ON OPERATOR-VALUED FOURIER MULTIPLIER THEOREMS

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ABSTRACT. The classical Fourier multiplier theorems of Marcinkiewicz and Mihlin are extended to vector-valued functions and operator-valued multiplier functions on $\mathbb{Z}^d$ or $\mathbb{R}^d$ which satisfy certain R -boundedness conditions.

1. INTRODUCTION AND MOTIVATION

Let X and Y be real (or complex) Banach spaces and $\mathcal{B}(X, Y)$ the Banach space of bounded linear operators $T : X \rightarrow Y$, endowed with the usual operator norm. By $\mathcal{S}(\mathbb{R}^d; X)$ we denote the Schwartz space of rapidly decreasing functions from $\mathbb{R}^d$ to X and by $\wedge, \vee$ we denote the Fourier transform and the inverse Fourier transform. For $1 \leq p < \infty$ let $L^p(\mathbb{T}^d; X)$ and $L^p(\mathbb{R}^d; X)$ be the usual Bochner spaces of p -integrable X -valued functions on the d -dimensional circle $\mathbb{T}^d$ and $\mathbb{R}^d$ respectively.

In the first part of this article we are interested in obtaining Fourier multiplier theorems on $L^p(\mathbb{T}^d; X)$ in the following sense.

For $z \in \mathbb{T}^d$ set $\mathbf{e}_{\mathbf{r}}(z) = z^{\mathbf{r}}$, $\mathbf{r} \in \mathbb{Z}^d$. We say that a function $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ is a **Fourier multiplier** on $L^p(\mathbb{T}^d; X)$ if the operator

$$(1.1) \quad f = \sum_{\mathbf{r} \in \mathbb{Z}^d} \hat{f}(\mathbf{r}) \otimes \mathbf{e}_{\mathbf{r}} \longmapsto \mathcal{K}_M f = \sum_{\mathbf{r} \in \mathbb{Z}^d} M(\mathbf{r}) \hat{f}(\mathbf{r}) \otimes \mathbf{e}_{\mathbf{r}},$$

first defined for f with a finitely valued Fourier transformation $\hat{f}$, extends uniquely to a bounded operator from $L^p(\mathbb{T}^d; X)$ to $L^p(\mathbb{T}^d; Y)$. We denote the set of such multipliers by $\mathcal{M}_p(\mathbb{Z}^d; X, Y)$.

For $d = 1$ the assumption of the Marcinkiewicz theorem requires that for the dyadic decomposition $I_n = \{\mathbf{r} \in \mathbb{Z} : 2^{n-1} < |\mathbf{r}| \leq 2^n\}$ we have

$$(1.2) \quad \text{var}(M_{I_n}) \leq C$$

for all $n \in \mathbb{N}$. For multipliers $M(\mathbf{r}) = m(\mathbf{r})I_X$ with a scalar function m it was shown in [Bou2] that (1.2) implies $M \in \mathcal{M}_p(\mathbb{Z}; X)$ if and only if X is a UMD-space. A UMD-space can be characterized by the fact that the special multiplier $m_0(\mathbf{r}) = \text{sign}(\mathbf{r})$ belongs to $\mathcal{M}_p(\mathbb{Z}; X)$ (see [Bou1], [Bu]). Indeed, Bourgain shows how “to built up” general scalar multipliers with (1.2) from modifications of m_0 . It is well known, that all subspaces and quotient spaces of $L^q(\Omega)$ -spaces with $1 < q < \infty$ are UMD-spaces.

For operator-valued multipliers $M(\mathbf{r}) \in \mathcal{B}(X)$ the variation (1.2), taken with respect to the operator norm, always implies that $M \in \mathcal{M}_p(\mathbb{Z}; X)$ if and only if

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X is a Hilbert space (Schwartz showed that (1.2) is sufficient in Hilbert spaces [BL], and Pisier observed the converse). So besides the UMD property for X and Y one needs additional assumptions on the multiplier function M . Recently it was shown in [We], in the context of the Mikhlin-multiplier theorem for operator-valued multipliers, that this additional condition can be expressed in terms of R -boundedness:

A subset $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R -bounded if there is a constant C such that for all $T_0, T_1, \dots, T_n \in \mathcal{T}$, $x_0, x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$

$$(1.3) \quad \int_0^1 \left\| \sum_{k=0}^n \varepsilon_k(t) T_k x_k \right\|_Y dt \leq C \int_0^1 \left\| \sum_{k=0}^n \varepsilon_k(t) x_k \right\|_X dt,$$

where (ε_k) is the sequence of Rademacher functions on $[0, 1]$. This concept was already used in [Bou2] and [BG] in connection with multiplier theorems, and more recently a detailed study was given in [CPSW]. If $X = Y = L^q(\Omega)$ for some $1 \leq q < \infty$, then (1.3) is equivalent to the square function estimate

$$(1.4) \quad \left\| \left(\sum_{k=0}^n |T_k x_k|^2 \right)^{1/2} \right\|_{L^q} \leq \tilde{C} \left\| \left(\sum_{k=0}^n |x_k|^2 \right)^{1/2} \right\|_{L^q}$$

known from harmonic analysis.

In this paper we use R -boundedness to give an appropriate form of the Marcinkiewicz condition for operator-valued multipliers. In place of (1.2) we assume that for some absolutely convex R -bounded set $\mathcal{T}$ we have

$$(1.5) \quad \sum_{k \in I_n} \|M(k+1) - M(k)\|_{\mathcal{T}} \leq C$$

for all $n \in \mathbb{N}$, where $\|\cdot\|_{\mathcal{T}}$ denotes the Minkowski functional of $\mathcal{T}$. If X and Y are UMD-spaces, M satisfies (1.5) and

$$(1.6) \quad \{M(\pm 2^{k-1}) : k \in \mathbb{N}\} \text{ is } R\text{-bounded,}$$

then we show in section 3 that $M \in \mathcal{M}_p(\mathbb{Z}; X, Y)$. We also give d -dimensional versions of this result. Our proof follows the techniques of [Zi], who proved d -dimensional generalizations of Bourgain's result for scalar multipliers. In Remark 1.1 below we point out that (1.6) is necessary for M to be in $\mathcal{M}_p(\mathbb{Z}; X, Y)$. If M belongs to $\mathcal{M}_p(\mathbb{Z}; X, Y)$ and (1.2) holds, it follows that $\{M(\mathfrak{x}) : \mathfrak{x} \in \mathbb{Z}\}$ is R -bounded, and this indicates that R -boundedness arises naturally in the context of multiplier theorems. The second part of the paper will treat the continuous case. In analogy to the discrete setting we say that a function $M : \mathbb{R}^d \setminus \{0\} \rightarrow \mathcal{B}(X, Y)$ is a Fourier multiplier, i.e. $M \in \mathcal{M}_p(\mathbb{R}^d; X, Y)$, if the operator

$$(1.7) \quad f \mapsto \mathcal{H}_M f = (M(\cdot) \hat{f}(\cdot))^\vee,$$

first defined for $f \in \mathcal{S}(\mathbb{R}^d; X)$, extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$.

For multiplier theorems of the Mikhlin type, one considers the sets

$$(1.8) \quad \{|x|^{|\gamma|} (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}$$

or

$$(1.9) \quad \{x^\gamma (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}.$$

Again the norm boundedness of these sets is only sufficient in Hilbert spaces. If X and Y are UMD-spaces and (1.8) is R -bounded, we show then in Theorem 4.4 that $M \in \mathcal{M}_p(\mathbb{R}^d; X, Y)$ holds. For the finer condition (1.9) we need besides the R -boundedness of (1.9) and the UMD property an additional assumption on X and Y , which is the property (α) , introduced by Pisier (see [Pi2]). In particular every q -concave Banach-lattice with $q < \infty$, or more generally, every Banach space with local unconditional structure and finite cotype has property (α) (cf. [Pi2], [DJT], Theorem 14.1). We reduce these theorems to the discrete case in section 3, again following the method in [Zi].

We also give a new criterion for the R -boundedness of a function $x \in I \rightarrow M(x)$. For one-dimensional intervals, R -boundedness follows if M is of bounded variation (see Theorem 2.7). For d -dimensional intervals we give an integrability criterion (see Theorem 4.1).

The next remark illustrates how the notion of R -boundedness is necessary if one considers operator-valued Fourier multipliers on vector-valued L^p -spaces.

1.1. Remark. Let us assume that we have a multiplier $M \in \mathcal{M}_p(\mathbb{Z}; X, Y)$. For trigonometric polynomials $f = \sum_{k=1}^n x_k \otimes \mathbf{e}_{2^k}$ we therefore obtain

$$(1.10) \quad \left\| \sum_{k=1}^n M(2^k) x_k \otimes \mathbf{e}_{2^k} \right\|_{L^p(\mathbb{T}; Y)} \leq C_1 \left\| \sum_{k=1}^n x_k \otimes \mathbf{e}_{2^k} \right\|_{L^p(\mathbb{T}; X)}.$$

Now, it is known (see [Pi1]) that there is a universal constant $C > 0$ such that for any Banach space E and any finite sequence $y_1, \dots, y_n$ in E :

$$\frac{1}{C} \left\| \sum_{k=1}^n \varepsilon_k y_k \right\|_{L^p([0,1]; E)} \leq \left\| \sum_{k=1}^n y_k \otimes \mathbf{e}_{2^k} \right\|_{L^p(\mathbb{T}; E)} \leq C \left\| \sum_{k=1}^n \varepsilon_k y_k \right\|_{L^p([0,1]; E)},$$

where $(\varepsilon_k)_{k \geq 1}$ denotes the sequence of Rademacher functions on $[0, 1]$. These inequalities in connection with (1.10) lead to

$$\left\| \sum_{k=1}^n \varepsilon_k M(2^k) x_k \right\|_{L^p([0,1]; Y)} \leq C_2 \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_{L^p([0,1]; X)}.$$

This means that the collection $\{M(2^k) : k \in \mathbb{N}\} \subset \mathcal{B}(X, Y)$ is R -bounded. By an application of Proposition 1.3 in [Bl] we get that $\mathcal{T} = \{M(r) : r \in \mathbb{Z}\}$ is R -bounded.

