ASYMPTOTIC DISTRIBUTION OF THE LARGEST OFF-DIAGONAL ENTRY OF CORRELATION MATRICES

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ABSTRACT. Suppose that we have n observations from a p -dimensional population. We are interested in testing that the p variates of the population are independent under the situation where p goes to infinity as $n \rightarrow \infty$. A test statistic is chosen to be $L_n = \max_{1 \leq i < j \leq p} |\rho_{ij}|$, where ρ_{ij} is the sample correlation coefficient between the i -th coordinate and the j -th coordinate of the population. Under an independent hypothesis, we prove that the asymptotic distribution of L_n is an extreme distribution of type G_1 , by using the Chen-Stein Poisson approximation method and the moderate deviations for sample correlation coefficients. As a statistically more relevant result, a limit distribution for $l_n = \max_{1 \leq i < j \leq p} |r_{ij}|$, where r_{ij} is Spearman's rank correlation coefficient between the i -th coordinate and the j -th coordinate of the population, is derived.

1. INTRODUCTION

Suppose $(X, Y), (X_i, Y_i), i = 1, \dots, n$ are independent and identically distributed (i.i.d.) random vectors from a bivariate distribution $F(x, y)$. For measuring the closeness of two random variables X and Y , we use Pearson's sample correlation coefficient, which is defined by

$$(1.1) \quad \rho_{XY} = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 \sum_{i=1}^n (Y_i - \bar{Y})^2}},$$

where $\bar{X} = \sum_{i=1}^n X_i/n$ and $\bar{Y} = \sum_{i=1}^n Y_i/n$. Obviously, ρ_{XY} is a natural estimator of the population correlation coefficient,

$$\rho = \mathbf{E}(X - \mathbf{E}X)(Y - \mathbf{E}Y) / \sqrt{\mathbf{E}(X - \mathbf{E}X)^2(Y - \mathbf{E}Y)^2}.$$

In the case of a bivariate normal distribution, ρ_{XY} is the maximum likelihood estimator of ρ . In addition, it is a commonly used statistic to test the null hypothesis that the two variables X, Y are independent, i.e.,

$$H_0 : \rho = 0,$$

$$H_1 : \rho \neq 0.$$

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Although many authors have studied the distribution of ρ_{XY} , see Hotelling [6] and the references therein, the best present-day usage in dealing with sample correlation coefficients is based on R. A. Fisher's work published in 1915; see [3].

Now suppose we have a p -dimensional distribution with mean μ and covariance matrix Σ . In the last three or four decades, due to the fast development of modern technology, in many research areas, including signal processing, network security, image processing, genetics, stock marketing and other economic problems, we are faced with the task of analyzing data with ever increasing dimension p . Naturally, one may ask how to test the independence among the p components of the population. For example, air pollutant levels are influenced by several meteorological conditions, such as wind speed, temperature, humidity, insolation, etc. The relationship of air pollutant levels to meteorological conditions can be described by regression models. But an essential part of constructing regression models is to check that all the interested meteorological conditions are independent or uncorrelated.

Under the additional normality assumption, Johnstone [9] derived the asymptotic distribution of the largest eigenvalue of the sample covariance matrices to study the test hypothesis $H_0: \Sigma = I$ assuming $\mu = 0$. On the other hand, Jiang [7] proposed to use a more intuitive independence test statistic, the largest off-diagonal entry of the sample correlation matrices, to test H_0 assuming at least a 30th moment of the population.

Before proceeding further, let us introduce some notation. Let $(X_{k1}, \dots, X_{kp})$, $k = 1, \dots, n$, be a sample of n independent observations taken from a nondegenerate p -variate population. Let $\mathbf{X}_i = (X_{1i}, \dots, X_{ni})^T$, $i = 1, \dots, p$, be columns of the $n \times p$ matrix $\mathcal{X}_n = (X_{ki})$, $k = 1, \dots, n$, $i = 1, \dots, p$. Hence $\mathcal{X}_n = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p)$. Introduce the average $\bar{X}_i = \sum_{k=1}^n X_{ki}/n$. We write $\mathbf{X}_i - \bar{X}_i$ for $\mathbf{X}_i - \bar{X}_i e$, where $e = (1, 1, \dots, 1)^T \in \mathbf{R}^n$. Then, ρ_{ij} , the sample correlation coefficient between $\mathbf{X}_i$ and $\mathbf{X}_j$ is defined by

$$(1.2) \quad \rho_{ij} = \frac{(\mathbf{X}_i - \bar{X}_i)^T (\mathbf{X}_j - \bar{X}_j)}{\|\mathbf{X}_i - \bar{X}_i\| \|\mathbf{X}_j - \bar{X}_j\|}, \quad 1 \leq i < j \leq p,$$

where $\|\cdot\|$ is the Euclidean norm. If we assume that the columns of $\mathcal{X}_n$ are independent, all the ρ_{ij} should be close to 0. In other words, the biggest off-diagonal entry of the sample correlation matrix $\Gamma_n = (\rho_{ij})$,

$$L_n = \max_{1 \leq i < j \leq p} |\rho_{ij}|,$$

has to be small. So we can use L_n as an independence test statistic. Under the i.i.d. assumption, Jiang [7] found an asymptotic distribution of L_n .

Theorem A (Jiang, [7]). *Suppose that $\mathbf{E}|X|^{30+\epsilon} < \infty$ for some $\epsilon > 0$. If $n/p \rightarrow \gamma$, then as $n \rightarrow \infty$ for any $y \in \mathbf{R}$,*

$$(1.3) \quad \mathbf{P}(nL_n^2 - a_n \leq y) \rightarrow \exp\left(-(\gamma^2 \sqrt{8\pi})^{-1} e^{-y/2}\right),$$

where $a_n = 4 \log n - \log \log n$.

Jiang's result makes it possible to test independence of components of vectors in large dimensional data problems when strong structural assumptions on the population, such as normality, are not present. But the conditions of his theorem are too restrictive. For example, a $(30+\epsilon)$ th moment is needed to get the asymptotic

distribution of L_n . So is it possible to reduce the high order moment requirements? On the other hand, we have no guarantee that the ratio of the sample size to the dimension will converge in practice. Sometimes it may happen that the dimension increases much slower than the sample size. Clearly, one may ask what we can do in those situations. In this paper, we try to answer the above mentioned questions through the following theorem.

Theorem 1.1. *Assume that the entries of the matrix $\mathcal{X}_n = (X_{ki})$ are i.i.d. Let $p = p_n$ satisfy*

$$(1.4) \quad \lim_{n \rightarrow \infty} p = \infty, \quad \limsup_{n \rightarrow \infty} p/n < \infty.$$

Write $b_n = 4 \log p - \log \log p$. Then

$$(1.5) \quad \lim_{n \rightarrow \infty} \mathbf{P}(nL_n^2 - b_n \leq y) = \exp\left(-Ke^{-y/2}\right), \quad K = (8\pi)^{-1/2},$$

provided that

$$(1.6) \quad \lim_{x \rightarrow \infty} x^6 \mathbf{P}\left(|XX'| > x\right) = 0,$$

where X and X' are independent copies of X_{11} . In particular, (1.6) holds if

$$(1.7) \quad \mathbf{E}X^6 < \infty.$$

We shall make the following remarks:

(1). Using the Chebyshev inequality, we can see that the condition $\mathbf{E}X^6 < \infty$ is stronger than the weak moment assumption (1.6) for the product XX' . Through Example 1 in Section 4, we explain that the weak moment condition (1.6) seems to be optimal.

(2). The limit distributions in (1.5) and (1.3) are different. This is due to the different choice of centering constants. Our choice of centering constants has an advantage that the limit distribution is independent of the unknown parameters, which is really useful in statistical applications. Of course, under Jiang's conditions, the relations (1.5) and (1.3) are mathematically equivalent, since in this case, $\lim_{n \rightarrow \infty} (a_n - b_n) = 4 \log \gamma$.

(3). To prove Theorem 1.1 we use the Poisson approximation technique, which was suggested by Barbour and Eagleson [2], and self-normalized moderate deviations studied by Shao [10], Wang and Jing [14], Shao [11], Jing, Shao and Wang [8].

Now we can see that the sample correlation coefficient is still useful in large dimensional statistical inference if high order moment assumptions are violated. But if strong structural assumptions on the population, such as normality, are not present, the sample correlation coefficient does not retain much interest because ρ is not invariant under rearrangement of the values of the independent random variables. More precisely, if $Z = \psi(X)$, where ψ is one-to-one and measurable, the correlation coefficient between X and Y may be very different from that between Z and Y . Another disadvantage of the sample correlation coefficient is that $\rho = 0$ does not necessarily imply the independence. Therefore other measures of independence can be of interest. For the independence test problems, it is Spearman [13] who introduced *Spearman's rank correlation coefficient* into statistics.

