EVEN DIMENSIONAL MANIFOLDS AND GENERALIZED ANOMALY CANCELLATION FORMULAS

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ABSTRACT. We give a direct proof of a cancellation formula raised by Han and Zhang (2004) on the level of differential forms. We also obtain more cancellation formulas for even dimensional Riemannian manifolds with a complex line bundle involved. Relations among these cancellation formulas are discussed.

1. INTRODUCTION

In 1983, the physicists Alvarez-Gaumé and Witten [1] discovered the “miraculous cancellation” formula for gravitational anomaly which reveals a beautiful relation between the top components of the Hirzebruch $\hat{L}$ -form and $\hat{A}$ -form of a 12-dimensional smooth Riemannian manifold M as follows:

$$(1.1) \quad \left\{ \hat{L}(TM, \nabla^{TM}) \right\}^{(12)} = \left\{ 8\hat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M}) - 32\hat{A}(TM, \nabla^{TM}) \right\}^{(12)},$$

where $T_{\mathbb{C}}M$ denotes the complexification of TM and $\nabla^{T_{\mathbb{C}}M}$ is canonically induced from ∇^{TM} , the Levi-Civita connection associated to the Riemannian structure of M .

Kefeng Liu [10] established higher dimensional “miraculous cancellation” formulas for $(8k + 4)$ -dimensional Riemannian manifolds by developing modular invariance properties of characteristic forms. In [10], he proved that for each $(8k + 4)$ -dimensional smooth Riemannian manifold M the following identity holds:

$$(1.2) \quad \left\{ \hat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} = 8 \sum_{j=0}^k 2^{6k-6j} \left\{ \hat{A}(TM, \nabla^{TM}) \text{ch} b_j \right\}^{(8k+4)},$$

where the b_j 's are elements in $KO(M) \otimes \mathbb{C}$. Liu's formula refines the argument of Landweber [9] to the level of differential forms and is a higher dimensional generalization of (1.1). One can also use (1.2) to deduce the Ochanine divisibility [12] from the Atiyah-Hirzebruch divisibility [3] for $(8k + 4)$ -dimensional smooth closed spin manifolds. In fact, the Atiyah-Hirzebruch divisibility guarantees that $\langle \hat{A}(TM, \nabla^{TM}) \text{ch}(E \otimes \mathbb{C}), [M] \rangle$ is even when M is a smooth closed $(8k + 4)$ -dimensional spin manifold and E is a real vector bundle on M . Thus (1.2) implies that the signature of M is divisible by 16. This is just the Ochanine divisibility, which generalizes the famous Rokhlin divisibility for spin 4-manifolds (when $k = 0$).

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In [6, 7], for each $(8k+4)$ -dimensional smooth Riemannian manifold M , a more general cancellation formula that involves a complex line bundle on M is established. To be precise, the authors proved that for each $(8k+4)$ -dimensional smooth Riemannian manifold M and a complex line bundle ξ on M , or equivalently a rank 2 real oriented bundle on M , the following identity holds:

$$(1.3) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{e}{2}\right)} \right\}^{(8k+4)} = 8 \sum_{j=0}^k 2^{6k-6j} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch} b'_j \cosh\left(\frac{e}{2}\right) \right\}^{(8k+4)},$$

where the b'_j s are elements in $KO(M) \otimes \mathbb{C}$, dependent on (M, ∇^{TM}) and (ξ, ∇^ξ) ; $e = e(\xi, \nabla^\xi)$ is the Euler form of (ξ, ∇^ξ) . Putting $k = 1$ in (1.3), one has

$$(1.4) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{e}{2}\right)} \right\}^{(12)} = \left\{ \left[8\widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M}) - 32\widehat{A}(TM, \nabla^{TM}) - 24 \widehat{A}(TM, \nabla^{TM}) (e^c + e^{-c} - 2) \right] \cosh\left(\frac{e}{2}\right) \right\}^{(12)}.$$

This is a twisted version of the original miraculous cancellation formula (1.1). When formula (1.3) is applied to spin^c manifolds, the authors are led directly to a refined version of [11, Theorem 4.2], which is a beautiful analytic version of the Ochanine congruence formula [12].

In [7], to obtain a direct proof of [11, Theorem 4.1], an analytic version of the Finashin congruence formula [5], the authors applied the following identity:

$$(1.5) \quad \begin{aligned} & \frac{1}{8} \int_B \widehat{L}(TB, \nabla^{TB}) \frac{\sinh\left(\frac{e}{2}\right)}{\cosh\left(\frac{e}{2}\right)} \\ &= \sum_{r=0}^k 2^{6k-6r} \int_B \widehat{A}(TB, \nabla^{TB}) \left(\text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) \right. \\ & \quad \left. - \cosh\left(\frac{e}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})) \right) \frac{1}{2 \sinh\left(\frac{e}{2}\right)}, \end{aligned}$$

where (B, ∇^{TB}) is an $(8k+2)$ -dimensional smooth Riemannian manifold, (N, ∇^N) is a rank two real oriented Euclidean vector bundle on B and $e = e(N, \nabla^N)$ is the associated Euler form of (N, ∇^N) . This identity is very crucial in their proof, and they proved it by using the cobordism argument. They also pointed out that (1.5) can be refined to the level of differential forms and one should be able to prove this directly by still using the modular invariance method without passing to the cobordism argument. One of the purposes of this article is to refine (1.5) to the level of differential forms (Theorem 3.1) and give such a direct proof. We also obtain an analogous formula for an $(8k+6)$ -dimensional Riemannian manifold (Theorem 3.2). One can view Theorem 3.1 and Theorem 3.2 as generalized miraculous cancellation formulas on $(8k+2)$ and $(8k+6)$ -dimensional smooth Riemannian manifolds respectively.

With a twisting complex line bundle, we also obtain a unified cancellation formula (Theorem 3.3) for each even dimensional smooth Riemannian manifold via

the same argument. When the manifold is of dimension $(8k + 4)$ and the bundle is trivial, our cancellation formula becomes Liu's cancellation formula (1.2). This unified formula is still a product of the modular invariance method developed in [10]. Finally, on the level of characteristic numbers, we discuss relations among cancellation formulas on manifolds of different dimensions by applying the method of integration along the fibre.

2. MODULAR INVARIANCE AND CHARACTERISTIC FORMS

The purpose of this section is to review the necessary knowledge on characteristic forms and modular forms that we are going to use. We also briefly review cancellation formulas obtained in [10] and [6, 7].