2. R -BOUNDEDNESS

In this section we list some important results about R -bounded collections $\mathcal{T}$ of bounded linear operators. Let $(\Omega, \mathfrak{A}, P)$ be a probability space and $(\varepsilon_k)_{k=0}^\infty$ a sequence of independent symmetric $\{-1, 1\}$ -valued random variables on $(\Omega, \mathfrak{A}, P)$. With $L^p(\Omega; X)$ we denote the Bochner space of p -integrable X -valued functions on $(\Omega, \mathfrak{A}, P)$.

2.1. Definition. A collection $\mathcal{T} \subset \mathcal{B}(X, Y)$ is said to be **R -bounded** if there exist a constant $C > 0$ such that for all $T_0, T_1, \dots, T_n \in \mathcal{T}$, $x_0, x_1, \dots, x_n \in X$ and all $n \in \mathbb{N}$

$$(2.1) \quad \left\| \sum_{k=0}^n \varepsilon_k T_k x_k \right\|_{L^1(\Omega; Y)} \leq C \left\| \sum_{k=0}^n \varepsilon_k x_k \right\|_{L^1(\Omega; X)}.$$

The smallest constant C , for which (2.1) holds is denoted by $R(\mathcal{T})$.

The notion of R -boundedness was already implicitly used in [Bou1], [Bou2] and was introduced in the paper [BG]. Detailed studies about collections of R -bounded operators can be found in [CPSW] and [We]. In [We] the reader will find as well a new characterization of maximal L^p -regularity of abstract differential equations using this notation.

Note. We want to emphasize that the definition of collections of R -bounded operators does not depend on the probability space $(\Omega, \mathfrak{A}, P)$ and the sequence of random variables (ε_k) .

2.2. *Remark.* Here are some rather known facts about R -boundedness:

(a) Using Kahane's inequality (see [LT]) we can replace (2.1) by $(1 \leq p < \infty)$

$$(2.2) \quad \left\| \sum_{k=0}^n \varepsilon_k T_k x_k \right\|_{L^p(\Omega; Y)} \leq C_p \left\| \sum_{k=0}^n \varepsilon_k x_k \right\|_{L^p(\Omega; X)}.$$

(b) It is easy to see that R -bounded collections $\mathcal{T} \subset \mathcal{B}(X, Y)$ are necessarily bounded in $\mathcal{B}(X, Y)$. If X and Y are both Hilbert spaces, (2.2) shows that the converse also holds.

(c) If X and Y are q -concave Banach lattices $(1 \leq q < \infty)$ (for the Definition see [LT] 1.d.3.(iii)) the definitions (2.1) and (2.2) are equivalent to (see [LT] Theorem 1.d.6.(i))

$$(2.3) \quad \left\| \left(\sum_{k=0}^n |T_k x_k|^2 \right)^{1/2} \right\|_Y \leq C \left\| \left(\sum_{k=0}^n |x_k|^2 \right)^{1/2} \right\|_X.$$

(d) If $\mathcal{T} := \{a_k I_X : k \in \mathbb{N}_0\} \subset \mathcal{B}(X)$, then $R(\mathcal{T}) \leq 2\|a\|_\infty$.

In the following we want to present four practical methods to enlarge an R -bounded collection $\mathcal{T}$. By $\text{aco}(\mathcal{T})$ we denote the real or complex absolute convex hull of a collection $\mathcal{T} \subset \mathcal{B}(X, Y)$. With this in mind, we can formulate the first statement.

2.3. Lemma. *Let $\mathcal{T} \subset \mathcal{B}(X, Y)$ be an R -bounded collection with R -bound $R(\mathcal{T})$. Then the absolute convex hull $\text{aco}(\mathcal{T})$ as well as the strong closure of $\mathcal{T}$ are R -bounded with R -bounds not larger than $2R(\mathcal{T})$.*

The statement is based on ideas introduced in [Bou2]. For the proof we refer to [CPSW]. The next two lemmas can also be found in [CPSW].

2.4. Lemma. *Let $\mathcal{S} \subset \mathcal{B}(X_2, X_3)$, $\mathcal{T} \subset \mathcal{B}(X_1, X_2)$ be two collections which are R -bounded. Then the collection*

$$\mathcal{ST} = \{ST : S \in \mathcal{S}, T \in \mathcal{T}\}$$

is R -bounded with an R -bound not greater than $R(\mathcal{S})R(\mathcal{T})$.

Let E be a Banach space, $X = L^p(\Lambda; E)$ for some σ -finite measure space $(\Lambda, \mathfrak{B}, \mu)$ and $1 \leq p < \infty$. For $\varphi \in L^\infty(\Lambda)$ we denote by M_φ the pointwise multiplication operator on X .

2.5. Lemma. *Let $X = L^p(\Lambda; E)$ and $\mathcal{T} \subset \mathcal{B}(X)$. If $\mathcal{T}$ is R -bounded, then the collection*

$$\{M_\varphi T M_\psi : \varphi, \psi \in L^\infty(\Lambda) \text{ with } \|\varphi\|_\infty, \|\psi\|_\infty \leq 1, T \in \mathcal{T}\}$$

is R -bounded as well.

The following extension result is also useful and is taken from [We], Proposition 2.11.

2.6. Lemma. *For $T \in \mathcal{B}(X, Y)$ define the operator $(\tilde{T}f)(\lambda) := T(f(\lambda))$, $f \in L^p(\Lambda; X)$, $\lambda \in \Lambda$, $1 \leq p < \infty$. Then, if $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R -bounded, the collection $\tilde{\mathcal{T}} = \{\tilde{T} : T \in \mathcal{T}\} \subset \mathcal{B}(L^p(\Lambda; X), L^p(\Lambda; Y))$ is also R -bounded.*

In the next theorem we give a sufficient condition on the regularity of operator-valued functions which ensures R -boundedness for their range collection. This result generalizes in particular Proposition 2.5 in [We]. Other statements of this type can be found in Corollary 3.5 and Theorem 4.1.

2.7. Theorem. *If X, Y are arbitrary Banach spaces and the function $M : I \rightarrow \mathcal{B}(X, Y)$ on an interval $I = [a, b] \subset \mathbb{R}$ is of bounded variation, then the collection*

$$\mathcal{M} := \{M(x) : x \in I\}$$

is R -bounded with $R(\mathcal{M}) \leq C(\|M(a)\| + \text{var}(M))$.

Proof. Assume that M has the form

$$M(t) = M(a) + \sum_{j=1}^m \chi_{A_j}(t) M_j$$

with $A_j \subset I$ and $M_j \in \mathcal{B}(X, Y)$. Then, using Lemma 2.4 in [We], we obtain

$$(2.4) \quad R(\{M(x) : x \in I\}) \leq \|M(a)\| + \sum_{j=1}^m \|M_j\|.$$

We will now show that for general M there is sequence of the form

$$(2.5) \quad M_k(t) = M(a) + \sum_j \chi_{A_{k,j}}(t) M_{k,j}$$

with

$$\sum_j \|M_{k,j}\| \leq C \text{var}(M) \quad \forall k \in \mathbb{N}$$

and

$$(2.6) \quad \|M_k(t) - M(t)\| \rightarrow 0$$

for $k \rightarrow \infty$ and all $t \in I$. With this approximation property for M the claim follows by an application of Lemma 2.3.

