$. The operator

$$A' = PAP|_{\mathcal{K}} \in L(\mathcal{K})$$

is called the P -compression of A [10]. Recall the function f defined in (3.2). The following statement may be found in, e.g., [26, formula (3.3)] and [28, Lemma 2.2].

Lemma 3. *With the above notations one has*

$$\langle (A - zI_{\mathcal{H}})^{-1} \mathbf{v}_n, \mathbf{v}_n \rangle = \frac{\det(A' - zI_{\mathcal{K}})}{\det(A - zI_{\mathcal{H}})} = -f(z) \text{ for } z \in \mathbb{C} \setminus \Sigma(A).$$

Proof. Let $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ be an orthonormal basis of $\mathcal{K}$. The matrix representation of A' in the basis $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ is given by the $(n-1) \times (n-1)$ upper left-hand principal submatrix of the matrix representation of A in the orthonormal basis $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n)$ of $\mathcal{H}$. For any $z \in \mathbb{C} \setminus \Sigma(A)$ the (n, n) entry of the matrix representation of the (normal) operator $(A - zI_{\mathcal{H}})^{-1}$ in the basis $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n)$ is given on the one hand by

$$\langle (A - zI_{\mathcal{H}})^{-1} \mathbf{v}_n, \mathbf{v}_n \rangle = \left\langle \sum_{i=1}^n \frac{\langle \mathbf{v}_n, \mathbf{e}_i \rangle}{z_i - z} \mathbf{e}_i, \sum_{i=1}^n \langle \mathbf{v}_n, \mathbf{e}_i \rangle \mathbf{e}_i \right\rangle = -f(z).$$

On the other hand, by Cramer's rule the (n, n) entry of the matrix representation of $(A - zI_{\mathcal{H}})^{-1}$ in the basis $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n)$ of $\mathcal{H}$ is given by the cofactor of the (n, n) entry of the matrix representation of $(A - zI_{\mathcal{H}})$ in the same basis. Thus

$$\langle (A - zI_{\mathcal{H}})^{-1} \mathbf{v}_n, \mathbf{v}_n \rangle = \frac{\det(A' - zI_{\mathcal{K}})}{\det(A - zI_{\mathcal{H}})}$$

for $z \in \mathbb{C} \setminus \Sigma(A)$, which proves the lemma. $\square$

Proof of Theorem 2. By Lemma 3 the zeros $w_1, \dots, w_{n-1}$ of f are the same as the eigenvalues of the P -compression A' . Let $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ be an orthonormal basis of $\mathcal{K}$ that triangularizes A' . Then

$$(4.2) \quad w_j = \mathbf{v}_j^* A' \mathbf{v}_j = \langle A' \mathbf{v}_j, \mathbf{v}_j \rangle = \langle A \mathbf{v}_j, \mathbf{v}_j \rangle = \sum_{i=1}^n z_i |\langle \mathbf{v}_j, \mathbf{e}_i \rangle|^2$$

since $\mathbf{v}_j = \sum_{i=1}^n \langle \mathbf{v}_j, \mathbf{e}_i \rangle \mathbf{e}_i$ for $1 \leq j \leq n-1$. Define the $(n-1) \times n$ matrix $R = (r_{ij})$ by setting $r_{ij} = |\langle \mathbf{v}_i, \mathbf{e}_j \rangle|^2$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Using the fact that $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n)$ is an orthonormal basis of $\mathcal{H}$ one can easily check that

$$\begin{aligned} \sum_{j=1}^n r_{ij} &= \sum_{j=1}^n |\langle \mathbf{v}_i, \mathbf{e}_j \rangle|^2 = \|\mathbf{v}_i\|^2 = 1, \quad 1 \leq i \leq n-1, \\ \sum_{i=1}^{n-1} r_{ij} &= \sum_{i=1}^{n-1} |\langle \mathbf{v}_i, \mathbf{e}_j \rangle|^2 = \|\mathbf{e}_j\|^2 - |\langle \mathbf{e}_j, \mathbf{v}_n \rangle|^2 = 1 - a_j, \quad 1 \leq j \leq n, \end{aligned}$$

so that $R \in \mathbb{M}_{n-1, n}^{rs}$ and $\mathbf{b} = \mathbf{a}R$. Note that by (4.2) one has

$$(w_1, \dots, w_{n-1})^T = R(z_1, \dots, z_n)^T.$$

Hence $(W(1, 0), \mathbf{a}) \prec (Z(1, 0), \mathbf{b})$, which completes the proof of the theorem in the case $\lambda = 1$ and $\mu = 0$. The general case follows from this one by composing f with a non-singular affine transformation of the plane. $\square$

Remark 16. The compression of a normal operator to a hyperplane is not necessarily normal. For instance, Fan and Pall [15] showed that principal submatrices of a given irreducible normal matrix B are normal if and only if B has collinear eigenvalues.

Remark 17. The proof of Theorem 2 given above is a slight modification of the corresponding arguments in [26] and [28].

We next prove Theorem 3. Clearly, conditions (ii), (iii) and (iv) in this theorem are equivalent by Definition 1 and Theorem 1. Let us first show that (v) $\Rightarrow$ (i). Set

$$q(z) = \prod_{j=1}^{n-1} (z - w_j), \quad p_i(z) = \prod_{\substack{j=1 \\ j \neq i}}^n (z - z_j), \quad 1 \leq i \leq n,$$

and note that condition (i) is equivalent to

$$q^{(n-1-k)}(0) = \sum_{i=1}^n a_i p_i^{(n-1-k)}(0), \quad 1 \leq k \leq n-1.$$

Elementary computations show that

$$\begin{aligned} q^{(n-1-k)}(0) &= (-1)^k (n-1-k)! \sum_{1 \leq r_1 < \dots < r_k \leq n-1} \prod_{j=1}^k w_{r_j}, \\ \sum_{i=1}^n a_i p_i^{(n-1-k)}(0) &= (-1)^k (n-1-k)! \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(1 - \sum_{i=1}^k a_{s_i}\right) \prod_{i=1}^k z_{s_i}, \end{aligned}$$

for $1 \leq k \leq n-1$. Thus condition (i) amounts to saying that if $1 \leq k \leq n-1$, then

$$(4.3) \quad \sum_{1 \leq r_1 < \dots < r_k \leq n-1} \prod_{j=1}^k w_{r_j} = \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(1 - \sum_{i=1}^k a_{s_i}\right) \prod_{i=1}^k z_{s_i}.$$

The fact that (v) $\Rightarrow$ (i) is now a consequence of the following lemma.

Lemma 4. *If condition (v) in Theorem 3 holds, then (4.3) is true for $1 \leq k \leq n-1$.*

Proof. Using, e.g., condition (iii) in Theorem 3 with $\lambda = 1$ and $\mu = 0$ one gets

$$\mathbf{a}^{[k]} W_k^{[k]}(1, 0)^T = \mathbf{b}^{[k]} Z_k^{[k]}(1, 0)^T, \quad 1 \leq k \leq n-1,$$

which is the same as the desired conclusion. $\square$

We shall also need the following extension to normal matrices of a well-known result of Schur for Hermitian matrices (cf. [1, 27]); see, e.g., [26, Proposition 3.9 (a)].