2.1. Characteristic forms. Let M be an even dimensional smooth Riemannian manifold. Let ∇^{TM} be the associated Levi-Civita connection and $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM} . Let $\hat{A}(TM, \nabla^{TM})$, $\hat{L}(TM, \nabla^{TM})$ be the Hirzebruch characteristic forms defined respectively by (cf. [15])

$$(2.1) \quad \begin{aligned} \hat{A}(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{4\pi} R^{TM}}{\sinh \left(\frac{\sqrt{-1}}{4\pi} R^{TM} \right)} \right), \\ \hat{L}(TM, \nabla^{TM}) &= \det^{1/2} \left(\frac{\frac{\sqrt{-1}}{2\pi} R^{TM}}{\tanh \left(\frac{\sqrt{-1}}{4\pi} R^{TM} \right)} \right). \end{aligned}$$

Let E, F be two Hermitian vector bundles over M carrying Hermitian connections ∇^E, ∇^F respectively. Let $R^E = (\nabla^E)^2$ (resp. $R^F = (\nabla^F)^2$) be the curvature of ∇^E (resp. ∇^F). If we set the formal difference $G = E - F$, then G carries an induced Hermitian connection ∇^G in an obvious sense. We define the associated Chern character form as (cf. [15])

$$(2.2) \quad \text{ch}(G, \nabla^G) = \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^E \right) \right] - \text{tr} \left[\exp \left(\frac{\sqrt{-1}}{2\pi} R^F \right) \right].$$

In the rest of this paper, for simplicity, when there is no confusion about the Hermitian connection ∇^E on a Hermitian vector bundle E , we will only write $\text{ch}(E)$ for the associated Chern character form.

For any complex number t , let

$$\Lambda_t(E) = \mathbb{C}|_M + tE + t^2\Lambda^2(E) + \cdots, \quad S_t(E) = \mathbb{C}|_M + tE + t^2S^2(E) + \cdots$$

denote respectively the total exterior and symmetric powers of E , which live in $K(M)[[t]]$. The following relations between these two operations [2, Chap. 3] hold:

$$(2.3) \quad S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}.$$

Moreover, if $\{\omega_i\}, \{\omega_j'\}$ are formal Chern roots for Hermitian vector bundles E, F respectively, then [8, Chap. 1]

$$(2.4) \quad \text{ch}(\Lambda_t(E)) = \prod_i (1 + e^{\omega_i t}).$$

Therefore, we have the following formulas for Chern character forms:

$$(2.5) \quad \text{ch}(S_t(E)) = \frac{1}{\text{ch}(\Lambda_{-t}(E))} = \frac{1}{\prod_i (1 - e^{\omega_i t})},$$

$$(2.6) \quad \text{ch}(\Lambda_t(E - F)) = \frac{\text{ch}(\Lambda_t(E))}{\text{ch}(\Lambda_t(F))} = \frac{\prod_i (1 + e^{\omega_i t})}{\prod_j (1 + e^{\omega_j' t})}.$$

If W is a real Euclidean vector bundle over M carrying a Euclidean connection ∇^W , its complexification $W_{\mathbb{C}} = W \otimes \mathbb{C}$ is a complex vector bundle over M carrying a canonically induced Hermitian metric from the Euclidean metric of W as well as a Hermitian connection $\nabla^{W_{\mathbb{C}}}$ induced from ∇^W .

2.2. Results needed on the Jacobi theta functions and modular forms.

The four Jacobi theta functions are defined as follows (cf. [4]):

$$(2.7) \quad \theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} \left[(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j) \right],$$

$$(2.8) \quad \theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} \left[(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j) \right],$$

$$(2.9) \quad \theta_2(v, \tau) = \prod_{j=1}^{\infty} \left[(1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 - e^{-2\pi\sqrt{-1}v} q^{j-1/2}) \right],$$

$$(2.10) \quad \theta_3(v, \tau) = \prod_{j=1}^{\infty} \left[(1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^{j-1/2})(1 + e^{-2\pi\sqrt{-1}v} q^{j-1/2}) \right],$$

where $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$, the upper half complex plane.

Let

$$(2.11) \quad \theta'(0, \tau) = \left. \frac{\partial \theta(v, \tau)}{\partial v} \right|_{v=0}.$$

Then the following Jacobi identity (cf. [4]) holds:

$$(2.12) \quad \theta'(0, \tau) = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau).$$

Denote $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$, the modular group.

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be the two generators of $SL_2(\mathbb{Z})$. They act on $\mathbb{H}$ by $S\tau = -1/\tau$, $T\tau = \tau + 1$. One has the following transformation laws of

theta functions under the actions of S and T (cf. [4]):

(2.13)

$$\theta(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta(v, \tau), \quad \theta(v, -1/\tau) = \frac{1}{\sqrt{-1}} \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta(\tau v, \tau);$$

(2.14)

$$\theta_1(v, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta_1(v, \tau), \quad \theta_1(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_2(\tau v, \tau);$$

$$(2.15) \quad \theta_2(v, \tau + 1) = \theta_3(v, \tau), \quad \theta_2(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_1(\tau v, \tau);$$

$$(2.16) \quad \theta_3(v, \tau + 1) = \theta_2(v, \tau), \quad \theta_3(v, -1/\tau) = \left(\frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi\sqrt{-1}\tau v^2} \theta_3(\tau v, \tau);$$

$$(2.17) \quad \theta'(0, \tau + 1) = e^{\frac{\pi\sqrt{-1}}{4}} \theta'(0, \tau), \quad \theta'(0, -1/\tau) = \frac{1}{\sqrt{-1}} \tau^{3/2} \theta'(0, \tau).$$

Definition 2.1. A modular form over Γ , a subgroup of $SL_2(\mathbb{Z})$, is a holomorphic function $f(\tau)$ on $\mathbb{H} \cup \{\infty\}$ such that

$$(2.18) \quad f(g\tau) := f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau), \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where $\chi : \Gamma \rightarrow \mathbb{C}^*$ is a character of Γ . k is called the weight of f .

Denote by $\theta_j = \theta_j(0, \tau)$, $1 \leq j \leq 3$, and define

$$(2.19) \quad \delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \quad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4,$$

$$(2.20) \quad \delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \quad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4.$$

They admit the Fourier expansions (cf. [9])

$$(2.21) \quad \delta_1(\tau) = \frac{1}{4} + 6q + \cdots, \quad \varepsilon_1(\tau) = \frac{1}{16} - q + \cdots,$$

$$(2.22) \quad \delta_2(\tau) = -\frac{1}{8} - 3q^{1/2} + \cdots, \quad \varepsilon_2(\tau) = q^{1/2} + \cdots,$$

where the “ $\cdots$ ” terms are higher degree terms all having integral coefficients. They also satisfy the following transformation laws under S (cf. [9] and [10]):

$$(2.23) \quad \delta_2(-1/\tau) = \tau^2 \delta_1(\tau), \quad \varepsilon_2(-1/\tau) = \tau^4 \varepsilon_1(\tau).$$

Let $\Gamma_0(2)$, $\Gamma^0(2)$ be the two subgroups of $SL_2(\mathbb{Z})$ defined by

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{2\mathbb{Z}} \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2\mathbb{Z}} \right\}.$$

Then T , ST^2ST are the two generators of $\Gamma_0(2)$, while STS , T^2STS are the two generators of $\Gamma^0(2)$.

The following weaker version of [10, Lemma 2] will be used in the next section.