Without loss of generality we assume that M is continuous from the left. If we put $\Delta(t) := \lim_{s \searrow t} M(s) - M(t)$, the bounded variation property on M states that for each $n \in \mathbb{N}$ the set

$$\mathcal{T}_n := \{t \in I : \frac{1}{n+1} \leq \|\Delta(t)\| < \frac{1}{n}\}$$

has to be finite, i.e. $\mathcal{T}_n = \{t_{n,1}, \dots, t_{n,k(n)}\}$. If we put

$$N(t) := \sum_{n=1}^{\infty} \sum_{x \in \mathcal{T}_n} \Delta(x) \chi_{(x,b)}(t),$$

one can show that N is continuous from the left and $\text{var}(N) \leq \text{var}(M)$ holds. Moreover the function $L := M - N$ is also of bounded variation with $\text{var}(L) \leq$

$2 \operatorname{var}(M)$, continuous and therefore uniformly continuous on $[a, b]$. Obviously the definition of N allows us to construct a sequence (N_k) of functions of the form

$$N_k = M(a) + \sum_{j=1}^{m_k} \chi_{A_{k,j}}(t) N_{k,j}$$

with $\|N_k(t) - N(t)\| \rightarrow 0$ for $k \rightarrow \infty$ and all $t \in [a, b]$ and $\operatorname{var}(N_k) \leq \operatorname{var}(N)$. To approximate L we choose for a given $\varepsilon > 0$ a $\delta > 0$ such that $|s - t| < \delta$ implies $\|L(s) - L(t)\| \leq \varepsilon$ and that for a partition (t_j) in (a, b) with $\sup_j |t_j - t_{j-1}| \leq \delta$ we have

$$\left| \sum_j \|L(t'_j) - L(t'_{j-1})\| - \operatorname{var}(L) \right| \leq \varepsilon$$

for all refinements (t'_j) of (t_j) . Now the function

$$\tilde{L}(t) := M(a) + \sum_j [L(t_j) - L(t_{j-1})] \chi_{[t_j, b)}(t)$$

satisfies $\operatorname{var}(\tilde{L}) \leq \operatorname{var}(L) + \varepsilon$ and $\|L(t) - \tilde{L}(t)\| \leq \varepsilon$. In this way we can find a sequence (L_k) of the form

$$L_k = M(a) + \sum_{j=1}^{m_k} \chi_{A_{k,j}}(t) M_j$$

satisfying $\|L_k(t) - L(t)\| \rightarrow 0$ for $k \rightarrow \infty$, $t \in [a, b]$ and $\operatorname{var}(L_k) \leq 2 \operatorname{var}(L)$. Now the sequence $M_k := N_k + L_k$ has the required properties (2.5) and (2.6). $\square$

Since outside the Hilbert space setting bounded sets of operators are usually not R -bounded anymore, we have to replace the operator norm in various estimates and definitions by the following norms “measuring” R -boundedness:

2.8. Notation. For a bounded collection $\mathcal{T} \subset \mathcal{B}(X, Y)$ we denote the Minkowski functional of $\operatorname{aco}(\mathcal{T})$ by

$$(2.7) \quad \|\cdot\|_{\mathcal{T}} : \begin{cases} \mathcal{B}(X, Y) & \longrightarrow \\ T & \longmapsto \|T\|_{\mathcal{T}} := \inf\{t > 0 : T \in t \cdot \operatorname{aco}(\mathcal{T})\}. \end{cases} [0, \infty],$$

Here are some obvious facts that will be used constantly in the next two sections.

2.9. Remark. a) If we set $\mathcal{T} := \{a_k I_X : k \in \mathbb{N}_0\} \subset \mathcal{B}(X)$, then we have

$$\operatorname{aco}(\mathcal{T}) = \{z I_X : |z| \leq \|a\|_{\infty}\}$$

and thus by Remark 2.2.(d) we get

$$R(\mathcal{T}) \|\alpha I_X\|_{\mathcal{T}} \leq 2 \|a\|_{\infty} \|\alpha I_X\|_{\mathcal{T}} \leq 2 |\alpha|, \quad \alpha \in \mathbb{C}.$$

b) Let $\mathcal{T} \subset \mathcal{B}(X, Y)$ be an R -bounded collection. Then the following holds:

- (i) $\|T_1 + T_2\|_{\mathcal{T}} \leq \|T_1\|_{\mathcal{T}} + \|T_2\|_{\mathcal{T}}$.
- (ii) If a collection $\mathcal{M} = \{M_n : n \in \mathbb{N}\}$ has the property that

$$C := \sup\{\|M_n\|_{\mathcal{T}} : n \in \mathbb{N}\} < \infty,$$

then $\mathcal{M}$ is also R -bounded and the R -bound is not greater than $4CR(\mathcal{T})$.

3. THE DISCRETE CASE

In this section we are interested in giving sufficient conditions on the function $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ so that the operator, defined in (1.1), extends to a bounded operator. In particular we want to generalize the results given in [Bou2] and [Zi].

The next two sections will rather follow the examinations given in section 1 and 2 of [Zi]. For that reason we will use the same kind of notation which appears in that work.

3.1. Definition. Let $\alpha, \beta \in \mathbb{Z}^d$, $\alpha \leq \beta$ (coordinatewise) and $[\alpha; \beta] := \{\mathfrak{x} \in \mathbb{Z}^d : \alpha \leq \mathfrak{x} \leq \beta\}$. For a function $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ we define the restriction of M to $G \subset \mathbb{Z}^d$ by

$$M_G(\mathfrak{x}) := \begin{cases} M(\mathfrak{x}) & : \mathfrak{x} \in G, \\ 0 & : \mathfrak{x} \notin G. \end{cases}$$

The difference operators Δ^{e_j} ($j = 1, \dots, d$) are defined for the unit vectors e_j of $\mathbb{Z}^d$ as

$$(\Delta^{e_j} M_{[\alpha; \beta]})(\mathfrak{x}) := \begin{cases} M_{[\alpha; \beta]}(\mathfrak{x}) - M_{[\alpha; \beta]}(\mathfrak{x} - e_j) & : \mathfrak{x}_j \neq \alpha_j, \\ 0 & : \mathfrak{x}_j = \alpha_j. \end{cases}$$

For arbitrary $\gamma = (\gamma_1, \dots, \gamma_d) = \sum_{j=1}^d \gamma_j e_j$ with $\gamma_j \in \{0, 1\}$ we set

$$\Delta^0 M_{[\alpha; \beta]} := M_{[\alpha; \beta]}, \quad \Delta^\gamma M_{[\alpha; \beta]} := \Delta^{\gamma_1 e_1} \circ \dots \circ \Delta^{\gamma_d e_d} M_{[\alpha; \beta]}.$$

Next we generalize the definition of a variation.

3.2. Definition. Let $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ be an arbitrary function and $\mathcal{T} \subset \mathcal{B}(X, Y)$ a bounded collection. We define the $\mathcal{T}$ -variation of M in the interval $[\alpha; \beta]$ by

$$(3.1) \quad \text{var}_{\mathcal{T}} M_{[\alpha; \beta]} := \sum_{\mathfrak{x} \in [\alpha; \beta]} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha; \beta]})(\mathfrak{x})\|_{\mathcal{T}},$$

where $\gamma_{\mathfrak{x}} = (\gamma_{\mathfrak{x}_1}, \dots, \gamma_{\mathfrak{x}_d})$ with

$$\gamma_{\mathfrak{x}_j} := \begin{cases} 1 & : \mathfrak{x}_j \neq \alpha_j, \\ 0 & : \mathfrak{x}_j = \alpha_j. \end{cases}$$

Of course if $\mathcal{T}$ is the unit ball of $\mathcal{B}(X, Y)$, we have the usual notation of bounded variation, which we simply denote by $\text{var}_{[\alpha; \beta]} M_{[\alpha; \beta]}$ without the subscript $\mathcal{T}$.

The next result is a practical tool to estimate the $\mathcal{T}$ -variation of a (discrete) function.

3.3. Lemma. Let $\alpha_n, \beta_n \in \mathbb{Z}^d$, $\alpha_n \leq \beta_n$ and $([\alpha_n; \beta_n])_{n \in \mathbb{N}}$ be a (disjoint) decomposition of $\mathbb{Z}^d$. If $F : \mathbb{R}^d \rightarrow \mathcal{B}(X, Y)$ is a sufficiently smooth function and the collection

$$\mathcal{T} := \bigcup_{n \in \mathbb{N}} \{(\beta_n - \alpha_n)^\gamma (D^\gamma F)(x) : x \in [\alpha_n, \beta_n], \gamma \leq (1, \dots, 1)\}$$

is bounded, then the restriction of F to $\mathbb{Z}^d$ satisfies

$$\sup_{n \in \mathbb{N}} \text{var}_{\mathcal{T}} F_{[\alpha_n; \beta_n]} \leq 2^d.$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. If we rearrange the sum in Definition 3.1 we obtain

$$(3.2) \quad \text{var}_{\mathcal{T}} F_{[\alpha_n; \beta_n]} = \sum_{[\alpha_n; \beta_n]} \sum_{\gamma \leq (1, \dots, 1)} \sum_{\{\mathfrak{x} : \gamma_{\mathfrak{x}} = \gamma\}} \|(\Delta^\gamma F_{[\alpha_n; \beta_n]})(\mathfrak{x})\|_{\mathcal{T}},$$

where the sum over γ has 2^d summands.