Lemma 5. *If $T = (t_{ij})$ is a normal $n \times n$ matrix with eigenvalues $\tau_1, \dots, \tau_n$, then*

$$(t_{11}, \dots, t_{nn}) \prec (\tau_1, \dots, \tau_n),$$

i.e., $(t_{11}, \dots, t_{nn})^T = S(\tau_1, \dots, \tau_n)^T$ for some doubly stochastic $n \times n$ matrix S .

Proof. Let U be a unitary matrix such that $T = U \text{diag}(\tau_1, \dots, \tau_n) U^*$ and set $S = U \circ \bar{U}$, where $\circ$ denotes the Hadamard-Schur (entrywise) product of matrices and $\bar{U}$ is the complex conjugate of U . Then $(t_{11}, \dots, t_{nn})^T = S(\tau_1, \dots, \tau_n)^T$. It remains to note that S is a unitarily stochastic, hence a doubly stochastic, matrix. $\square$

To complete the proof of Theorem 3 it will be convenient to work with certain suitably chosen matrix representations of the operators A and A' defined above. As in the proof of Theorem 2 let us fix an orthonormal basis $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ of $\mathcal{K}$ in which the matrix representation of the P -compression A' of A is upper triangular. For simplicity, denote the matrix representation of A in the basis $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n)$ again by A and let A_i , $1 \leq i \leq n$, be the degeneracy one principal submatrix of A obtained by deleting its i -th row and i -th column. Then clearly $A_n = A'$, and so by Lemma 3 the spectrum of A_n is $\Sigma(A_n) = \{w_1, \dots, w_{n-1}\}$, where $\Sigma(A_n)$ is viewed as a multiset whose elements occur as many times as their algebraic multiplicities as eigenvalues of A_n . Thus A is an $n \times n$ normal matrix with $\Sigma(A) = \{z_1, \dots, z_n\}$ such that

$$(4.4) \quad A = \begin{pmatrix} A_n & * \\ * & \zeta \end{pmatrix},$$

where A_n is upper triangular with $\Sigma(A_n) = \{w_1, \dots, w_{n-1}\}$ and $\zeta = \langle A\mathbf{v}_n, \mathbf{v}_n \rangle = \sum_{i=1}^n a_i z_i$ is the $(a_1, \dots, a_n)$ -barycenter of $Z(1, 0)$ (cf. Remark 8). Note that since the w_j appear on the diagonal of A_n Lemma 5 actually provides a new proof of Theorem 2. Let $U = (u_{ij})$ be the (unitary) change of basis matrix from $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ to $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ so that we may write

$$(4.5) \quad A = U \text{diag}(z_1, \dots, z_n) U^*.$$

Proof of Theorem 3. Let us first show that if (3.6) holds and $k \in \{1, \dots, n-1\}$, then one can find an $\binom{n-1}{k} \times \binom{n}{k}$ row stochastic matrix R_k such that

$$(4.6) \quad W_k^{[k]}(1, 0)^T = R_k Z_k^{[k]}(1, 0)^T \text{ and } \mathbf{b}^{[k]} = \mathbf{a}^{[k]} R_k.$$

This is clear for $k = 1$ by Theorem 2 and so we may assume that $k \geq 2$. Let

$$A^{(k)} = (\det A(\mathbf{i}, \mathbf{j}))_{\mathbf{i}, \mathbf{j} \in Q_{k,n}}$$

be the k -th compound matrix of A acting on the k -th Grassmann power $\bigwedge^k \mathcal{H}$ (recall that the elements of $Q_{k,n}$ and therefore also the $\binom{n}{k} \times \binom{n}{k}$ determinants

appearing in $A^{(k)}$ are arranged in lexicographic order). The Binet-Cauchy formula for compound matrices [27, Theorem 19.F.2] and (4.5) imply that

$$A^{(k)} = U^{(k)} \operatorname{diag}(z_1, \dots, z_n)^{(k)} U^{*(k)},$$

which by an argument similar to the proof of Lemma 5 leads to

$$(4.7) \quad (\det A(\mathbf{i}, \mathbf{i}))_{\mathbf{i} \in Q_{k,n}}^T = S_k Z_k^{[k]}(1, 0)^T, \quad S_k = U^{(k)} \circ \bar{U}^{(k)},$$

where $\circ$ denotes as before the Hadamard-Schur product. Note that S_k is an $\binom{n}{k} \times \binom{n}{k}$ doubly stochastic matrix since $U^{(k)}$ is unitary. Clearly, the diagonal elements of $A^{(k)}$ corresponding to the $k \times k$ principal minors of A_n are of the form $\det A(\mathbf{i}, \mathbf{i})$, $\mathbf{i} \in Q_{k,n-1}$. There are a total of $\binom{n-1}{k}$ such diagonal elements, and these must actually coincide with the coordinates of the vector $W_k^{[k]}(1, 0)^T$ since A_n is upper triangular. Let R_k be the $\binom{n-1}{k} \times \binom{n}{k}$ row stochastic matrix obtained from S_k by deleting all elements $S_k(\mathbf{i}, \mathbf{j})$ with $\mathbf{i} \in Q_{k,n} \setminus Q_{k,n-1}$. We show that R_k satisfies (4.6). For this one has to prove that for any $\mathbf{j} = (j_1, \dots, j_k) \in Q_{k,n}$ the identity

$$\sum_{1 \leq i_1 < \dots < i_{k-1} \leq n-1} S_k \begin{pmatrix} i_1, \dots, i_{k-1}, n \\ j_1, \dots, j_k \end{pmatrix} = \sum_{l=1}^k a_{j_l}$$

holds, which is equivalent to

$$(4.8) \quad \sum_{1 \leq i_1 < \dots < i_{k-1} \leq n-1} \left| \det U \begin{pmatrix} i_1, \dots, i_{k-1}, n \\ j_1, \dots, j_k \end{pmatrix} \right|^2 = \sum_{l=1}^k a_{j_l}$$

or

$$(4.9) \quad \sum_{\mathbf{i} \in Q_{k,n-1}} |\det U(\mathbf{i}, \mathbf{j})|^2 = 1 - \sum_{l=1}^k a_{j_l}.$$

By (4.5) one has $u_{nj} = \sqrt{a_j}$, $1 \leq j \leq n$, so that by expanding each determinant with respect to the last row we get

$$\det U \begin{pmatrix} i_1, \dots, i_{k-1}, n \\ j_1, \dots, j_k \end{pmatrix} = \sum_{m=1}^k (-1)^{k+m} \sqrt{a_{j_m}} \det U \begin{pmatrix} i_1, \dots, i_{k-1} \\ j_1, \dots, \widehat{j_m}, \dots, j_k \end{pmatrix}$$

and thus

$$\begin{aligned} & \left| \det U \begin{pmatrix} i_1, \dots, i_{k-1}, n \\ j_1, \dots, j_k \end{pmatrix} \right|^2 \\ &= \sum_{r,s=1}^k (-1)^{r+s} \sqrt{a_{j_r} a_{j_s}} \det U \begin{pmatrix} i_1, \dots, i_{k-1} \\ j_1, \dots, \widehat{j_r}, \dots, j_k \end{pmatrix} \det \bar{U} \begin{pmatrix} i_1, \dots, i_{k-1} \\ j_1, \dots, \widehat{j_s}, \dots, j_k \end{pmatrix} \end{aligned}$$

for $1 \leq i_1 < \dots < i_{k-1} \leq n-1$. Therefore

$$\sum_{1 \leq i_1 < \dots < i_{k-1} \leq n-1} \left| \det U \begin{pmatrix} i_1, \dots, i_{k-1}, n \\ j_1, \dots, j_k \end{pmatrix} \right|^2 = \sum_{r,s=1}^k (-1)^{r+s} \sqrt{a_{j_r} a_{j_s}} \alpha_{r,s}(\mathbf{j}),$$

where

$$\alpha_{r,s}(\mathbf{j}) = \sum_{1 \leq i_1 < \dots < i_{k-1} \leq n-1} \det U \begin{pmatrix} i_1, \dots, i_{k-1} \\ j_1, \dots, \widehat{j_r}, \dots, j_k \end{pmatrix} \det \bar{U} \begin{pmatrix} i_1, \dots, i_{k-1} \\ j_1, \dots, \widehat{j_s}, \dots, j_k \end{pmatrix}.$$

