COMBINATORIAL CONGRUENCES MODULO PRIME POWERS

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ABSTRACT. Let p be any prime, and let α and n be nonnegative integers. Let $r \in \mathbb{Z}$ and $f(x) \in \mathbb{Z}[x]$. We establish the congruence

$$p^{\deg f} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f\left(\frac{k-r}{p^\alpha}\right) \equiv 0 \pmod{p^{\sum_{i=\alpha}^\infty \lfloor n/p^i \rfloor}}$$

(motivated by a conjecture arising from algebraic topology) and obtain the following vast generalization of Lucas' theorem: If α is greater than one, and l, s, t are nonnegative integers with $s, t < p$, then

$$\begin{aligned} & \frac{1}{\lfloor n/p^{\alpha-1} \rfloor!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{pn+s}{pk+t} (-1)^{pk} \left(\frac{k-r}{p^{\alpha-1}}\right)^l \\ & \equiv \frac{1}{\lfloor n/p^{\alpha-1} \rfloor!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} \binom{s}{t} (-1)^k \left(\frac{k-r}{p^{\alpha-1}}\right)^l \pmod{p}. \end{aligned}$$

We also present an application of the first congruence to Bernoulli polynomials and apply the second congruence to show that a p -adic order bound given by the authors in a previous paper can be attained when $p = 2$.

1. INTRODUCTION

In this paper we establish a number of new congruences for sums involving binomial coefficients with the summation index restricted in a residue class modulo a prime power; some of them are vast extensions of some classical congruences. We begin by providing some historical background for these results.

Let p be a prime, and let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the field of p -adic numbers and the ring of p -adic integers, respectively. For $\omega \in \mathbb{Q}_p \setminus \{0\}$ we define its p -adic order by $\text{ord}_p(\omega) = \max\{a \in \mathbb{Z} : \omega/p^a \in \mathbb{Z}_p\}$; in addition, we set $\text{ord}_p(0) = +\infty$.

In 1913, A. Fleck (cf. [D, p. 274]) proved that for any $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and $r \in \mathbb{Z}$ we have the congruence

$$\sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \equiv 0 \pmod{p^{\lfloor \frac{n-1}{p-1} \rfloor}}$$

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(where $\lfloor \cdot \rfloor$ is the greatest integer function, and we regard $\binom{x}{k} = 0$ for $k = -1, -2, -3, \dots$); that is,

$$\text{ord}_p \left(\sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \right) \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor.$$

In 1977, C. S. Weisman [We] showed further that if $\alpha, n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $r \in \mathbb{Z}$, then

$$\text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \right) \geq \left\lfloor \frac{n - p^{\alpha-1}}{\varphi(p^\alpha)} \right\rfloor,$$

where φ is the well-known Euler function. Weisman remarked that this kind of work is closely related to p -adic continuation.

In 2005, motivated by Fontaine's theory of (ϕ, Γ) -modules, D. Wan got an extension of Fleck's result in his lecture notes by giving a lower bound for the p -adic order of the sum $\sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \binom{(k-r)/p}{l}$, where $l, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Soon after this, Z. W. Sun [S06] obtained a common generalization of Weisman's and Wan's extensions of Fleck's congruence by studying the p -adic order of the sum

$$\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(k-r)/p^\alpha}{l}$$

(with $\alpha \in \mathbb{N}$) via a combinatorial approach. This direction is closely related to the α th iteration of the basic ψ -operator in Iwasawa theory and p -adic Langlands correspondence (cf. P. Colmez [C] and Wan [W]). However, when $l \geq n/p^\alpha$, any result along this line yields no nontrivial lower bound for the p -adic order of the last sum.

Unlike the above development of Fleck's congruence, there is another direction motivated by algebraic topology. In order to obtain a strong lower bound for homotopy exponents of the special unitary group $\text{SU}(n)$, the authors [DS] were led to show that if $\alpha, n \in \mathbb{N}$ and $r \in \mathbb{Z}$, then

$$\min_{f(x) \in \mathbb{Z}[x]} \text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f \left(\frac{k-r}{p^\alpha} \right) \right) \geq \text{ord}_p \left(\left\lfloor \frac{n}{p^\alpha} \right\rfloor! \right).$$

As shown in [DS], this inequality implies a subtle divisibility property of Stirling numbers of the second kind. Note that if $n \in \mathbb{Z}^+$, then

$$\text{ord}_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{n}{p^i} = \frac{n}{p(1-p^{-1})} = \frac{n}{p-1}$$

and hence $\text{ord}_p(n!) \leq (n-1)/(p-1)$.

Now we introduce some conventions used throughout this paper. As usual, the degree of the zero polynomial is regarded as $-\infty$. For an integer a and a positive real number m , we let $\{a\}_m$ denote the fractional part of a/m times m (i.e., $\{a\}_m$ is the unique number in the interval $[0, m)$ with $a - \{a\}_m \in m\mathbb{Z}$). For a prime p , if $a, b \in \mathbb{N}$, then $\tau_p(a, b)$ stands for the number of carries when adding a and b in base p ; a theorem of E. Kummer states that $\tau_p(a, b) = \text{ord}_p \binom{a+b}{a}$.

Here is our first theorem.

Theorem 1.1. Let p be a prime and $f(x) \in \mathbb{Z}_p[x]$. Let $\alpha, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then

$$p^{\deg f} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f\left(\frac{k-r}{p^\alpha}\right) \\ \equiv 0 \pmod{p^{\sum_{i=\alpha}^{\infty} \lfloor n/p^i \rfloor + \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}})}},$$

i.e.,

$$(1.1) \quad \text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f\left(\frac{k-r}{p^\alpha}\right) \right) \\ \geq \text{ord}_p \left(\left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! \right) - \deg f + \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}).$$

Remark 1.1. It is interesting to compare (1.1) with the following inequality ([DS, Theorem 5.1]) established for a certain topological purpose:

$$(1.2) \quad \text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f\left(\frac{k-r}{p^\alpha}\right) \right) \\ \geq \text{ord}_p \left(\left\lfloor \frac{n}{p^\alpha} \right\rfloor! \right) + \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}).$$

Note that $\text{ord}_p(\lfloor n/p^\alpha \rfloor!) = \text{ord}_p(\lfloor n/p^{\alpha-1} \rfloor!) - \lfloor n/p^\alpha \rfloor$ and

$$0 \leq \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}) - \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}) \leq 1.$$

In general, when $\deg f < \lfloor n/p^\alpha \rfloor$, the term $\tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}})$ in (1.1) cannot be replaced by $\tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha})$. A principal motivation for the development of Theorem 1.1 is that the inequality (1.2), although quite sharp when $\deg f \geq \lfloor n/p^\alpha \rfloor$, is not a good estimate for smaller values of $\deg f$.

Example 1.1. Let $p = \alpha = 2$, $r = 1$ and $n = 20$. Then, for each $0 < l < \lfloor n/p^\alpha \rfloor = 5$, equality in (1.1) with $f(x) = x^l$ is attained while

$$\tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}) = \tau_2(1, 3) = \tau_2(1, 1) + 1 = \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}) + 1.$$

Example 1.2. Let $p = 3$, $\alpha = r = 2$, $90 \leq n \leq 98$ and $0 \leq l < \lfloor n/p^\alpha \rfloor = 10$. In Table 1, $\delta_n(l)$ denotes the left hand side of (1.1) minus the right hand side with $f(x) = x^l$.

TABLE 1. Values of $\delta_n(l)$ with $0 \leq l \leq 9$ and $90 \leq n \leq 98$

$n \setminus l$	0	1	2	3	4	5	6	7	8	9
90	1	0	2	0	1	0	1	0	3	0
91	1	0	1	0	3	0	1	0	1	0
92	0	1	0	3	0	1	0	1	0	2
93	0	2	0	1	0	1	0	4	0	1
94	0	1	0	2	0	1	0	1	0	3
95	1	0	0	0	0	1	1	0	0	0
96	1	0	0	0	0	1	2	0	0	0
97	1	0	0	0	0	1	1	0	0	0
98	1	3	0	1	0	1	1	4	0	1

Example 1.3. The combination of using Theorem 1.1 when $\deg f \leq \lfloor n/p^\alpha \rfloor$ and the inequality (1.2) for larger values of $\deg f$ provides an excellent estimate for

$$\text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f \left(\frac{k-r}{p^\alpha} \right) \right).$$

For example, if $p = 2$, $\alpha = 1$, $r = 0$, $n = 20$ and $f(x) = x^l$, then the actual values of the expression for $l = 0, \dots, 21$ are

$$19, 19, 17, 17, 14, 14, 12, 12, 10, 10, 8, 8, 8, 8, 11, 9, 9, 9, 8, 8, 8, 8.$$

The inequality (1.2) guarantees that each of the 22 numbers should be at least 8, while Theorem 1.1 gives the lower bounds

$$18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8$$

for the first 11 of the above 22 values, respectively. The bound given by (1.1) is attained in this example with $l = 4$, while $\text{ord}_p(\lfloor (n/p^{\alpha-1})! \rfloor) = 18 < 19 = \lfloor (n - p^{\alpha-1})/\varphi(p^\alpha) \rfloor$; this shows that for a general $f(x) \in \mathbb{Z}_p[x]$ we cannot replace $\text{ord}_p(\lfloor (n/p^{\alpha-1})! \rfloor)$ in (1.1) by Weisman's bound $\lfloor (n - p^{\alpha-1})/\varphi(p^\alpha) \rfloor$ even if $r = 0$ (and hence the τ -term vanishes).

Here is an application of Theorem 1.1 to Bernoulli polynomials. (The reader is referred to [IR] and [S03] for some basic properties and known congruences concerning Bernoulli polynomials.)

Corollary 1.1. *Let p be a prime, and let $\alpha \in \mathbb{N}$, $m, n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. Then*

$$(1.3) \quad \begin{aligned} & \text{ord}_p \left(\frac{p^{m-1}}{m} \sum_{k=0}^n \binom{n}{k} (-1)^k B_m \left(\left\lfloor \frac{k-r}{p^\alpha} \right\rfloor \right) \right) \\ & \geq \sum_{i=\alpha}^{\infty} \left\lfloor \frac{n-1}{p^i} \right\rfloor + \tau_p(\{r-1\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}). \end{aligned}$$

Proof. Set $\bar{r} = r + p^\alpha - 1$. In view of [S06, Lemma 2.1] and the known identity $B_m(x+1) - B_m(x) = mx^{m-1}$, we have

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{m} B_m \left(\left\lfloor \frac{k-r}{p^\alpha} \right\rfloor \right) \\ &= \sum_{k \equiv \bar{r} \pmod{p^\alpha}} \binom{n-1}{k} \frac{(-1)^{k-1}}{m} \left(B_m \left(\frac{k-\bar{r}}{p^\alpha} + 1 \right) - B_m \left(\frac{k-\bar{r}}{p^\alpha} \right) \right) \\ &= \sum_{k \equiv \bar{r} \pmod{p^\alpha}} \binom{n-1}{k} (-1)^{k-1} \left(\frac{k-\bar{r}}{p^\alpha} \right)^{m-1}. \end{aligned}$$

This, together with Theorem 1.1, yields that

$$\begin{aligned} & \text{ord}_p \left(\sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{m} B_m \left(\left\lfloor \frac{k-r}{p^\alpha} \right\rfloor \right) \right) \\ & \geq \text{ord}_p \left(\left\lfloor \frac{n-1}{p^{\alpha-1}} \right\rfloor! \right) - (m-1) + \tau_p(\{\bar{r}\}_{p^{\alpha-1}}, \{n-1-\bar{r}\}_{p^{\alpha-1}}). \end{aligned}$$

So (1.3) follows. $\square$

Let p be any prime, $f(x) \in \mathbb{Q}_p[x]$ and $\deg f \leq l \in \mathbb{N}$. It is well known that $f(a) \in \mathbb{Z}_p$ for all $a \in \mathbb{Z}$ if and only if $f(x) = \sum_{j=0}^l a_j \binom{x}{j}$ for some $a_0, \dots, a_l \in \mathbb{Z}_p$. Also, $f(x) \in \mathbb{Z}_p[x]$ if and only if it can be written in the form $\sum_{j=0}^l b_j (x)_j$ with $b_j \in \mathbb{Z}_p$ and $(x)_j = j! \binom{x}{j} \in \mathbb{Z}[x]$. (Recall that $x^l = \sum_{j=0}^l S(l, j)(x)_j$ where $S(l, j)$ ($0 \leq j \leq l$) are Stirling numbers of the second kind.) Thus, we can reformulate Theorem 1.1 as follows.