1. CASE: $\gamma = (0, \dots, 0)$

In this case only $\mathfrak{x} = \alpha$ has the property $\gamma_{\mathfrak{x}} = \gamma$ and therefore

$$\sum_{\{\mathfrak{x}:\gamma_{\mathfrak{x}}=\gamma\}} \|(\Delta^\gamma F_{[\alpha_n;\beta_n]})(\mathfrak{x})\|_{\mathcal{T}} = \|F_{[\alpha_n;\beta_n]}(\alpha_n)\|_{\mathcal{T}} = \|F(\alpha_n)\|_{\mathcal{T}} \leq 1.$$

2. CASE: $(0, \dots, 0) \neq \gamma \leq (1, \dots, 1)$

Take $\mathfrak{x} \in [\alpha_n; \beta_n]$ with $\gamma_{\mathfrak{x}} = \gamma$. Let $q_1, \dots, q_r$ be the coordinate directions for which $\gamma_{\mathfrak{x}_{q_i}} \neq 0$. Now by definition of the difference operator and the fundamental theorem of calculus we get

$$\begin{aligned} (\Delta^{\gamma_{\mathfrak{x}}} F_{[\alpha_n;\beta_n]})(\mathfrak{x}) &= \int_{[\mathfrak{x}_{q_1}-1, \mathfrak{x}_{q_1}]} \cdots \int_{[\mathfrak{x}_{q_r}-1, \mathfrak{x}_{q_r}]} (D^{\gamma} F)(\xi) d\xi_{q_r} \cdots d\xi_{q_1} \\ &= \frac{1}{(\beta_n - \alpha_n)^{\gamma}} \int_{[\mathfrak{x}_{q_1}-1, \mathfrak{x}_{q_1}]} \cdots \int_{[\mathfrak{x}_{q_r}-1, \mathfrak{x}_{q_r}]} (\beta_n - \alpha_n)^{\gamma} (D^{\gamma} F)(\xi) d\xi_{q_r} \cdots d\xi_{q_1}. \end{aligned}$$

This immediately implies $(\Delta^{\gamma} F_{[\alpha_n;\beta_n]})(\mathfrak{x}) \in \frac{1}{(\beta_n - \alpha_n)^{\gamma}} \cdot \text{aco}(\mathcal{T})$ and thus

$$(3.3) \quad \sum_{\{\mathfrak{x}:\gamma_{\mathfrak{x}}=\gamma\}} \|(\Delta^{\gamma} F_{[\alpha_n;\beta_n]})(\mathfrak{x})\|_{\mathcal{T}} \leq \sum_{\{\mathfrak{x}:\gamma_{\mathfrak{x}}=\gamma\}} \frac{1}{(\beta_n - \alpha_n)^{\gamma}} = 1.$$

The last equality holds because there are exactly $(\beta_n - \alpha_n)^{\gamma}$ different $\mathfrak{x}$ in $[\alpha_n; \beta_n]$ with $\gamma_{\mathfrak{x}} = \gamma$. Thus the first case, (3.3) and (3.2) yield to the desired result. $\square$

The proof of the following result, which extends Stečkin's multiplier theorem, illustrates how the notion of bounded variation allows us to write a multiplier function as a sum of characteristic functions (cf. (3.6) below).

Let $I_n := [\alpha_n; \beta_n]$ with $\alpha_n = (-n, -n, \dots, -n)$, $\beta_n = (n, n, \dots, n)$ and

$$I(\gamma, n) := \{\mathfrak{x} \in I_n : \mathfrak{x}_i = -n \text{ if } \gamma_i = 0\}.$$

3.4. Theorem. *Let X be a UMD-space, Y an arbitrary Banach space and $1 < p < \infty$. Assume that the function $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ satisfies*

$$(3.4) \quad \sum_{\mathfrak{x} \in I(\gamma, n)} \|(\Delta^{\gamma} M_{I_n})(\mathfrak{x})\| \leq C < \infty$$

for all $\gamma \leq (1, \dots, 1)$ and all $n \in \mathbb{N}$. Then $M \in \mathcal{M}_p(\mathbb{Z}^d; X, Y)$.

Proof. Using the same rearrangement as in (3.2) and the assumption (3.4) we have that

$$(3.5) \quad \text{var}_{I_n} M_{I_n} = \sum_{\gamma \leq (1, \dots, 1)} \sum_{\mathfrak{x} \in I(\gamma, n)} \|(\Delta^{\gamma} M_{I_n})(\mathfrak{x})\| \leq 2^d C.$$

The point of this notation of bounded variation is that M can be written as

$$(3.6) \quad M_{[\alpha_n;\beta_n]} = \sum_{\mathfrak{x} \in [\alpha_n;\beta_n]} ((\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_n;\beta_n]})(\mathfrak{x})) \chi_{[\mathfrak{x};\beta_n]}.$$

For scalar valued M this was checked in [Zi], Lemma 1.3 (ii) and we apply this identity to $y^*(M_{[\alpha_n;\beta_n]}(\mathfrak{x})x)$ for all $x \in X, y^* \in Y^*$.

Now for an arbitrary $f : \mathbb{T}^d \rightarrow X$ with $\text{supp } \hat{f} \subset I_n$ we obtain from (3.6)

$$(3.7) \quad \mathcal{H}_{M_{I_n}} f = \sum_{\mathfrak{x} \in I_n} ((\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x}))^{\sim} \circ \mathcal{H}_{\chi_{[\mathfrak{x};\beta_n]}} f,$$

where $((\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x}))^\sim$ is the operator which arises from $(\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x})$ in the same way as it was done in Lemma 2.6. Since X is a UMD-space we know by [Zi] that there exists a constant D such that

$$\|\mathcal{K}_{\chi_{[\alpha;\beta]}} f\| \leq D\|f\|$$

for all intervals $[\alpha; \beta]$. Hence by (3.7) and (3.5)

$$\begin{aligned} \|\mathcal{K}_M f\| &= \|\mathcal{K}_{M_{I_n}} f\| \leq D\|f\| \sum_{\mathfrak{x} \in I_n} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{I_n})(\mathfrak{x})\| \\ &= D \operatorname{var}_{I_n} M_{I_n} \|f\| \leq 2^d CD \|f\|. \end{aligned}$$

Since the functions f with compact support $\hat{f}$ are dense in $L^p(\mathbb{T}^d; X)$, the claim follows. $\square$

3.5. Corollary. *If X and Y are arbitrary Banach spaces, and $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ satisfies (3.4), then $\{M(\mathfrak{x}) : \mathfrak{x} \in \mathbb{Z}^d\}$ is R -bounded.*

Proof. This follows from [We], Lemma 2.4, and the representations (3.6) and (3.7). $\square$

To obtain more refined multiplier theorems that generalize the Marcinkiewicz multiplier theorem, we assume that M is not of bounded variation on all of $\mathbb{Z}^d$ but only uniformly on certain partitions of $\mathbb{Z}^d$. As one might guess from Corollary 3.5, the R -boundedness will then be needed. The partitions we will use are the following ones:

(a) *The coarse decomposition.* Set $\mathbf{D}_0 := \{0\} \subset \mathbb{Z}^d$ and for $n = dr + j$, $r \in \mathbb{N}_0$, $j \in \{1, \dots, d\}$

$$\begin{aligned} \mathbf{D}_n := \{\mathfrak{x} = (\mathfrak{x}_1, \dots, \mathfrak{x}_d) \in \mathbb{Z}^d : |\mathfrak{x}_1|, \dots, |\mathfrak{x}_{j-1}| < 2^{r+1}, 2^r \leq |\mathfrak{x}_j| < 2^{r+1}, \\ |\mathfrak{x}_{j+1}|, \dots, |\mathfrak{x}_d| < 2^r\}. \end{aligned}$$

(b) *The fine decomposition.* For $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$ we define

$$\mathbf{D}_\nu := I_{\nu_1} \times \dots \times I_{\nu_d},$$

where $I_0 = \{0\}$ and $I_n = \{k \in \mathbb{Z} : 2^{n-1} \leq |k| < 2^n\}$ ($n \in \mathbb{N}$).

Since $\mathbf{D} = \mathbf{D}_n$ (resp. $\mathbf{D}_\nu$) are unions of $s = 2$ (resp. $s = 2^d$) intervals, we can moreover define the $\mathcal{T}$ -variation of M with respect to the decompositions by

$$\operatorname{var}_{\mathbf{D}}^{\mathcal{T}} M := \sum_{i=1}^s \operatorname{var}_{[\alpha_i; \beta_i]}^{\mathcal{T}} M.$$

We are now able to define the generalized Marcinkiewicz conditions.

3.6. Definition. A function $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ is said to be of bounded $\mathcal{T}$ -variation with respect to the decompositions $(\mathbf{D}_k)$ (resp. $(\mathbf{D}_\nu)$), if there exist an R -bounded collection $\mathcal{T} \subset \mathcal{B}(X, Y)$ such that the condition $(\mathcal{M}_{cD}^{\mathcal{T}})$

$$\sup_{k \in \mathbb{N}_0} \operatorname{var}_{\mathbf{D}_k}^{\mathcal{T}} M < \infty,$$

respectively $(\mathcal{M}_{fD}^{\mathcal{T}})$

$$\sup_{\nu \in \mathbb{N}_0^d} \operatorname{var}_{\mathbf{D}_\nu}^{\mathcal{T}} M < \infty$$

is fulfilled.