Let us consider the $k \times (n-1)$ matrix

$$M = U \begin{pmatrix} 1, \dots, n-1 \\ j_1, \dots, j_k \end{pmatrix}^T$$

and the $k \times k$ matrix $B = (b_{ij}) = MM^*$ as well as the $k \times k$ matrix given by its $k-1$ Grassmann power $B^{(k-1)} = (b_{ij}^{(k-1)})$. The Binet-Cauchy formula again implies that

$$B^{(k-1)} = M^{(k-1)} M^{*(k-1)}$$

from which we deduce that

$$b_{k+1-r, k+1-s}^{(k-1)} = \alpha_{r,s}(\mathbf{j}), \quad 1 \leq r, s \leq k.$$

Thus the left-hand side of (4.8) equals

$$\sum_{r,s=1}^k (-1)^{r+s} \sqrt{a_{j_r} a_{j_s}} b_{k+1-r, k+1-s}^{(k-1)}$$

and by further manipulating this expression we obtain

$$\sum_{1 \leq i_1 < \dots < i_{k-1} \leq n-1} \left| \det U \begin{pmatrix} i_1, \dots, i_{k-1}, n \\ j_1, \dots, j_k \end{pmatrix} \right|^2 = \sum_{\mathbf{i} \in Q_{k-1,k}} \det B(\mathbf{i}, \mathbf{i}) - k \det B$$

hence

$$\sum_{\mathbf{i} \in Q_{k,n-1}} |\det U(\mathbf{i}, \mathbf{j})|^2 = 1 - \sum_{\mathbf{i} \in Q_{k-1,k}} \det B(\mathbf{i}, \mathbf{i}) + k \det B.$$

To show that the right-hand side of the above identity equals the right-hand side of (4.9) we proceed as follows. Define the vector $\mathbf{u} = (\sqrt{a_{j_1}}, \dots, \sqrt{a_{j_k}})$. Since U is unitary an easy computation shows that $B = I_k - \mathbf{u}^T \mathbf{u}$, so that B depends only on the numbers a_{j_l} , $1 \leq l \leq k$. It follows that the expression

$$\sum_{1 \leq i_1 < \dots < i_k \leq n-1} \left| \det U \begin{pmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{pmatrix} \right|^2 = \sum_{\mathbf{i} \in Q_{k,n-1}} |\det U(\mathbf{i}, \mathbf{j})|^2$$

also depends exclusively on the numbers a_{j_l} , $1 \leq l \leq k$, and is therefore independent of the z_i , $1 \leq i \leq n$. On the other hand, by summing all coordinates corresponding to $\mathbf{i} \in Q_{k,n-1}$ in each side of (4.7) we get

$$\begin{aligned} (4.10) \quad & \sum_{1 \leq r_1 < \dots < r_k \leq n-1} \prod_{j=1}^k w_{r_j} \\ &= \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(\sum_{1 \leq i_1 < \dots < i_k \leq n-1} \left| \det U \begin{pmatrix} i_1, \dots, i_k \\ s_1, \dots, s_k \end{pmatrix} \right|^2 \right) \prod_{i=1}^k z_{s_i}. \end{aligned}$$

Obviously, (3.6) defines the w_j , $1 \leq j \leq n-1$, as continuous functions of the z_i , $1 \leq i \leq n$. By letting $z_i \rightarrow 0$ for $i \in \{1, \dots, n\} \setminus \{j_1, \dots, j_k\}$ and $z_{j_l} \rightarrow 1$ for $1 \leq l \leq k$ in both (4.10) and (4.3) and comparing coefficients in the resulting identities we get (4.9), which proves (4.6).

Finally, to prove the last statement of the theorem it is enough to consider the function $f(\lambda^{-1}(z - \mu))$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. The zeros and poles of this function are the eigenvalues of $\lambda A_n + \mu$ and $\lambda A + \mu$, respectively. These matrices

satisfy relations similar to (4.4) and (4.5), while the unitary matrix U is unchanged. One may therefore use the same arguments as above to get (3.7). $\square$

Remark 18. Modulo some technical modifications that have to deal with unequal weights, the arguments given in the proof of Theorem 3 are the same as those given by S. Malamud in his proof of [26, Theorem 4.7].

Remark 19. An interesting byproduct of the proof of Theorem 3 is the following identity for certain types of M -matrices. Let $m \in \mathbb{N}$, $\mathbf{x} = (x_1, \dots, x_m) \in (0, 1)^m$ with $\|\mathbf{x}\|^2 = \sum_{i=1}^m x_i^2 < 1$ and define the $m \times m$ matrix $X = I_m - \mathbf{x}^T \mathbf{x}$. Clearly, $\mathbf{x}^T \mathbf{x}$ is a rank one matrix whose eigenvalues 0 (with multiplicity $m - 1$) and $\|\mathbf{x}\|^2$ have modulus less than 1, so that X is an M -matrix (see, e.g., [27]). By the arguments in the last part of the proof of Theorem 3 we see that

$$\sum_{\mathbf{i} \in Q_{m-1, m}} \det X(\mathbf{i}, \mathbf{i}) - m \det X = \operatorname{tr}(I_m - X),$$

both sides being equal to $\|\mathbf{x}\|^2$.

Recall that if C is an $n \times n$ matrix and $1 \leq k \leq n$ one defines the k -th additive compound matrix $\Delta_k(C)$ as the $\binom{n}{k} \times \binom{n}{k}$ given by

$$\Delta_k(C) = \left. \frac{d}{dt} (I_n + tC)^{(k)} \right|_{t=0}.$$

Equivalently, $\Delta_k(C)$ is the coefficient matrix of t in the expansion $(I_n + tC)^{(k)} = I_n + t\Delta_k(C) + \mathcal{O}(t^2)$. As an operator, this is the same as the restriction to $\bigwedge^k \mathcal{H}$ of the operator

$$C \otimes I_n \otimes \cdots \otimes I_n + I_n \otimes C \otimes I_n \otimes \cdots \otimes I_n + \cdots + I_n \otimes I_n \otimes \cdots \otimes C$$

defined on $\otimes^k \mathcal{H}$. It is well known that if C has eigenvalues $\lambda_1, \dots, \lambda_n$, then the eigenvalues of $\Delta_k(C)$ are $\lambda_{i_1} + \dots + \lambda_{i_k}$, $1 \leq i_1 < \dots < i_k \leq n$ (cf., e.g., [27]).