Lemma 2.1. *One has that δ_2 (resp. ε_2) is a modular form of weight 2 (resp. 4) over $\Gamma^0(2)$. Furthermore, $\mathcal{M}_{\mathbb{R}}(\Gamma^0(2)) = \mathbb{R}[\delta_2(\tau), \varepsilon_2(\tau)]$, where $\mathcal{M}_{\mathbb{R}}(\Gamma)$ denotes the ring of modular forms over Γ with real Fourier coefficients.*

2.3. Cancellation formulas for $(8k+4)$ -dimensional Riemannian manifolds. Let M be an $(8k+4)$ -dimensional smooth Riemannian manifold with Levi-Civita connection ∇^{TM} . Let $\nabla^{T_{\mathbb{C}}M}$ be the canonically induced Hermitian connection on $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. Let V be a rank $2l$ Euclidean vector bundle over M carrying a Euclidean connection ∇^V . Let ξ be a rank two oriented Euclidean vector bundle carrying a Euclidean connection ∇^ξ . Let $\nabla^{\xi_{\mathbb{C}}}$ be the canonically induced Hermitian connection on $\xi_{\mathbb{C}} = \xi \otimes \mathbb{C}$. Let $c = e(\xi, \nabla^\xi)$ be the Euler form of ξ canonically associated to ∇^ξ . If W is a complex vector bundle over M , denote $\widetilde{W} = W - \mathbb{C}^{\dim_{\mathbb{C}} W}|_M$ in $K(M)$.

Using the same notation as in Section 2.1, we construct two formal power series in $q^{1/2}$ with coefficients in the semigroup generated by complex vector bundles over M , which are introduced in [6, 7] to prove Theorem 2.1 in this text,

$$(2.24) \quad \Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{V_{\mathbb{C}}} - 2\widetilde{\xi_{\mathbb{C}}}) \\ \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}),$$

$$(2.25) \quad \Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}M}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(\widetilde{V_{\mathbb{C}}} - 2\widetilde{\xi_{\mathbb{C}}}) \\ \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\widetilde{\xi_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{\xi_{\mathbb{C}}}).$$

$\Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ and $\Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$ admit formal Fourier expansions in $q^{1/2}$ as

$$(2.26) \quad \Theta_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = A_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) + A_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})q^{1/2} + \cdots,$$

$$(2.27) \quad \Theta_2(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) = B_0(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}}) + B_1(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})q^{1/2} + \cdots,$$

where the A_j 's and B_j 's are elements in the semigroup formally generated by Hermitian vector bundles over M . Moreover, they carry canonically induced Hermitian connections denoted by ∇^{A_j} and ∇^{B_j} respectively, and $\nabla^{\Theta_i(M, V, \xi)}$ are the induced Hermitian connections with $q^{1/2}$ -coefficients on Θ_i from the ∇^{A_j} and ∇^{B_j} .

Now, we can state a cancellation formula, which is obtained in [6, 7].

Theorem 2.1 (Han and Zhang, 2003). *If the equality for the first Pontrjagin forms $p_1(TM, \nabla^{TM}) = p_1(V, \nabla^V)$ holds, then one has an equality for $(8k+4)$ -forms,*

$$(2.28) \quad \left\{ \frac{\widehat{A}(TM, \nabla^{TM}) \det^{1/2} \left(2 \cosh \left(\frac{\sqrt{-1}}{4\pi} R^V \right) \right)}{\cosh^2 \left(\frac{e}{2} \right)} \right\}^{(8k+4)} \\ = 2^{l+2k+1} \sum_{r=0}^k 2^{-6r} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})) \cosh \left(\frac{e}{2} \right) \right\}^{(8k+4)},$$

where each $b_r(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $0 \leq r \leq k$, is a canonical integral linear combination of $B_j(T_{\mathbb{C}}M, V_{\mathbb{C}}, \xi_{\mathbb{C}})$, $0 \leq j \leq r$.

From now on, denote $\Theta_i(T_{\mathbb{C}}M, T_{\mathbb{C}}M, \xi_{\mathbb{C}})$ as $\Theta_i(T_{\mathbb{C}}M, \xi_{\mathbb{C}})$ and $b_r(T_{\mathbb{C}}M, T_{\mathbb{C}}M, \xi_{\mathbb{C}})$ as $b_r(T_{\mathbb{C}}M, \xi_{\mathbb{C}})$.

Taking $V = TM$ in (2.28), we have

Corollary 2.1. *The following identity of characteristic forms holds:*

$$(2.29) \quad \left\{ \frac{\widehat{L}(TM, \nabla^{TM})}{\cosh^2\left(\frac{e}{2}\right)} \right\}^{(8k+4)} \\ = 8 \sum_{r=0}^k 2^{6k-6r} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbb{C}}M, \xi_{\mathbb{C}})) \cosh\left(\frac{e}{2}\right) \right\}^{(8k+4)}.$$

Moreover, taking $\xi = \mathbb{R}^2$ in (2.29), we have

Corollary 2.2. *The following identity of characteristic forms holds:*

$$(2.30) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} \\ = 8 \sum_{r=0}^k 2^{6k-6r} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch}(b_r(T_{\mathbb{C}}M, \mathbb{C}^2)) \right\}^{(8k+4)}.$$

(2.30) is exactly the cancellation formula obtained in [10].

3. CANCELLATION FORMULAS FOR EVEN DIMENSIONAL RIEMANNIAN MANIFOLDS

Let B be an $(8k+2)$ -dimensional smooth oriented Riemannian manifold with Levi-Civita connection ∇^{TB} . Let $\pi : N \rightarrow B$ be a rank two real oriented Euclidean vector bundle over B carrying the Euclidean connection ∇^N . Let $R^N = (\nabla^N)^2$ be the curvature of ∇^N and $e = e(N, \nabla^N)$ be the Euler form of (N, ∇^N) . Then we have the following cancellation formula for characteristic numbers:

$$(3.1) \quad \frac{1}{8} \int_B \widehat{L}(TB, \nabla^{TB}) \frac{\sinh\left(\frac{e}{2}\right)}{\cosh\left(\frac{e}{2}\right)} \\ = \sum_{r=0}^k 2^{6k-6r} \int_B \widehat{A}(TB, \nabla^{TB}) \left(\text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) \right. \\ \left. - \cosh\left(\frac{e}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})) \right) \frac{1}{2 \sinh\left(\frac{e}{2}\right)},$$

which is applied in [7] to give the analytic Finashin congruence [11] a direct analytic proof via a beautiful Rokhlin type congruence formula obtained in [14]. In fact, formula (3.1) holds on the level of differential forms. In §3.1 we prove the form-level version of (3.1) directly by applying the modular invariance argument. Also in this subsection, we give the $(8k+6)$ -dimensional analogue without proof.

In §3.2, for all even dimensional smooth Riemannian manifolds, we obtain a general type of cancellation formulas, which imply Liu's formula (2.30) as a special case.

3.1. The direct proof for the form-level version of (3.1). We make the same assumptions and use the same notation as in §2.3. Define

(3.2)

$$\begin{aligned} \Theta_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}B + N_{\mathbb{C}}}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbb{C}}B + N_{\mathbb{C}} - 2\widetilde{N_{\mathbb{C}}}}) \\ &\quad \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{(r-\frac{1}{2})}}(\widetilde{N_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{(s-\frac{1}{2})}}(\widetilde{N_{\mathbb{C}}}), \end{aligned}$$

(3.3)

$$\begin{aligned} \Theta_2(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}) &= \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}B + N_{\mathbb{C}}}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{(m-\frac{1}{2})}}(\widetilde{T_{\mathbb{C}}B + N_{\mathbb{C}} - 2\widetilde{N_{\mathbb{C}}}}) \\ &\quad \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{(r-\frac{1}{2})}}(\widetilde{N_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^s}(\widetilde{N_{\mathbb{C}}}), \end{aligned}$$

and denote by $A_i(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})$ and $B_i(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})$ the coefficients in their Fourier expansions respectively. Then we have the following equality for $(8k+2)$ -forms associated to B and N .