3.7. Theorem (operator-valued Marcinkiewicz theorem for $(\mathbf{D}_k)$). *Let X, Y be UMD-spaces and $1 < p < \infty$. If the function $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ possesses condition $(\mathcal{M}_{cD}^{\mathcal{T}})$, then $M \in \mathcal{M}_p(\mathbb{Z}^d; X, Y)$.*

3.8. Theorem (operator-valued Marcinkiewicz theorem for $(\mathbf{D}_\nu)$). *Let X, Y be UMD-spaces with the property (α) and $1 < p < \infty$. If the function $M : \mathbb{Z}^d \rightarrow \mathcal{B}(X, Y)$ possesses condition $(\mathcal{M}_{fD}^{\mathcal{T}})$, then $M \in \mathcal{M}_p(\mathbb{Z}^d; X, Y)$.*

3.9. Example. The Marcinkiewicz multiplier theorems in [Bou2] and [Zi] are special cases of these theorems. In both papers one considers functions M of the form $M(\mathbf{x}) = a(\mathbf{x})I$, $a(\mathbf{x}) \in \mathbb{C}$. If we set $\mathcal{T} := \{M(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^d\}$, then using Remark 2.9 a) we get

$$\begin{aligned} R(\mathcal{T}) \operatorname{var}_{\mathbf{D}} M &= R(\mathcal{T}) \sum_i \operatorname{var}_{[\alpha_i; \beta_i]} M_{[\alpha_i; \beta_i]} = R(\mathcal{T}) \sum_i \sum_{\mathbf{x} \in [\alpha_i; \beta_i]} \|(\Delta^{\gamma_{\mathbf{x}}} M_{[\alpha_i; \beta_i]})(\mathbf{x})\|_{\mathcal{T}} \\ &= R(\mathcal{T}) \sum_i \sum_{\mathbf{x} \in [\alpha_i; \beta_i]} \|(\Delta^{\gamma_{\mathbf{x}}} a_{[\alpha_i; \beta_i]})(\mathbf{x})I\|_{\mathcal{T}} \leq 2 \sum_i \sum_{\mathbf{x} \in [\alpha_i; \beta_i]} |(\Delta^{\gamma_{\mathbf{x}}} a_{[\alpha_i; \beta_i]})(\mathbf{x})| = 2 \operatorname{var}_{\mathbf{D}} a. \end{aligned}$$

3.10. Remark. Since the elements $\mathbf{D}_k$ are unions of two intervals $[\alpha_{1k}; \beta_{1k}]$, $[\alpha_{2k}; \beta_{2k}]$, the assumption of Theorem 3.7 implies in particular that the collection $\mathcal{S} = \{M(\alpha_{ik}) : i = 1, 2; k \in \mathbb{N}_0\}$ has to be R -bounded (use Remark 2.9 b, (ii)).

Before we start to prove both theorems we list some further results, which will simplify the later arguments. For $G \subset \mathbb{Z}^d$ we define a characteristic multiplier function from $\mathbb{Z}^d$ to $\mathcal{B}(E)$ by

$$\chi_G(\mathbf{x}) := \begin{cases} I & : \mathbf{x} \in G, \\ 0 & : \mathbf{x} \notin G. \end{cases}$$

Now the following results hold

3.11. Theorem. *Let E be a UMD-space, $1 < p < \infty$ and $S_k^E := \mathcal{K}_{\chi_{\mathbf{D}_k}}$, $k \in \mathbb{N}_0$. Then there exist a $C_p > 0$ such that for every trigonometric polynomial f*

$$(3.8) \quad \frac{1}{C_p} \|f\|_{L^p(\mathbb{T}^d; E)} \leq \left\| \sum_{k=0}^{\infty} \varepsilon_k S_k^E f \right\|_{L^p(\Omega; L^p(\mathbb{T}^d; E))} \leq C_p \|f\|_{L^p(\mathbb{T}^d; E)}.$$

A similar result is true for the fine decomposition. By $(\varepsilon_\nu)_{\nu \in \mathbb{N}_0^d}$ we denote an arbitrary d -dimensional renumeration of $(\varepsilon_k)_{k=0}^\infty$.

3.12. Theorem. *Let E be a UMD-space with the property (α) , $1 < p < \infty$ and $S_\nu^E := \mathcal{K}_{\chi_{\mathbf{D}_\nu}}$, $\nu \in \mathbb{N}_0^d$. Then there exist a $C_p > 0$ such that for every trigonometric polynomial f*

$$(3.9) \quad \frac{1}{C_p} \|f\|_{L^p(\mathbb{T}^d; E)} \leq \left\| \sum_{\nu \in \mathbb{N}_0^d} \varepsilon_\nu S_\nu^E f \right\|_{L^p(\Omega; L^p(\mathbb{T}^d; E))} \leq C_p \|f\|_{L^p(\mathbb{T}^d; E)}.$$

The proofs of both lemmas can be found in [Bou2] and [Zi]. Actually Zimmermann assumes for Theorem 3.12 that E is a UMD-space with local unconditional structure. But his proof works also for our weaker assumption. The next result is a consequence of the Definition of a UMD-space and Lemma 2.5. The proof is implicitly in [Zi] (see also Lemma 7 in [Bou2] or Lemma 3.5 in [BG]).

3.13. Lemma. *Let E be a UMD-space and $1 < p < \infty$. Then the collection*

$$\mathcal{K} := \{\mathcal{K}_{\chi_G} : G \text{ is an interval in } \mathbb{Z}^d\} \subset \mathcal{B}(L^p(\mathbb{T}^d; E))$$

is R -bounded.

3.14. Remark. The same statement also holds in the case $\mathbb{R}^d$.

Proof of Theorem 3.7. Let $\mathcal{K}_n := \mathcal{K}_{M_{D_n}}$, $n \in \mathbb{N}_0$. Now for any trigonometric polynomial we get

$$S_n^Y \circ \mathcal{K}_M f = \mathcal{K}_n \circ S_n^X f$$

and thus, using Theorem 3.11,

$$\|\mathcal{K}_M f\|_{L^p(\mathbb{T}^d; Y)} \leq C_p \left\| \sum_{n=0}^{\infty} \varepsilon_n \mathcal{K}_n \circ S_n^X f \right\|_{L^p(\Omega; L^p(\mathbb{T}^d; Y))}.$$

Now, if we can prove that the collection $\{\mathcal{K}_n : n \in \mathbb{N}_0\}$ is R -bounded, an additional application of Theorem 3.11 would complete this proof. $\square$

3.15. Lemma. *The collection*

$$\{\mathcal{K}_n : n \in \mathbb{N}_0\} \subset \mathcal{B}(L^p(\mathbb{T}^d; X), L^p(\mathbb{T}^d; Y))$$

is R -bounded.

Proof of Lemma 3.15. By assumption there exist an R -bounded collection $\mathcal{T}$ with

$$(3.10) \quad \text{var}_{\mathcal{D}_k} M = \sum_{i=1}^2 \sum_{\mathfrak{x} \in [\alpha_{i,k}; \beta_{i,k}]} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,k}; \beta_{i,k}]}) (\mathfrak{x})\|_{\mathcal{T}} \leq C < \infty \quad \forall k \in \mathbb{N}_0.$$

For the operator $\mathcal{K}_n$ we have the representation

$$(3.11) \quad \mathcal{K}_n = \mathcal{K}_{M_{[\alpha_{1,n}; \beta_{1,n}]}} + \mathcal{K}_{M_{[\alpha_{2,n}; \beta_{2,n}]}}.$$

Let us define the collection

$$\mathcal{S} := \text{aco}(\tilde{\mathcal{T}})\mathcal{K},$$

where $\tilde{\mathcal{T}}$ is the collection from Lemma 2.6 and $\mathcal{K}$ is the one from Lemma 3.13. Using Lemma 2.4, Lemma 2.6 and Lemma 3.13 we get that $\mathcal{S}$ is R -bounded. Now using the representation formulas from (3.6) and (3.7) we obtain

$$(3.12) \quad \mathcal{K}_{M_{[\alpha_{i,n}; \beta_{i,n}]}} = \sum_{\mathfrak{x} \in [\alpha_{i,n}; \beta_{i,n}]} ((\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x}))^{\sim} \circ \mathcal{K}_{\chi_{[\mathfrak{x}; \beta_{i,n}]}}.$$

Since

$$\begin{aligned} & \|((\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x}))^{\sim} \circ \mathcal{K}_{\chi_{[\mathfrak{x}; \beta_{i,n}]}}\|_{\mathcal{S}} \\ &= \inf\{t > 0 : ((\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x}))^{\sim} \circ \mathcal{K}_{\chi_{[\mathfrak{x}; \beta_{i,n}]}} \in t \cdot \text{aco}(\mathcal{S})\} \\ &\leq \inf\{t > 0 : ((\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x}))^{\sim} \circ \mathcal{K}_{\chi_{[\mathfrak{x}; \beta_{i,n}]}} \in t \cdot \text{aco}(\tilde{\mathcal{T}})\mathcal{K}\} \\ &\leq \inf\{t > 0 : ((\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x}))^{\sim} \in t \cdot \text{aco}(\tilde{\mathcal{T}})\} \\ &= \inf\{t > 0 : (\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x}) \in t \cdot \text{aco}(\mathcal{T})\} \\ &= \|(\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x})\|_{\mathcal{T}}, \end{aligned}$$

we thus obtain from (3.12) and Remark 2.9 b,(i) that

$$\|\mathcal{K}_{M_{[\alpha_{i,n}; \beta_{i,n}]}}\|_{\mathcal{S}} \leq \sum_{\mathfrak{x} \in [\alpha_{i,n}; \beta_{i,n}]} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}) (\mathfrak{x})\|_{\mathcal{T}}.$$

Taking (3.11), (3.10) and again Remark 2.9 b,(i) this yields

$$\|\mathcal{K}_n\|_s \leq C < \infty \quad \forall n \in \mathbb{N}_0.$$

Applying Remark 2.9 b,(ii) the proof is complete. $\square$

3.16. *Remark.* The proofs of Theorem 3.7 and Lemma 3.15 showed that the operator norm of $\mathcal{K}_M$ can be estimated by

$$\|\mathcal{K}_M\|_{\mathcal{B}(L^p(\mathbb{T}^d; X), L^p(\mathbb{T}^d; Y))} \leq CR(\mathcal{T}) \sup_{k \in \mathbb{N}_0} \sup_{\mathbf{D}_k} \text{var}_{\mathcal{T}} M,$$

where the constant C only depends on p and the dimension d , but not on the collection $\mathcal{T}$ and the multiplier M .