Proof of Corollary 4. Using (4.5) and arguing as in the first part of the proof of Theorem 3 we arrive at

$$\left. \frac{d}{dt} (I_n + tA)^{(k)} \right|_{t=0} = U^{(k)} \left. \frac{d}{dt} \left(\operatorname{diag}(1 + tz_1, \dots, 1 + tz_n)^{(k)} \right) \right|_{t=0} U^{*(k)}.$$

Since A_n is upper triangular the diagonal elements of $(I_n + tA)^{(k)}$ corresponding to the $k \times k$ principal minors indexed by multi-indices $\mathbf{i} = (i_1, \dots, i_k) \in Q_{k, n-1}$ have the form $\prod_{j=1}^k (1 + tw_{i_j})$. The conclusion now follows from the obvious identities

$$\left. \frac{d}{dt} \prod_{j=1}^k (1 + tw_{i_j}) \right|_{t=0} = \sum_{j=1}^k w_{i_j}, \quad \left. \frac{d}{dt} \prod_{l=1}^k (1 + tz_{s_l}) \right|_{t=0} = \sum_{l=1}^k z_{s_l}$$

combined with the arguments in the second part of the proof of Theorem 3. $\square$

Corollary 4 may alternatively be proved by using (3.7) with $\lambda = t$ and $\mu = 1$ and differentiating with respect to t . Using this alternative method we can actually prove Corollary 3.

Proof of Corollary 3. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$. By (3.7) and Remark 11 there exists an $\binom{n-1}{k} \times \binom{n}{k}$ row stochastic matrix R_k such that the relations

$$W_k^{[k]}(t\lambda, t\mu + 1)^T = R_k Z_k^{[k]}(t\lambda, t\mu + 1)^T \quad \text{and} \quad \mathbf{b}^{[k]} = \mathbf{a}^{[k]} R_k$$

hold for all $t \in \mathbb{C}$. Thus

$$\left. \frac{d^m}{dt^m} W_k^{[k]}(t\lambda, t\mu + 1)^T \right|_{t=0} = R_k \left. \frac{d^m}{dt^m} Z_k^{[k]}(t\lambda, t\mu + 1)^T \right|_{t=0}$$

for $m \in \{1, \dots, k\}$, which is the same as (3.8). $\square$

Remark 20. The assumption that the points z_i , $1 \leq i \leq n$, are *distinct* that was made at the beginning of §3.1 was not actually used in the above proofs. However, this condition is natural when viewing – as we did – these points as charged particles generating a resultant electrostatic force given by (3.2). Clearly, the latter can be appropriately rewritten even in the case of multiple points.

4.2. Proof of Theorems 4–5. The inequality $h(\mathcal{W}, \mathcal{Z}) \leq h(\mathcal{W}_e, \mathcal{Z})$ is trivially true since $\mathcal{W} \subseteq \mathcal{W}_e$. To prove the interesting part of Theorem 4 we make use of a result on spectral variations of normal matrices which is usually referred to as the Bauer-Fike theorem (cf., e.g., [1, Theorem VI.3.3]).

Lemma 6. *Let A and B be $n \times n$ complex matrices. If A is normal, then*

$$h(\Sigma(B), \Sigma(A)) \leq \|A - B\|,$$

where h denotes the relative Hausdorff distance from $\Sigma(B)$ to $\Sigma(A)$.

Recall the orthonormal basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ of $\mathcal{H}$ where the matrix representation of the normal operator A with $\Sigma(A) = \mathcal{Z}$ is given by (4.4), that is,

$$(4.11) \quad A = \begin{pmatrix} A_n & \mathbf{x} \\ \mathbf{y}^* & \zeta \end{pmatrix}.$$

In the above decomposition A_n is an $(n-1) \times (n-1)$ upper triangular matrix with $\Sigma(A_n) = \mathcal{W}$, $\zeta = \langle A\mathbf{v}_n, \mathbf{v}_n \rangle = \sum_{i=1}^n a_i z_i$, and $\mathbf{x}, \mathbf{y} \in \mathbb{C}^{n-1}$. Set $B = A_n \oplus \zeta$. An easy computation now shows that

$$\begin{aligned} \|A - B\|^2 &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \sum_{i=1}^{n-1} |\langle A\mathbf{v}_n, \mathbf{v}_i \rangle|^2 = \|A\mathbf{v}_n\|^2 - |\langle A\mathbf{v}_n, \mathbf{v}_n \rangle|^2 \\ &= \sum_{i=1}^n a_i |z_i|^2 - |\zeta|^2 = \sigma_2(\mathbf{z}; \mathbf{a})^2, \end{aligned}$$

which combined with Lemma 6 yields the inequality stated in Theorem 4.

To prove Theorem 5 note first that by Theorem 4 it remains to check that $h(\mathcal{Z}, \mathcal{W}_e) \leq \sigma_2(\mathbf{z}; \mathbf{a})$. The latter inequality is invariant under nonsingular affine transformations of the complex plane, and it is therefore enough to show that $h(\mathcal{Z}, \mathcal{W}_e) \leq \sigma_2(\mathbf{z}; \mathbf{a})$ whenever $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$. Since in this case A is a Hermitian operator we get from (4.11) that $\mathbf{x} := (x_1, \dots, x_{n-1})^t = \mathbf{y}$ and $A_n = \text{diag}(w_1, \dots, w_{n-1})$. Suppose z_i is such that

$$(4.12) \quad \min_{1 \leq j \leq n-1} |z_i - w_j| > \sigma_2(\mathbf{z}; \mathbf{a})$$

and let $\mathbf{u} = (u_1, \dots, u_n)^t \in \mathbb{C}^n$ be a unit eigenvector of A with eigenvalue z_i . Then by (4.11) we may rewrite the identity $A\mathbf{u} = z_i\mathbf{u}$ as

$$(4.13) \quad (\zeta - z_i)u_n = - \sum_{j=1}^{n-1} \bar{x}_j u_j \text{ and } (w_j - z_i)u_j = x_j u_n, \quad 1 \leq j \leq n-1.$$

Note that $\sigma_2(\mathbf{z}; \mathbf{a}) > 0$ since $z_1, \dots, z_n$ are assumed to be pairwise distinct. Moreover, the fact that $\|\mathbf{u}\| = 1$ combined with (4.13) implies that $u_n \neq 0$. From (4.12) and the last $n - 1$ identities in (4.13) we get

$$|u_j| \leq \sigma_2(\mathbf{z}; \mathbf{a})^{-1} |x_j| |u_n|, \quad 1 \leq j \leq n - 1,$$

and then using the first relation in (4.13) we deduce that

$$|\zeta - z_i| \leq \sigma_2(\mathbf{z}; \mathbf{a})^{-1} \|\mathbf{x}\|^2 = \sigma_2(\mathbf{z}; \mathbf{a}).$$

This shows that $h(\{z_i\}, \mathcal{W}_e) \leq \sigma_2(\mathbf{z}; \mathbf{a})$, which completes the proof of Theorem 5.

4.3. Proof of Theorems 6–7. It is clear that up to a rescaling we may assume that $\rho = 1$ in both Theorem 6 and Theorem 7. Below we shall use the Hilbert space of square summable complex sequences $\mathcal{H} = l^2(\mathbb{N})$ with standard scalar product $\langle \cdot, \cdot \rangle$ and standard orthonormal basis $\{\mathbf{e}_i\}_{i \in \mathbb{N}}$.