Theorem 3.1. *The following identity holds:*

$$(3.4) \quad \left\{ \widehat{L}(TB, \nabla^{TB}) \frac{\sinh\left(\frac{e}{2}\right)}{\cosh\left(\frac{e}{2}\right)} \right\}^{(8k+2)} = 8 \sum_{r=0}^k 2^{6k-6r} h_r,$$

where each h_r , $0 \leq r \leq k$, is a canonical integral linear combination of the characteristic forms

$$\left\{ \widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \cosh\left(\frac{e}{2}\right) \text{ch}(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{2 \sinh\left(\frac{e}{2}\right)} \right\}^{(8k+2)},$$

$0 \leq j \leq r$. Actually, h_r is just

$$\left\{ \widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \cosh\left(\frac{e}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{2 \sinh\left(\frac{e}{2}\right)} \right\}^{(8k+2)}.$$

Proof. As in [10], we use the formal Chern roots $\{\pm 2\pi\sqrt{-1}x_j\}$ for $(T_{\mathbb{C}}B, \nabla^{T_{\mathbb{C}}B})$. Let $e = 2\pi\sqrt{-1}u$, $q = e^{2\pi\sqrt{-1}\tau}$ with $\tau \in \mathbb{H}$, the upper half complex plane. Set

$$\begin{aligned} (3.5) \quad Q_1(\tau) &= \widehat{L}(TB, \nabla^{TB}) \frac{\cosh\left(\frac{e}{2}\right)}{\sinh\left(\frac{e}{2}\right)} \\ &\quad \cdot \left(\text{ch}(\Theta_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \frac{\text{ch}(\Theta_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{\cosh^2\left(\frac{e}{2}\right)} \right) \\ &= 2^{4k+1} \left(\prod_{j=1}^{4k+1} \frac{\pi x_j}{\sin(\pi x_j)} \right) \left(\prod_{j=1}^{4k+1} \cos(\pi x_j) \right) \frac{\cos(\pi u)}{\sin(\pi u)} \\ &\quad \cdot \left(\text{ch}(\Theta_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \frac{\text{ch}(\Theta_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{\cos^2(\pi u)} \right), \end{aligned}$$

$$\begin{aligned}
(3.6) \quad Q_2(\tau) &= \widehat{A}(TB, \nabla^{TB}) \frac{1}{2 \sinh\left(\frac{e}{2}\right)} \left(\text{ch}(\Theta_2(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) \right. \\
&\quad \left. - \cosh\left(\frac{e}{2}\right) \text{ch}(\Theta_2(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})) \right) \\
&= \left(\prod_{j=1}^{4k+1} \frac{\pi x_j}{\sin(\pi x_j)} \right) \frac{1}{2 \sin(\pi u)} \left(\text{ch}(\Theta_2(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) \right. \\
&\quad \left. - \cos(\pi u) \text{ch}(\Theta_2(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})) \right).
\end{aligned}$$

We can actually write $Q_1(\tau)$ and $Q_2(\tau)$ in terms of the Jacobi theta-functions as

$$\begin{aligned}
(3.7) \quad Q_1(\tau) &= 2^{4k+1} \left(\prod_{j=1}^{4k+1} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_1(x_j, \tau)}{\theta_1(0, \tau)} \right) \frac{\theta'(0, \tau)}{\theta(u, \tau)} \\
&\quad \cdot \left(\frac{\theta_1(u, \tau)}{\theta_1(0, \tau)} - \frac{\theta_1(0, \tau)}{\theta_1(u, \tau)} \frac{\theta_3(u, \tau)}{\theta_3(0, \tau)} \frac{\theta_2(u, \tau)}{\theta_2(0, \tau)} \right)
\end{aligned}$$

and

$$\begin{aligned}
(3.8) \quad Q_2(\tau) &= \frac{1}{2} \left(\prod_{j=1}^{4k+1} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_2(x_j, \tau)}{\theta_2(0, \tau)} \right) \frac{\theta'(0, \tau)}{\theta(u, \tau)} \\
&\quad \cdot \left(\frac{\theta_2(u, \tau)}{\theta_2(0, \tau)} - \frac{\theta_2(0, \tau)}{\theta_2(u, \tau)} \frac{\theta_3(u, \tau)}{\theta_3(0, \tau)} \frac{\theta_1(u, \tau)}{\theta_1(0, \tau)} \right).
\end{aligned}$$

In fact, by (3.2), (2.5) and (2.6),

$$\begin{aligned}
(3.9) \quad \text{ch}(\Theta_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})) &= \prod_{n=1}^{\infty} \frac{\text{ch} \Lambda_{-q^n}(\mathbb{C}^{8k+2})}{\text{ch} \Lambda_{-q^n}(T_{\mathbb{C}}B)} \prod_{n=1}^{\infty} \frac{\text{ch} \Lambda_{-q^n}(\mathbb{C}^2)}{\text{ch} \Lambda_{-q^n}(N_{\mathbb{C}})} \\
&\quad \cdot \prod_{m=1}^{\infty} \frac{\text{ch} \Lambda_{q^m}(T_{\mathbb{C}}B)}{\text{ch} \Lambda_{q^m}(\mathbb{C}^{8k+2})} \prod_{m=1}^{\infty} \frac{\text{ch} \Lambda_{q^m}(\mathbb{C}^2)}{\text{ch} \Lambda_{q^m}(N_{\mathbb{C}})} \prod_{r=1}^{\infty} \frac{\text{ch} \Lambda_{q^{(r-\frac{1}{2})}}(N_{\mathbb{C}})}{\text{ch} \Lambda_{q^{(r-\frac{1}{2})}}(\mathbb{C}^2)} \prod_{s=1}^{\infty} \frac{\text{ch} \Lambda_{-q^{(s-\frac{1}{2})}}(N_{\mathbb{C}})}{\text{ch} \Lambda_{-q^{(s-\frac{1}{2})}}(\mathbb{C}^2)}.
\end{aligned}$$

From (2.4), the Jacobi identity (2.12) and (2.7), one deduces directly that

$$\begin{aligned}
(3.10) \quad &\prod_{j=1}^{4k+1} \frac{\pi x_j}{\sin(\pi x_j)} \prod_{n=1}^{\infty} \frac{\text{ch} \Lambda_{-q^n}(\mathbb{C}^{8k+2})}{\text{ch} \Lambda_{-q^n}(T_{\mathbb{C}}B)} \\
&= \prod_{j=1}^{4k+1} x_j \frac{\pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)}{\theta(x_j, \tau)} = \prod_{j=1}^{4k+1} x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)}.
\end{aligned}$$