Theorem 1.2. *Let p be a prime, $\alpha, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Let $f(x) \in \mathbb{Q}_p[x]$, $\deg f \leq l \in \mathbb{N}$, and $f(a) \in \mathbb{Z}_p$ for all $a \in \mathbb{Z}$. Then we have*

$$(1.4) \quad \begin{aligned} & \text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k f \left(\frac{k-r}{p^\alpha} \right) \right) \\ & \geq \text{ord}_p \left(\left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! \right) - l - \text{ord}_p(l!) + \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}). \end{aligned}$$

Remark 1.2. This theorem has topological background. In the case $p = \alpha = r = 2$ and $f(x) = \binom{x}{l}$, it first arose as a conjecture of the second author in his study of algebraic topology.

Let $[x^n]F(x)$ denote the coefficient of x^n in the power series expansion of $F(x)$. Theorem 1.1 also has the following equivalent form.

Theorem 1.3. *Let p be a prime, and let $\alpha, l, n, r \in \mathbb{N}$. If $r > n - (l+1)p^\alpha$, then*

$$(1.5) \quad \begin{aligned} & \text{ord}_p \left([x^r] \frac{(1-x)^n}{(1-x^{p^\alpha})^{l+1}} \right) \\ & \geq \text{ord}_p \left(\left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! \right) - l - \text{ord}_p(l!) + \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}). \end{aligned}$$

Proof. Let $r > n - (l+1)m$ where $m = p^\alpha$. Observe that

$$\begin{aligned} [x^r] \frac{(1-x)^n}{(1-x^m)^{l+1}} &= \sum_{k=0}^r \binom{n}{k} (-1)^k [x^{r-k}] (1-x^m)^{-l-1} \\ &= \sum_{k=0}^r \binom{n}{k} (-1)^k [x^{r-k}] \sum_{j \geq 0} \binom{l+j}{l} (x^m)^j \\ &= \sum_{\substack{0 \leq k \leq r \\ k \equiv r \pmod{m}}} \binom{n}{k} (-1)^k \binom{l - (k-r)/m}{l}. \end{aligned}$$

If $r < k \leq n$ and $k \equiv r \pmod{m}$, then $0 < (k-r)/m \leq (n-r)/m < l+1$ and hence $\binom{l - (k-r)/m}{l} = 0$. Therefore

$$\begin{aligned} [x^r] \frac{(1-x)^n}{(1-x^m)^{l+1}} &= \sum_{\substack{0 \leq k \leq n \\ k \equiv r \pmod{m}}} \binom{n}{k} (-1)^k \binom{l - (k-r)/m}{l}, \\ &= (-1)^l \sum_{k \equiv r' \pmod{m}} \binom{n}{k} (-1)^k \binom{(k-r')/m}{l}, \end{aligned}$$

where $r' = r + m$. Applying Theorem 1.1 or 1.2 we immediately get the inequality (1.5). $\square$

Here is another equivalent version of Theorem 1.1.

Theorem 1.4. *Let p be a prime and α be a nonnegative integer. Let $r \in \mathbb{Z}$ and $f(x) \in \mathbb{Z}_p[x]$ with $\deg f = l \in \mathbb{N}$. Then, there is a sequence $\{a_k\}_{k \in \mathbb{N}}$ of p -adic integers such that for any $n \in \mathbb{N}$ we have*

$$(1.6) \quad \sum_{k=0}^n \binom{n}{k} (-1)^k \left\lfloor \frac{k}{p^{\alpha-1}} \right\rfloor! \binom{\{r\}_{p^{\alpha-1}} + \{k-r\}_{p^{\alpha-1}}}{\{r\}_{p^{\alpha-1}}} a_k \\ = \begin{cases} p^l f\left(\frac{n-r}{p^\alpha}\right) & \text{if } n \equiv r \pmod{p^\alpha}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The binomial inversion formula (cf. [GKP, (5.48)]), states that $\sum_{k=0}^n \binom{n}{k} (-1)^k b_k = d_n$ for all $n \in \mathbb{N}$ if and only if $\sum_{k=0}^n \binom{n}{k} (-1)^k d_k = b_n$ for all $n \in \mathbb{N}$. Thus the desired result has the following equivalent form: There exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of p -adic integers such that for all $n \in \mathbb{N}$ we have

$$\left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! \binom{\{r\}_{p^{\alpha-1}} + \{n-r\}_{p^{\alpha-1}}}{\{r\}_{p^{\alpha-1}}} a_n \\ = \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k p^l f\left(\frac{k-r}{p^\alpha}\right).$$

This is essentially what Theorem 1.1 says. $\square$

A famous theorem of E. Lucas states that if p is a prime and n, r, s, t are non-negative integers with $s, t < p$, then

$$\binom{pn+s}{pr+t} \equiv \binom{n}{r} \binom{s}{t} \pmod{p}.$$

Now we present our following analogue of Lucas' theorem.

Theorem 1.5. *Let p be a prime and $\alpha \geq 2$ be an integer. Then, for any $l, n \in \mathbb{N}$ and $r \in \mathbb{Z}$, we have the congruence*

$$(1.7) \quad T_{l,\alpha+1}^{(p)}(n, r) \equiv (-1)^{\{r\}_p} \binom{\{n\}_p}{\{r\}_p} T_{l,\alpha}^{(p)}\left(\left\lfloor \frac{n}{p} \right\rfloor, \left\lfloor \frac{r}{p} \right\rfloor\right) \pmod{p},$$

where

$$T_{l,\alpha}^{(p)}(n, r) := \frac{l! p^l}{[n/p^{\alpha-1}]!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(k-r)/p^\alpha}{l}.$$

Remark 1.3. Theorem 1.2 guarantees that $T_{l,\alpha}^{(p)}(n, r) \in \mathbb{Z}_p$ (and our proof of Theorem 1.2 given later is based on an analysis of this T). Theorem 1.5 provides further information on $T_{l,\alpha}^{(p)}(n, r)$ modulo p .

Since $f(x) = l! \binom{x}{l} - x^l \in \mathbb{Z}[x]$ has degree smaller than l , we have

$$T_{l,\alpha}^{(p)}(n, r) \equiv \frac{p^l}{[n/p^{\alpha-1}]!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k-r}{p^\alpha}\right)^l \pmod{p}$$

by Theorem 1.1. Thus, Theorem 1.5 has the following equivalent version.

Theorem 1.6. *Let p be any prime, and let $l, n \in \mathbb{N}$, $r, s, t \in \mathbb{Z}$ and $0 \leq s, t < p$. Then, for every $\alpha = 2, 3, \dots$, we have*

$$(1.8) \quad \frac{1}{[n/p^{\alpha-1}]!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{pn+s}{pk+t} (-1)^{pk} \left(\frac{k-r}{p^{\alpha-1}} \right)^l \\ \equiv \frac{1}{[n/p^{\alpha-1}]!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} \binom{s}{t} (-1)^k \left(\frac{k-r}{p^{\alpha-1}} \right)^l \pmod{p}.$$

Remark 1.4. Theorem 1.6 is a vast generalization of Lucas' theorem. Given a prime p and nonnegative integers n, r, s, t with $r \leq n$ and $s, t < p$, if we apply (1.8) with $\alpha > \log_p(\max\{n, p\})$ and $l = 0$, then we obtain Lucas' congruence $\binom{pn+s}{pr+t} \equiv \binom{n}{r} \binom{s}{t} \pmod{p}$. We conjecture that (1.8) (or its equivalent form (1.7)) also holds when $\alpha = 1$.

Let $p > 3$ be a prime. A well-known theorem of Wolstenholme asserts that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$, i.e., $\binom{2p}{p} \equiv 2 \pmod{p^3}$. In 1952 W. Ljunggren (cf. [B], [G]) generalized this as follows: $\binom{pn}{pr} \equiv \binom{n}{r} \pmod{p^3}$ for any $n, r \in \mathbb{N}$. Our following conjecture extends Ljunggren's result greatly.

Conjecture 1.1. *Let p be an odd prime, and let $\alpha \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then, for all $l \in \mathbb{N}$ we have*

$$T_{l, \alpha+1}^{(p)}(pn, pr) - T_{l, \alpha}^{(p)}(n, r) \equiv \begin{cases} 0 \pmod{p^3} & \text{if } p > 3, \\ 0 \pmod{p^2} & \text{if } p = 3. \end{cases}$$

Equivalently, for any $f(x) \in \mathbb{Z}_p[x]$ we have

$$\frac{1}{[n/p^{\alpha-1}]!} \sum_{k \equiv r \pmod{p^\alpha}} \left(\binom{pn}{pk} - \binom{n}{k} \right) (-1)^k f\left(\frac{k-r}{p^{\alpha-1}}\right) \\ \equiv \begin{cases} 0 \pmod{p^3} & \text{if } p > 3, \\ 0 \pmod{p^2} & \text{if } p = 3. \end{cases}$$

Remark 1.5. The reason for the equivalence of the two parts in Conjecture 1.1 is as follows: For any $l \in \mathbb{N}$ we have

$$p^l x^l = \sum_{j=0}^l S(l, j) p^{l-j} j! p^j \binom{x}{j} \text{ and } l! p^l \binom{x}{l} = \sum_{j=0}^l (-1)^{l-j} s(l, j) p^{l-j} (p^j x^j),$$

where $s(l, j)$ ($0 \leq j \leq l$) are Stirling numbers of the first kind.

The proof of Theorem 1.5 involves our following refinement of Weisman's result.

Theorem 1.7. *Let p be any prime, and let $\alpha, n \in \mathbb{N}$, $\alpha \geq 2$, $r, s, t \in \mathbb{Z}$ and $0 \leq s, t < p^{\alpha-2}$. Then*

$$(1.9) \quad p^{-\left\lfloor \frac{p^{\alpha-2}n+s-p^{\alpha-1}}{\varphi(p^\alpha)} \right\rfloor} \sum_{k \equiv p^{\alpha-2}r+t \pmod{p^\alpha}} \binom{p^{\alpha-2}n+s}{k} (-1)^k \\ \equiv (-1)^t \binom{s}{t} \left(p^{-\left\lfloor \frac{n-p}{\varphi(p^2)} \right\rfloor} \sum_{k \equiv r \pmod{p^2}} \binom{n}{k} (-1)^k \right) \pmod{p}.$$

Here is a consequence of this theorem.