For a column $\mathbf{X}_i = (X_{1i}, \dots, X_{ni})^T$, we order the elements $X_{1i}, \dots, X_{ni}$ of $\mathbf{X}_i$ from least to greatest, and let Q_{ki} denote the rank of X_{ki} . Replacing the matrix

$\mathcal{X}_n = (X_{ki})$ by the matrix (Q_{ki}) , similar to (1.2), we can define the so-called Spearman rank correlation coefficient between $\mathbf{X}_i$ and $\mathbf{X}_j$ as

$$(1.8) \quad \begin{aligned} r_{ij} &= \frac{\sum_{k=1}^n Q_{ki}Q_{kj} - \frac{1}{n}(\sum_{k=1}^n Q_{ki})(\sum_{k=1}^n Q_{kj})}{\sqrt{\sum_{k=1}^n (Q_{ki} - \bar{Q}_i)^2 \sum_{k=1}^n (Q_{kj} - \bar{Q}_j)^2}} \\ &= \frac{12}{n(n^2 - 1)} \sum_{k=1}^n Q_{ki}Q_{kj} - \frac{3(n+1)}{n-1}, \end{aligned}$$

where $\bar{Q}_i = \bar{Q}_j = (n+1)/2$. All the r_{ij} constitute Spearman's rank correlation matrix

$$\mathcal{R}_n = (r_{ij})_{1 \leq i, j \leq p}.$$

The most important point is that the distribution of r_{ij} does not depend on nuisance parameters. Much information on r_{ij} can be found in Hájek, Šidák and Sen [4].

As a statistically more relevant result than Theorem 1.1, we give the following Theorem 1.2 for the largest off-diagonal entry of $\mathcal{R}_n$,

$$l_n = \max_{1 \leq i < j \leq p} |r_{ij}|.$$

Although l_n is distribution-free, it is hard to get the exact distribution of l_n when n is large. In order to make statistical inference based on l_n , it is essential to obtain the asymptotic distribution of l_n .

Theorem 1.2. *Assume that $\mathbf{X}_1, \dots, \mathbf{X}_p$ are independent, and all the entries of $\mathcal{X}_n$ are continuous random variables. If $p = p_n$ satisfy*

$$\lim_{n \rightarrow \infty} p = \infty, \quad \limsup_{n \rightarrow \infty} p/n < \infty,$$

then

$$(1.9) \quad \lim_{n \rightarrow \infty} \mathbf{P}(nl_n^2 - b_n \leq y) = \exp\left(-Ke^{-y/2}\right), \quad K = (8\pi)^{-1/2},$$

with $b_n = 4 \log p - \log \log p$.

In Theorem 1.2, we assume that all the entries of $\mathcal{X}_n$ are continuous random variables in order to avoid ties among the observations. Otherwise, the number of ties must be taken into account. As a consequence, the expectation and the variance under the null (independence) hypothesis need adjustment. We refer to Hollander and Wolfe [5] for more details. For the adjusted r_{ij} , Theorem 1.2 still holds.

The paper is organized as follows. We will prove the main theorems in Section 2. Some technical lemmas are deferred to Section 3. A counterexample is presented in Section 4 to explain that the weak moment condition (1.6) seems to be optimal.

2. PROOFS OF MAIN THEOREMS

We first present a lemma about Poisson approximation, which is a special case of Theorem 1 in Arratia, Goldstein and Gordon [1]. In the proofs of the main theorems, we rely heavily on it.

Lemma 2.1. *Let $\mathbf{B}$ be an index set and $\{B_\alpha, \alpha \in \mathbf{B}\}$ be a set of subsets of $\mathbf{B}$, i.e., $B_\alpha \subset \mathbf{B}$ for each $\alpha \in \mathbf{B}$. Also let $\{\eta_\alpha, \alpha \in \mathbf{B}\}$ be random variables. For a given $t \in \mathbf{R}$, set $\lambda = \sum_{\alpha \in \mathbf{B}} \mathbf{P}(\eta_\alpha > t)$. Then*

$$\left| \mathbf{P}(\max_{\alpha \in \mathbf{B}} \eta_\alpha \leq t) - e^{-\lambda} \right| \leq (1 \wedge \lambda^{-1})(b_1 + b_2 + b_3)$$

where

$$\begin{aligned} b_1 &= \sum_{\alpha \in \mathbf{B}} \sum_{\beta \in B_\alpha} \mathbf{P}(\eta_\alpha > t) \mathbf{P}(\eta_\beta > t), \\ b_2 &= \sum_{\alpha \in \mathbf{B}} \sum_{\alpha \neq \beta \in B_\alpha} \mathbf{P}(\eta_\alpha > t, \eta_\beta > t), \\ b_3 &= \sum_{\alpha \in \mathbf{B}} \mathbf{E} \left| \mathbf{P}(\eta_\alpha > t | \sigma(\eta_\beta, \beta \notin B_\alpha)) - \mathbf{P}(\eta_\alpha > t) \right|, \end{aligned}$$

where $\sigma(\eta_\beta, \beta \notin B_\alpha)$ is the σ -algebra generated by $\{\eta_\beta, \beta \notin B_\alpha\}$. In particular, if η_α is independent of $\{\eta_\beta, \beta \notin B_\alpha\}$ for each α , then $b_3 = 0$.

Before we prove Theorem 1.1, let us illustrate the main steps of the proof. We first show that it suffices to consider L_n based on the truncated random variables. See (2.1). Then we replace sample variances, the denominator of each correlation coefficient, by population variances. See (2.4). The next step is to truncate the products $\hat{X}_{ki} \hat{X}_{kj}$. See (2.5). Finally, we apply Lemma 2.1 to get the asymptotic distribution.

For notational convenience, we write

$$t_p = \sqrt{4 \log p - \log \log p + y}$$

and

$$t_{pn} = \sqrt{4 \log p - \log \log p + y + \epsilon_n}$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Throughout the paper, for an arbitrary set Ω , Ω^c denotes the complementary set of Ω .

Proof of Theorem 1.1. Since the sample correlation coefficient is scale and shift invariant, we can assume that

$$\mathbf{E}X = 0, \quad \mathbf{E}X^2 = 1.$$

Define

$$\hat{X}_{ki} = X_{ki} \mathbf{I}(|X_{ki}| \leq \sqrt{n}/\log^2 n), \quad \hat{\mathbf{X}}_i = (\hat{X}_{1i}, \dots, \hat{X}_{ni})^T, \quad \overline{\hat{X}}_i = \sum_{k=1}^n \hat{X}_{ki}/n,$$

and

$$\hat{\rho}_{ij} = \frac{(\hat{\mathbf{X}}_i - \overline{\hat{X}}_i)^T (\hat{\mathbf{X}}_j - \overline{\hat{X}}_j)}{\|\hat{\mathbf{X}}_i - \overline{\hat{X}}_i\| \|\hat{\mathbf{X}}_j - \overline{\hat{X}}_j\|}, \quad \hat{L}_n = \max_{1 \leq i < j \leq p} |\hat{\rho}_{ij}|,$$

where $1 \leq i < j \leq p$, $1 \leq k \leq n$. Then, for $\kappa < 1$,

$$\begin{aligned} (2.1) \quad \left| \mathbf{P}(nL_n^2 \leq t_p^2) - \mathbf{P}(n\hat{L}_n^2 \leq t_p^2) \right| &\leq \mathbf{P} \left(\max_{1 \leq k \leq n, 1 \leq i \leq p} |X_{ki}| > \sqrt{n}/\log^2 n \right) \\ &\leq np \mathbf{P}(|X_{12}| > \sqrt{n}/\log^2 n) \\ &\leq np (\log n)^{12-2\kappa} \mathbf{E}|X|^{6-\kappa}/n^{3-\kappa/2} \rightarrow 0 \end{aligned}$$

as $p \rightarrow \infty$, where in the last step we used Lemma 3.1. Let us deal with $\hat{L}_n$.

Write

$$\hat{L}_{1n} = \max_{1 \leq i < j \leq p} \left| \frac{(\hat{\mathbf{X}}_i - \overline{\hat{X}}_i)^T (\hat{\mathbf{X}}_j - \overline{\hat{X}}_j)}{n(1 - 1/(\log n)^2)} \right|,$$

and

$$\hat{L}_{2n} = \max_{1 \leq i < j \leq p} \left| \frac{(\hat{\mathbf{X}}_i - \overline{\hat{X}}_i)^T (\hat{\mathbf{X}}_j - \overline{\hat{X}}_j)}{n(1 + 1/(\log n)^2)} \right|.$$

Using the basic inequality

$$\mathbf{P}(\Theta_1) \leq \mathbf{P}(\Theta_1 \cap \Theta_2) + \mathbf{P}(\Theta_1 \cap \Theta_2^c),$$

with

$$\Theta_1 = (n\hat{L}_n^2 \leq t_p^2) \quad \text{and} \quad \Theta_2 = \left(\max_{1 \leq i \leq p} \left| \|\hat{\mathbf{X}}_i - \overline{\hat{X}}_i\|^2 - n \right| \leq n/(\log n)^2 \right),$$

and noting that $\hat{L}_{2n}^2 \leq \hat{L}_n^2$ if Θ_2 is true, we have

$$\mathbf{P}(n\hat{L}_n^2 \leq t_p^2) \leq \mathbf{P}(n\hat{L}_{2n}^2 \leq t_p^2) + \mathbf{P} \left(\max_{1 \leq i \leq p} \left| \|\hat{\mathbf{X}}_i - \overline{\hat{X}}_i\|^2 - n \right| > n/(\log n)^2 \right).$$

On the other hand, since $\hat{L}_{1n}^2 \geq \hat{L}_n^2$ if Θ_2 is true, we have

$$\mathbf{P}(\Theta_1) \geq \mathbf{P}(\Theta_1 \cap \Theta_2) \geq \mathbf{P}(\Theta_3 \cap \Theta_2) \geq \mathbf{P}(\Theta_3) - \mathbf{P}(\Theta_2^c),$$

where $\Theta_3 = (n\hat{L}_{1n}^2 \leq t_p^2)$.