Similarly, from (2.4) and (2.7)–(2.10), one deduces that

$$(3.11) \quad \prod_{j=1}^{4k+1} \cos(\pi x_j) \prod_{m=1}^{\infty} \frac{\text{ch} \Lambda_{q^m}(T_{\mathbb{C}} B)}{\text{ch} \Lambda_{q^m}(\mathbb{C}^{8k+2})} = \prod_{j=1}^{4k+1} \frac{\theta_1(x_j, \tau)}{\theta_1(0, \tau)}, \quad \prod_{r=1}^{\infty} \frac{\text{ch} \Lambda_{q^{(r-\frac{1}{2})}}(N_{\mathbb{C}})}{\text{ch} \Lambda_{q^{(r-\frac{1}{2})}}(\mathbb{C}^2)} = \frac{\theta_3(u, \tau)}{\theta(0, \tau)},$$

$$\frac{1}{\sin(\pi u)} \prod_{n=1}^{\infty} \frac{\text{ch} \Lambda_{-q^n}(\mathbb{C}^2)}{\text{ch} \Lambda_{-q^n}(N_{\mathbb{C}})} = \frac{\theta'(0, \tau)}{\theta(u, \tau)}, \quad \frac{1}{\cos(\pi u)} \prod_{m=1}^{\infty} \frac{\text{ch} \Lambda_{q^m}(\mathbb{C}^2)}{\text{ch} \Lambda_{q^m}(N_{\mathbb{C}})} = \frac{\theta_1(0, \tau)}{\theta_1(u, \tau)},$$

$$\prod_{r=1}^{\infty} \frac{\text{ch} \Lambda_{-q^{(r-1/2)}}(N_{\mathbb{C}})}{\text{ch} \Lambda_{-q^{(r-1/2)}}(\mathbb{C}^2)} = \frac{\theta_1(u, \tau)}{\theta_1(0, \tau)}.$$

Putting (3.5) and (3.9)–(3.11) together, we get (3.7). By doing similar computations, one also gets (3.8).

Let $P_1(\tau) = \{Q_1(\tau)\}^{(8k+2)}$, $P_2(\tau) = \{Q_2(\tau)\}^{(8k+2)}$ be the $(8k+2)$ -components of $Q_1(\tau)$, $Q_2(\tau)$ respectively. Applying the transformation laws (2.13)–(2.17) to $P_1(\tau)$ and $P_2(\tau)$, we find that $P_1(\tau)$ is a modular form of weight $4k+2$ over $\Gamma_0(2)$, while $P_2(\tau)$ is a modular form of weight $4k+2$ over $\Gamma^0(2)$. Moreover, the following identity holds:

$$(3.12) \quad P_1(-1/\tau) = 2^{4k+2} P_2(\tau).$$

Observe that at any point $x \in M$, up to the volume form determined by the metric on $T_x M$, both $P_i(\tau)$, $i = 1, 2$, can be viewed as a power series of $q^{1/2}$ with real Fourier coefficients. Thus, one can apply Lemma 2.1 to $P_2(\tau)$ to get, at x , that

$$(3.13) \quad P_2(\tau) = h_0(8\delta_2)^{2k+1} + h_0(8\delta_2)^{2k-1}\varepsilon_2 + \cdots + h_k(8\delta_2)\varepsilon_2^k,$$

where each h_r , $0 \leq r \leq k$, is a (canonically) finite integral linear combination of the forms

$$\left\{ \widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}(B_j(T_{\mathbb{C}} B + N_{\mathbb{C}}, \mathbb{C}^2)) - \cosh\left(\frac{\varepsilon}{2}\right) \text{ch}(B_j(T_{\mathbb{C}} B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{2 \sinh\left(\frac{\varepsilon}{2}\right)} \right\}^{(8k+2)},$$

$0 \leq j \leq r$.

By (2.23) and (3.12), we have

$$(3.14) \quad \begin{aligned} P_1(\tau) &= 2^{4k+2} \frac{1}{\tau^{4k+2}} P_2(-1/\tau) \\ &= 2^{4k+2} \frac{1}{\tau^{4k+2}} \left[h_0(8\delta_2(-1/\tau))^{2k+1} + h_1(8\delta_2(-1/\tau))^{2k-1} \varepsilon_2(-1/\tau) + \cdots \right. \\ &\quad \left. + h_k(8\delta_2(-1/\tau)) (\varepsilon_2(-1/\tau))^k \right] \\ &= 2^{4k+2} [h_0(8\delta_1)^{2k+1} + h_1(8\delta_1)^{2k-1} \varepsilon_1 + \cdots + h_k(8\delta_1) \varepsilon_1^k]. \end{aligned}$$

By (2.21) and (3.4) and by setting $q = 0$ in (3.14), we have

$$(3.15) \quad \left\{ \widehat{L}(TB, \nabla^{TB}) \frac{\cosh\left(\frac{\varepsilon}{2}\right)}{\sinh\left(\frac{\varepsilon}{2}\right)} \left(1 - \frac{1}{\cosh^2\left(\frac{\varepsilon}{2}\right)}\right) \right\}^{(8k+2)} = 2^{6k+3} \sum_{r=0}^k 2^{-6r} h_r.$$

Therefore,

$$(3.16) \quad \left\{ \widehat{L}(TB, \nabla^{TB}) \frac{\sinh\left(\frac{\epsilon}{2}\right)}{\cosh\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+2)} = 8 \sum_{r=0}^k 2^{6k-6r} h_r.$$

We also need to show that each $h_r, 0 \leq r \leq k$, can be expressed through a canonical integral linear combination of

$$(3.17) \quad \left\{ \widehat{A}(TB, \nabla^{TB}) \frac{1}{2 \sinh\left(\frac{\epsilon}{2}\right)} \left(\text{ch}\left(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)\right) - \cosh\left(\frac{\epsilon}{2}\right) \text{ch}\left(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})\right) \right) \right\}^{(8k+2)},$$

$0 \leq j \leq r$, with coefficients not depending on $x \in M$. As in [10], one can use the induction method to prove this fact easily by comparing the coefficients of $q^{j/2}, j \geq 0$, between the two sides of (3.13). For consideration of the volume of this paper, we do not give details here but only write down the explicit expressions for h_0 and h_1 as follows:

$$(3.18) \quad h_0 = - \left\{ \widehat{A}(TB, \nabla^{TB}) \frac{1}{2 \sinh\left(\frac{\epsilon}{2}\right)} \left(1 - \cosh\left(\frac{\epsilon}{2}\right) \right) \right\}^{(8k+2)},$$

$$(3.19) \quad h_1 = - \left\{ \widehat{A}(TB, \nabla^{TB}) \frac{1}{2 \sinh\left(\frac{\epsilon}{2}\right)} \left[\text{ch}\left(B_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)\right) - \cosh\left(\frac{\epsilon}{2}\right) \text{ch}\left(B_1(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})\right) - 24(2k+1) \left(1 - \cosh\left(\frac{\epsilon}{2}\right) \right) \right] \right\}^{(8k+2)}.$$

In summary, we get

$$(3.20) \quad \left\{ \widehat{L}(TB, \nabla^{TB}) \frac{\sinh\left(\frac{\epsilon}{2}\right)}{\cosh\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+2)} = 8 \sum_{r=0}^k 2^{6k-6r} h_r,$$

where each $h_r, 0 \leq r \leq k$, is a canonical integral linear combination of the characteristic forms

$$\left\{ \widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}\left(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)\right) - \cosh\left(\frac{\epsilon}{2}\right) \text{ch}\left(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})\right)}{2 \sinh\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+2)},$$

$0 \leq j \leq r$.