Corollary 1.2. *Let $\alpha \in \{2, 3, \dots\}$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then*

$$2^{-\lfloor \frac{n-2^{\alpha-1}}{\varphi(2^\alpha)} \rfloor} \sum_{k \equiv r \pmod{2^\alpha}} \binom{n}{k} \equiv 1 \pmod{2}$$

if and only if $\binom{n}{r}_{2^{\alpha-2}} \equiv 1 \pmod{2}$ and the following (a) or (b) holds.

(a) $n_* > 2$ and $n_* \not\equiv 2r_* + 2 \pmod{4}$, (b) $n_* = 2$ and $2 \mid r_*$,
where $n_* = \lfloor n/2^{\alpha-2} \rfloor$ and $r_* = \lfloor r/2^{\alpha-2} \rfloor$.

As a complement to Theorem 1.7, we have the following conjecture.

Conjecture 1.2. *Let p be any prime, and let $n \in \mathbb{N}$, $r \in \mathbb{Z}$ and $s \in \{0, \dots, p-1\}$. If $p \mid n$ or $p-1 \nmid n-1$, then*

$$\begin{aligned} & p^{-\lfloor \frac{pn+s-p}{\varphi(p^2)} \rfloor} \sum_{k \equiv pr+t \pmod{p^2}} \binom{pn+s}{k} (-1)^k \\ & \equiv (-1)^t \binom{s}{t} p^{-\lfloor \frac{n-1}{p-1} \rfloor} \sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \pmod{p} \end{aligned}$$

for every $t = 0, \dots, p-1$. When $s \neq p-1$, $p \nmid n$ and $p-1 \mid n-1$, the least nonnegative residue of

$$p^{-\lfloor \frac{pn+s-p}{\varphi(p^2)} \rfloor} \sum_{k \equiv pr+t \pmod{p^2}} \binom{pn+s}{k} (-1)^k$$

modulo p does not depend on r for $t = s+1, \dots, p-1$; moreover, these residues form a permutation of $1, \dots, p-1$ if $s = 0$ and $n \neq 1$.

Conjecture 5.2 of [DS] states that the bound in the inequality (1.2) is attained if $f(x) = x^l$ and l satisfies a certain congruence equation. In the following result, we prove this conjecture in the case $p = 2$ and $r = 0$.

Theorem 1.8. *Let $\alpha, n \in \mathbb{N}$, $l \geq \lfloor n/2^\alpha \rfloor \geq 1$ and*

$$l \equiv \left\lfloor \frac{n}{2^\alpha} \right\rfloor \pmod{2^{\lfloor \log_2(n/2^\alpha) \rfloor}}.$$

Then

$$(1.10) \quad \text{ord}_2 \left(\sum_{k \equiv 0 \pmod{2^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k}{2^\alpha} \right)^l \right) = \text{ord}_2 \left(\left\lfloor \frac{n}{2^\alpha} \right\rfloor! \right).$$

Remark 1.6. Theorem 1.8 in the case $\alpha = 0$ essentially asserts that if $n \in \mathbb{Z}^+$ and $q \in \mathbb{N}$, then $S(n + 2^{\lfloor \log_2 n \rfloor} q, n)$ is odd. This is because $\sum_{k=0}^n \binom{n}{k} (-1)^k k^l = (-1)^n n! S(l, n)$ for $l \in \mathbb{N}$ (cf. [LW, pp. 125–126]).

The following conjecture is a refinement of [DS, Conjecture 5.2].

Conjecture 1.3. *Let p be a prime and r be an integer. Let $\alpha, n \in \mathbb{N}$, $n \geq 2p^\alpha - 1$, $l \geq \lfloor n/p^\alpha \rfloor$ and*

$$l \equiv \left\lfloor \frac{r}{p^\alpha} \right\rfloor + \left\lfloor \frac{n-r}{p^\alpha} \right\rfloor \pmod{(p-1)p^{\lfloor \log_p(n/p^\alpha) \rfloor}}.$$

Then

$$\frac{1}{[n/p^\alpha]! \binom{n_*}{r_*}} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k-r}{p^\alpha} \right)^l \equiv \pm 1 \pmod{p},$$

where $r_* = \{r\}_{p^\alpha}$ and $n_* = r_* + \{n-r\}_{p^\alpha}$.

For convenience, throughout this paper we use $\llbracket A \rrbracket$ to denote the characteristic function of an assertion A , i.e., the value of $\llbracket A \rrbracket$ is 1 or 0 according to whether A holds or not. For $m, n \in \mathbb{N}$ the Kronecker symbol $\delta_{m,n}$ stands for $\llbracket m = n \rrbracket$.

The next section is devoted to the proof of an equivalent version of Theorems 1.1–1.4. In Section 3 we will prove Theorem 1.7 and Corollary 1.2 on the basis of Theorem 3.1, and in Section 4 we establish Theorems 1.5 and 1.8.

2. PROOF OF THEOREM 1.2

In this section we prove the following equivalent version of Theorem 1.2.

Theorem 2.1. *Let p be a prime, and let $\alpha, l, n \in \mathbb{N}$. Then, for all $r \in \mathbb{Z}$, we have*

$$(2.1) \quad \text{ord}_p(T_l(n, r)) \geq \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}),$$

where $T_l(n, r)$ stands for $T_{l,\alpha}^{(p)}(n, r)$ given in Theorem 1.5.

Lemma 2.1. *Theorem 2.1 holds in the case $\alpha = 0$.*

Proof. Clearly $\tau_p(\{r\}_{p^{-1}}, \{n-r\}_{p^{-1}}) = \tau_p(0, 0) = 0$. Provided $\alpha = 0$ we have

$$T_l(n, r) = \frac{l! p^l}{[n/p^{-1}]!} \sum_{k=0}^n \binom{n}{k} (-1)^k \binom{k-r}{l} = \frac{l! p^l}{(pn)!} (-1)^n \binom{-r}{l-n},$$

where we have applied a known identity (cf. [GKP, (5.24)]) to get the last equality. If $l < n$, then $T_l(n, r) = 0 \in \mathbb{Z}_p$. When $l \geq n$, we also have

$T_l(n, r) \in \mathbb{Z}_p$ because $\text{ord}_p((pn)!) = n + \text{ord}_p(n!) \leq l + \text{ord}_p(l!)$. This ends the proof. $\square$

For convenience, in the following lemmas we let p be a fixed prime and α be a positive integer.

Lemma 2.2. *Let $l \in \mathbb{N}$, $n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. Then*

$$(2.2) \quad T_l(n-1, r) - T_l(n-1, r-1) = \begin{cases} T_l(n, r) & \text{if } p^{\alpha-1} \nmid n, \\ \frac{n}{p^{\alpha-1}} T_l(n, r) & \text{otherwise.} \end{cases}$$

When $l > 0$, we also have

$$(2.3) \quad \begin{aligned} & T_l(n, r) + \frac{r}{p^{\alpha-1}} T_{l-1}(n, r + p^\alpha) \\ &= \begin{cases} -T_{l-1}(n-1, r + p^\alpha - 1) & \text{if } p^{\alpha-1} \mid n, \\ -\frac{n}{p^{\alpha-1}} T_{l-1}(n-1, r + p^\alpha - 1) & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Clearly

$$\begin{aligned}
& \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(k-r)/p^\alpha}{l} \\
& + \sum_{k \equiv r-1 \pmod{p^\alpha}} \binom{n-1}{k} (-1)^k \binom{(k-(r-1))/p^\alpha}{l} \\
& = \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(k-r)/p^\alpha}{l} \\
& - \sum_{k \equiv r \pmod{p^\alpha}} \binom{n-1}{k-1} (-1)^k \binom{(k-r)/p^\alpha}{l} \\
& = \sum_{k \equiv r \pmod{p^\alpha}} \binom{n-1}{k} (-1)^k \binom{(k-r)/p^\alpha}{l}.
\end{aligned}$$

Therefore

$$\left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! T_l(n, r) + \left\lfloor \frac{n-1}{p^{\alpha-1}} \right\rfloor! T_l(n-1, r-1) = \left\lfloor \frac{n-1}{p^{\alpha-1}} \right\rfloor! T_l(n-1, r),$$

and hence (2.2) follows.

Now let $l > 0$. Note that

$$\begin{aligned}
& \left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! \frac{T_l(n, r)}{(l-1)! p^{l-1}} \\
& = lp \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \frac{(k-r)/p^\alpha}{l} \binom{(k-r)/p^\alpha - 1}{l-1} \\
& = \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k}{p^{\alpha-1}} - \frac{r}{p^{\alpha-1}} \right) \binom{(k-r-p^\alpha)/p^\alpha}{l-1}.
\end{aligned}$$

Using the identity $\binom{n}{k} k = n \binom{n-1}{k-1}$, we find that

$$\begin{aligned}
& \left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! \frac{T_l(n, r)}{(l-1)! p^{l-1}} \\
& = \frac{n}{p^{\alpha-1}} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n-1}{k-1} (-1)^k \binom{(k-r-p^\alpha)/p^\alpha}{l-1} \\
& - \frac{r}{p^{\alpha-1}} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(k-r-p^\alpha)/p^\alpha}{l-1} \\
& = \frac{n}{p^{\alpha-1}} \sum_{k \equiv r-1 \pmod{p^\alpha}} \binom{n-1}{k} (-1)^{k+1} \binom{(k-(r-1)-p^\alpha)/p^\alpha}{l-1} \\
& - \frac{r}{p^{\alpha-1}} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(k-r-p^\alpha)/p^\alpha}{l-1}.
\end{aligned}$$

So we have

$$\begin{aligned} \left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! T_l(n, r) &= -\frac{n}{p^{\alpha-1}} \left\lfloor \frac{n-1}{p^{\alpha-1}} \right\rfloor! T_{l-1}(n-1, r+p^\alpha-1) \\ &\quad - \frac{r}{p^{\alpha-1}} \left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor! T_{l-1}(n, r+p^\alpha), \end{aligned}$$

which is equivalent to (2.3). $\square$

Remark 2.1. Lemma 2.2 is not sufficient for an induction proof of Theorem 2.1; in fact, we immediately encounter difficulty when $p^{\alpha-1} \mid n$ and $p^{\alpha-1} \nmid r$.