Combining the above inequalities gives

$$\begin{aligned} & \mathbf{P}(n\hat{L}_{1n}^2 \leq t_p^2) - \mathbf{P} \left(\max_{1 \leq i \leq p} \left| \|\hat{\mathbf{X}}_i - \overline{\hat{X}}_i\|^2 - n \right| > n/(\log n)^2 \right) \\ (2.2) \quad & \leq \mathbf{P}(n\hat{L}_n^2 \leq t_p^2) \\ & \leq \mathbf{P}(n\hat{L}_{2n}^2 \leq t_p^2) + \mathbf{P} \left(\max_{1 \leq i \leq p} \left| \|\hat{\mathbf{X}}_i - \overline{\hat{X}}_i\|^2 - n \right| > n/(\log n)^2 \right). \end{aligned}$$

By (3.20) of Lemma 3.5,

$$\begin{aligned} & \mathbf{P} \left(\max_{1 \leq i \leq p} \left| \|\hat{\mathbf{X}}_i - \overline{\hat{X}}_i\|^2 - n \right| > n/(\log n)^2 \right) \\ (2.3) \quad & \leq p \mathbf{P} \left(\left| \|\hat{\mathbf{X}}_1 - \overline{\hat{X}}_1\|^2 - n \right| > n/(\log n)^2 \right) \\ & \leq pn^{-M} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Also note that

$$t_p^2 (1 \pm 1/(\log n)^2) = t_p^2 + o(1)$$

as $n \rightarrow \infty$. It follows from (2.2)–(2.3), in order to prove the theorem, that it suffices to prove the following:

$$(2.4) \quad \mathbf{P}(n\overline{L}_n^2 \leq t_p^2) \rightarrow \exp \left(-Ke^{-y/2} \right)$$

as $p \rightarrow \infty$, where

$$\overline{L}_n = \max_{1 \leq i < j \leq p} \left| (\hat{\mathbf{X}}_i - \overline{\hat{X}}_i)^T (\hat{\mathbf{X}}_j - \overline{\hat{X}}_j) / n \right|.$$

Now for ϵ satisfying (3.33), define

$$(2.5) \quad \begin{aligned} \xi_{kij} &= \hat{X}_{ki} \hat{X}_{kj} \mathbf{I} \left(|\hat{X}_{ki} \hat{X}_{kj}| \leq \epsilon \sqrt{n} \right), \quad 1 \leq k \leq n, \\ y_{ij} &= \frac{\sum_{k=1}^n \xi_{kij} - n \overline{\hat{X}_i} \overline{\hat{X}_j}}{\sqrt{n}}, \quad 1 \leq i < j \leq p, \end{aligned}$$

and

$$(2.6) \quad W_n = \max_{1 \leq i < j \leq p} |y_{ij}|.$$

For W_n , we have

$$(2.7) \quad \begin{aligned} \left| \mathbf{P}(n \overline{L}_n^2 \leq t_p^2) - \mathbf{P}(W_n^2 \leq t_p^2) \right| &\leq \mathbf{P} \left(\max_{1 \leq k \leq n, 1 \leq i < j \leq p} |X_{ki} X_{kj}| > \epsilon \sqrt{n} \right) \\ &\leq np^2 \mathbf{P}(|X_{12} X_{13}| > \epsilon \sqrt{n}) \\ &\rightarrow 0 \end{aligned}$$

as $p \rightarrow \infty$, where in the last inequality, we used the conditions (1.6) and (1.4). So we may turn to consider W_n .

We will apply Lemma 2.1 to prove

$$(2.8) \quad \mathbf{P} \left(\max_{1 \leq i < j \leq p} |y_{ij}| \leq t_p \right) \rightarrow \exp \left(-K e^{-y/2} \right)$$

as $p \rightarrow \infty$.

Take $\mathbf{B} = \{(i, j) : 1 \leq i < j \leq p\}$. For $\alpha = (i, j) \in \mathbf{B}$, set $\eta_\alpha = |y_{ij}|$ and $B_\alpha = \{(k, l) : \text{one of } k \text{ and } l = i \text{ or } j, \text{ but } (k, l) \neq \alpha\}$. We first calculate $\lambda = \lambda_n$ in Lemma 2.1. Since $\{y_{ij} : (i, j) \in \mathbf{B}\}$ are identically distributed,

$$\lambda_n = \frac{p^2 - p}{2} \mathbf{P}(|y_{12}| > t_p) = \frac{\exp(-y/2)}{\sqrt{8\pi}} (1 + o(1))$$

as $p \rightarrow \infty$, where the last equality is from Lemma 3.2.

Notice that y_α is independent of $\{y_\beta : \beta \in \mathbf{B} \setminus B_\alpha\}$ for any $\alpha \in \mathbf{B}$. To complete the proof of (2.8), by Lemma 2.1, we need to show that $b_1 \rightarrow 0$ and $b_2 \rightarrow 0$. But

$$b_1 \leq \frac{1}{2}(p^2 - p) 2p \mathbf{P}(|y_{12}| > t_p)^2 = O(1/n)$$

by Lemma 3.2. Also Lemma 3.4 gives

$$\begin{aligned} b_2 &\leq p(p^2 - p) \mathbf{P}(|y_{12}| > t_p, |y_{13}| > t_p) \\ &\leq p(p^2 - p) \left(\mathbf{P}(|y_{12} + y_{13}| > 2t_p) + \mathbf{P}(|y_{12} - y_{13}| > 2t_p) \right) \\ &\rightarrow 0. \end{aligned}$$

Hence (2.8) is verified. So we have completed the proof. $\square$

Proof of Theorem 1.2. We apply Lemma 2.1 directly to prove Theorem 1.2.

Take $\mathbf{B} = \{(i, j) : 1 \leq i < j \leq p\}$. For $\alpha = (i, j) \in \mathbf{B}$, set $\eta_\alpha = \sqrt{n}|r_{ij}|$ and $B_\alpha = \{(k, l) : \text{one of } k \text{ and } l = i \text{ or } j, \text{ but } (k, l) \neq \alpha\}$. We first calculate $\lambda = \lambda_n$ in Lemma 2.1. Since $\{r_{ij} : (i, j) \in \mathbf{B}\}$ are identically distributed,

$$\lambda_n = \sum_{1 \leq i < j \leq p} \mathbf{P}(\sqrt{n}|r_{ij}| > t_p) = \frac{p^2 - p}{2} \mathbf{P}(\sqrt{n}|r_{12}| > t_p).$$

By (1.8),

$$r_{12} = \frac{12}{n(n^2-1)} \sum_{k=1}^n Q_{k1} Q_{k2} - \frac{3(n+1)}{n-1},$$

which can be reexpressed as

$$(2.9) \quad r_{12} = \frac{12}{n(n^2-1)} \sum_{i=1}^n i R_i - \frac{3(n+1)}{n-1},$$

where R_i is some Q_{k2} such that $Q_{k1} = i$. Now applying Lemma 3.6 to (2.9), we have

$$\lambda_n = \frac{\exp(-y/2)}{\sqrt{8\pi}} (1 + o(1))$$

as $n \rightarrow \infty$.