Proof of Theorem 6. We may assume wlog that $\{z_i\}_{i \in \mathbb{N}}$ is an increasing sequence. The fact that f has infinitely many zeros $\{w_j\}_{j \in \mathbb{N}}$ that interlace the poles $\{z_i\}_{i \in \mathbb{N}}$ on the real axis follows by noticing that all zeros of f are necessarily real and that f is strictly decreasing on each interval (z_i, z_{i+1}) , $i \in \mathbb{N}$. Alternatively, this may also be viewed as a consequence of the operator theoretic interpretation given below combined with Hochstadt's interlacing theorem for the eigenvalues of rank one perturbations of compact selfadjoint Hilbert space operators [20].

Let A denote the unique bounded operator on $\mathcal{H}$ satisfying $A\mathbf{e}_i = (1 - z_i)\mathbf{e}_i$, $i \in \mathbb{N}$. Note that A is injective, selfadjoint and compact since the z_i are real and $z_i \rightarrow 1$ as $i \rightarrow \infty$. By assumption one has $\sum_{i=1}^{\infty} a_i = 1$, and so we may define a unit vector $\mathbf{v} \in \mathcal{H}$ in the following way:

$$(4.14) \quad \mathbf{v} = \sum_{i=1}^{\infty} \sqrt{a_i} \mathbf{e}_i.$$

Set $\mathcal{K} = \mathbf{v}^\perp$ and let P be the orthoprojection on $\mathcal{K}$. Recall from [22, §IV.6] that the Weinstein-Aronszajn determinant of the second kind associated with A and P is given by $\langle (A - zI)^{-1} \mathbf{v}, \mathbf{v} \rangle$. A simple computation shows that

$$(4.15) \quad \langle (A - zI)^{-1} \mathbf{v}, \mathbf{v} \rangle = f(1 - z), \quad z \in \mathbb{C} \setminus \{1 - z_i\}_{i \in \mathbb{N}}.$$

Let us now consider the compression A' of A to $\mathcal{K}$, that is, the operator on $\mathcal{K}$ given by $A' = PAP|_{\mathcal{K}}$. Note that A' is itself a compact selfadjoint operator. We claim that the point spectrum of A' consists precisely of the zeros of $f(1 - z)$.

Lemma 7. *The set of eigenvalues of A' is $\{1 - w_j\}_{j \in \mathbb{N}}$.*

Proof. If $\mathbf{u} \in \mathcal{K}$ is an eigenvector of A' with eigenvalue $\zeta \in \mathbb{C}$ (actually $\mathbb{R}$), then

$$(4.16) \quad (A - \zeta I)\mathbf{u} = (I - P)A\mathbf{u} + (A' - \zeta I)\mathbf{u} = \langle A\mathbf{u}, \mathbf{v} \rangle \mathbf{v}.$$

From (4.16) we see that $\langle A\mathbf{u}, \mathbf{v} \rangle \neq 0$ since otherwise $\mathbf{u}$ would be an eigenvector of A , say $\mathbf{u} = \mathbf{e}_i$ for some $i \in \mathbb{N}$, which would then lead to the contradiction $\langle A\mathbf{u}, \mathbf{v} \rangle = \langle A\mathbf{e}_i, \mathbf{v} \rangle = (1 - z_i)\sqrt{a_i} \neq 0$. By (4.15) and (4.16) we get

$$f(1 - \zeta) = \langle (A - \zeta I)^{-1} \mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{u}, \mathbf{v} \rangle^{-1} \langle \mathbf{u}, \mathbf{v} \rangle = 0$$

so that $1 - \zeta = w_j$ for some $j \in \mathbb{N}$. Conversely, let w be such that $f(1 - w) = 0$, that is, $\langle (A - wI)^{-1} \mathbf{v}, \mathbf{v} \rangle = 0$, and set $\mathbf{u} = (A - wI)^{-1} \mathbf{v}$. Then clearly $\mathbf{u} \neq 0$ and one has

$$\mathbf{v} = (A - wI)\mathbf{u} = \langle A\mathbf{u}, \mathbf{v} \rangle \mathbf{v} + (A' - wI)\mathbf{u}.$$

Since $(A' - wI)\mathbf{u} \in \mathcal{K}$ and $\mathbf{v} \in \mathcal{K}^\perp$ it follows that

$$(A' - wI)\mathbf{u} = (1 - \langle A\mathbf{u}, \mathbf{v} \rangle)\mathbf{v} = 0.$$

Thus $\mathbf{u}$ is an eigenvector of A' with eigenvalue w . $\square$

It is not difficult to see that A' is injective. Indeed, if $0 \neq \mathbf{u} \in \mathcal{K}$ is such that $A'\mathbf{u} = 0$, then from $PA\mathbf{u} = 0$ and the fact that A is injective we get $A\mathbf{u} = \alpha\mathbf{v}$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. Hence

$$0 = \langle \mathbf{u}, \mathbf{v} \rangle = \alpha \langle A^{-1}\mathbf{v}, \mathbf{v} \rangle = \alpha f(1),$$

which is a contradiction since the assumptions of the theorem (more specifically, condition (3.14) with $\rho = 1$) imply that $f(1) > 0$.

By the spectral theorem for compact selfadjoint operators [11, Corollary X.3.3.5] there exists a complete orthonormal system $\{\mathbf{f}_j\}_{j \in \mathbb{N}}$ in $\mathcal{K}$ consisting of eigenvectors for A' . Set $s_{jk} = |\langle \mathbf{f}_j, \mathbf{e}_k \rangle|^2$ for $j, k \in \mathbb{N}$. Then

$$1 - w_j = \langle A'\mathbf{f}_j, \mathbf{f}_j \rangle = \langle A\mathbf{f}_j, \mathbf{f}_j \rangle = \sum_{k=1}^{\infty} s_{jk}(1 - z_k).$$

Both $\{\mathbf{v}\} \cup \{\mathbf{f}_j\}_{j \in \mathbb{N}}$ and $\{\mathbf{e}_k\}_{k \in \mathbb{N}}$ are complete orthonormal systems in $\mathcal{H}$, so that

$$\begin{aligned} \sum_{k=1}^{\infty} s_{jk} &= \sum_{k=1}^{\infty} |\langle \mathbf{f}_j, \mathbf{e}_k \rangle|^2 = \|\mathbf{f}_j\|^2 = 1, \quad j \in \mathbb{N}, \\ \sum_{j=1}^{\infty} s_{jk} &= \sum_{j=1}^{\infty} |\langle \mathbf{f}_j, \mathbf{e}_k \rangle|^2 = \|\mathbf{e}_k\|^2 - |\langle \mathbf{v}, \mathbf{e}_k \rangle|^2 = 1 - a_k, \quad k \in \mathbb{N}, \end{aligned}$$

which completes the proof of the theorem. $\square$

Proof of Theorem 7. Let $\{z_i\}_{i \in \mathbb{N}}$ be a sequence of distinct complex numbers satisfying (3.14) with $\rho = 1$ and let A be the injective compact normal operator on $\mathcal{H}$ defined by $A\mathbf{e}_i = (1 - z_i)\mathbf{e}_i$, $i \in \mathbb{N}$. As before we denote by A' the compression $PAP|_{\mathcal{K}}$ of A , where $\mathcal{K} = \mathbf{v}^\perp$ and $\mathbf{v}$ is given by (4.14). Clearly, A' is a compact operator on $\mathcal{K}$. Moreover, since $\Re f(1) > 0$ the arguments in the proof of Theorem 6 show that A' is injective. Note that Lemma 7 remains valid in the present setting and that by assumption f has a discrete set of zeros $\{w_j\}_{j \in \mathbb{N}}$. It follows that the point spectrum of A' is $\{1 - w_j\}_{j \in \mathbb{N}}$. Denote by $\mathcal{K}_{A'}$ the closed linear hull of the root subspaces of A' and let $\widehat{A'}$ be the operator on $\mathcal{K}_{A'}$ induced by A' . Recall the operator version of Schur's lemma given in [17, Lemma 4.1].