Lemma 2.3. *Let $d, m \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$, and let $f(x)$ be a function from $\mathbb{Z}$ to the complex field. Then we have*

$$\begin{aligned} (2.4) \quad & \sum_{k \equiv r \pmod{d}} \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right) \\ &= \sum_{j=0}^n \binom{n}{j} \left(\sum_{i \equiv r \pmod{d}} \binom{j}{i} (-1)^i \right) \sum_{i=0}^{m-1} \sigma_{ij}, \end{aligned}$$

where

$$(2.5) \quad \sigma_{ij} = \sum_{k \equiv r+i-j \pmod{m}} \binom{n-j}{k} (-1)^k f\left(\frac{k-(r+i-j)}{m}\right).$$

Proof. Let ζ be a primitive d th root of unity. Given $j \in \{0, \dots, n\}$, we have

$$\begin{aligned} \sum_{i \equiv r \pmod{d}} \binom{j}{i} (-1)^i &= \sum_{i=0}^j \binom{j}{i} \frac{(-1)^i}{d} \sum_{s=0}^{d-1} \zeta^{(i-r)s} \\ &= \frac{1}{d} \sum_{s=0}^{d-1} \zeta^{-rs} \sum_{i=0}^j \binom{j}{i} (-\zeta^s)^i \\ &= \frac{1}{d} \sum_{s=0}^{d-1} \zeta^{-rs} (1 - \zeta^s)^j. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{i=0}^{m-1} \sigma_{ij} &= \sum_{i=0}^{m-1} \sum_{k-(r-j) \equiv i \pmod{m}} \binom{n-j}{k} (-1)^k f\left(\frac{k-(r-j)-i}{m}\right) \\ &= \sum_{i=0}^{m-1} \sum_{k-(r-j) \equiv i \pmod{m}} \binom{n-j}{k} (-1)^k f\left(\left\lfloor \frac{k-(r-j)}{m} \right\rfloor\right) \\ &= \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^k f\left(\left\lfloor \frac{k+j-r}{m} \right\rfloor\right). \end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{j=0}^n \binom{n}{j} \left(\sum_{i \equiv r \pmod{d}} \binom{j}{i} (-1)^i \right) \sum_{i=0}^{m-1} \sigma_{ij} \\
&= \sum_{j=0}^n \binom{n}{j} \frac{1}{d} \sum_{s=0}^{d-1} \zeta^{-rs} (1 - \zeta^s)^j \sum_{k=j}^n \binom{n-j}{k-j} (-1)^{k-j} f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right) \\
&= \frac{1}{d} \sum_{s=0}^{d-1} \zeta^{-rs} \sum_{k=0}^n \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right) \sum_{j=0}^k \binom{k}{j} (\zeta^s - 1)^j \\
&= \sum_{k=0}^n \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right) \frac{1}{d} \sum_{s=0}^{d-1} \zeta^{(k-r)s} \\
&= \sum_{k \equiv r \pmod{d}} \binom{n}{k} (-1)^k f\left(\left\lfloor \frac{k-r}{m} \right\rfloor\right).
\end{aligned}$$

This proves (2.4). $\square$

Lemma 2.4. *Let $l, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then*

$$(2.6) \quad T_l(n, r) = \sum_{i=0}^{p^\alpha-1} \sum_{j=0}^n c_\alpha(n, j) T_0(j, r) T_l(n-j, r+i-j),$$

where

$$(2.7) \quad c_\alpha(n, j) := \binom{n}{j} \frac{\lfloor j/p^{\alpha-1} \rfloor! \lfloor (n-j)/p^{\alpha-1} \rfloor!}{\lfloor n/p^{\alpha-1} \rfloor!}.$$

Proof. It suffices to apply Lemma 2.3 with $d = m = p^\alpha$ and $f(x) = \binom{x}{l}$. $\square$

Lemma 2.5. *We have $c_\alpha(n, j) \in \mathbb{Z}_p$ for all $j = 0, \dots, n$.*

Proof. Clearly

$$\begin{aligned}
\text{ord}_p(c_\alpha(n, j)) &= \text{ord}_p(n!) - \text{ord}_p\left(\left\lfloor \frac{n}{p^{\alpha-1}} \right\rfloor!\right) \\
&\quad - \left(\text{ord}_p(j!) - \text{ord}_p\left(\left\lfloor \frac{j}{p^{\alpha-1}} \right\rfloor!\right) \right) \\
&\quad - \left(\text{ord}_p((n-j)!) - \text{ord}_p\left(\left\lfloor \frac{n-j}{p^{\alpha-1}} \right\rfloor!\right) \right) \\
&= \sum_{0 < s < \alpha} \left(\left\lfloor \frac{n}{p^s} \right\rfloor - \left\lfloor \frac{j}{p^s} \right\rfloor - \left\lfloor \frac{n-j}{p^s} \right\rfloor \right) \geq 0.
\end{aligned}$$

This concludes the proof. $\square$

Proof of Theorem 2.1. Lemma 2.1 indicates that Theorem 2.1 holds when $\alpha = 0$. Below we let $\alpha > 0$.

Step I. Use induction on $l + n$ to show that $T_l(n, r) \in \mathbb{Z}_p$ for any $r \in \mathbb{Z}$.

The case $l = n = 0$ is trivial since $T_0(0, r) \in \mathbb{Z}$.

Now let $l + n > 0$, and assume that $T_{l_*}(n_*, r_*) \in \mathbb{Z}_p$ whenever $l_*, n_* \in \mathbb{N}$, $l_* + n_* < l + n$ and $r_* \in \mathbb{Z}$.

Case 1. $l = 0$. By Weisman's result mentioned in the first section (see also [S06]),

$$\begin{aligned} & \text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \right) \\ & \geq \left\lfloor \frac{n - p^{\alpha-1}}{\varphi(p^\alpha)} \right\rfloor = \left\lfloor \frac{n/p^{\alpha-1} - 1}{p-1} \right\rfloor = \left\lfloor \frac{n_0 - 1}{p-1} \right\rfloor, \end{aligned}$$

where $n_0 = \lfloor n/p^{\alpha-1} \rfloor$. If $n_0 > 0$, then $\text{ord}_p(n_0!) \leq \lfloor (n_0 - 1)/(p-1) \rfloor$ (as mentioned in the first section); hence

$$T_0(n, r) = \frac{1}{n_0!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \in \mathbb{Z}_p.$$

Clearly this also holds when $n_0 = 0$.

Case 2. $l > 0$ and $p^\alpha \nmid r$. In this case, $T_0(0, r)$ vanishes. Thus, by Lemmas 2.4–2.5 and the induction hypothesis, we have

$$T_l(n, r) = \sum_{i=0}^{p^\alpha-1} \sum_{0 < j \leq n} c_\alpha(n, j) T_0(j, r) T_l(n-j, r+i-j) \in \mathbb{Z}_p.$$

Case 3. $l > 0$ and $p^\alpha \mid r$. If $p^{\alpha-1} \nmid n$, then

$$T_l(n, r) = T_l(n-1, r) - T_l(n-1, r-1) \in \mathbb{Z}_p$$

by (2.2) and the induction hypothesis. If $p^{\alpha-1} \mid n$ and $n \neq 0$, then

$$T_l(n, r) = -\frac{r}{p^{\alpha-1}} T_{l-1}(n, r+p^\alpha) - T_{l-1}(n-1, r+p^\alpha-1) \in \mathbb{Z}_p$$

by (2.3) and the induction hypothesis. Note also that $T_l(0, r) \in \mathbb{Z}_p$.

In view of the above, we have finished the first step.

Step II. Use induction on n to prove (2.1) for any $r \in \mathbb{Z}$.

If $p^\alpha \nmid r$, then $T_l(0, r) = 0$; if $p^\alpha \mid r$, then $\tau_p(\{r\}_{p^{\alpha-1}}, \{-r\}_{p^{\alpha-1}}) = 0$. So (2.1) holds when $n = 0$.

Now let $n > 0$ and $\tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}) \neq 0$. Then both $\text{ord}_p(r)$ and $\text{ord}_p(n-r)$ are smaller than $\alpha-1$. Assume that (2.1) with n replaced by $n-1$ holds for all $r \in \mathbb{Z}$. For $r' = n-r - (l-1)p^\alpha$, we have

$$\begin{aligned} \tau_p(\{r'\}_{p^{\alpha-1}}, \{n-r'\}_{p^{\alpha-1}}) &= \tau_p(\{n-r\}_{p^{\alpha-1}}, \{r\}_{p^{\alpha-1}}) \\ &= \tau_p(\{r\}_{p^{\alpha-1}}, \{n-r\}_{p^{\alpha-1}}) \end{aligned}$$

and

$$\begin{aligned} T_l(n, r') &= \frac{l!p^l}{\lfloor n/p^{\alpha-1} \rfloor!} \sum_{k \equiv r' \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(n-r' - (n-k))/p^\alpha}{l} \\ &= \frac{l!p^l}{\lfloor n/p^{\alpha-1} \rfloor!} \sum_{k \equiv n-r' \pmod{p^\alpha}} \binom{n}{k} (-1)^{n-k} \binom{(n-r' - k)/p^\alpha}{l} \\ &= \frac{(-1)^{l+n} l!p^l}{\lfloor n/p^{\alpha-1} \rfloor!} \sum_{k \equiv r \pmod{p^\alpha}} \binom{n}{k} (-1)^k \binom{(k - (n-r'))/p^\alpha + l - 1}{l} \\ &= (-1)^{l+n} T_l(n, r); \end{aligned}$$

also, $\text{ord}_p(r') = \text{ord}_p(n - r) = \text{ord}_p(n)$ if $\text{ord}_p(r) > \text{ord}_p(n)$. Thus, without loss of generality, below we simply let $\text{ord}_p(r) \leq \text{ord}_p(n)$.

In view of Lemma 2.2,

$$T_{l+1}(n, r - p^\alpha) + \frac{r - p^\alpha}{p^{\alpha-1}} T_l(n, r) = \begin{cases} -T_l(n - 1, r - 1) & \text{if } p^{\alpha-1} \mid n, \\ -\frac{n}{p^{\alpha-1}} T_l(n - 1, r - 1) & \text{otherwise.} \end{cases}$$

As $T_{l+1}(n, r - p^\alpha) \in \mathbb{Z}_p$, and

$$\text{ord}_p(T_l(n - 1, r - 1)) \geq \tau_p(\{r - 1\}_{p^{\alpha-1}}, \{n - 1 - (r - 1)\}_{p^{\alpha-1}})$$

by the induction hypothesis, we have

$$\begin{aligned} \text{ord}_p(r T_l(n, r)) &\geq \min\{\alpha - 1, \text{ord}_p(n T_l(n - 1, r - 1))\} \\ &\geq \min\{\alpha - 1, \text{ord}_p(n) + \tau_p(\{r - 1\}_{p^{\alpha-1}}, \{n - r\}_{p^{\alpha-1}})\}. \end{aligned}$$

By the definition of τ_p and the inequality $\text{ord}_p(r) \leq \text{ord}_p(n)$,

$$\tau_p(\{r\}_{p^{\alpha-1}}, \{n - r\}_{p^{\alpha-1}}) \leq \alpha - 1 - \text{ord}_p(r)$$

and also

$$\tau_p(\{r\}_{p^{\alpha-1}}, \{n - r\}_{p^{\alpha-1}}) \leq \tau_p(\{r - 1\}_{p^{\alpha-1}}, \{n - r\}_{p^{\alpha-1}}) + \text{ord}_p(n) - \text{ord}_p(r).$$

(Note that $\{r\}_{p^{\alpha-1}} + \{n - r\}_{p^{\alpha-1}} \equiv n \pmod{p^{\alpha-1}}$.) So, (2.1) follows from the above.

The proof of Theorem 2.1 is now complete. $\square$

3. PROOFS OF THEOREM 1.7 AND COROLLARY 1.2

To prove Theorem 1.7 and Corollary 1.2, we need to establish an auxiliary theorem first.

Lemma 3.1. *Let $m \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$ be relatively prime. Then, for any $n \in \mathbb{Z}^+$, we have*

$$(3.1) \quad \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{km + r} \equiv \frac{1}{r} + \frac{m}{2} \llbracket 2 \mid n \rrbracket \pmod{m}.$$

Proof. We use induction on n .