Since both h_r 's and b_r 's are canonically determined by induction, one easily finds that,

$$h_r = \left\{ \widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}\left(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)\right) - \cosh\left(\frac{\epsilon}{2}\right) \text{ch}\left(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})\right)}{2 \sinh\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+2)}.$$

□

Remark 3.1. It's not hard to see that each

$$\widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}\left(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)\right) - \cosh\left(\frac{\epsilon}{2}\right) \text{ch}\left(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}})\right)}{2 \sinh\left(\frac{\epsilon}{2}\right)}$$

makes sense as a differential form.

For $(8k + 6)$ -dimensional manifolds, we have an analogue of Theorem 3.1. Let B be an $(8k + 6)$ -dimensional smooth oriented Riemannian manifold, and let all of

the notation in the following theorem have the same sense as above. Then we can argue verbatim to get

Theorem 3.2. *The following cancellation formula holds:*

$$\left\{ \widehat{L}(TB, \nabla^{TB}) \frac{\sinh\left(\frac{\epsilon}{2}\right)}{\cosh\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+6)} = 64 \sum_{r=0}^k 2^{6k-6r} h_r,$$

where each h_r , $0 \leq r \leq k$, is a canonical integral linear combination of the characteristic forms

$$\left\{ \widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \cosh\left(\frac{\epsilon}{2}\right) \text{ch}(B_j(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{2 \sinh\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+6)},$$

$0 \leq j \leq r$. Actually, h_r is just

$$\left\{ \widehat{A}(TB, \nabla^{TB}) \frac{\text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \cosh\left(\frac{\epsilon}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{2 \sinh\left(\frac{\epsilon}{2}\right)} \right\}^{(8k+6)}.$$

3.2. A general type of cancellation formula for even dimensional Riemannian manifolds. In this subsection, let's continue to discuss a general type of cancellation formula. Let B be a $2d$ -dimensional smooth oriented Riemannian manifold, m be a nonnegative integer and N be a complex line bundle on B . Define

$$(3.21) \quad \Theta'_1(T_{\mathbb{C}}B, m, N_{\mathbb{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}B} - m\widetilde{N_{\mathbb{C}}}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\widetilde{T_{\mathbb{C}}B} - m\widetilde{N_{\mathbb{C}}}),$$

$$(3.22) \quad \Theta'_2(T_{\mathbb{C}}B, m, N_{\mathbb{C}}) = \bigotimes_{n=1}^{\infty} S_{q^n}(\widetilde{T_{\mathbb{C}}B} - m\widetilde{N_{\mathbb{C}}}) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-\frac{1}{2}}}(\widetilde{T_{\mathbb{C}}B} - m\widetilde{N_{\mathbb{C}}}),$$

and assume they admit a Fourier expansion as follows:

$$(3.23) \quad \Theta'_1(T_{\mathbb{C}}B, m, N_{\mathbb{C}}) = A'_0(T_{\mathbb{C}}B, m, N_{\mathbb{C}}) + A'_1(T_{\mathbb{C}}B, m, N_{\mathbb{C}})q^{1/2} + \dots,$$

$$(3.24) \quad \Theta'_2(T_{\mathbb{C}}B, m, N_{\mathbb{C}}) = B'_0(T_{\mathbb{C}}B, m, N_{\mathbb{C}}) + B'_1(T_{\mathbb{C}}B, m, N_{\mathbb{C}})q^{1/2} + \dots.$$

Set

$$(3.25) \quad Q'_1(\tau) = \widehat{L}(TB, \nabla^{TB}) \frac{\sinh^{2n+\frac{1-(-1)^d}{2}}\left(\frac{\epsilon}{2}\right)}{\cosh^{2n+\frac{1-(-1)^d}{2}}\left(\frac{\epsilon}{2}\right)} \text{ch}(\Theta'_1(T_{\mathbb{C}}B, 2n + \frac{1-(-1)^d}{2}, N_{\mathbb{C}})),$$

$$(3.26) \quad Q'_2(\tau) = \widehat{A}(TB, \nabla^{TB}) \sinh^{2n+\frac{1-(-1)^d}{2}}\left(\frac{\epsilon}{2}\right) \text{ch}(\Theta'_2(T_{\mathbb{C}}B, 2n + \frac{1-(-1)^d}{2}, N_{\mathbb{C}})),$$

where n is a nonnegative integer that satisfies $d - \left(2n + \frac{1-(-1)^d}{2}\right) > 0$.

By similar computations to those in the proof of Theorem 3.1, we have

$$(3.27) \quad Q'_1(\tau) = 2^d \left(\prod_{j=1}^d x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_1(x_j, \tau)}{\theta_1(0, \tau)} \right) \cdot \left(\frac{\theta(u, \tau)}{\theta'(0, \tau)} \right)^{2n + \frac{1-(-1)^d}{2}} \left(\frac{\theta_1(0, \tau)}{\theta_1(u, \tau)} \right)^{2n + \frac{1-(-1)^d}{2}},$$

$$(3.28) \quad Q'_2(\tau) = \left(\prod_{j=1}^d x_j \frac{\theta'(0, \tau)}{\theta(x_j, \tau)} \frac{\theta_2(x_j, \tau)}{\theta_2(0, \tau)} \right) \cdot \left(\frac{\theta(u, \tau)}{\theta'(0, \tau)} \right)^{2n + \frac{1-(-1)^d}{2}} \left(\frac{\theta_2(0, \tau)}{\theta_2(u, \tau)} \right)^{2n + \frac{1-(-1)^d}{2}}.$$

Let $P'_1(\tau) = \{Q'_1(\tau)\}^{(2d)}$, $P'_2(\tau) = \{Q'_2(\tau)\}^{(2d)}$ be the $(2d)$ -components of $Q'_1(\tau)$, $Q'_2(\tau)$ respectively. $P'_1(\tau)$ is a modular form of weight $d - \left(2n + \frac{1-(-1)^d}{2}\right)$ over $\Gamma_0(2)$ and $P'_2(\tau)$ is a modular form of weight $d - \left(2n + \frac{1-(-1)^d}{2}\right)$ over $\Gamma^0(2)$. Playing the same game as in the proof of Theorem 3.1, we obtain

Theorem 3.3. *The following identity holds:*

$$(3.29) \quad \frac{1}{2^{\frac{3d}{2} - \frac{1-(-1)^d}{4} - n}} \left\{ \widehat{L}(TB, \nabla^{TB}) \frac{\sinh^{2n + \frac{1-(-1)^d}{2}} \left(\frac{e}{2}\right)}{\cosh^{2n + \frac{1-(-1)^d}{2}} \left(\frac{e}{2}\right)} \right\}^{(2d)} \\ = \sum_{r=0}^m 2^{-6r} \left\{ d_r(B, 2n + \frac{1-(-1)^d}{2}, N) \sinh^{2n + \frac{1-(-1)^d}{2}} \left(\frac{e}{2}\right) \right\}^{(2d)},$$

where each $d_r(B, 2n + \frac{1-(-1)^d}{2}, N)$, $0 \leq r \leq k$, is a finite and canonical linear combination of characteristic forms $\widehat{A}(TB, \nabla^{TB}) \text{ch} \left(B'_i(T_{\mathbb{C}}B, 2n + \frac{1-(-1)^d}{2}, N_{\mathbb{C}}) \right)$, $0 \leq i \leq r$ and $m = \left\lceil \frac{d-2n-\frac{1-(-1)^d}{2}}{4} \right\rceil$.