Notice that r_α is independent of $\{r_\beta : \beta \in \mathbf{B} \setminus B_\alpha\}$ for any $\alpha \in \mathbf{B}$. To complete the proof, by Lemma 2.1, we need to show that $b_1 \rightarrow 0$ and $b_2 \rightarrow 0$. But

$$b_1 \leq \frac{1}{2}(p^2 - p) \, 2p \, \mathbf{P}^2(\sqrt{n}|r_{12}| > t_p) = O(1/n)$$

by Lemma 3.6, and

$$\begin{aligned} b_2 &\leq p(p^2 - p) \mathbf{P}(\sqrt{n}|r_{12}| > t_p, \sqrt{n}|r_{13}| > t_p) \\ &= p(p^2 - p) \mathbf{E} \mathbf{P}_1^2(\sqrt{n}|r_{12}| > t_p), \end{aligned}$$

where $\mathbf{P}_1$ means the conditional probability given $\mathbf{X}_1$. For Spearman's correlation coefficient, $\mathbf{P}_1(\sqrt{n}|r_{12}| > t_p) = \mathbf{P}(\sqrt{n}|r_{12}| > t_p)$ *a.s.* Applying Lemma 3.6 again gives $b_2 \rightarrow 0$. So we have completed the proof. $\square$

3. TECHNICAL LEMMAS

Let $\{\xi, \eta, \xi_k, \eta_k, \eta'_k, k = 1, 2, \dots, n\}$ be i.i.d. random variables with mean 0 and variance 1. Also suppose

$$(3.1) \quad \sup_{x>0} x^6 \mathbf{P}(|\xi\eta| > x) < \infty.$$

Lemma 3.1. *Under conditions (3.1), we have*

$$\mathbf{E}|\xi|^{6-\kappa} < \infty, \quad \mathbf{E}|\xi\eta|^{6-\kappa} < \infty, \quad \text{for all } 0 < \kappa < 1.$$

The above Lemma 3.1 is well known for one random variable. For the product, just note that $\mathbf{E}|\xi\eta|^{6-\kappa} = \mathbf{E}|\xi|^{6-\kappa} \mathbf{E}|\eta|^{6-\kappa}$, due to the i.i.d. assumption. So we omit its proof.

Now we introduce the following truncated random variables:

$$\hat{\xi} = \xi \mathbf{I}(|\xi| \leq d_n), \quad \hat{\eta} = \eta \mathbf{I}(|\eta| \leq d_n), \quad \hat{\eta}' = \eta' \mathbf{I}(|\eta'| \leq d_n)$$

with

$$d_n = \sqrt{n}/(\log n)^2 \quad \text{or} \quad d_n = n^{1/3},$$

and

$$\hat{\xi}_i = \xi_i \mathbf{I}(|\xi_i| \leq d_n), \quad \hat{\eta}_i = \eta_i \mathbf{I}(|\eta_i| \leq d_n), \quad \hat{\eta}'_i = \eta'_i \mathbf{I}(|\eta'_i| \leq d_n), \quad \text{for } i = 1, 2, \dots, n.$$

So we truncate all the variables at two levels. One is $\sqrt{n}/(\log n)^2$, which is used in the proof of Theorem 1.1. The other one is $n^{1/3}$, which we use in Example 1 of Section 4. The means of $\hat{\xi}_i$, $\hat{\eta}_i$ and $\hat{\eta}'_i$ are denoted by

$$\bar{\xi} = \sum_{i=1}^n \hat{\xi}_i/n, \quad \bar{\eta} = \sum_{i=1}^n \hat{\eta}_i/n, \quad \bar{\eta}' = \sum_{i=1}^n \hat{\eta}'_i/n.$$

Define, for ϵ satisfying (3.33),

$$z = \hat{\xi}\hat{\eta}\mathbf{I}(|\hat{\xi}\hat{\eta}| \leq \epsilon\sqrt{n}), \quad z' = \hat{\xi}\hat{\eta}'\mathbf{I}(|\hat{\xi}\hat{\eta}'| \leq \epsilon\sqrt{n}), \\ z_i = \hat{\xi}_i\hat{\eta}_i\mathbf{I}(|\hat{\xi}_i\hat{\eta}_i| \leq \epsilon\sqrt{n}), \quad z'_i = \hat{\xi}_i\hat{\eta}'_i\mathbf{I}(|\hat{\xi}_i\hat{\eta}'_i| \leq \epsilon\sqrt{n}), \quad \text{for } i = 1, 2, \dots, n.$$

Truncating z and z_i again, we have

$$\hat{z} = z\mathbf{I}(|z| \leq \sqrt{n}/(\log n)^2), \quad \hat{z}_i = z_i\mathbf{I}(|z_i| \leq \sqrt{n}/(\log n)^2), \quad \text{for } i = 1, 2, \dots, n.$$

The sums of $z_i, z'_i, \hat{z}_i$ are denoted by

$$S = \sum_{i=1}^n z_i, \quad S' = \sum_{i=1}^n z'_i, \quad \hat{S} = \sum_{i=1}^n \hat{z}_i.$$

The following Lemma 3.2 is used to calculate λ and b_1 of Lemma 2.1. The main idea of proving Lemma 3.2 is to replace the population variance by the sample variance so that we can use the self-normalized moderate deviations.

Lemma 3.2. *For the above truncated random variables,*

$$(3.2) \quad \mathbf{P}\left(\left|(S - n\bar{\xi}\bar{\eta})/\sqrt{n}\right| > t_p\right) / \left(\frac{e^{-y/2}}{\sqrt{2\pi}}p^{-2}\right) \rightarrow 1,$$

and for any $A > 1$,

$$(3.3) \quad \mathbf{P}\left(|S/\sqrt{n}| > A\sqrt{\log p}\right) = o(p^{-A^2/4})$$

as $n \rightarrow \infty$.

Proof. The Bernstein inequality implies, for any $0 < \delta_0 < 1/2$, that

$$\mathbf{P}\left(\left|\bar{\xi} - \mathbf{E}\bar{\xi}\right| > n^{\delta_0-1/2}\right) \leq 2\exp\left(-\frac{n^{2\delta_0}}{2\text{Var}\hat{\xi} + \frac{2}{3}\epsilon\sqrt{n}n^{\delta_0-1/2}}\right) \leq n^{-c}$$

with $c > 2$ for n sufficiently large, which combined with $|\mathbf{E}\hat{\xi}| \leq n^{-(5-\kappa)/3}\mathbf{E}|\xi|^{6-\kappa}$ implies that

$$(3.4) \quad \begin{aligned} \mathbf{P}\left(|S/\sqrt{n}| - 4n^{2\delta_0-1/2} > t_p\right) - n^{-c} &\leq \mathbf{P}\left(|(S - n\bar{\xi}\bar{\eta})/\sqrt{n}| > t_p\right) \\ &\leq \mathbf{P}\left(|S/\sqrt{n}| + 4n^{2\delta_0-1/2} > t_p\right) + n^{-c} \end{aligned}$$

for n sufficiently large and $\kappa < 1$. Let $\delta_0 \in (0, 1/4)$. Then $2\delta_0 - 1/2 < 0$, which implies that

$$(3.5) \quad (t_p \pm 4n^{2\delta_0-1/2})^2 - t_p^2 \rightarrow 0$$

as $n \rightarrow \infty$. Hence, in order to prove (3.2), by (3.4) and (3.5) it suffices to prove

$$(3.6) \quad \mathbf{P}\left(|S/\sqrt{n}| > t_{pn}\right) / \left(\frac{e^{-y/2}}{\sqrt{2\pi}}p^{-2}\right) \rightarrow 1$$

as $n \rightarrow \infty$. Since

$$\begin{aligned} & \mathbf{P}(|S/\sqrt{n}| > t_{pn}) - \mathbf{P}(|\hat{S}/\sqrt{n}| > t_{pn}) \\ &= \mathbf{P}\left(|S/\sqrt{n}| > t_{pn}, \max_{1 \leq i \leq n} |z_i| > \sqrt{n}/(\log n)^2\right) \\ &- \mathbf{P}\left(|\hat{S}/\sqrt{n}| > t_{pn}, \max_{1 \leq i \leq n} |z_i| > \sqrt{n}/(\log n)^2\right) \end{aligned}$$

by Lemma 3.3, it remains to prove

$$(3.7) \quad \mathbf{P}(|\tilde{S}/\sqrt{n}| > t_{pn}, \max_{1 \leq i \leq n} |z_i| > \sqrt{n}/(\log n)^2) = o(p^{-2}) \text{ with } \tilde{S} = S \text{ or } \hat{S}$$

as $n \rightarrow \infty$. Let us write $\mathbf{Z}_1 = (|z_1| > \sqrt{n}/(\log n)^2)$, $\mathbf{Z}_2 = (|z_2| > \sqrt{n}/(\log n)^2)$, $\mathbf{Z}_{2n} = (\max_{2 \leq i \leq n} |z_i| > \sqrt{n}/(\log n)^2)$. Then, noting $z_i = \hat{z}_i$ for $2 \leq i \leq n$ if $\mathbf{Z}_{2n}^c$ holds, we have

$$\begin{aligned} (3.7) &\leq n\mathbf{P}(\mathbf{Z}_1 \cap \mathbf{Z}_{2n}) + n\mathbf{P}\left(|\tilde{S}/\sqrt{n}| > t_{pn} \cap \mathbf{Z}_1 \cap \mathbf{Z}_{2n}^c\right) \\ &\leq n^2\mathbf{P}(\mathbf{Z}_1 \cap \mathbf{Z}_2) + n\mathbf{P}(|z_1/\sqrt{n}| > \delta t_{pn}) \\ &\quad + n\mathbf{P}\left(\left|\sum_{i=2}^n z_i/\sqrt{n}\right| > (1-\delta)t_{pn} \cap \mathbf{Z}_1 \cap \mathbf{Z}_{2n}^c\right) \\ (3.8) \quad &\leq n^2\mathbf{P}(\mathbf{Z}_1 \cap \mathbf{Z}_2) + n\mathbf{P}(|z_1/\sqrt{n}| > \delta t_{pn}) \\ &\quad + n\mathbf{P}(\mathbf{Z}_1)\mathbf{P}\left(\left|\sum_{i=2}^n \hat{z}_i/\sqrt{n}\right| > (1-\delta)t_{pn}\right) \end{aligned}$$

where δ can be arbitrarily small. In the above inequality, applying Condition (3.1) and the Chebyshev inequality, we have

$$n\mathbf{P}(|z_1/\sqrt{n}| > \delta t_{pn}) = o(n^{-2}) \quad \text{and} \quad n^2\mathbf{P}(\mathbf{Z}_1 \cap \mathbf{Z}_2) = o(n^{-2}).$$