If $n \in \mathbb{Z}^+$ is relatively prime to m (e.g., $n = 1$), then 2 cannot divide both m and n ; hence

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{km + r} \equiv \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{r} \equiv \frac{1}{r} + \frac{m}{2} \llbracket 2 \mid n \rrbracket \pmod{m}.$$

Now suppose that p is the least common prime divisor of m and n , and set $n_0 = n/p$. Then

$$\sum_{k=0}^{n-1} \frac{1}{km + r} = \sum_{i=0}^{n_0-1} \sum_{j=0}^{p-1} \frac{1}{(i + jn_0)m + r} = \sum_{i=0}^{n_0-1} \sum_{j=0}^{p-1} \frac{1}{im + jmn_0 + r}.$$

For any $i = 0, \dots, n_0 - 1$, clearly

$$\begin{aligned} & \frac{1}{n} \sum_{j=0}^{p-1} \left(\frac{1}{im + jmn_0 + r} - \frac{1}{im + r} \right) \\ &= \frac{1}{n} \sum_{j=0}^{p-1} \frac{-jmn/p}{(im + jmn_0 + r)(im + r)} \\ &\equiv - \sum_{j=0}^{p-1} \frac{jn/p}{r^2} = -\frac{p-1}{2} \cdot \frac{m}{r^2} \equiv \delta_{p,2} \frac{m}{2} \pmod{m}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{km + r} &\equiv \sum_{i=0}^{n_0-1} \left(\frac{1}{n} \sum_{j=0}^{p-1} \frac{1}{im + r} + \delta_{p,2} \frac{m}{2} \right) \\ &\equiv \frac{1}{n_0} \sum_{i=0}^{n_0-1} \frac{1}{im + r} + \delta_{p,2} \frac{m}{2} \cdot \frac{n}{2} \pmod{m}. \end{aligned}$$

Note that $n_0 < n$. If

$$\frac{1}{n_0} \sum_{i=0}^{n_0-1} \frac{1}{im + r} \equiv \frac{1}{r} + \frac{m}{2} \llbracket 2 \mid n_0 \rrbracket \pmod{m},$$

then (3.1) holds by the above, because

$$\delta_{p,2} \frac{m}{2} \cdot \frac{n}{2} + \frac{m}{2} \llbracket 2 \mid n_0 \rrbracket \equiv \delta_{p,2} \frac{m}{2} (\llbracket 4 \mid n - 2 \rrbracket + \llbracket 4 \mid n \rrbracket) \equiv \frac{m}{2} \llbracket 2 \mid n \rrbracket \pmod{m}.$$

This concludes the induction proof. $\square$

Lemma 3.2. *Let p be a prime, and let $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. If p is odd, then*

$$(3.2) \quad \binom{pn}{pk} \equiv \binom{n}{k} \pmod{p^{2\text{ord}_p(n)+2}}.$$

For $p = 2$ we have

$$(3.3) \quad \binom{2n}{2k} \equiv (-1)^k \binom{n}{k} \pmod{2^{2\text{ord}_2(n)+1}}.$$

Proof. The case $k = 0$ or $k \geq n$ is trivial. Below we let $0 < k < n$.

By a result of Jacobsthal (see, e.g., [G]), if $p > 3$, then

$$\binom{pn}{pk} \bigg/ \binom{n}{k} = 1 + p^3 nk(n-k)q$$

for some $q \in \mathbb{Z}_p$, and hence

$$\begin{aligned} \binom{pn}{pk} - \binom{n}{k} &= \binom{n}{k} p^3 nk(n-k)q = p^3 n^2 \binom{n-1}{k-1} (n-k)q \\ &\equiv 0 \pmod{p^{3+2\text{ord}_p(n)}}. \end{aligned}$$

Now we handle the case $p = 3$. Observe that

$$\begin{aligned} \binom{3n}{3k} / \binom{n}{k} &= \frac{(3n-1)(3n-2)}{1 \cdot 2} \times \frac{(3n-4)(3n-5)}{4 \cdot 5} \\ &\quad \times \dots \times \frac{(3n-(3k-2))(3n-(3k-1))}{(3k-2)(3k-1)} \end{aligned}$$

and

$$(3n - (3i+1))(3n - (3i+2)) = 9n^2 - 9n(2i+1) + (3i+1)(3i+2)$$

for any $i \in \mathbb{Z}$. So we have

$$\begin{aligned} \binom{3n}{3k} / \binom{n}{k} &= \prod_{i=0}^{k-1} \left(1 + 9n \frac{n-2i-1}{(3i+1)(3i+2)} \right) \\ &\equiv 1 + 9n \sum_{i=0}^{k-1} \frac{n-2i-1}{(3i+1)(3i+2)} \pmod{(3^{2+\text{ord}_3(n)})^2}. \end{aligned}$$

Clearly,

$$\begin{aligned} \sum_{i=0}^{k-1} \frac{n-2i-1}{(3i+1)(3i+2)} &= \sum_{i=0}^{k-1} \left(\frac{n-2i-1}{3i+1} - \frac{n-2i-1}{3i+2} \right) \\ &= \sum_{i=0}^{k-1} \left(\frac{n-2i-1}{3i+1} - \frac{n-2(k-1-i)-1}{3(k-1-i)+2} \right) \end{aligned}$$

and hence

$$\begin{aligned} &\frac{1}{k} \sum_{i=0}^{k-1} \frac{n-2i-1}{(3i+1)(3i+2)} \\ &= \frac{1}{k} \sum_{i=0}^{k-1} \left(\frac{n-(2i+1)}{3i+1} - \frac{n+(2i+1)-2k}{3k-(3i+1)} \right) \\ &= \frac{1}{k} \sum_{i=0}^{k-1} \frac{3k(n-(2i+1)) + 2k(3i+1) - 2n(3i+1)}{(3i+1)(3k-3i-1)} \\ &= \sum_{i=0}^{k-1} \frac{3n-1}{(3i+1)(3k-3i-1)} - \frac{2n}{k} \sum_{i=0}^{k-1} \frac{1}{3(k-1-i)+2} \\ &\equiv \sum_{i=0}^{k-1} \frac{-1}{-1} - 2n \times \frac{1}{2} = k - n \pmod{3}, \end{aligned}$$

where we have applied Lemma 3.1 with $m = 3$ to get the last congruence. Therefore

$$\begin{aligned} \binom{3n}{3k} &\equiv \binom{n}{k} + 9n \frac{n}{k} \binom{n-1}{k-1} \sum_{i=0}^{k-1} \frac{n-2i-1}{(3i+1)(3i+2)} \pmod{3^{2+\text{ord}_3(n)+4}} \\ &\equiv \binom{n}{k} + 9n^2 \binom{n-1}{k-1} (k-n) \pmod{3^{2+\text{ord}_3(n)+3}} \\ &\equiv \binom{n}{k} \pmod{3^{2+\text{ord}_3(n)+2}}. \end{aligned}$$

Finally we consider the case $p = 2$. Observe that

$$\begin{aligned} \frac{\binom{2n}{2k}}{\binom{n}{k}} &= \prod_{j=0}^{k-1} \frac{2n - (2j+1)}{2j+1} = (-1)^k \prod_{j=0}^{k-1} \left(1 - \frac{2n}{2j+1}\right) \\ &\equiv (-1)^k \left(1 - 2n \sum_{j=0}^{k-1} \frac{1}{2j+1}\right) \pmod{(2^{\text{ord}_2(n)+1})^2}. \end{aligned}$$

This, together with Lemma 3.1 in the case $m = 2$, yields that

$$\begin{aligned} \binom{2n}{2k} &\equiv (-1)^k \left(\binom{n}{k} - \frac{2n^2}{k} \binom{n-1}{k-1} \sum_{j=0}^{k-1} \frac{1}{2j+1} \right) \pmod{2^{2\text{ord}_2(n)+2}} \\ &\equiv (-1)^k \binom{n}{k} - (-1)^k 2n^2 \binom{n-1}{k-1} (1 + \llbracket 2 \mid k \rrbracket) \pmod{2^{2\text{ord}_2(n)+2}} \\ &\equiv (-1)^k \binom{n}{k} \pmod{2^{2\text{ord}_2(n)+1}}. \end{aligned}$$

We are done. $\square$

If p is a prime, $\alpha \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$, then

$$\text{ord}_p \left(\sum_{k \equiv r \pmod{p^\alpha}} \binom{p^{\alpha-1}n}{k} (-1)^k \right) \geq \left\lfloor \frac{p^{\alpha-1}n - p^{\alpha-1}}{\varphi(p^\alpha)} \right\rfloor = \left\lfloor \frac{n-1}{p-1} \right\rfloor$$

by Weisman's result, and hence

$$(3.4) \quad S_\alpha^{(p)}(n, r) := p^{-\lfloor \frac{n-1}{p-1} \rfloor} \sum_{k \equiv r \pmod{p^\alpha}} \binom{p^{\alpha-1}n}{k} (-1)^k$$

is an integer.

Theorem 3.1. *Let p be a prime. Then, for each $\alpha = 2, 3, \dots$, whenever $n \in \mathbb{N}$ and $r \in \mathbb{Z}$ we have*

$$(3.5) \quad S_\alpha^{(p)}(n, r) \equiv \begin{cases} S_{\alpha-1}^{(p)}(n, r/p) \pmod{p^{(2-\delta_{p,2})(\alpha-2)}} & \text{if } p \mid r, \\ 0 \pmod{p^{\alpha-2}} & \text{otherwise,} \end{cases}$$

where $S_\alpha^{(p)}(n, r)$ is defined by (3.4).

Proof. We use induction on α and write $S_\alpha(n, r)$ for $S_\alpha^{(p)}(n, r)$.

When $\alpha = 2$ we need do nothing.

Now let $\alpha > 2$ and assume the desired result for $\alpha - 1$. We use induction on n to prove that (3.5) holds for any $r \in \mathbb{Z}$.

First we consider the case $n \in \{0, \dots, p-1\}$. Note that $p^{\alpha-1}n < p^\alpha$ and $\delta = -\lfloor (n-1)/(p-1) \rfloor \in \{0, 1\}$. Thus

$$S_\alpha(n, r) = p^\delta \binom{p^{\alpha-1}n}{\{r\}_{p^\alpha}} (-1)^{\{r\}_{p^\alpha}}.$$

If $p \nmid r$, then

$$\binom{p^{\alpha-1}n}{\{r\}_{p^\alpha}} = \frac{p^{\alpha-1}n}{\{r\}_{p^\alpha}} \binom{p^{\alpha-1}n-1}{\{r\}_{p^\alpha}-1} \equiv 0 \pmod{p^{\alpha-1}},$$

and hence $S_\alpha(n, r) \equiv 0 \pmod{p^{\alpha-1}}$. When $p \mid r$, by Lemma 3.2 we have

$$\begin{aligned} & (-1)^{\{r\}_{p^\alpha} - \{r/p\}_{p^{\alpha-1}}} \binom{p^{\alpha-2}n}{\{r/p\}_{p^{\alpha-1}}} \\ &= (-1)^{(p-1)\{r\}_{p^\alpha}/p} \binom{p^{\alpha-2}n}{\{r\}_{p^\alpha}/p} \equiv \binom{p^{\alpha-1}n}{\{r\}_{p^\alpha}} \pmod{p^{2(\alpha-2)+2-\delta_{p,2}}}; \end{aligned}$$

thus $\text{ord}_p(S_\alpha(n, r) - S_{\alpha-1}(n, r/p)) \geq 2\alpha - 2 - \delta_{p,2} > (2 - \delta_{p,2})(\alpha - 2)$.