3.17. *Remark.* The proof of Theorem 3.8 works in the same way.

4. THE CONTINUOUS CASE

In the beginning of this section we'd like to present a criterion for the R -boundedness of an operator-valued function on $\mathbb{R}^d$. This will be done by using Corollary 3.5 of the preceding section, which already gave a tool on how to decide whether an operator-valued function on $\mathbb{Z}^d$ is R -bounded. Before stating the result, we need some additional notation.

Let $\xi \in \mathbb{R}^d$, γ be a multiindex with $0 \neq \gamma \leq (1, \dots, 1)$ and $q_1, \dots, q_r$ be the coordinate directions for which $\gamma_{q_i} = 1$. In this case we set $\xi_\gamma = (\xi_{q_1}, \dots, \xi_{q_r}) \in \mathbb{R}^r$.

4.1. **Theorem.** *Let X, Y be arbitrary Banach spaces and $M : \mathbb{R}^d \rightarrow \mathcal{B}(X, Y)$ a bounded function with continuous derivatives $D^\gamma M$, $\gamma \leq (1, \dots, 1)$. If moreover*

$$(4.1) \quad \int_{\mathbb{R}^r} \|(D^\gamma M)(\xi_\gamma)\| d\xi_\gamma \leq C < \infty$$

for each $0 \neq \gamma \leq (1, \dots, 1)$, then the collection

$$\mathcal{M} := \{M(x) : x \in \mathbb{R}^d\}$$

is R -bounded.

Proof. Set $M_k(x) := M(x/2^k)$, $k \in \mathbb{N}$, and restrict M_k to $\mathbb{Z}^d$. Since M is bounded, we know from Corollary 3.5 that the collection

$$\mathcal{M}_k := \{M_k(\mathfrak{x}) : \mathfrak{x} \in \mathbb{Z}^d\}$$

is R -bounded, if

$$\sum_{\mathfrak{x} \in I(\gamma, n)} \|(\Delta^\gamma M_{k, I_n})(\mathfrak{x})\| \leq \tilde{C}_k < \infty$$

for each $0 \neq \gamma \leq (1, \dots, 1)$ holds. Now, in analogy to the proof of Lemma 3.3, the fundamental theorem of calculus states that for $\mathfrak{x} \in I(\gamma, n)$ we have

$$(\Delta^\gamma M_{k, I_n})(\mathfrak{x}) = \int_{[\mathfrak{x}_{q_1}-1, \mathfrak{x}_{q_1}]} \cdots \int_{[\mathfrak{x}_{q_r}-1, \mathfrak{x}_{q_r}]} (D^\gamma M_k)(\xi) d\xi_{q_r} \cdots d\xi_{q_1}$$

and moreover

$$\begin{aligned} \sum_{\mathfrak{x} \in I(\gamma, n)} \|(\Delta^\gamma M_{k, I_n})(\mathfrak{x})\| &\leq \int_{[-n, n]^r} \|(D^\gamma M_k)(\xi_\gamma)\| d\xi_\gamma \\ &= \int_{[-n/2^k, n/2^k]^r} \|(D^\gamma M)(\xi_\gamma)\| d\xi_\gamma \\ &\leq \int_{\mathbb{R}^r} \|(D^\gamma M)(\xi_\gamma)\| d\xi_\gamma \leq C. \end{aligned}$$

So by now we have proved that

$$\sum_{\mathfrak{x} \in I(\gamma, n)} \|(\Delta^\gamma M_{k, I_n})(\mathfrak{x})\| \leq C,$$

where the constant is independent of k . Since $\mathcal{M}_k \subset \mathcal{M}_{k+1}$, we obtain from Corollary 3.5 that the collection

$$\bigcup_{k=1}^{\infty} \mathcal{M}_k = \{M(\mathfrak{x}/2^k) : \mathfrak{x} \in \mathbb{Z}^d, k \in \mathbb{N}\}$$

is R -bounded. An application of Lemma 2.3 completes the proof. $\square$

The remaining part of this section is concerned with extensions of the Mikhlin multiplier theorems from [Bou2], [McC] and [Zi] for scalar-valued multipliers to new theorems with operator-valued multiplier functions. In [We] the second author already considered the one-dimensional case and used it to give a new characterization of maximal L^p -regularity of abstract differential equations. In this paragraph we will generalize this result to the higher-dimensional setting using the Marcinkiewicz theorems from section 3.

To be able to use the results of the preceeding section we apply the Poisson summation formula, as in section 2 of [Zi], and the following two lemmas.

4.2. Lemma. *Let E be a Banach space, $1 \leq p < \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^d; E)$. Then we have*

$$\|\varphi\|_{L^p(\mathbb{R}^d; E)} = \lim_{k \rightarrow \infty} \|\varphi_{k, p}\|_{L^p(\mathbb{T}^d; E)},$$

where

$$\varphi_{k, p}(\mathfrak{x}) = 2^{-\frac{dk}{p'}} \sum_{\mathfrak{r} \in \mathbb{Z}^d} \hat{\varphi}(\mathfrak{r}/2^k) \otimes \mathfrak{e}_{\mathfrak{r}} \quad (1/p + 1/p' = 1).$$

4.3. Lemma. *Let $(M_n)_{n \in \mathbb{N}} \subset \mathcal{M}_p(\mathbb{R}^d; X, Y)$ be a sequence of Fourier multipliers that converges almost everywhere to M . Then*

$$\|\mathcal{K}_M\|_{\mathcal{B}(L^p(\mathbb{T}^d; X), L^p(\mathbb{T}^d; Y))} \leq \sup\{\|\mathcal{K}_{M_n}\|_{\mathcal{B}(L^p(\mathbb{T}^d; X), L^p(\mathbb{T}^d; Y))} : n \in \mathbb{N}\}.$$

We now state the first of two Mikhlin-type Fourier multiplier theorems

4.4. Theorem (First operator-valued Mikhlin theorem). *Let X, Y be UMD spaces and $1 < p < \infty$. If the function $M : \mathbb{R}^d \setminus \{0\} \rightarrow \mathcal{B}(X, Y)$ has the property that their distributional derivatives $D^\gamma M$ of order $\gamma \leq (1, \dots, 1)$ are represented by functions and moreover*

$$R(\{|x|^{|\gamma|} (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}) < \infty$$

holds, then $M \in \mathcal{M}_p(\mathbb{R}^d; X, Y)$.

Proof. We will divide the proof into three steps.

Step 1: $M \in \mathcal{S}(\mathbb{R}^d; \mathcal{B}(X, Y))$

For $f \in \mathcal{S}(\mathbb{R}^d; X)$ we now have

$$\mathcal{H}_M f = (M(\cdot) \hat{f}(\cdot))^\vee = \check{M} * f \in \mathcal{S}(\mathbb{R}^d; Y).$$

By applying Lemma 4.2 we thus obtain

$$(4.2) \quad \begin{aligned} \|\mathcal{H}_M f\|_{L^p(\mathbb{R}^d; Y)} &= \lim_k \|\mathcal{H}_{M_k} f_k\|_{L^p(\mathbb{T}^d; Y)} \\ &\leq \sup\{\|\mathcal{H}_{M_k}\|_{\mathcal{B}(L^p(\mathbb{T}^d; X), L^p(\mathbb{T}^d; Y))} : k \in \mathbb{N}\} \|f\|_{L^p(\mathbb{R}^d; X)}, \end{aligned}$$

where

$$M_k(\mathfrak{x}) := M(\mathfrak{x}/2^k), \quad f_k(\mathfrak{x}) = 2^{-\frac{dk}{p'}} \sum_{\mathfrak{x} \in \mathbb{Z}^d} \hat{f}(\mathfrak{x}/2^k) \otimes \mathfrak{e}_{\mathfrak{x}}, \quad \mathfrak{x} \in \mathbb{Z}^d.$$

The goal of the next calculation is to show that

$$\|\mathcal{H}_{M_k}\|_{\mathcal{B}(L^p(\mathbb{T}^d; X), L^p(\mathbb{T}^d; Y))} \leq C \quad \forall k \in \mathbb{N}.$$

This will be done by using Theorem 3.7. For that reason we have to secure that the Marcinkiewicz conditions $(\mathcal{M}_{cD}^{\mathcal{T}_k})$ hold for each M_k with Marcinkiewicz constants that can be estimated independently of k (see Remark 3.16). For the R -bounded collection $\mathcal{T}_k$ we choose

$$\mathcal{T}_k := \bigcup_{i \in \{1, 2\}, n \in \mathbb{N}_0} \{(\beta_{i,n} - \alpha_{i,n})^\gamma (D^\gamma M(\cdot/2^k))(x) : x \in [\alpha_{i,n}, \beta_{i,n}], \gamma \leq (1, \dots, 1)\}.$$