For the second probability in (3.8), by (3.13) of Lemma 3.3, we have

$$\mathbf{P}\left(\left|\sum_{i=2}^n \hat{z}_i/\sqrt{n}\right| > (1-\delta)t_{pn}\right) = o(p^{-2+6\delta})$$

for n sufficiently large, which combined with

$$\mathbf{P}(\mathbf{Z}_1) \leq \mathbf{E}|z_1|^{6-\kappa}(\log n)^{12-2\kappa}/n^{3-\kappa/2}$$

shows that, for sufficiently small κ and δ ,

$$n\mathbf{P}(\mathbf{Z}_1)\mathbf{P}\left(\left|\sum_{i=2}^n \hat{z}_i/\sqrt{n}\right| > (1-\delta)t_{pn}\right) = o(p^{-2})$$

as $n \rightarrow \infty$. Hence (3.7) is proved, and we have obtained (3.2).

Now let us prove (3.3). Note that

$$(3.9) \quad \sqrt{n}|\mathbf{E}z| = \sqrt{n}|\mathbf{E}\hat{\xi}\hat{\eta} - \mathbf{E}\hat{\xi}\hat{\eta}\mathbf{I}(|\hat{\xi}\hat{\eta}| > \epsilon\sqrt{n})| = o(n^{-3/2}).$$

Hence for n sufficiently large,

$$(3.10) \quad \mathbf{P}\left(|S/\sqrt{n}| > A\sqrt{\log p}\right) \leq \mathbf{P}\left(|(S - n\mathbf{E}z)/\sqrt{n}| > 0.9A\sqrt{\log p}\right).$$

So by (3.24) of Lemma 3.5, we have

$$(3.11) \quad \begin{aligned} & \mathbf{P} \left(\left| (S - n\mathbf{E}z)/\sqrt{n} \right| > 0.9A\sqrt{\log p} \right) \\ & \leq \mathbf{P} \left(|T_z| > 0.9A\sqrt{\log p}/\sqrt{1+\epsilon_1} \right) + n^{-M}, \end{aligned}$$

where $T_z = (S - n\mathbf{E}z)/\sqrt{\sum_{i=1}^n (z_i - \mathbf{E}z_i)^2}$ and $M > A^2/4$. Now applying Theorem 2.3 of Jing, Shao and Wang [8],

$$\mathbf{P} \left(|T_z| > 0.9A\sqrt{\log p}/\sqrt{1+\epsilon_1} \right) / \left(2 \left(1 - \Phi(0.9A\sqrt{\log p}/\sqrt{1+\epsilon_1}) \right) \right) \rightarrow 1$$

as $n \rightarrow \infty$, which combined with (3.10), (3.11) and the fact $1 - \Phi(s) \sim \frac{1}{\sqrt{2\pi}s} e^{-s^2/2}$ as $s \rightarrow \infty$, implies (3.3). $\square$

Lemma 3.3. For the $\hat{z}_i$, $i = 1, 2, \dots, n$, we have

$$(3.12) \quad \mathbf{P} \left(|\hat{S}/\sqrt{n}| > t_p \right) / \left(\frac{e^{-y/2}}{\sqrt{2\pi}} p^{-2} \right) \rightarrow 1,$$

and for sufficiently small but fixed $\delta > 0$,

$$(3.13) \quad \mathbf{P} \left(\left| \sum_{i=2}^n \hat{z}_i/\sqrt{n} \right| > (1-\delta)t_{pn} \right) = o(p^{-2+6\delta}),$$

as $n \rightarrow \infty$.

Proof. The proofs of (3.12) and (3.13) are the same. We only prove (3.12). Note that

$$(3.14) \quad \begin{aligned} \sqrt{n}|\mathbf{E}\hat{z}| &= \sqrt{n}|\mathbf{E}\hat{\xi}\hat{\eta} - \mathbf{E}\hat{\xi}\hat{\eta}\mathbf{I}(|\hat{\xi}\hat{\eta}| > \epsilon\sqrt{n}) - \mathbf{E}z\mathbf{I}(|z| > \sqrt{n}/(\log n)^2)| \\ &= o(n^{-3/2}) \end{aligned}$$

as $n \rightarrow \infty$ by the inequality $\mathbf{E}|z|\mathbf{I}(|z| > x) \leq x^{-5+\kappa}\mathbf{E}|z|^{6-\kappa}$ and Lemma 3.1. Hence in order to show (3.12), it suffices to prove

$$(3.15) \quad \mathbf{P} \left(|(\hat{S} - n\mathbf{E}\hat{z})/\sqrt{n}| > t_{pn} \right) / \left(\frac{e^{-y/2}}{\sqrt{2\pi}} p^{-2} \right) \rightarrow 1$$

as $n \rightarrow \infty$. Applying (3.21) of Lemma 3.5, we have

$$\begin{aligned} & \mathbf{P} \left(|T_{\hat{z}}| > (1 + (\log n)^{-2})^{1/2} t_{pn} \right) - n^{-M} \\ & \leq \mathbf{P} \left(|(\hat{S} - n\mathbf{E}\hat{z})/\sqrt{n}| > t_{pn} \right) \\ & \leq \mathbf{P} \left(|T_{\hat{z}}| > (1 - (\log n)^{-2})^{1/2} t_{pn} \right) + n^{-M}, \end{aligned}$$

where $T_{\hat{z}} = (\hat{S} - n\mathbf{E}\hat{z})/\sqrt{\sum_{i=1}^n (\hat{z}_i - \mathbf{E}\hat{z})^2}$ and $M > 3$. Note that

$$(1 \pm (\log n)^{-2}) t_{pn}^2 = t_{pn}^2 + o(1)$$

as $n \rightarrow \infty$. So in order to get (3.15), we need to prove

$$(3.16) \quad \mathbf{P} (|T_{\hat{z}}| > t_{pn}) / \left(\frac{e^{-y/2}}{\sqrt{2\pi}} p^{-2} \right) \rightarrow 1$$

as $n \rightarrow \infty$. Now we can invoke Theorem 2.3 of Jing, Shao and Wang [8] to obtain (3.16). Hence, we have completed the proof of (3.12). $\square$

The following Lemma 3.4 is useful when we estimate b_2 of Lemma 2.1.

Lemma 3.4. *For the above truncated random variables,*

$$(3.17) \quad \mathbf{P} \left(\left| (S + S' - n\bar{\xi}(\bar{\eta} + \bar{\eta}')) / \sqrt{n} \right| > 2t_p \right) = o(p^{-3})$$

and

$$(3.18) \quad \mathbf{P} \left(\left| (S - S' - n\bar{\xi}(\bar{\eta} - \bar{\eta}')) / \sqrt{n} \right| > 2t_p \right) = o(p^{-3})$$

as $n \rightarrow \infty$

Proof. Similarly as in the derivation of (3.4), and noting (3.9), in order to prove (3.17), we need to prove

$$(3.19) \quad \mathbf{P} \left(\left| (S + S' - n(\mathbf{E}z + \mathbf{E}z')) / \sqrt{n} \right| > 2t_{pn} \right) = o(p^{-3})$$

as $n \rightarrow \infty$. By (3.22) of Lemma 3.5, we have

$$\mathbf{P} \left(\left| (S + S' - n(\mathbf{E}z + \mathbf{E}z')) \right| > 2t_{pn} \right) \leq \mathbf{P} \left(|T_{zz'}| > (1 - \epsilon_1/2)^{1/2} \sqrt{2}t_{pn} \right) + n^{-M},$$

where $T_{zz'} = (S + S' - n(\mathbf{E}z + \mathbf{E}z')) / (\sum_{i=1}^n (z_i + z'_i - \mathbf{E}z_i - \mathbf{E}z'_i)^2)^{1/2}$ and $M > 4$. Now applying Theorem 2.3 of Jing, Shao and Wang [8] to

$$\mathbf{P} \left(|T_{zz'}| > (1 - \epsilon_1/2)^{1/2} \sqrt{2}t_{pn} \right)$$

gives (3.19). So the proof of (3.17) is complete, and we can get (3.18) similarly. $\square$

Lemma 3.5 guarantees that in the proof of Theorem 1.1, the sample variance and the population variance are equivalent.