Lemma 8. *With the above notations there exists an orthonormal basis $\{\mathbf{f}_j\}_{j \in \mathbb{N}}$ of $\mathcal{K}_{A'}$ in which the matrix of the operator $\widehat{A'}$ has triangular form and $\langle \widehat{A'}\mathbf{f}_j, \mathbf{f}_j \rangle = 1 - w_j$, $j \in \mathbb{N}$.*

Set $s_{jk} = |\langle \mathbf{f}_j, \mathbf{e}_k \rangle|^2$ for $j, k \in \mathbb{N}$. By Lemma 8 we have

$$1 - w_j = \langle \widehat{A'}\mathbf{f}_j, \mathbf{f}_j \rangle = \langle A\mathbf{f}_j, \mathbf{f}_j \rangle = \sum_{k=1}^{\infty} s_{jk}(1 - z_k).$$

It is clear that

$$(4.17) \quad \sum_{k=1}^{\infty} s_{jk} = \sum_{k=1}^{\infty} |\langle \mathbf{f}_j, \mathbf{e}_k \rangle|^2 = \|\mathbf{f}_j\|^2 = 1, \quad j \in \mathbb{N}.$$

On the other hand $\{\mathbf{v}\} \cup \{\mathbf{f}_j\}_{j \in \mathbb{N}}$ is an orthonormal system in $\mathcal{H}$ so that by Bessel's inequality we get

$$(4.18) \quad \sum_{j=1}^{\infty} s_{jk} = \sum_{j=1}^{\infty} |\langle \mathbf{f}_j, \mathbf{e}_k \rangle|^2 \leq \|\mathbf{e}_k\|^2 - |\langle \mathbf{v}, \mathbf{e}_k \rangle|^2 = 1 - a_k, \quad k \in \mathbb{N}.$$

The inequality stated in the theorem by means of nonnegative convex functions is an immediate consequence of (4.17)–(4.18). $\square$

5. A HIERARCHY OF DE BRUIJN-SPRINGER RELATIONS AND INERTIA LAWS

Theorems 2–3 suggest even deeper physical connections between the set of equilibrium points and the given set of charged particles in the case of finite planar systems, such as a whole new hierarchy of inertia laws for planar solid bodies associated with these sets. In order to make a precise statement let us identify the complex plane $\mathbb{C}$ with the x_1x_2 -plane in $\mathbb{R}^3$ equipped with the standard coordinate system $(O; x_1, x_2, x_3)$. For an integer $k \geq 2$ we denote by Ξ_k the standard simplex

$$\Xi_k = \{(t_1, \dots, t_k) \in [0, 1]^k \mid t_1 + \dots + t_k = 1\}.$$

Definition 3. Let L be a line in $\mathbb{R}^3$. Given a convex k -gon $\mathcal{K}$ in $\mathbb{C}$ with vertices $v_1, \dots, v_k$ and $\alpha \geq 1$ we define the α -moment of $\mathcal{K}$ with respect to L to be

$$I_{\alpha, L}(\mathcal{K}) = \int \cdots \int_{\Xi_k} d(t_1 v_1 + \dots + t_k v_k, L)^\alpha dt_1 \cdots dt_k,$$

where $d(\cdot, L)$ denotes the distance function associated with L .

Remark 21. If $\mathcal{K}$ is viewed as a planar solid body with uniform mass distribution, then $I_{\alpha, L}(\mathcal{K})$ may be interpreted as a higher moment of inertia (or generalized angular momentum) of $\mathcal{K}$ with respect to the rotation axis L .

Let Σ_m denote the symmetric group on m elements. It is not difficult to see that if $\mathcal{K}$ is as above and $u_1, \dots, u_m \in \mathbb{C}$ are such that $\mathcal{K} = \text{conv}(u_1, \dots, u_m)$, then

$$(5.1) \quad \int \cdots \int_{\Sigma_m} d(t_{\pi(1)} u_1 + \dots + t_{\pi(m)} u_m, L)^\alpha dt_1 \cdots dt_m = I_{\alpha, L}(\mathcal{K})$$

for any $\pi \in \Sigma_m$, so that $I_{\alpha, L}(\mathcal{K})$ depends only on the extreme points of $\mathcal{K}$.

The following conjecture is a powerful generalization of Theorems 2–3 and suggests a “hierarchy” of weighted de Bruijn-Springer relations and inertia laws for planar polygons constructed from the equilibrium points and the charged particles.

Conjecture 1. Let z_i , $1 \leq i \leq n$, be distinct points in $\mathbb{C}$ and $a_i > 0$, $1 \leq i \leq n$, be such that $\sum_{i=1}^n a_i = 1$, where $n \geq 2$. Denote by w_j , $1 \leq j \leq n-1$, the zeros of

$$f(z) = \sum_{i=1}^n \frac{a_i}{z - z_i}$$

and let $\Phi : \mathbb{C} \rightarrow \mathbb{R}$ be an arbitrary convex function. Then for any $k \in \{1, \dots, n-1\}$, $1 \leq m \leq k$, and $(t_1, \dots, t_k) \in \mathbb{C}^k$ one has

$$(5.2) \quad \begin{aligned} & \sum_{1 \leq r_1 < \dots < r_k \leq n-1} \sum_{\pi \in \Sigma_k} \Phi(\Pi_{k,m}(t_{\pi(1)} w_{r_1}, \dots, t_{\pi(k)} w_{r_k})) \\ & \leq \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(1 - \sum_{i=1}^k a_{s_i}\right) \sum_{\pi \in \Sigma_k} \Phi(\Pi_{k,m}(t_{\pi(1)} z_{s_1}, \dots, t_{\pi(k)} z_{s_k})), \end{aligned}$$

where $\Pi_{k,m}$ is the m -th elementary symmetric function on k symbols. In particular, for any line L in $\mathbb{R}^3$ and $\alpha \geq 1$ one has

$$(5.3) \quad \sum_{1 \leq r_1 < \dots < r_k \leq n-1} I_{\alpha,L}(\text{conv}(w_{r_1}, \dots, w_{r_k})) \leq \sum_{1 \leq s_1 < \dots < s_k \leq n} \left(1 - \sum_{i=1}^k a_{s_i}\right) I_{\alpha,L}(\text{conv}(z_{s_1}, \dots, z_{s_k})).$$

The fact that (5.2) implies (5.3) follows from (5.1) by taking $m = 1$ and integrating over the simplex Ξ_k . Arguing as in the proof of Corollary 3 one can see that it is actually enough to prove (5.2) for $m = k$ and $k \in \{1, \dots, n-1\}$. Note also that in the case of equal charges Conjecture 1 would provide a whole new series of inequalities for averages of generalized inertia moments involving the critical points and the zeros of an arbitrary complex polynomial. These would substantially improve both Corollary 5 and the inequalities conjectured in [30, Remark 2.3.9].