Here $([\alpha_{i,n}; \beta_{i,n}])_{i,n}$ is the coarse decomposition. By definition of the $\mathcal{T}$ -variation we have $(n \in \mathbb{N}_0)$

$$\text{var}_{\mathcal{T}_k} M_k = \sum_{\mathbf{D}_n} \sum_{i=1}^2 \sum_{\mathfrak{x} \in [\alpha_{i,n}; \beta_{i,n}]} \|(\Delta^{\gamma_{\mathfrak{x}}} M_{[\alpha_{i,n}; \beta_{i,n}]}(\cdot/2^k))(\mathfrak{x})\|_{\mathcal{T}_k}.$$

Using Lemma 3.3, we obtain

$$(4.3) \quad \sup_{n \in \mathbb{N}_0} \text{var}_{\mathcal{T}_k} M_k \leq 2^d.$$

To apply Theorem 3.7 and Remark 3.16 we have to estimate $R(\mathcal{T}_k)$. If we define $\Delta_0 = \{0\} \subset \mathbb{R}^d$ and for $n = dr + j$, $r \in \mathbb{N}_0$, $j \in \{1, \dots, d\}$

$$\Delta_n := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : |x_1|, \dots, |x_{j-1}| < 2^{r+1}, 2^r \leq |x_j| < 2^{r+1}, |x_{j+1}|, \dots, |x_d| < 2^r\},$$

then from the definition of the coarse decomposition we know that (note that the following constant r depends on n)

- The sizes of the edges of the two subcubes $[\alpha_{1,n}; \beta_{1,n}]$, $[\alpha_{2,n}; \beta_{2,n}]$ of $\mathbf{D}_n$ are not greater than 2^{r+1} ,
- $|x|_\infty \geq 2^r$ for all $x \in \Delta_n$ ($n \geq 1$).

Now let $x \in [\alpha_{i,n}; \beta_{i,n}]$ be arbitrary. Thus

$$(\beta_{i,n} - \alpha_{i,n})^\gamma (D^\gamma M(\cdot/2^k))(x) = 2^{|\gamma|} \frac{(\beta_{i,n} - \alpha_{i,n})^\gamma}{2^{(r+1)|\gamma|}} 2^{(r-k)|\gamma|} (D^\gamma M)(x/2^k)$$

and therefore (use Remark 2.2 (d))

$$\begin{aligned}
R(\mathcal{T}_k) &\leq C_1 R\left(\bigcup_{i,n} \{2^{(r-k)|\gamma|} (D^\gamma M)(x/2^k) : x \in [\alpha_{i,n}, \beta_{i,n}], \gamma \leq (1, \dots, 1)\}\right) \\
&= C_1 R\left(\bigcup_n \{2^{(r-k)|\gamma|} (D^\gamma M)(x/2^k) : x \in \Delta_n, \gamma \leq (1, \dots, 1)\}\right) \\
&= C_1 R\left(\bigcup_n \left\{ \frac{2^{(r-k)|\gamma|}}{|x|^{|\gamma|}} \cdot |x|^{|\gamma|} (D^\gamma M)(x) : x \in \frac{1}{2^k} \Delta_n, \gamma \leq (1, \dots, 1) \right\}\right) \\
&\leq C_2 R\left(\bigcup_n \{|x|^{|\gamma|} (D^\gamma M)(x) : x \in \frac{1}{2^k} \Delta_n, \gamma \leq (1, \dots, 1)\}\right) \\
&\leq C_2 R(\{|x|^{|\gamma|} (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}).
\end{aligned}$$

So (4.3), (4.2) and Remark 3.16 yield

$$(4.4) \quad \|\mathcal{H}_M\| \leq CR(\{|x|^{|\gamma|} (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}),$$

where C does not depend on the multiplier function M .

Step 2: M is infinitely often differentiable.

Fix an infinitely often differentiable (scalar) function ϱ with compact support such that $\varrho(0) = 1$. Define for all $\varepsilon > 0$, $\varrho_\varepsilon(\cdot) := \varrho(\varepsilon \cdot)$. Now $M_\varepsilon := \varrho_\varepsilon M \in \mathcal{S}(\mathbb{R}^d; \mathcal{B}(X, Y))$ converges pointwise to M as ε goes to 0. By Lemma 4.3 and (4.4) we get

$$\begin{aligned}
(4.5) \quad \|\mathcal{H}_M\| &\leq \sup\{\|\mathcal{H}_{M_\varepsilon}\| : \varepsilon > 0\} \\
&\leq C \sup_\varepsilon R(\{|x|^{|\gamma|} (D^\gamma M_\varepsilon)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}).
\end{aligned}$$

By Leibniz's formula we have

$$|x|^{|\gamma|} (D^\gamma M_\varepsilon)(x) = \sum_{\alpha+\beta=\gamma} C_{\alpha,\beta} |x|^{|\alpha|} (D^\alpha M)(x) |x|^{|\beta|} (D^\beta \varrho_\varepsilon)(x).$$

The R -bound of each term in the sum can be estimated by

$$\begin{aligned}
&R(\{|x|^{|\alpha|} (D^\alpha M)(x) |x|^{|\beta|} (D^\beta \varrho_\varepsilon)(x) : x \in \mathbb{R}^d \setminus \{0\}\}) \\
&\leq R(\{|x|^{|\alpha|} (D^\alpha M)(x) : x \neq 0\}) \cdot \sup_\varepsilon \{|x|^{|\beta|} (D^\beta \varrho_\varepsilon)(x) : x \neq 0\},
\end{aligned}$$

where the supremum is independent of ε . Therefore Remark 2.2 (d) and (4.5) yield

$$\begin{aligned}
\|\mathcal{H}_M\| &\leq C \sup_\varepsilon R(\{|x|^{|\gamma|} (D^\gamma M_\varepsilon)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}) \\
&\leq C_1 \sup_{\varepsilon, \gamma} \sum_{\alpha+\beta=\gamma} C_{\alpha,\beta} R(\{|x|^{|\alpha|} (D^\alpha M)(x) |x|^{|\beta|} (D^\beta \varrho_\varepsilon)(x) : x \in \mathbb{R}^d \setminus \{0\}\}) \\
&\leq C_2 R(\{|x|^{|\gamma|} (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}).
\end{aligned}$$

Here C_2 is again independent of M .

Step 3: M arbitrary as in the assumption.

Choose an infinitely often differentiable (scalar) function ϱ with $\varrho \geq 0$, $\|\varrho\|_1 = 1$ and $\text{supp } \varrho \subset [-1, 1]^d$. For $\varepsilon > 0$ define $\varrho_\varepsilon(\cdot) := \varepsilon^{-d} \varrho(\cdot/\varepsilon)$. Now $M_\varepsilon := M * \varrho_\varepsilon$ is

infinitely often differentiable and converges almost everywhere to M as ε converges to 0. The result from the second step in connection with Lemma 4.3 lead to

$$(4.6) \quad \|\mathcal{K}_M\| \leq \sup\{\|\mathcal{K}_{M_\varepsilon}\| : \varepsilon > 0\} \\ \leq C \sup_{\varepsilon} R(\{|x|^{|\gamma|}(D^\gamma[M * \varrho_\varepsilon])(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}).$$

Since for arbitrary $\gamma \leq (1, \dots, 1)$

$$\begin{aligned} & R(\{|x|^{|\gamma|}(D^\gamma[M * \varrho_\varepsilon])(x) : 0 < |x|_\infty \leq 2\varepsilon\}) \\ &= R(\{\frac{|x|^{|\gamma|}}{(2\varepsilon)^{|\gamma|}} \cdot (2\varepsilon)^{|\gamma|} \cdot \|D^\gamma \varrho_\varepsilon\|_1 \cdot (M * \frac{D^\gamma \varrho_\varepsilon}{\|D^\gamma \varrho_\varepsilon\|_1})(x) : 0 < |x|_\infty \leq 2\varepsilon\}) \\ &\leq C_1 (2\varepsilon)^{|\gamma|} \cdot \|D^\gamma \varrho_\varepsilon\|_1 \cdot R(\{M(x) : x \in \mathbb{R}^d \setminus \{0\}\}) \\ &= C_1 2^{|\gamma|} \cdot \|D^\gamma \varrho\|_1 \cdot R(\{M(x) : x \in \mathbb{R}^d \setminus \{0\}\}), \end{aligned}$$

we thus in particular obtain

$$(4.7) \quad R(\{|x|^{|\gamma|}(D^\gamma[M * \varrho_\varepsilon])(x) : 0 < |x|_\infty \leq 2\varepsilon, \gamma \leq (1, \dots, 1)\}) \\ \leq C_2 R(\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}).$$

The above estimations are consequences of Remark 2.2 (d) and Lemma 2.3.