Lemma 3.5. *For the above truncated random variables $\hat{\xi}$, z , z' , $\hat{\xi}_i$, z_i , z'_i , $i = 1, 2, \dots, n$, we have*

$$(3.20) \quad \mathbf{P} \left(\left| \sum_{i=1}^n \hat{\xi}_i^2 / n - \bar{\xi}^2 - 1 \right| > 1/(\log n)^2 \right) \leq n^{-M},$$

$$(3.21) \quad \mathbf{P} \left(\left| \sum_{i=1}^n (\hat{z}_i - \mathbf{E}\hat{z})^2 / n - 1 \right| > 1/(\log n)^2 \right) \leq n^{-M},$$

$$(3.22) \quad \mathbf{P} \left(\left| \frac{1}{n} \sum_{i=1}^n (z_i + z'_i - \mathbf{E}z_i - \mathbf{E}z'_i)^2 - 2 \right| > \epsilon_1 \right) \leq n^{-M},$$

$$(3.23) \quad \mathbf{P} \left(\left| \frac{1}{n} \sum_{i=1}^n (z_i - z'_i - \mathbf{E}z_i + \mathbf{E}z'_i)^2 - 2 \right| > \epsilon_1 \right) \leq n^{-M},$$

$$(3.24) \quad \mathbf{P} \left(\left| \frac{1}{n} \sum_{i=1}^n (z_i - \mathbf{E}z_i)^2 - 1 \right| > \epsilon_1 \right) \leq n^{-M},$$

with arbitrarily large M (say, greater than 100) and arbitrarily small but fixed ϵ_1 .

Proof. The Bernstein inequality implies that

$$\mathbf{P} \left(\left| \bar{\xi} - \mathbf{E}\bar{\xi} \right| > n^{-1/6} \right) \leq n^{-M}.$$

Noting

$$\begin{aligned} 1 &\geq \mathbf{E}\hat{\xi}^2 \\ &= 1 - \mathbf{E}\xi^2 \mathbf{I}(|\xi| > d_n) \geq 1 - n^{-(4-\kappa)/3} \mathbf{E}|\xi|^{6-\kappa} \end{aligned}$$

and $|\mathbf{E}\hat{\xi}| \leq n^{-(5-\kappa)/3} \mathbf{E}|\xi|^{6-\kappa}$, we have

$$\begin{aligned} (3.25) \quad &\mathbf{P}\left(\left|\sum_{i=1}^n \hat{\xi}_i^2/n - \bar{\xi}^2 - 1\right| > 1/(\log n)^2\right) \\ &\leq \mathbf{P}\left(\left|\sum_{i=1}^n \hat{\xi}_i^2/n - \mathbf{E}\hat{\xi}^2\right| > 1/(2(\log n)^2)\right) + n^{-M}, \end{aligned}$$

for n sufficiently large. For the probability term in (3.25), we have

$$\begin{aligned} &\mathbf{P}\left(\left|\sum_{i=1}^n \hat{\xi}_i^2/n - \mathbf{E}\hat{\xi}^2\right| > 1/(2(\log n)^2)\right) \\ &\leq \exp\left(-\frac{t}{2(\log n)^2}\right) \left(\mathbf{E}\exp\left(tn^{-1}(\hat{\xi}^2 - \mathbf{E}\hat{\xi}^2)\right)\right)^n \\ &\quad + \exp\left(-\frac{t}{2(\log n)^2}\right) \left(\mathbf{E}\exp\left(tn^{-1}(\mathbf{E}\hat{\xi}^2 - \hat{\xi}^2)\right)\right)^n \\ &\leq 2\exp\left(-\frac{t}{2(\log n)^2}\right) \left(1 + \frac{1}{2}t^2n^{-2}\mathbf{E}\xi^4 \exp(2t/(\log n)^4)\right)^n \\ &\leq n^{-M}, \quad (\text{let } t = 2M \log^3 n) \end{aligned}$$

where the above second inequality follows from the Taylor expansion of the exponential function and the fact that $\hat{\xi}$ is bounded by $d_n (\leq \sqrt{n}/(\log n)^2)$, and the last one follows from the fact that $1 + x \leq e^x$ for $x > 0$. Hence we can complete the proof of (3.20).

Now let us prove (3.21). Recalling that $z = \hat{\xi}\hat{\eta} \mathbf{I}(|\hat{\xi}\hat{\eta}| \leq \epsilon\sqrt{n})$, we have

$$(3.26) \quad z + \xi \mathbf{I}(|\xi| > d_n) \hat{\eta} \mathbf{I}(|\hat{\xi}\hat{\eta}| \leq \epsilon\sqrt{n}) + \xi \eta \mathbf{I}(|\eta| > d_n) \mathbf{I}(|\hat{\xi}\hat{\eta}| \leq \epsilon\sqrt{n})$$

$$(3.27) = \xi\eta - \xi\eta \mathbf{I}(|\xi\eta| > \epsilon\sqrt{n}) - \xi\eta \mathbf{I}_2 - \xi\eta \mathbf{I}_3 - \xi\eta \mathbf{I}_4,$$

where

$$\begin{aligned} \mathbf{I}_2 &= \mathbf{I}\left(|\xi\eta| \mathbf{I}(|\xi| > d_n, |\eta| \leq d_n) \leq \epsilon\sqrt{n}\right), \\ \mathbf{I}_3 &= \mathbf{I}\left(|\xi\eta| \mathbf{I}(|\xi| \leq d_n, |\eta| > d_n) \leq \epsilon\sqrt{n}\right) \end{aligned}$$

and

$$\mathbf{I}_4 = \mathbf{I}\left(|\xi\eta| \mathbf{I}(|\xi| > d_n, |\eta| > d_n) \leq \epsilon\sqrt{n}\right).$$

For the last two terms in (3.26), noting $\mathbf{E}\xi^2 = \mathbf{E}\eta^2 = 1$, we have

$$\begin{aligned} (3.28) \quad &E\left(\xi \mathbf{I}(|\xi| > d_n) \hat{\eta} \mathbf{I}(|\hat{\xi}\hat{\eta}| \leq \epsilon\sqrt{n})\right)^2 \\ &\leq \mathbf{E}\xi^2 \mathbf{I}(|\xi| > d_n) \leq n^{-(4-\kappa)/3} \mathbf{E}|\xi|^{6-\kappa}, \end{aligned}$$

and

$$(3.29) \quad \mathbf{E} \left(\xi \eta \mathbf{I}(|\eta| > d_n) \mathbf{I}(|\hat{\xi} \hat{\eta}| \leq \epsilon \sqrt{n}) \right)^2 \leq n^{-(4-\kappa)/3} \mathbf{E} |\eta|^{6-\kappa}.$$

For the last three terms in (3.27), we have

$$(3.30) \quad \begin{aligned} \mathbf{E} \xi^2 \eta^2 \mathbf{I}_2 &\leq \mathbf{E} \xi^2 \mathbf{I}(|\xi| > d_n) \eta^2 \mathbf{I}(|\eta| \leq d_n) \\ &\leq n^{-(4-\kappa)/3} \mathbf{E} |\xi|^{6-\kappa}, \end{aligned}$$

and

$$(3.31) \quad \mathbf{E} \xi^2 \eta^2 \mathbf{I}_3 \leq n^{-(4-\kappa)/3} \mathbf{E} |\eta|^{6-\kappa}, \quad \mathbf{E} \xi^2 \eta^2 \mathbf{I}_4 = 0$$

for n sufficiently large. Combining (3.26)–(3.31) and making use of the elementary fact

$$\mathbf{E} a^2 + \mathbf{E} b^2 - 2\sqrt{\mathbf{E} a^2 \mathbf{E} b^2} \leq \mathbf{E} (a+b)^2 \leq \mathbf{E} a^2 + \mathbf{E} b^2 + 2\sqrt{\mathbf{E} a^2 \mathbf{E} b^2},$$

we have

$$(3.32) \quad \mathbf{E} z^2 = 1 + O(1/\sqrt{n}).$$

But

$$\begin{aligned} \mathbf{E}(\hat{z} - \mathbf{E} \hat{z})^2 &= \mathbf{E} z^2 - \mathbf{E} z^2 \mathbf{I}(|z| > \sqrt{n}/(\log n)^2) - (\mathbf{E} \hat{z})^2 \\ &= 1 + O(1/\sqrt{n}) \end{aligned}$$

as $n \rightarrow \infty$, where the last equality comes from (3.14) and (3.32). Hence we can repeat the proof of (3.20) to get (3.21).