Remark 22. In view of the proofs of [26, Theorem 4.7] and Theorem 3 above, a natural way of attacking Conjecture 1 would be to interpret the arguments of Φ in (5.2) as eigenvalues of some appropriately defined multilinear power “ $\bigwedge^k A$ ” of a normal matrix A and one of its principal submatrices, where “ $\bigwedge$ ” is a suitable multilinear functional on $\bigotimes^k \mathcal{H}$. However, defining such a functional appears to be a nontrivial question. Note for instance that an obvious polarization of the usual antisymmetric tensor power, i.e., setting

$$A \wedge B = \frac{1}{2} \left[(A+B)^{(2)} - A^{(2)} - B^{(2)} \right]$$

for square matrices A, B of the same order, cannot lead to the desired conclusion since to begin with one does not get the correct dimensions (these should agree with the number of k -tuples in the two sides of (5.2)).

6. ZEROS OF BOREL SERIES WITH POSITIVE l^1 -COEFFICIENTS

Theorem 7 applies essentially to any bounded discrete set of positively charged particles with finite total charge for which equilibrium points are known to exist. We conjecture that if the particles accumulate at a point on the boundary of a disk containing all the particles, then there exist infinitely many equilibrium points.

Conjecture 2. Let $\{a_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers whose sum is finite and $\{z_i\}_{i \in \mathbb{N}}$ a sequence of distinct complex numbers satisfying

$$|z_i| < \rho \text{ for } i \in \mathbb{N} \text{ and } \lim_{i \rightarrow \infty} z_i = \xi \text{ with } |\xi| = \rho$$

for some fixed positive number ρ . Then the function

$$(6.1) \quad f(z) = \sum_{i=1}^{\infty} \frac{a_i}{z - z_i}$$

has an infinite set of zeros in the disk $\mathbb{D}(\rho) = \{z \in \mathbb{C} : |z| < \rho\}$.

Note that the assumptions of Conjecture 2 imply that condition (3.14) holds and thus f is meromorphic in $\mathbb{D}(\rho)$. Functions representable as a series of the form (6.1) are known as Borel series [33], and so Conjecture 2 may be rephrased as follows: every Borel series with positive l^1 -coefficients whose poles lie in an open disk and accumulate at a point on the boundary of this disk has infinitely many zeros.

Conjecture 2 may also be translated in operator theoretic terms by using the methods employed in the proofs of Theorems 6–7. Note first that up to a similarity transformation of the complex plane we may assume that $\xi = \rho = 1$. Then one can show that the above conjecture is equivalent to the following statement.

Conjecture 3. *Let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence of distinct complex numbers in the open unit disk converging to 1 and denote by A the (bounded) normal operator on a separable infinite-dimensional complex Hilbert space $\mathcal{H}$ satisfying $A\mathbf{e}_n = z_n\mathbf{e}_n$, $n \in \mathbb{N}$, where $\{\mathbf{e}_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. If P is a bounded operator on $\mathcal{H}$ such that $I - P$ is a rank one orthoprojection, then the compression $PAP|_{P\mathcal{H}}$ has infinite point spectrum.*

Remark 23. The hypotheses in Conjecture 3 imply that the normal operator $I - A$ is compact, which may facilitate its study. However, it seems likely that Conjecture 2 should actually hold even under the less restrictive conditions $|z_i| < \rho$ for all $i \in \mathbb{N}$ and $\lim_{i \rightarrow \infty} |z_i| = \rho$. In other words, it is reasonable to believe that the convergence condition in Conjecture 3 may be relaxed as follows: $|z_n| < 1$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} |z_n| = 1$.

7. OPERATOR VERSIONS OF THE CLUNIE-EREMENKO-ROSSI CONJECTURE

Various results concerning the existence of equilibrium points for Newtonian and logarithmic potentials have been obtained in, e.g., [8], [14] and [24]. It should be emphasized though that these deal almost exclusively with *unbounded* discrete charge configurations. Note that Conjecture 2 may actually be viewed as a natural analog for *bounded* discrete charge configurations of the following well-known conjecture of Clunie-Eremenko-Rossi; see, e.g., [8, Conjecture 2.7] and [24, Conjecture 1.1].

Conjecture 4 (Clunie-Eremenko-Rossi conjecture). *Let $\{a_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers and $\{z_i\}_{i \in \mathbb{N}}$ a sequence of distinct complex numbers such that*

$$z_i \rightarrow \infty \text{ as } i \rightarrow \infty \text{ and } \sum_{z_i \neq 0} \frac{a_i}{|z_i|} < \infty.$$

Then the meromorphic function

$$f(z) = \sum_{i=1}^{\infty} \frac{a_i}{z - z_i}$$

has infinitely many zeros.

A catchy albeit somewhat loose rephrasing of this conjecture is as follows: *every flat universe has infinitely many resting points.*

Let us use the framework of §4.3 to give an operator theoretic interpretation and generalization of Conjecture 4. Note first that up to a translation of the variable z and a rescaling of the coefficients a_n we may assume wlog that all z_n are nonzero and satisfy the normalization condition

$$(7.1) \quad \sum_{i=1}^{\infty} \frac{a_i}{|z_i|} = 1.$$

As in §4.3 we use the Hilbert space $\mathcal{H} = l^2(\mathbb{N})$ with standard inner product $\langle \cdot, \cdot \rangle$ and standard complete orthonormal system $\{\mathbf{e}_n\}_{n \in \mathbb{N}}$. Set $z_n = r_n e^{i\theta_n}$, $n \in \mathbb{N}$, and

let A and B be the bounded normal operators on $\mathcal{H}$ satisfying

$$(7.2) \quad A\mathbf{e}_n = e^{i\theta_n}\mathbf{e}_n, \quad B_n = r_n^{-1}\mathbf{e}_n, \quad n \in \mathbb{N}.$$

Clearly, A is unitary and B is compact. Condition (7.1) allows us to define a unit vector $\mathbf{v} \in \mathcal{H}$ by setting

$$\mathbf{v} = \sum_{n=1}^{\infty} \sqrt{\frac{a_n}{|z_n|}} \mathbf{e}_n,$$

and one can easily verify that

$$(7.3) \quad \langle (A - zB)^{-1}\mathbf{v}, \mathbf{v} \rangle = -f(z), \quad z \in \mathbb{C} \setminus \{z_n\}_{n \in \mathbb{N}}.$$

Set $\mathcal{K} = \mathbf{v}^\perp$ and let P be the orthoprojection on $\mathcal{K}$. We use the notation A' and B' for the compressions to $\mathcal{K}$ of A and B , respectively, that is,

$$A' = PAP|_{\mathcal{K}}, \quad B' = PBP|_{\mathcal{K}}.$$

By definition, the set of generalized eigenvalues of A' with respect to B' is

$$\Sigma_{B'}(A') = \{z \in \mathbb{C} \mid N(A' - zB') \neq \{0\}\},$$

where $N(T)$ denotes as usual the null space of an operator T . Let $\mathcal{Z}(f)$ be the zero set of f .