For arbitrary $\varepsilon > 0$, ϱ_ε has its support in $[-\varepsilon, \varepsilon]^d$ and so for any $\alpha \leq (1, \dots, 1)$ and each $|x|_\infty \geq 2\varepsilon$

$$\begin{aligned} |x|^{|\alpha|}(D^\alpha[M * \varrho_\varepsilon])(x) &= |x|^{|\alpha|} \int_{\{\xi: |x-\xi| \leq \varepsilon\}} (D^\alpha M)(\xi) \varrho_\varepsilon(x - \xi) d\xi \\ &= \int_{\{\xi: |x-\xi| \leq \varepsilon\}} |\xi|^{|\alpha|} (D^\alpha M)(\xi) \cdot \frac{|x|^{|\alpha|}}{|\xi|^{|\alpha|}} \varrho_\varepsilon(x - \xi) d\xi \\ &= C_1 \int_{\{\xi: |x-\xi| \leq \varepsilon\}} |\xi|^{|\alpha|} (D^\alpha M)(\xi) \cdot \frac{|x|^{|\alpha|}}{C_1 |\xi|^{|\alpha|}} \varrho_\varepsilon(x - \xi) d\xi, \end{aligned}$$

where C_1 is a constant which fulfills for all α and $\varepsilon > 0$

$$\sup\{\frac{|x|^{|\alpha|}}{|\xi|^{|\alpha|}} : |x - \xi| \leq \varepsilon, |x|_\infty \geq 2\varepsilon\} \leq C_1.$$

Since $\|\varrho_\varepsilon\|_1 = 1$ we obtain that for each x with $|x|_\infty \geq 2\varepsilon$

$$\begin{aligned} & \int_{\{\xi: |x-\xi| \leq \varepsilon\}} |\xi|^{|\alpha|} (D^\alpha M)(\xi) \cdot \frac{|x|^{|\alpha|}}{C_1 |\xi|^{|\alpha|}} \varrho_\varepsilon(x - \xi) d\xi \\ & \in \text{aco}(\{|x|^{|\alpha|}(D^\alpha M)(x) : x \in \mathbb{R}^d \setminus \{0\}\}) \end{aligned}$$

and thus by Lemma 2.3

$$(4.8) \quad R(\{|x|^{|\gamma|}(D^\gamma M * \varrho_\varepsilon)(x) : |x|_\infty \geq 2\varepsilon, \gamma \leq (1, \dots, 1)\}) \\ \leq C_2 R(\{|x|^{|\gamma|}(D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}).$$

Using (4.6), (4.7) the theorem is proved. $\square$

4.5. Theorem (Second operator-valued Mikhlin theorem). *If X and Y are UMD-spaces with the property (α) and where the function $M : \mathbb{R}^d \setminus \{0\} \rightarrow \mathcal{B}(X, Y)$ has*

the property that their distributional derivatives $D^\gamma M$ of order $\gamma \leq (1, \dots, 1)$ are represented by functions which fulfill

$$R(\{x^\gamma (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}) < \infty,$$

then $M \in \mathcal{M}_p(\mathbb{R}^d; X, Y)$ ($1 < p < \infty$).

Proof. Again we divide the proof into three steps.

Step 1: $M \in \mathcal{S}(\mathbb{R}^d; \mathcal{B}(X, Y))$

Arguing as in the first step of the proof of Theorem 4.4 we obtain in analogy to (4.3)

$$\sup_{\nu \in \mathbb{N}_0^d} \text{var}_{\mathcal{T}_k} M_k \leq 2^d,$$

where the collection $\mathcal{T}_k$ is defined by

$$\mathcal{T}_k := \bigcup_{\substack{i \in \{1, \dots, 2^d\} \\ \nu \in \mathbb{N}_0^d}} \{(\beta_{i,\nu} - \alpha_{i,\nu})^\gamma (D^\gamma M(\cdot/2^k))(x) : x \in [\alpha_{i,\nu}, \beta_{i,\nu}], \gamma \leq (1, \dots, 1)\}.$$

Here $([\alpha_{i,\nu}; \beta_{i,\nu}])_{i,\nu}$ is the fine decomposition. Again it remains to show that $\mathcal{T}_k$ is R -bounded. Now we define for $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{Z}^d$

$$\Delta_\nu := \{x \in \mathbb{R}^d \setminus \{0\} : 2^{\nu_i-1} \leq |x_i| < 2^{\nu_i} \text{ for } i \in \{1, \dots, d\}\}.$$

For the fine decomposition we know

- The sizes of the edges of the 2^d subcuboids $[\alpha_{i,\nu}; \beta_{i,\nu}]$ of $\mathbf{D}_\nu$ in the j -th coordinate direction are not larger than 2^{ν_j} .
- For $x \in \Delta_\nu$, ($\nu \in \mathbb{N}^d$) we have $|x_j| \geq 2^{\nu_j-1}$.

If $x \in [\alpha_{i,\nu}; \beta_{i,\nu}]$ is arbitrary, we use the identity $((2^\alpha) := (2^{\alpha_1}, \dots, 2^{\alpha_d}), \alpha \in \mathbb{Z}^d)$

$$(\beta_{i,\nu} - \alpha_{i,\nu})^\gamma (D^\gamma M(\cdot/2^k))(x) = \frac{(\beta_{i,\nu} - \alpha_{i,\nu})^\gamma}{(2^\nu)^\gamma} \cdot (2^\nu)^\gamma \cdot 2^{-k|\gamma|} (D^\gamma M)(x/2^k)$$

and Remark 2.2 (d) to get

$$\begin{aligned} R(\mathcal{T}_k) &\leq C_1 R\left(\bigcup_{i,\nu} \left\{ \frac{(2^\nu)^\gamma}{2^{k|\gamma|}} (D^\gamma M)(x/2^k) : x \in [\alpha_{i,\nu}, \beta_{i,\nu}], \gamma \leq (1, \dots, 1) \right\}\right) \\ &= C_1 R\left(\bigcup_{\nu} \left\{ (2^\nu)^\gamma \cdot 2^{-k|\gamma|} (D^\gamma M)(x/2^k) : x \in \Delta_\nu, \gamma \leq (1, \dots, 1) \right\}\right) \\ &= C_1 R\left(\bigcup_{\nu} \left\{ \frac{(2^\nu)^\gamma \cdot 2^{-k|\gamma|}}{x^\gamma} \cdot x^\gamma (D^\gamma M)(x) : x \in \frac{1}{2^k} \Delta_\nu, \gamma \leq (1, \dots, 1) \right\}\right) \\ &\leq C_2 R\left(\bigcup_{\nu} \left\{ x^\gamma (D^\gamma M)(x) : x \in \frac{1}{2^k} \Delta_\nu, \gamma \leq (1, \dots, 1) \right\}\right) \\ &\leq C_2 R(\{x^\gamma (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}) \end{aligned}$$

and therefore

$$\|\mathcal{H}_M\| \leq CR(\{x^\gamma (D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}),$$

where C does not depend on the multiplier function M .

Step 2: The case where M is infinitely often differentiable can be treated in the same way as in Theorem 4.4.

Step 3: M fulfills the assumption of Theorem 4.5.

For this step we modify the first part of the proof of Theorem 4.4 and obtain similarly to (4.7) for all $0 < \varepsilon \leq 1$

$$(4.9) \quad R(\{x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) : 0 < |x|_\infty \leq \varepsilon, \gamma \leq (1, \dots, 1)\}) \\ \leq C R(\{x^\gamma(D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}).$$

The remaining part will be treated in the following way:

Choose $\gamma \leq (1, \dots, 1)$ arbitrary. Applying the binomial formula we get

$$x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) = \sum_{\alpha+\beta \leq \gamma} C_{\alpha,\beta}((\xi^\alpha(D^\alpha M)(\xi)) * (\xi^\beta(D^\beta \varrho_\varepsilon)(\xi)))(x).$$

Since $\text{supp } \varrho_\varepsilon \subset [-\varepsilon, \varepsilon]^d$, our assumption on the function M enables us to write the above convolution for each $|x|_\infty > \varepsilon$ as follows:

$$x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) = \sum_{\alpha+\beta=\gamma} C_{\alpha,\beta} \int_{[-\varepsilon, \varepsilon]^d} (x - \xi)^\alpha (D^\alpha M)(x - \xi) \xi^\beta (D^\beta \varrho_\varepsilon)(\xi) d\xi.$$

Now a similar argument as used in step 3 of the proof of Theorem 4.3 yields

$$\begin{aligned} & R(\{x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) : |x|_\infty > \varepsilon\}) \\ & \leq C_1 R(\{x^\alpha(D^\alpha M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \alpha \leq (1, \dots, 1)\}) \\ & \quad \cdot \sup_{\varepsilon} \{\|\xi^\beta(D^\beta \varrho_\varepsilon)(\xi)\| : \beta \leq (1, \dots, 1)\} \\ & = C_1 R(\{x^\alpha(D^\alpha M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \alpha \leq (1, \dots, 1)\}) \\ & \quad \cdot \sup\{\|\xi^\beta(D^\beta \varrho)(\xi)\| : \beta \leq (1, \dots, 1)\} \end{aligned}$$

and thus also

$$\begin{aligned} & R(\{x^\gamma(D^\gamma[M * \varrho_\varepsilon])(x) : |x|_\infty > \varepsilon, \gamma \leq (1, \dots, 1)\}) \\ & \leq C_2 R(\{x^\gamma(D^\gamma M)(x) : x \in \mathbb{R}^d \setminus \{0\}, \gamma \leq (1, \dots, 1)\}). \end{aligned}$$

Together with (4.9) this completes the proof. $\square$

Note. Observe that the weight function $|x|^{|\gamma|}$ of Theorem 4.4 is larger than $|x^\gamma|$ from Theorem 4.5.

4.6. *Remark.* i) Theorem 4.4 and Theorem 4.5 are generalizations of Proposition 3 in [Zi].

ii) If X and Y are Hilbert spaces, then the unit ball of $\mathcal{B}(X, Y)$ is R -bounded and both theorems reduce to the result of Schwartz, which assumes that

$$\|x^\gamma(D^\gamma M)(x)\| \leq C < \infty$$

for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\gamma \leq (1, \dots, 1)$.

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