Remark 3.2. The condition $d - \left(2n + \frac{1-(-1)^d}{2}\right) > 0$ is put to make Theorem 3.3 nontrivial. If $d - \left(2n + \frac{1-(-1)^d}{2}\right) = 0$, then both sides are $\frac{e^{2d}}{2^{\frac{3d}{2} - \frac{1-(-1)^d}{4} - n}}$. If $d - \left(2n + \frac{1-(-1)^d}{2}\right) < 0$, then both sides of Theorem 3.3 are zeros since $\sinh$ is an odd function and the degrees of the top components of both sides are greater than $2d$.

Remark 3.3. When $n = 0$ and $\frac{1-(-1)^d}{2} = 1$, i.e. $d = 4a + 1$ or $4a + 3$, the integral of the left-hand side of Theorem 3.3 against the fundamental class of B is, up to a constant, the signature of a submanifold of B which is the smooth zero locus of a generic section of the bundle N . Thus when B is a spin^c manifold and N is dual to $w_2(B)$, Theorem 3.3 shows that the signature of the smooth submanifolds of B dual to $e(N)$ can be given by indexes of Dirac operators on the spin^c manifold B .

Putting $d = 4k + 2$ and $n = 0$ in Theorem 3.3, we get

Corollary 3.1. *The following cancellation formula holds:*

$$\left\{ \widehat{L}(TM, \nabla^{TM}) \right\}^{(8k+4)} = 8 \sum_{j=0}^k 2^{6k-6j} \left\{ \widehat{A}(TM, \nabla^{TM}) \text{ch} b_j \right\}^{(8k+4)},$$

where the b_j 's are elements in $KO(M) \otimes \mathbb{C}$.

This is just Liu's original cancellation formula [10]. So Theorem 3.3 is a generalization of Liu's cancellation formula to all even dimensional oriented Riemannian manifolds with a complex line bundle involved. In particular, when $d = 6, n = 0$, we get the Alvarez-Gaumé–Witten miraculous cancellation formula (1.1).

Looking at Theorem 3.3, let's get some interesting cancellation formulas for special d and n . Putting $d = 6$ and $n = 1$, i.e. for a 12-dimensional manifold M , we have

Corollary 3.2. *The following formula holds:*

$$(3.30) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \frac{\sinh^2 \left(\frac{c}{2} \right)}{\cosh^2 \left(\frac{c}{2} \right)} \right\}^{(12)} = \left\{ \left[-4\widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M}) \right. \right. \\ \left. \left. + 112\widehat{A}(TM, \nabla^{TM}) + 8 \widehat{A}(TM, \nabla^{TM}) (e^c + e^{-c} - 2) \right] \sinh^2 \left(\frac{c}{2} \right) \right\}^{(12)}.$$

Putting $d = 6$ and $n = 2$, i.e. for a 12-dimensional manifold M , we have

Corollary 3.3. *The following formula holds:*

$$(3.31) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \frac{\sinh^4 \left(\frac{c}{2} \right)}{\cosh^4 \left(\frac{c}{2} \right)} \right\}^{(12)} = -128 \left\{ \widehat{A}(TM, \nabla^{TM}) \sinh^4 \left(\frac{c}{2} \right) \right\}^{(12)}.$$

Corollary 3.2 and 3.3 are both analogous to the Alvarez-Gaumé–Witten original miraculous cancellation formula (1.1) with a complex line bundle involved.

Putting $d = 5$ and $n = 0$, i.e. for a 10-dimensional manifold M , we have

Corollary 3.4. *The following formula holds:*

$$(3.32) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \frac{\sinh \left(\frac{c}{2} \right)}{\cosh \left(\frac{c}{2} \right)} \right\}^{(10)} = \left\{ \left[-2\widehat{A}(TM, \nabla^{TM}) \text{ch}(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M}) \right. \right. \\ \left. \left. + 52\widehat{A}(TM, \nabla^{TM}) + 2 \widehat{A}(TM, \nabla^{TM}) (e^c + e^{-c} - 2) \right] \sinh \left(\frac{c}{2} \right) \right\}^{(10)}.$$

Putting $d = 5$ and $n = 1$, i.e. for a 10-dimensional manifold M , we have

Corollary 3.5. *The following formula holds:*

$$(3.33) \quad \left\{ \widehat{L}(TM, \nabla^{TM}) \frac{\sinh^3 \left(\frac{c}{2} \right)}{\cosh^3 \left(\frac{c}{2} \right)} \right\}^{(10)} = -64 \left\{ \widehat{A}(TM, \nabla^{TM}) \sinh^3 \left(\frac{c}{2} \right) \right\}^{(10)}.$$

4. RELATIONS AMONG THE CANCELLATION FORMULAS

In this section, by applying the method of integration along the fibre, we show some relations on the level of characteristic numbers among cancellation formulas obtained in [10], [7] and §3 of this article.

First let's get (3.1) from Theorem 2.1 by integration along the fibre. Let $u \in H_{cv}^2(N)$, the second compact vertical supports cohomology of N , be the Thom class of the bundle (N, π, B) with fibre L . By the Thom isomorphism theorem, we have the following identity of cohomology classes in $H^*(B)$:

$$\left[\int_L u^{2i} \right] = [e^{2i-1}], \quad i = 1, 2, \dots$$

By integration along the fibre, on the one hand, we have

$$\begin{aligned} & \int_N \widehat{L}(TN) \left(1 - \frac{1}{\cosh^2\left(\frac{u}{2}\right)} \right) \\ (4.1) \quad &= \int_B \widehat{L}(TB) \int_L \frac{u}{\sinh\left(\frac{u}{2}\right)} \cosh\left(\frac{u}{2}\right) \frac{\sinh^2\left(\frac{u}{2}\right)}{\cosh^2\left(\frac{u}{2}\right)} \\ &= \int_B \widehat{L}(TB) \frac{\sinh\left(\frac{e}{2}\right)}{\cosh\left(\frac{e}{2}\right)} \end{aligned}$$

and on the other hand, $\forall 0 \leq r \leq k$, we have

$$\begin{aligned} & \int_N \widehat{A}(TN) \left(\text{ch}(b_r(T_{\mathbb{C}}N, \mathbb{C}^2)) - \cosh\left(\frac{u}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}N, \xi_{\mathbb{C}})) \right) \\ (4.2) \quad &= \int_B \widehat{A}(TB) \int_L \frac{\sinh\left(\frac{u}{2}\right)}{\cosh\left(\frac{u}{2}\right)} \left\{ \text{ch}(b_r(T_{\mathbb{C}}N, \mathbb{C}^2)) - \cosh\left(\frac{u}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}N, \xi_{\mathbb{C}})) \right\} \\ &= \int_B \widehat{A}(TB) \frac{\text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \cosh\left(\frac{e}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{2 \sinh\left(\frac{e}{2}\right)}. \end{aligned}$$

Then by Theorem 2.1, Corollaries 2.1, Corollary 2.2 and (4.1), (4.2), we obtain

$$\begin{aligned} & \frac{1}{8} \int_B \widehat{L}(TB) \frac{\sinh\left(\frac{e}{2}\right)}{\cosh\left(\frac{e}{2}\right)} \\ &= \sum_{r=0}^k 2^{6k-6r} \int_B \widehat{A}(TB) \frac{\text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, \mathbb{C}^2)) - \cosh\left(\frac{e}{2}\right) \text{ch}(b_r(T_{\mathbb{C}}B + N_{\mathbb{C}}, N_{\mathbb{C}}))}{2 \sinh\left(\frac{e}{2}\right)}. \end{aligned}$$

Remark 4.1. One can also obtain the characteristic number version of Theorem 3.2 from an $8k$ -analogue of Theorem 2.1 stated in [7, Theorem A.1] by integration along the fibre as above.