Finally we will prove (3.22). The proofs of (3.23) and (3.24) are the same as that of (3.22). Noting (3.32) and the basic inequality

$$(\mathbf{E} a)^2 \leq \mathbf{E} a^2,$$

we have

$$\mathbf{E}(z + z' - \mathbf{E} z - \mathbf{E} z')^2 = 2 - 2\mathbf{E} z z' + O(1/\sqrt{n}).$$

Let us turn to the calculation of $\mathbf{E} z z'$. Recalling the definitions of z and z' , we have

$$\begin{aligned} z z' &+ \hat{\xi}^2 \hat{\eta} \hat{\eta}' \mathbf{I}(|\hat{\xi} \hat{\eta}| > \epsilon \sqrt{n}) \mathbf{I}(|\hat{\xi} \hat{\eta}'| \leq \epsilon \sqrt{n}) \\ &+ \hat{\xi}^2 \hat{\eta} \hat{\eta}' \mathbf{I}(|\hat{\xi} \hat{\eta}| \leq \epsilon \sqrt{n}) \mathbf{I}(|\hat{\xi} \hat{\eta}'| > \epsilon \sqrt{n}) \\ &+ \hat{\xi}^2 \hat{\eta} \hat{\eta}' \mathbf{I}(|\hat{\xi} \hat{\eta}| > \epsilon \sqrt{n}) \mathbf{I}(|\hat{\xi} \hat{\eta}'| > \epsilon \sqrt{n}) = \hat{\xi}^2 \hat{\eta} \hat{\eta}', \end{aligned}$$

which implies

$$\mathbf{E} z z' = O(n^{-1/2})$$

as $n \rightarrow \infty$. Hence

$$\mathbf{E}(z + z' - \mathbf{E} z - \mathbf{E} z')^2 = 2 + O(1/\sqrt{n}).$$

Write $\omega_i = (z_i + z'_i - \mathbf{E}z_i - \mathbf{E}z'_i)^2$, $i = 1, \dots, n$. We have $|\omega_i| \leq 16\epsilon^2 n$ and $\mathbf{E}\omega_i = 2 + O(1/\sqrt{n})$. Hence for n large enough,

$$\begin{aligned} \mathbf{P}\left(\left|\sum_{i=1}^n \omega_i/n - 2\right| > \epsilon_1\right) &\leq \mathbf{P}\left(\left|\sum_{i=1}^n (\omega_i - \mathbf{E}\omega_i)/n\right| > \epsilon_1/2\right) \\ &\leq \exp(-t\epsilon_1/2) \left(\mathbf{E} \exp(t(\omega_1 - \mathbf{E}\omega_1)/n)\right)^n \\ &\quad + \exp(-t\epsilon_1/2) \left(\mathbf{E} \exp(t(\mathbf{E}\omega_1 - \omega_1)/n)\right)^n \\ &\leq 2 \exp(-t\epsilon_1/2) \exp\left(\frac{1}{2}t^2 n^{-1} \mathbf{E}\omega_1^2 \exp(32\epsilon^2 t)\right) \\ &\leq n^{-M}, \quad (\text{let } t = 3M \log n/\epsilon_1) \end{aligned}$$

where ϵ satisfies

$$(3.33) \quad 96M\epsilon^2/\epsilon_1 < 1. \quad \square$$

The following lemma is a special case of Theorem 2.1 in Seoh, Ralescu and Puri [12].

Lemma 3.6. *For Spearman's correlation coefficient (1.8), $1 \leq i < j \leq p$, if all the entries of $\mathcal{X}_n$ are continuous random variables, we have as $n \rightarrow \infty$, uniformly in the range $0 < x \leq c_n n^{1/6}$, $c_n = o(1)$,*

$$\mathbf{P}\left(\frac{|r_{ij}|}{\sqrt{\text{Var } r_{ij}}} > x\right) = 2(1 - \Phi(x))(1 + o(1)),$$

where $\text{Var } r_{ij} = 1/(n-1)$.

4. EXAMPLE

Now let us give an example explaining that if

$$(4.1) \quad \lim_{x \rightarrow \infty} x^6 \mathbf{P}(|XX'| > x) > 0,$$

then (1.5) may not hold, or the limit could be different. So our weak moment condition (1.6) seems to be optimal for (1.5). In this section, the results are not rigorous.

Example 1. Let X be a symmetric random variable such that

$$(4.2) \quad \mathbf{P}(X = 0) = 1 - 2c_0 a^{-6} \log^{-1/2} a, \quad \mathbf{P}(X > x) = c_0 x^{-6} \log^{-1/2} x, \quad x > a > 1,$$

where c_0 is determined by $\mathbf{E}X^2 = 1$. The density function of the absolute continuous part of X is

$$f(x) = 6c_0 |x|^{-7} \log^{-1/2} |x| + 2^{-1} c_0 |x|^{-7} \log^{-3/2} |x|, \quad |x| > a.$$

Let X', X_{ki} , $1 \leq k \leq n, 1 \leq i \leq p$, be i.i.d. random variables with the same distribution as X . Then for $x > a^2$,

$$\begin{aligned} \mathbf{P}(|XX'| > x) &= 4 \int_a^{x/a} \mathbf{P}(X > x/x_1) f(x_1) dx_1 + 4\mathbf{P}(X > a)\mathbf{P}(X' > x/a) \\ &= 24c_0^2 x^{-6} \int_{(\log a)/\log x}^{1-(\log a)/\log x} y^{-1/2}(1-y)^{-1/2} dy \\ &\quad + 2c_0^2 x^{-6} \log^{-1} x \int_{(\log a)/\log x}^{1-(\log a)/\log x} y^{-3/2}(1-y)^{-1/2} dy \\ &\quad + 4\mathbf{P}(X > a)\mathbf{P}(X' > x/a), \end{aligned}$$

which implies that

$$(4.3) \quad \lim_{x \rightarrow \infty} x^6 \mathbf{P}(|XX'| > x) = 24c_0^2 \int_0^1 y^{-1/2}(1-y)^{-1/2} dy.$$

Note that (4.3) is weaker than the moment condition (1.6).

Choose $p = n$. We will use an indirect argument, not so rigorously, to show that for random variables X_{ki} , $1 \leq k \leq n, 1 \leq i \leq p$,

$$\mathbf{P}(nL_n^2 \leq t_p^2) \not\rightarrow \exp(-Ke^{-y/2})$$

as $n \rightarrow \infty$.

Now we assume for random variables X_{ki} , $1 \leq k \leq n, 1 \leq i \leq p$, that

$$(4.4) \quad \mathbf{P}(nL_n^2 \leq t_p^2) \rightarrow \exp(-Ke^{-y/2})$$

as $n \rightarrow \infty$. Based on the assumption (4.4), we may get a contradiction.

Define

$$\tilde{X}_{ki} = X_{ki} \mathbf{I}(|X_{ki}| \leq n^{1/3}), \quad \tilde{\mathbf{X}}_i = (\tilde{X}_{1i}, \dots, \tilde{X}_{ni})^T, \quad \overline{\tilde{X}}_i = \sum_{k=1}^n \tilde{X}_{ki}/n,$$

and

$$\tilde{\rho}_{ij} = \frac{(\tilde{\mathbf{X}}_i - \overline{\tilde{X}}_i)^T (\tilde{\mathbf{X}}_j - \overline{\tilde{X}}_j)}{\|\tilde{\mathbf{X}}_i - \overline{\tilde{X}}_i\| \|\tilde{\mathbf{X}}_j - \overline{\tilde{X}}_j\|}, \quad \tilde{L}_n = \max_{1 \leq i < j \leq p} |\tilde{\rho}_{ij}|,$$

where $1 \leq i < j \leq p, 1 \leq k \leq n$. Then,

$$\begin{aligned} (4.5) \quad \left| \mathbf{P}(nL_n^2 \leq t_p^2) - \mathbf{P}(n\tilde{L}_n^2 \leq t_p^2) \right| &\leq \mathbf{P} \left(\max_{1 \leq k \leq n, 1 \leq i \leq p} |X_{ki}| > n^{1/3} \right) \\ &\leq np \mathbf{P}(|X_{12}| > n^{1/3}) \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Write

$$\tilde{L}_{1n} = \max_{1 \leq i < j \leq p} \left| \frac{(\tilde{\mathbf{X}}_i - \overline{\tilde{X}}_i)^T (\tilde{\mathbf{X}}_j - \overline{\tilde{X}}_j)}{n(1 - 1/(\log n)^2)} \right|$$

and

$$\tilde{L}_{2n} = \max_{1 \leq i < j \leq p} \left| \frac{(\tilde{\mathbf{X}}_i - \overline{\tilde{X}}_i)^T (\tilde{\mathbf{X}}_j - \overline{\tilde{X}}_j)}{n(1 + 1/(\log n)^2)} \right|.$$