Below we let $n \geq p$, and assume that (3.5) is valid for all $r \in \mathbb{Z}$ if we replace n in (3.5) by a smaller nonnegative integer. Set $n' = n - (p - 1) \geq 1$. Then, by Vandermonde's identity $\binom{x+y}{k} = \sum_{j \in \mathbb{N}} \binom{x}{j} \binom{y}{k-j}$ (cf. [GKP, (5.22)]), we have

$$\begin{aligned} S_\alpha(n, r) &= p^{-\lfloor \frac{n-1}{p-1} \rfloor} \sum_{k \equiv r \pmod{p^\alpha}} \sum_{j=0}^{\varphi(p^\alpha)} \binom{\varphi(p^\alpha)}{j} \binom{p^{\alpha-1}n'}{k-j} (-1)^k \\ &= \sum_{j=0}^{\varphi(p^\alpha)} \frac{(-1)^j}{p} \binom{\varphi(p^\alpha)}{j} p^{-\lfloor \frac{n'-1}{p-1} \rfloor} \sum_{k \equiv r-j \pmod{p^\alpha}} \binom{p^{\alpha-1}n'}{k} (-1)^k. \end{aligned}$$

If $0 \leq j \leq \varphi(p^\alpha)$ and $p \nmid j$, then

$$\frac{1}{p} \binom{\varphi(p^\alpha)}{j} = \frac{p^{\alpha-1}(p-1)}{pj} \binom{\varphi(p^\alpha)-1}{j-1} \equiv 0 \pmod{p^{\alpha-2}},$$

and also $S_\alpha(n', r-j) \equiv 0 \pmod{p^{\alpha-2}}$ providing $p \mid r$ (by the induction hypothesis). Thus

$$S_\alpha(n, r) \equiv \sum_{j=0}^{\varphi(p^{\alpha-1})} \frac{(-1)^{pj}}{p} \binom{\varphi(p^\alpha)}{pj} S_\alpha(n', r - pj) \pmod{p^{(1+\lfloor p \rfloor)(\alpha-2)}}.$$

Note that when $j \not\equiv 0 \pmod{p^{\alpha-2}}$ we have

$$\frac{1}{p} \binom{\varphi(p^\alpha)}{pj} = \frac{p^{\alpha-1}(p-1)}{p^2j} \binom{\varphi(p^\alpha)-1}{pj-1} \in \mathbb{Z}_p.$$

Case I. $p \nmid r$. By the induction hypothesis, $p^{\alpha-2} \mid S_\alpha(n', r - pj)$ for all $j \in \mathbb{Z}$. Thus, by the above,

$$S_\alpha(n, r) \equiv \sum_{j=0}^{p-1} \frac{(-1)^{p^{\alpha-1}j}}{p} \binom{\varphi(p^\alpha)}{p^{\alpha-1}j} S_\alpha(n', r - p^{\alpha-1}j) \pmod{p^{\alpha-2}}.$$

In view of Lucas' theorem or Lemma 3.2,

$$\binom{p^{\alpha-1}(p-1)}{p^{\alpha-1}j} \equiv \binom{p^{\alpha-2}(p-1)}{p^{\alpha-2}j} \equiv \cdots \equiv \binom{p-1}{j} \equiv (-1)^j \pmod{p}$$

for every $j = 0, \dots, p-1$. Note also that $(-1)^j \equiv 1 \pmod{2}$. So we have

$$S_\alpha(n, r) \equiv \frac{1}{p} \sum_{j=0}^{p-1} S_\alpha(n', r - p^{\alpha-1}j) \pmod{p^{\alpha-2}}.$$

Observe that

$$\begin{aligned}
& \sum_{j=0}^{p-1} S_{\alpha}(n', r - p^{\alpha-1}j) \\
&= p^{-\lfloor \frac{n'-1}{p-1} \rfloor} \sum_{k \equiv r \pmod{p^{\alpha-1}}} \binom{p^{\alpha-2}pn'}{k} (-1)^k \\
&= p^{-\lfloor \frac{n'-1}{p-1} \rfloor} p^{\lfloor \frac{pn'-1}{p-1} \rfloor} S_{\alpha-1}(pn', r) = p^{n'} S_{\alpha-1}(pn', r).
\end{aligned}$$

Therefore

$$S_{\alpha}(n, r) \equiv p^{n'-1} S_{\alpha-1}(pn', r) \pmod{p^{\alpha-2}}.$$

By the induction hypothesis for $\alpha - 1$, we have $p^{\alpha-3} \mid S_{\alpha-1}(pn', r)$; hence $S_{\alpha}(n, r) \equiv 0 \pmod{p^{\alpha-2}}$ if $n' > 1$. In the case $n' = 1$, we need to show that $S_{\alpha-1}(p, r) \equiv 0 \pmod{p^{\alpha-2}}$. In fact,

$$\begin{aligned}
S_{\alpha-1}(p, r) &= p^{-\lfloor \frac{p-1}{p-1} \rfloor} \sum_{k \equiv r \pmod{p^{\alpha-1}}} \binom{p^{\alpha-2}p}{k} (-1)^k \\
&= \sum_{k \equiv r \pmod{p^{\alpha-1}}} \frac{p^{\alpha-2}}{k} \binom{p^{\alpha-1}-1}{k-1} (-1)^k \equiv 0 \pmod{p^{\alpha-2}}.
\end{aligned}$$

(Note that if $k \equiv r \pmod{p^{\alpha-1}}$, then $p \nmid k$ since $p \nmid r$.)

Case II. $p \mid r$. By the paragraph before Case I,

$$\begin{aligned}
& S_{\alpha}(n, r) - S_{\alpha-1}\left(n, \frac{r}{p}\right) \\
&\equiv \sum_{j=0}^{\varphi(p^{\alpha-1})} \frac{(-1)^{pj}}{p} \binom{\varphi(p^{\alpha})}{pj} S_{\alpha}(n', r - pj) \\
&\quad - \sum_{j=0}^{\varphi(p^{\alpha-1})} \frac{(-1)^j}{p} \binom{\varphi(p^{\alpha-1})}{j} S_{\alpha-1}\left(n', \frac{r}{p} - j\right) \pmod{p^{2(\alpha-2)}}.
\end{aligned}$$

In view of Lemma 3.2,

$$\binom{p^{\alpha-1}(p-1)}{pj} \equiv (-1)^{(p-1)j} \binom{p^{\alpha-2}(p-1)}{j} \pmod{p^{2(\alpha-2)+1}}.$$

Thus $S_{\alpha}(n, r) - S_{\alpha-1}(n, r/p)$ is congruent to

$$\sum_{j=0}^{\varphi(p^{\alpha-1})} \frac{(-1)^j}{p} \binom{\varphi(p^{\alpha-1})}{j} \left(S_{\alpha}(n', r - pj) - S_{\alpha-1}\left(n', \frac{r}{p} - j\right) \right)$$

modulo $p^{2(\alpha-2)}$. If $p^{\alpha-2} \nmid j$, then

$$\frac{1}{p} \binom{\varphi(p^{\alpha-1})}{j} = \frac{p^{\alpha-2}(p-1)}{pj} \binom{\varphi(p^{\alpha-1})-1}{j-1} \in \mathbb{Z}_p.$$

Since $S_\alpha(n', r - pj) \equiv S_{\alpha-1}(n', r/p - j) \pmod{p^{(2-\delta_{p,2})(\alpha-2)}}$ for any $j \in \mathbb{Z}$ (by the induction hypothesis), and $-1 \equiv 1 \pmod{2}$, we have

$$\begin{aligned} & S_\alpha(n, r) - S_{\alpha-1}\left(n, \frac{r}{p}\right) \\ & \equiv \sum_{j=0}^{p-1} \frac{(-1)^j}{p} \binom{\varphi(p^{\alpha-1})}{p^{\alpha-2}j} \left(S_\alpha(n', r - p^{\alpha-1}j) - S_{\alpha-1}\left(n', \frac{r}{p} - p^{\alpha-2}j\right) \right) \\ & \equiv \sum_{j=0}^{p-1} \frac{S_\alpha(n', r - p^{\alpha-1}j) - S_{\alpha-1}(n', r/p - p^{\alpha-2}j)}{p} \pmod{p^{(2-\delta_{p,2})(\alpha-2)}}, \end{aligned}$$

where we have used the fact that $\binom{p^{\alpha-2}(p-1)}{p^{\alpha-2}j} \equiv \binom{p-1}{j} \equiv (-1)^j \pmod{p}$ for $j = 0, \dots, p-1$. As in Case I,

$$\sum_{j=0}^{p-1} S_\alpha(n', r - p^{\alpha-1}j) = p^{n'} S_{\alpha-1}(pn', r)$$

and

$$\sum_{j=0}^{p-1} S_{\alpha-1}\left(n', \frac{r}{p} - p^{\alpha-2}j\right) = p^{n'} S_{\alpha-2}\left(pn', \frac{r}{p}\right).$$

Thus

$$\begin{aligned} & S_\alpha(n, r) - S_{\alpha-1}\left(n, \frac{r}{p}\right) \\ & \equiv p^{n'-1} \left(S_{\alpha-1}(pn', r) - S_{\alpha-2}\left(pn', \frac{r}{p}\right) \right) \pmod{p^{(2-\delta_{p,2})(\alpha-2)}}. \end{aligned}$$

By the induction hypothesis for $\alpha-1$, $S_{\alpha-1}(pn', r)$ and $S_{\alpha-2}(pn', r/p)$ are congruent modulo $p^{(2-\delta_{p,2})(\alpha-3)}$. Therefore, if $n' > 2$, then

$$S_\alpha(n, r) \equiv S_{\alpha-1}\left(n, \frac{r}{p}\right) \pmod{p^{(2-\delta_{p,2})(\alpha-2)}}.$$

When $n' \in \{1, 2\}$, we have $n' - 1 - \lfloor (pn' - 1)/(p - 1) \rfloor \geq -2$ and also

$$\begin{aligned} & p^{\lfloor \frac{pn'-1}{p-1} \rfloor} \left(S_{\alpha-1}(pn', r) - S_{\alpha-2}\left(pn', \frac{r}{p}\right) \right) \\ & = \sum_{k \equiv r \pmod{p^{\alpha-1}}} \binom{p^{\alpha-1}n'}{k} (-1)^k - \sum_{k \equiv r/p \pmod{p^{\alpha-2}}} \binom{p^{\alpha-2}n'}{k} (-1)^k \\ & = \sum_{k \equiv r/p \pmod{p^{\alpha-2}}} \left(\binom{p^{\alpha-1}n'}{pk} (-1)^{pk} - \binom{p^{\alpha-2}n'}{k} (-1)^k \right). \end{aligned}$$

By Lemma 3.2, the last sum is a multiple of

$$p^{2(\alpha-2)+2-\delta_{p,2}} = p^{(2-\delta_{p,2})(\alpha-2)} p^{2+\delta_{p,2}(\alpha-3)}.$$

So we also have the desired result in the case $n' \leq 2$.