Next let's say something about the relations among formulas in Theorem 3.3 for different d and n .

Let $u \in H_{cv}^2(N)$, the second compact vertical supports cohomology of N , be the Thom class of the bundle (N, π, B) with fibre L . By the Thom isomorphism theorem, we have the following identity of cohomology classes in $H^*(B)$:

$$\left[\int_L u^{2i} \right] = [e^{2i-1}], \quad i = 1, 2, \dots$$

Let M be an $(8k + 4)$ -dimensional closed oriented Riemannian manifold and (E, π, M) be a complex line bundle on M with fibre L . Let $u \in H_{cv}^2(E)$ be the Thom class of this bundle. Then from Theorem 3.3 in the case $d = 4k + 3$ and $n = 0$, one has

$$\left\{ \widehat{L}(TE, \nabla^{TE}) \frac{\sinh\left(\frac{u}{2}\right)}{\cosh\left(\frac{u}{2}\right)} \right\}^{(8k+6)} = 16 \sum_{r=0}^k 2^{6k-6r} \left\{ d_r(E, 1, \pi^*E) \sinh\left(\frac{u}{2}\right) \right\}^{(8k+6)},$$

where each $d_r(E, 1, \pi^*E)$, $0 \leq r \leq k$, is a finite and canonical linear combination of characteristic forms $\widehat{A}(TE, \nabla^{TE}) \text{ch}(B'_i(T_{\mathbb{C}}E, 1, (\pi^*E)_{\mathbb{C}}))$, $0 \leq i \leq r$. Performing integration along the fibre, we have

$$\int_E \widehat{L}(TE) \frac{\sinh\left(\frac{u}{2}\right)}{\cosh\left(\frac{u}{2}\right)} = \int_M \widehat{L}(TM) \int_L \frac{u}{\tanh\left(\frac{u}{2}\right)} \frac{\sinh\left(\frac{u}{2}\right)}{\cosh\left(\frac{u}{2}\right)} = \int_M \widehat{L}(TM)$$

and

$$\begin{aligned} & \int_E \widehat{A}(TE) \text{ch}((B'_i(T_{\mathbb{C}}E, 1, (\pi^*E)_{\mathbb{C}}))) \sinh\left(\frac{u}{2}\right) \\ &= \int_M \widehat{A}(TM) \int_L \frac{\frac{u}{2}}{\sinh\left(\frac{u}{2}\right)} \text{ch}((B'_i(T_{\mathbb{C}}E, 1, (\pi^*E)_{\mathbb{C}}))) \sinh\left(\frac{u}{2}\right) \\ &= \frac{1}{2} \int_M \widehat{A}(TM) \text{ch}((B'_i(T_{\mathbb{C}}M, 0, E_{\mathbb{C}}))). \end{aligned}$$

Hence we obtain

$$\int_M \widehat{L}(TM) = 8 \sum_{j=0}^k 2^{6k-6j} \int_M \widehat{A}(TM) \text{ch} d_j(M, 0, E),$$

which is just the characteristic number version of the case of $d = 4k + 2$ and $n = 0$ in Theorem 3.3.

More generally, with the same pattern, we can apply integration along the fibre to get the formula in the case of (d, n) from the formula in the case of $(d+1, n + \frac{1-(-1)^d}{2})$ on the level of characteristic numbers. This phenomena looks very interesting since it beautifully relates different cancellation formulas in Theorem 3.3 which are all products of modular invariance.

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REFERENCES

1. L. Alvarez-Gaumé and E. Witten, Gravitational anomalies. *Nucl. Phys.* B234 (1983), 269-330. MR736803 (85j:81062)
2. M. F. Atiyah, *K-theory*. Benjamin, New York, 1967. MR0224083 (36:7130)
3. M. F. Atiyah and F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds. *Bull. Amer. Math. Soc.* 65 (1959), 276-281. MR0110106 (22:989)
4. K. Chandrasekharan, *Elliptic Functions*. Springer-Verlag, 1985. MR808396 (87e:11058)

5. S. M. Finashin, A Pin^- -cobordism invariant and a generalization of Rokhlin signature congruence. *Leningrad Math. J.* 2 (1991), 917-924. MR1080207 (91i:57016)
6. F. Han and W. Zhang, Spin^c -manifold and elliptic genera. *C. R. Acad. Sci. Paris, Série I.* 336 (2003), 1011-1014. MR1993972 (2004j:58024)
7. F. Han and W. Zhang, Modular invariance, characteristic numbers and η invariants. *Journal of Differential Geometry.* 67 (2004), 257-288. MR2153079 (2006k:58038)
8. F. Hirzebruch, *Topological Methods in Algebraic Geometry.* Springer-Verlag, 1966. MR0202713 (34:2573)
9. P. S. Landweber, Elliptic cohomology and modular forms. in *Elliptic Curves and Modular Forms in Algebraic Topology*, pp. 55-68. Ed. P. S. Landweber. Lecture Notes in Mathematics Vol. 1326, Springer-Verlag (1988). MR970281
10. K. Liu, Modular invariance and characteristic numbers. *Commun. Math. Phys.* 174 (1995), 29-42. MR1372798 (96m:57034)
11. K. Liu and W. Zhang, Elliptic genus and η -invariants. *Inter. Math. Res. Notices* No. 8 (1994), 319-328. MR1289577 (96b:57030)
12. S. Ochanine, Signature modulo 16, invariants de Kervaire généralisés et nombres caractéristiques dans la K -théorie réelle. *Mémoire Soc. Math. France*, Tom. 109 (1987), 1-141.
13. W. Zhang, Spin^c -manifolds and Rokhlin congruences. *C. R. Acad. Sci. Paris, Série I*, 317 (1993), 689-692. MR1245100 (94i:57042)
14. W. Zhang, Circle bundles, adiabatic limits of η -invariants and Rokhlin congruences. *Ann. Inst. Fourier* 44 (1994), 249-270. MR1262887 (95h:58127)
15. W. Zhang, *Lectures on Chern-Weil Theory and Witten Deformations.* Nankai Tracts in Mathematics Vol. 4, World Scientific, Singapore, 2001. MR1864735 (2002m:58032)

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