Lemma 9. *In the above notation one has $\mathcal{Z}(f) = \Sigma_{B'}(A')$.*

Proof. We make an appropriate adaption of the argument used in the proof of Lemma 7. Using the fact that the z_n are distinct one can check that if $0 \neq \mathbf{u} \in \mathcal{H}$, then $\mathbf{u} \in N(A - \zeta B)$ if and only if $\zeta = z_n$ and $\mathbf{u} = \mathbf{e}_n$ for some $n \in \mathbb{N}$. However

$$\langle \mathbf{e}_n, \mathbf{v} \rangle = \sqrt{\frac{a_n}{|z_n|}} \neq 0,$$

which shows that $N(A - \zeta B) \cap \mathcal{K} = \{0\}$ for $\zeta \in \mathbb{C}$. Now let $\mathbf{u} \in \mathcal{K}$ be such that $(A' - \zeta B')\mathbf{u} = 0$ for some $\zeta \in \mathbb{C}$. From the identity

$$(A - \zeta B)\mathbf{u} = (I - P)(A - \zeta B)\mathbf{u} + (A' - \zeta B')\mathbf{u} = \langle (A - \zeta B)\mathbf{u}, \mathbf{v} \rangle \mathbf{v}$$

and $\langle (A - \zeta B)\mathbf{u}, \mathbf{v} \rangle \neq 0$ it follows that $(A - \zeta B)^{-1}\mathbf{v} = \langle (A - \zeta B)\mathbf{u}, \mathbf{v} \rangle^{-1}\mathbf{u}$. By (7.3) one gets

$$f(\zeta) = -\langle (A - \zeta B)\mathbf{u}, \mathbf{v} \rangle^{-1} \langle \mathbf{u}, \mathbf{v} \rangle = 0,$$

so that $\zeta \in \mathcal{Z}(f)$. Conversely, let $w \in \mathcal{Z}(f)$. Then $w \notin \{z_n\}_{n \in \mathbb{N}}$ and so $A - wB$ is invertible by (7.2). The vector $\mathbf{u} := (A - wB)^{-1}\mathbf{v}$ is clearly nonzero and one has $\langle \mathbf{u}, \mathbf{v} \rangle = f(w) = 0$, hence $\mathbf{u} \in \mathcal{K}$. Thus

$$\mathbf{v} = (A - wB)\mathbf{u} = \langle (A - wB)\mathbf{u}, \mathbf{v} \rangle \mathbf{v} + (A' - wB')\mathbf{u},$$

which shows that $(A' - wB')\mathbf{u} \in \mathbf{v}^\perp \cap \{\mathbf{v}\} = \{0\}$ and therefore $w \in \Sigma_{B'}(A')$. $\square$

Recall that a vector with all nonzero coordinates in the standard basis $\{\mathbf{e}_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$ is called totally nonzero. The above discussion shows that the Clunie-Eremenko-Rossi conjecture (Conjecture 4) is in fact equivalent to the following statement.

Conjecture 5. *Let $\{\alpha_n\}_{n \in \mathbb{N}}$ and $\{\beta_n\}_{n \in \mathbb{N}}$ be a sequence of unimodular complex numbers and a sequence of positive numbers converging to 0, respectively. Assume that $\alpha_n \beta_n^{-1}$ are distinct, and define a unitary operator A and a compact selfadjoint operator B on $\mathcal{H}$ by setting*

$$A\mathbf{e}_n = \alpha_n \mathbf{e}_n, \quad B\mathbf{e}_n = \beta_n \mathbf{e}_n, \quad n \in \mathbb{N}.$$

Let $\mathbf{v}$ be a totally nonzero unit vector in $\mathcal{H}$, denote by P the orthoprojection on $\mathcal{K} = \mathbf{v}^\perp$ and let $A' = PAP|_{\mathcal{K}}$ and $B' = PBP|_{\mathcal{K}}$. Then $|\Sigma_{B'}(A')| = \infty$, i.e., there exist infinitely many $z \in \mathbb{C}$ such that $N(A' - zB') \neq \{0\}$.

If the vector $\mathbf{v}$ in Conjecture 5 is represented in the standard orthonormal basis of $\mathcal{H}$ as $\mathbf{v} = \sum_{n=1}^{\infty} v_n \mathbf{e}_n$, then Keldysh's value distribution theorem for meromorphic functions implies that Conjecture 5 is true provided that $\sum_{n=1}^{\infty} \beta_n^{-1} |v_n|^2 < \infty$ (see, e.g., [8] and [18, Ch. 5, Theorem 6.2]).

Conjecture 5 suggests the following more general question: let A be an injective compact normal Hilbert space operator and let B and C be finite-rank operators. Is it true that $|\Sigma_{I+C}(A+B)| = \infty$, that is, are there infinitely many $z \in \mathbb{C}$ such that $N(A+B-z(I+C)) \neq \{0\}$? It is not difficult to see that an affirmative answer to this question would actually imply the validity of the operator theoretic version of the Clunie-Eremenko-Rossi conjecture (Conjecture 5). Indeed, in the notation of Conjecture 5 we may write

$$A' - zB' = -zA\{A^{-1}B - A^{-1}[PB(I-P) - (I-P)B] \\ - z^{-1}[I - A^{-1}(PA(I-P) + (I-P)A)]\}.$$

It remains to note that under the assumptions of Conjecture 5 the operator $A^{-1}B$ is injective, compact and normal while both $A^{-1}[PB(I-P) - (I-P)B]$ and $A^{-1}[PA(I-P) + (I-P)A]$ have rank at most two. However, the answer to the question raised above is negative, as one can see from the following example.

Example 1. Let J be the simple dissipative Volterra operator on $L^2(0,1)$ defined by

$$(Jf)(x) = 2i \int_0^x f(t) dt$$

and set $A = J + J^*$, $B = J - J^*$ and $C = 0$. Then A is injective compact selfadjoint, B is a one-dimensional operator while the point spectrum of $A + B = 2J$ is empty (cf. [17, p. 187]), so that in this case $A + B - z(I + C)$ is injective for all $z \in \mathbb{C}$.

Let us finally mention that the existence of equilibrium points for some discrete potentials generated by complex charges was established in, e.g., [24]. We propose the following analog of Conjecture 4 for certain potentials of this type.

Conjecture 6. Let $\{a_n\}_{n \in \mathbb{N}}$ and $\{z_n\}_{n \in \mathbb{N}}$ be sequences of complex numbers such that the z_n are distinct and satisfy

$$a_n \bar{z}_n > 0 \text{ for } n \in \mathbb{N}, \quad z_n \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \sum_{n=1}^{\infty} \left| \frac{a_n}{z_n} \right| < \infty.$$

Then the meromorphic function

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z - z_n}$$

has infinitely many zeros.

As in the case of Conjectures 2 and 4, one can reduce the conjecture stated above to a purely operator theoretic question. Indeed, in the notations of Conjecture 6

let A denote the injective compact normal operator on $\mathcal{H}$ satisfying $A\mathbf{e}_n = z_n^{-1}\mathbf{e}_n$, $n \in \mathbb{N}$. Define a vector

$$\mathbf{v} = \sum_{n=1}^{\infty} \sqrt{\left| \frac{a_n}{z_n} \right|} \mathbf{e}_n \in \mathcal{H},$$

let P be the orthoprojection on $\mathbf{v}^\perp$ and set $A' = PAP|_{\mathbf{v}^\perp}$. Then

$$\langle (A - z^{-1}I)^{-1}\mathbf{v}, \mathbf{v} \rangle = zf(z), \quad z \in \mathbb{C}^* \setminus \{z_n\}_{n \in \mathbb{N}}.$$

Arguing as in the proof of Lemma 9 one can show that $\zeta^{-1} \in \mathcal{Z}(f)$ for any $\zeta \in \Sigma_p(A') \setminus \{0\}$, where $\mathcal{Z}(f)$ is the zero set of f and $\Sigma_p(A')$ denotes the point spectrum of A' . For Conjecture 6 it would therefore be enough to show that $|\Sigma_p(A')| = \infty$.

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