The induction proof of Theorem 3.1 is now complete. $\square$

We believe that (3.5) in the case $p \mid r$ can be improved slightly. Here is our conjecture.

Conjecture 3.1. *Let p be an odd prime, and let $\alpha \geq 2$ be an integer. If $n \in \mathbb{N}$ and $r \in \mathbb{Z}$, then*

$$S_{\alpha}^{(p)}(n, pr) \equiv S_{\alpha-1}^{(p)}(n, r) \pmod{p^{2\alpha-2-\delta_{p,3}}}.$$

Now we give a useful consequence of Theorem 3.1.

Corollary 3.1. *Let p be a prime, and let $\alpha, \beta, n \in \mathbb{N}$ with $\alpha > \beta$. Given $r \in \mathbb{Z}$ we have*

$$(3.6) \quad S_{\alpha}^{(p)}(n, p^{\beta}r) \equiv S_{\alpha-\beta}^{(p)}(n, r) \pmod{p^{(2-\delta_{p,2})(\alpha-\beta-1)}}.$$

Provided that $r \not\equiv 0 \pmod{p}$, we also have

$$(3.7) \quad \text{ord}_p \left(S_{\alpha}^{(p)}(n, p^{\beta}r) \right) \geq \alpha - \beta - 2.$$

Proof. By Theorem 3.1, if $0 \leq j < \beta$, then

$$S_{\alpha-j}^{(p)}(n, p^{\beta-j}r) \equiv S_{\alpha-j-1}^{(p)}(n, p^{\beta-j-1}r) \pmod{p^{(2-\delta_{p,2})(\alpha-j-2)}}.$$

So (3.6) follows.

When $\alpha = \beta + 1$, (3.7) holds trivially. If $p \nmid r$ and $\alpha - \beta \geq 2$, then $S_{\alpha-\beta}^{(p)}(n, r) \equiv 0 \pmod{p^{\alpha-\beta-2}}$ by Theorem 3.1; combining this with (3.6) we immediately obtain (3.7). $\square$

Proof of Theorem 1.7. Write $n = pn_0 + s_0$ with $n_0, s_0 \in \mathbb{N}$ and $s_0 < p$. Then

$$\begin{aligned} \left\lfloor \frac{p^{\alpha-2}n + s - p^{\alpha-1}}{\varphi(p^{\alpha})} \right\rfloor &= \left\lfloor \frac{n + s/p^{\alpha-2} - p}{\varphi(p^2)} \right\rfloor = \left\lfloor \frac{n - p}{\varphi(p^2)} \right\rfloor \\ &= \left\lfloor \frac{n/p - 1}{p - 1} \right\rfloor = \left\lfloor \frac{n_0 - 1}{p - 1} \right\rfloor. \end{aligned}$$

By Vandermonde's identity,

$$\begin{aligned} &\sum_{k \equiv p^{\alpha-2}r+t \pmod{p^{\alpha}}} \binom{p^{\alpha-2}n + s}{k} (-1)^k \\ &= \sum_{k \equiv p^{\alpha-2}r+t \pmod{p^{\alpha}}} \sum_{j \in \mathbb{N}} \binom{p^{\alpha-2}s_0 + s}{j} \binom{p^{\alpha-1}n_0}{k-j} (-1)^k \\ &= \sum_{j \in \mathbb{N}} \binom{p^{\alpha-2}s_0 + s}{j} (-1)^j \sum_{k \equiv p^{\alpha-2}r+t-j \pmod{p^{\alpha}}} \binom{p^{\alpha-1}n_0}{k} (-1)^k. \end{aligned}$$

In light of Corollary 3.1, for any $r' \in \mathbb{Z}$ we have

$$S_{\alpha}(n_0, r') \equiv \begin{cases} S_2(n_0, r'/p^{\alpha-2}) \pmod{p^{2-\delta_{p,2}}} & \text{if } r' \equiv 0 \pmod{p^{\alpha-2}}, \\ 0 \pmod{p} & \text{if } \text{ord}_p(r') < \alpha - 2. \end{cases}$$

Thus

$$\begin{aligned}
& p^{-\lfloor \frac{n_0-1}{p-1} \rfloor} \sum_{k \equiv p^{\alpha-2}r+t \pmod{p^\alpha}} \binom{p^{\alpha-2}n+s}{k} (-1)^k \\
& \equiv \sum_{i \in \mathbb{N}} \binom{p^{\alpha-2}s_0+s}{p^{\alpha-2}i+t} (-1)^{p^{\alpha-2}i+t} p^{-\lfloor \frac{n_0-1}{p-1} \rfloor} \sum_{k \equiv r-i \pmod{p^2}} \binom{pn_0}{k} (-1)^k \\
& \equiv \sum_{i \in \mathbb{N}} \binom{s_0}{i} \binom{s}{t} (-1)^{i+t} p^{-\lfloor \frac{n_0-1}{p-1} \rfloor} \sum_{k \equiv r-i \pmod{p^2}} \binom{pn_0}{k} (-1)^k \pmod{p},
\end{aligned}$$

where we have applied Lucas' theorem to get the last congruence. (Note that $s = t = 0$ if $\alpha = 2$; also, $-1 \equiv 1 \pmod{2}$.) Since

$$\begin{aligned}
& \sum_{i=0}^{s_0} \binom{s_0}{i} (-1)^i \sum_{k \equiv r-i \pmod{p^2}} \binom{pn_0}{k} (-1)^k \\
& = \sum_{i=0}^{s_0} \binom{s_0}{i} \sum_{k \equiv r \pmod{p^2}} \binom{pn_0}{k-i} (-1)^k \\
& = \sum_{k \equiv r \pmod{p^2}} \binom{pn_0+s_0}{k} (-1)^k = \sum_{k \equiv r \pmod{p^2}} \binom{n}{k} (-1)^k,
\end{aligned}$$

the desired result follows from the above. $\square$

Proof of Corollary 1.2. Let $s = \{n\}_{2^{\alpha-2}}$ and $t = \{r\}_{2^{\alpha-2}}$. Then $n = 2^{\alpha-2}n_* + s$ and $r = 2^{\alpha-2}r_* + t$. By Theorem 1.7,

$$\begin{aligned}
& 2^{-\lfloor \frac{n-2^{\alpha-1}}{\varphi(2^\alpha)} \rfloor} \sum_{k \equiv r \pmod{2^\alpha}} \binom{n}{k} (-1)^k \equiv 1 \pmod{2} \\
& \iff \binom{s}{t} \equiv 2^{-\lfloor \frac{n_*-2}{\varphi(4)} \rfloor} \sum_{k \equiv r_* \pmod{4}} \binom{n_*}{k} (-1)^k \equiv 1 \pmod{2}.
\end{aligned}$$

In the case $n_* \equiv 1 \pmod{2}$, by [S02, (3.3)] we have

$$2 \sum_{\substack{0 < k < n_* \\ 4 \mid k-r_*}} \binom{n_*}{k} - (2^{n_*-1} - 1) = (-1)^{\frac{r_*(n_*-r_*)}{2}} \left((-1)^{\frac{n_*^2-1}{8}} 2^{\frac{n_*-1}{2}} - 1 \right);$$

thus

$$\begin{aligned}
2 \sum_{k \equiv r_* \pmod{4}} \binom{n_*}{k} & = 2(\llbracket 4 \mid r_* \rrbracket + \llbracket 4 \mid n_* - r_* \rrbracket) + 2^{n_*-1} - 1 \\
& \quad + (-1)^{\frac{r_*(n_*-r_*)}{2}} \left((-1)^{\frac{n_*^2-1}{8}} 2^{\frac{n_*-1}{2}} - 1 \right) \\
& = 2^{n_*-1} + (-1)^{\frac{r_*(n_*-r_*)}{2} + \frac{n_*^2-1}{8}} 2^{\frac{n_*-1}{2}},
\end{aligned}$$

and hence

$$2^{-\lfloor \frac{n_*}{2} \rfloor + 1} \sum_{k \equiv r_* \pmod{4}} \binom{n_*}{k} \equiv 1 \pmod{2} \iff n_* > 1.$$

When $n_* \equiv 0 \pmod{2}$, if $n_* > 0$, then by the above we have

$$\begin{aligned}
& 2 \sum_{k \equiv r_* \pmod{4}} \binom{n_*}{k} = 2 \sum_{k \equiv r_* \pmod{4}} \left(\binom{n_*-1}{k} + \binom{n_*-1}{k-1} \right) \\
&= 2 \sum_{k \equiv r_* \pmod{4}} \binom{n_*-1}{k} + 2 \sum_{k \equiv r_*-1 \pmod{4}} \binom{n_*-1}{k} \\
&= 2^{(n_*-1)-1} + (-1)^{\frac{r_*(n_*-1-r_*)}{2} + \frac{(n_*-1)^2-1}{8}} 2^{\frac{(n_*-1)-1}{2}} \\
&\quad + 2^{(n_*-1)-1} + (-1)^{\frac{(r_*-1)(n_*-1-(r_*-1))}{2} + \frac{(n_*-1)^2-1}{8}} 2^{\frac{(n_*-1)-1}{2}} \\
&= 2^{n_*-1} + (-1)^{\binom{n_*/2}{2}} 2^{n_*/2-1} \left((-1)^{\frac{r_*(n_*-1-r_*)}{2}} + (-1)^{\frac{(r_*-1)(n_*-r_*)}{2}} \right) \\
&= 2^{n_*-1} + (-1)^{\binom{n_*/2}{2} + \lfloor \frac{r_*}{2} \rfloor} \left(1 + (-1)^{\frac{n_*}{2} + r_*} \right) 2^{n_*/2-1};
\end{aligned}$$

therefore

$$\begin{aligned}
& 2^{-\lfloor \frac{n_*}{2} \rfloor + 1} \sum_{k \equiv r_* \pmod{4}} \binom{n_*}{k} \equiv 1 \pmod{2} \\
&\iff n_* > 2 \ \& \ \frac{n_*}{2} \equiv r_* \pmod{2}, \text{ or } n_* = 2 \ \& \ 2 \mid r_*.
\end{aligned}$$

□

Combining the above we obtain the desired result.

4. PROOFS OF THEOREMS 1.5 AND 1.8

Proof of Theorem 1.5. For convenience, we set $T_{l,\alpha}(n, r) := T_{l,\alpha}^{(p)}(n, r)$.

The case $n = 0$ is easy, because

$$\begin{aligned}
T_{l,\alpha+1}(0, r) &= \llbracket p^{\alpha+1} \mid r \rrbracket l! p^l \binom{-r/p^{\alpha+1}}{l} \\
&= \llbracket p \mid r \ \& \ p^\alpha \mid \lfloor r/p \rfloor \rrbracket l! p^l \binom{-\lfloor r/p \rfloor / p^\alpha}{l} \\
&= (-1)^{\{r\}_p} \binom{0}{\{r\}_p} T_{l,\alpha} \left(0, \left\lfloor \frac{r}{p} \right\rfloor \right).
\end{aligned}$$

Below we use induction on $l + n$ to prove the desired result.

If $l + n = 0$, then $n = 0$ and hence we are done.