By similar arguments leading to (2.2), we have

$$\begin{aligned}
 & \mathbf{P}(n\tilde{L}_{1n}^2 \leq t_p^2) - \mathbf{P}\left(\max_{1 \leq i \leq p} \left| \|\tilde{\mathbf{X}}_i - \overline{\tilde{\mathbf{X}}_i}\|^2 - n \right| > n/(\log n)^2\right) \\
 (4.6) \quad & \leq \mathbf{P}(n\tilde{L}_n^2 \leq t_p^2) \\
 & \leq \mathbf{P}(n\tilde{L}_{2n}^2 \leq t_p^2) + \mathbf{P}\left(\max_{1 \leq i \leq p} \left| \|\tilde{\mathbf{X}}_i - \overline{\tilde{\mathbf{X}}_i}\|^2 - n \right| > n/(\log n)^2\right).
 \end{aligned}$$

By (3.20) of Lemma 3.5,

$$(4.7) \quad \mathbf{P}\left(\max_{1 \leq i \leq p} \left| \|\tilde{\mathbf{X}}_i - \overline{\tilde{\mathbf{X}}_i}\|^2 - n \right| > n/(\log n)^2\right) \leq pn^{-M} \rightarrow 0$$

as $n \rightarrow \infty$. Also note that

$$t_p^2 (1 \pm 1/(\log n)^2) = t_p^2 + o(1)$$

as $n \rightarrow \infty$. Combining (4.5)–(4.7) and the assumption (4.4) gives

$$(4.8) \quad \mathbf{P}(n\overline{L}_n^2 \leq t_p^2) \rightarrow \exp\left(-Ke^{-y/2}\right)$$

as $n \rightarrow \infty$, where

$$\overline{L}_n = \max_{1 \leq i < j \leq p} \left| (\tilde{\mathbf{X}}_i - \overline{\tilde{\mathbf{X}}_i})^T (\tilde{\mathbf{X}}_j - \overline{\tilde{\mathbf{X}}_j}) / n \right|.$$

Now for ϵ satisfying (3.33), define

$$\begin{aligned}
 (4.9) \quad & \xi_{kij} = \tilde{X}_{ki} \tilde{X}_{kj} \mathbf{I}\left(|\tilde{X}_{ki} \tilde{X}_{kj}| \leq \epsilon \sqrt{n}\right), \quad 1 \leq k \leq n, \\
 & \tilde{y}_{ij} = \frac{\sum_{k=1}^n \xi_{kij} - n \overline{\tilde{X}_i} \overline{\tilde{X}_j}}{\sqrt{n}}, \quad 1 \leq i < j \leq p,
 \end{aligned}$$

and

$$(4.10) \quad \tilde{W}_n = \max_{1 \leq i < j \leq p} |\tilde{y}_{ij}|.$$

Now following the same proof of (2.8), we have

$$(4.11) \quad \mathbf{P}(n\tilde{W}_n^2 \leq t_p^2) \rightarrow \exp\left(-Ke^{-y/2}\right)$$

as $n \rightarrow \infty$. Noting (4.8), we have

$$(4.12) \quad \mathbf{P}(n\overline{L}_n^2 \leq t_p^2) - \mathbf{P}(n\tilde{W}_n^2 \leq t_p^2) \rightarrow 0$$

as $n \rightarrow \infty$. Using the inequality $\mathbf{P}(\bigcup_i \Theta_i) \geq \sum_i \mathbf{P}(\Theta_i) - \sum_{i \neq j} \mathbf{P}(\Theta_i \cap \Theta_j)$, we conjecture

$$\begin{aligned}
 (4.13) \quad & |\mathbf{P}(n\overline{L}_n^2 \leq t_p^2) - \mathbf{P}(n\tilde{W}_n^2 \leq t_p^2)| \\
 & \geq 2^{-1}np(p-1)\mathbf{P}(\mathbf{L}_{112})
 \end{aligned}$$

$$(4.14) \quad -np^3\mathbf{P}(\mathbf{L}_{112} \cap \mathbf{L}_{113}) - (np(p-1))^2\mathbf{P}(\mathbf{L}_{112} \cap \mathbf{L}_{212})$$

$$(4.15) \quad -\mathbf{P}(n\overline{L}_n^2 > t_p^2),$$

where $\mathbf{L}_{kij} = (|\tilde{X}_{ki}\tilde{X}_{kj}| > A_0\sqrt{n})$, and A_0 is a constant specified in (4.20). For (4.13),

$$\begin{aligned} np(p-1)\mathbf{P}(\mathbf{L}_{112}) &= np(p-1)\mathbf{P}(|X_{11}X_{12}| > A_0\sqrt{n}, |X_{11}| \leq n^{1/3}, |X_{12}| \leq n^{1/3}) \\ &= 4np(p-1) \int_{A_0n^{1/6}}^{n^{1/3}} \mathbf{P}(A_0\sqrt{n}/x < X_{12} \leq n^{1/3})f(x)dx \\ (4.16) \quad &\rightarrow 24c_0^2A_0^{-6}c \end{aligned}$$

with $c = \int_{1/3}^{2/3} y^{-1/2}(1-y)^{-1/2}dy$, as $n \rightarrow \infty$. For the first term in (4.14),

$$\begin{aligned} np^3\mathbf{P}(\mathbf{L}_{112} \cap \mathbf{L}_{113}) &= np^3\mathbf{P}(|X_{11}X_{12}| > A_0\sqrt{n}, |X_{11}X_{13}| > A_0\sqrt{n}, |X_{11}| \leq n^{1/3}, \\ &\quad |X_{12}| \leq n^{1/3}, |X_{13}| \leq n^{1/3}) \\ &= 8np^3 \int_{A_0n^{1/6}}^{n^{1/3}} \mathbf{P}^2(A_0\sqrt{n}/x < X_{12} \leq n^{1/3})f(x)dx. \end{aligned}$$

Since

$$\mathbf{P}^2(A_0\sqrt{n}/x < X_{12} \leq n^{1/3}) \leq 2c_0^2 \left((A_0\sqrt{n}/x)^{-12} \log^{-1}(A_0\sqrt{n}/x) + n^{-4} \log^{-1}n^{1/3} \right),$$

we have

$$\begin{aligned} &8np^3 \int_{A_0n^{1/6}}^{n^{1/3}} \mathbf{P}^2(A_0\sqrt{n}/x < X_{12} \leq n^{1/3})f(x)dx \\ &\leq 16np^3c_0^2 \int_{A_0n^{1/6}}^{n^{1/3}} (A_0\sqrt{n}/x)^{-12} \log^{-1}(A_0\sqrt{n}/x)f(x)dx \\ &\quad + 16np^3c_0^2 \int_{A_0n^{1/6}}^{n^{1/3}} n^{-4} \log^{-1}(n^{1/3})f(x)dx \\ &=: I_1 + I_2. \end{aligned}$$

For I_1 ,

$$\begin{aligned} I_1 &= 16np^3c_0^3(A_0\sqrt{n})^{-12} \int_{A_0n^{1/6}}^{n^{1/3}} \log^{-1}(A_0\sqrt{n}/x)x^5(6\log^{-1/2}x + 2^{-1}\log^{-3/2}x)dx \\ &= 16np^3c_0^3(A_0\sqrt{n})^{-12} \int_{\log(A_0n^{1/6})/\log(A_0\sqrt{n})}^{\log(n^{1/3})/\log(A_0\sqrt{n})} (1-y)^{-1} \exp(6y\log(A_0\sqrt{n})) \\ &\quad \times \left(6y^{-1/2}\log^{-1/2}(A_0\sqrt{n}) + 2^{-1}y^{-3/2}\log^{-3/2}(A_0\sqrt{n}) \right) dy \\ &= o(1) \end{aligned}$$

as $n \rightarrow \infty$. For I_2 , we also have $I_2 = o(1)$ as $n \rightarrow \infty$. Hence

$$(4.17) \quad np^3\mathbf{P}(\mathbf{L}_{112} \cap \mathbf{L}_{113}) = o(1)$$

as $n \rightarrow \infty$.

For the second term in (4.14),

$$(4.18) \quad (np(p-1))^2 \mathbf{P}(\mathbf{L}_{112} \cap \mathbf{L}_{212}) \rightarrow (24c_0^2A_0^{-6}c)^2$$

as $n \rightarrow \infty$.

Hence combining (4.12)–(4.18) and our assumption (4.4), we have

$$(4.19) \quad \begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left(\mathbf{P}(n\bar{L}_n^2 \leq t_p^2) - \mathbf{P}(n\tilde{W}_n^2 \leq t_p^2) \right) \\ &\geq 12c_0^2 A_0^{-6} c - (24c_0^2 A_0^{-6} c)^2 - \left(1 - \exp(-K e^{-y/2}) \right). \end{aligned}$$

Let

$$(4.20) \quad A_0 = \max \left((48c_0^2 c)^{1/6}, \epsilon \right).$$

Select y so that

$$12c_0^2 A_0^{-6} c - (24c_0^2 A_0^{-6} c)^2 > 1 - \exp(-K e^{-y/2}).$$

Then (4.19) yields $0 > 0$, a contradiction if we have (4.13)–(4.15). Hence for the random variable (4.2), we may not have (4.4).

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