Now let $n > 0$, and assume that

$$T_{l_*,\alpha+1}(n_*, r_*) \equiv (-1)^{\{r_*\}_p} \binom{\{n_*\}_p}{\{r_*\}_p} T_{l_*,\alpha} \left(\left\lfloor \frac{n_*}{p} \right\rfloor, \left\lfloor \frac{r_*}{p} \right\rfloor \right) \pmod{p}$$

whenever $l_*, n_* \in \mathbb{N}$, $l_* + n_* < l + n$ and $r_* \in \mathbb{Z}$. Write $n = pn_0 + s$ and $r = pr_0 + t$, where $n_0, r_0 \in \mathbb{Z}$ and $s, t \in \{0, \dots, p-1\}$.

Case 1. $p^\alpha \nmid n$. By Lemma 2.2 and the induction hypothesis, if $s \neq 0$, then

$$\begin{aligned}
 T_{l,\alpha+1}(n, r) &= T_{l,\alpha+1}(pn_0 + s - 1, pr_0 + t) \\
 &\quad - T_{l,\alpha+1}(pn_0 + s - 1, pr_0 + t - 1) \\
 &\equiv (-1)^t \binom{s-1}{t} T_{l,\alpha}(n_0, r_0) \\
 &\quad - \begin{cases} (-1)^{t-1} \binom{s-1}{t-1} T_{l,\alpha}(n_0, r_0) & \text{if } t > 0, \\ (-1)^{p-1} \binom{s-1}{p-1} T_{l,\alpha}(n_0, r_0 - 1) & \text{if } t = 0, \end{cases} \\
 &\equiv (-1)^t \binom{s-1}{t} T_{l,\alpha}(n_0, r_0) + (-1)^t \binom{s-1}{t-1} T_{l,\alpha}(n_0, r_0) \\
 &\equiv (-1)^t \binom{s}{t} T_{l,\alpha}(n_0, r_0) \pmod{p}.
 \end{aligned}$$

Now we assume that $s = 0$. In a similar way,

$$\begin{aligned}
 T_{l,\alpha+1}(n, r) &= T_{l,\alpha+1}(p(n_0 - 1) + p - 1, pr_0 + t) \\
 &\quad - T_{l,\alpha+1}(p(n_0 - 1) + p - 1, pr_0 + t - 1) \\
 &\equiv (-1)^t \binom{p-1}{t} T_{l,\alpha}(n_0 - 1, r_0) \\
 &\quad - \begin{cases} (-1)^{t-1} \binom{p-1}{t-1} T_{l,\alpha}(n_0 - 1, r_0) \pmod{p} & \text{if } t > 0, \\ (-1)^{p-1} \binom{p-1}{p-1} T_{l,\alpha}(n_0 - 1, r_0 - 1) \pmod{p} & \text{if } t = 0. \end{cases}
 \end{aligned}$$

Thus, if $0 < t < p$, then

$$T_{l,\alpha+1}(n, r) \equiv (-1)^t \binom{p}{t} T_{l,\alpha}(n_0 - 1, r_0) \equiv 0 \pmod{p},$$

and hence $T_{l,\alpha+1}(n, r) \equiv (-1)^t \binom{0}{t} T_{l,\alpha}(n_0, r_0) \pmod{p}$; if $t = 0$, then

$$T_{l,\alpha+1}(n, r) \equiv T_{l,\alpha}(n_0 - 1, r_0) - T_{l,\alpha}(n_0 - 1, r_0 - 1) = T_{l,\alpha}(n_0, r_0) \pmod{p},$$

with the help of Lemma 2.2. (Note that $p^{\alpha-1}$ does not divide $n_0 = n/p$.)

Case 2. $p^\alpha \mid n$ and $p^\alpha \nmid r$. In this case,

$$\text{ord}_p(T_{l,\alpha+1}(n, r)) \geq \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}) > 0$$

by Theorem 2.1. If $p \nmid r$ (i.e., $t > 0$), then $\binom{\{n\}_p}{\{r\}_p} = \binom{0}{t} = 0$; if $p \mid r$, then

$$\begin{aligned}
 \text{ord}_p(T_{l,\alpha}(n_0, r_0)) &\geq \tau_p(\{r_0\}_{p^{\alpha-1}}, \{n_0 - r_0\}_{p^{\alpha-1}}) \\
 &= \tau_p(p\{r_0\}_{p^{\alpha-1}}, p\{n_0 - r_0\}_{p^{\alpha-1}}) = \tau_p(\{r\}_{p^\alpha}, \{n-r\}_{p^\alpha}).
 \end{aligned}$$

So we have

$$T_{l,\alpha+1}(n, r) \equiv 0 \equiv (-1)^{\{r\}_p} \binom{\{n\}_p}{\{r\}_p} T_{l,\alpha}(n_0, r_0) \pmod{p}.$$

Case 3. $n \equiv r \equiv 0 \pmod{p^\alpha}$. In this case, $s = t = 0$. When $l = 0$, by Theorem 3.1 we have

$$\begin{aligned}
 T_{0,\alpha+1}(n, r) &= \frac{p^{\lfloor \frac{n/p^\alpha - 1}{p-1} \rfloor}}{(n/p^\alpha)!} S_{\alpha+1}^{(p)} \left(\frac{n}{p^\alpha}, r \right) \\
 &\equiv \frac{p^{\lfloor \frac{n_0/p^{\alpha-1} - 1}{p-1} \rfloor}}{(n_0/p^{\alpha-1})!} S_\alpha^{(p)} \left(\frac{n_0}{p^{\alpha-1}}, \frac{r}{p} \right) = T_{0,\alpha}(n_0, r_0) \pmod{p^{(2-\delta_{p,2})(\alpha-1)}}.
 \end{aligned}$$

As $\alpha \geq 2$ this implies that $T_{0,\alpha+1}(n, r) \equiv T_{0,\alpha}(n_0, r_0) \pmod{p}$. In view of Lemma 2.2 and the induction hypothesis, if $l > 0$, then

$$\begin{aligned} T_{l,\alpha+1}(n, r) &= -\frac{r}{p^\alpha} T_{l-1,\alpha+1}(n, r + p^{\alpha+1}) - T_{l-1,\alpha+1}(n-1, r + p^{\alpha+1} - 1) \\ &= -\frac{r_0}{p^{\alpha-1}} T_{l-1,\alpha+1}(pn_0, p(r_0 + p^\alpha)) \\ &\quad - T_{l-1,\alpha+1}(p(n_0 - 1) + p - 1, p(r_0 + p^\alpha - 1) + p - 1) \\ &\equiv -\frac{r_0}{p^{\alpha-1}} T_{l-1,\alpha}(n_0, r_0 + p^\alpha) \\ &\quad - (-1)^{p-1} \binom{p-1}{p-1} T_{l-1,\alpha}(n_0 - 1, r_0 + p^\alpha - 1) \\ &\equiv T_{l,\alpha}(n_0, r_0) \pmod{p}. \end{aligned}$$

Combining the above we have completed the induction proof. $\square$

To establish Theorem 1.8 we need some auxiliary results.

Lemma 4.1. *Let p be a prime, and let $n = p^\beta(pq + r)$ with $\beta, q, r \in \mathbb{N}$ and $\{-q\}_{p-1} < r < p$. Then*

$$\text{ord}_p(n!) = \left\lfloor \frac{n-1}{p-1} \right\rfloor \iff q = 0.$$

Proof. For $s = \{-q\}_{p-1}$ we clearly have $pq + s \equiv 0 \pmod{p-1}$. Observe that

$$\begin{aligned} \left\lfloor \frac{n-1}{p-1} \right\rfloor &= \frac{p^\beta - 1}{p-1}(pq + r) + \left\lfloor \frac{pq + r - 1}{p-1} \right\rfloor \\ &= \frac{p^\beta - 1}{p-1}(pq + r) + \frac{pq + s}{p-1}. \end{aligned}$$

Also,

$$\begin{aligned} \text{ord}_p(n!) &= \sum_{0 < i \leq \beta} \frac{p^\beta(pq + r)}{p^i} + \sum_{i=1}^{\infty} \left\lfloor \frac{p^\beta(pq + r)}{p^{\beta+i}} \right\rfloor \\ &= (pq + r) \sum_{0 < i \leq \beta} p^{\beta-i} + \text{ord}_p((pq + r)!) \\ &= \frac{p^\beta - 1}{p-1}(pq + r) + \text{ord}_p((pq)!). \end{aligned}$$

In the case $q = 0$, we have $s = 0$ and hence

$$\left\lfloor \frac{n-1}{p-1} \right\rfloor = \frac{p^\beta - 1}{p-1} r = \text{ord}_p(n!)$$

by the above.

Now let $q > 0$. Then

$$\text{ord}_p((pq)!) = \sum_{i=1}^{\infty} \left\lfloor \frac{pq}{p^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{pq}{p^i} = \frac{q}{1-p^{-1}} = \frac{pq}{p-1} \leq \frac{pq + s}{p-1},$$

and therefore $\text{ord}_p(n!) < \lfloor (n-1)/(p-1) \rfloor$. This ends the proof. $\square$

Remark 4.1. By Lemma 4.1, if p is a prime and n is a positive integer with $\text{ord}_p(n) = \lfloor \log_p n \rfloor$, then $\text{ord}_p(n!) = \lfloor (n-1)/(p-1) \rfloor$.

Using Lemma 4.1 and Corollary 1.2, we can deduce the following lemma.

Lemma 4.2. *Let $\alpha \in \mathbb{N}$, $n \in \mathbb{Z}^+$, $r \in \mathbb{Z}$ and $n \equiv r \equiv 0 \pmod{2^\alpha}$. Then*

$$T_{0,\alpha+1}^{(2)}(n, r) \equiv 1 \pmod{2} \iff n \text{ is a power of } 2.$$

Proof. Clearly

$$T_{0,\alpha+1}^{(2)}(n, r) = \frac{(-1)^r 2^{n/2^\alpha - 1}}{(n/2^\alpha)!} 2^{-\lfloor \frac{n-2^\alpha}{\varphi(2^{\alpha+1})} \rfloor} \sum_{k \equiv r \pmod{2^{\alpha+1}}} \binom{n}{k}.$$

By Lemma 4.1 in the case $p = 2$,

$$\text{ord}_2 \left(\frac{n}{2^\alpha}! \right) = \frac{n}{2^\alpha} - 1 \iff \frac{n}{2^\alpha} \text{ is a power of } 2.$$

If $\alpha = 0$, then

$$2^{-\lfloor \frac{n-2^\alpha}{\varphi(2^{\alpha+1})} \rfloor} \sum_{k \equiv r \pmod{2^{\alpha+1}}} \binom{n}{k} = 2^{-(n-1)} \sum_{k \equiv r \pmod{2}} \binom{n}{k} = 1.$$

Now we let $\alpha \geq 1$. Note that both $n_* = n/2^{\alpha-1}$ and $r_* = r/2^{\alpha-1}$ are even. Also, $\{n\}_{2^{\alpha-1}} = \{r\}_{2^{\alpha-1}} = 0$. Thus, Corollary 1.2 implies that

$$\begin{aligned} 2^{-\lfloor \frac{n-2^\alpha}{\varphi(2^{\alpha+1})} \rfloor} \sum_{k \equiv r \pmod{2^{\alpha+1}}} \binom{n}{k} &\equiv 1 \pmod{2} \\ \iff n_* = 2 \text{ or } n_* \equiv 2r_* \equiv 0 \pmod{4}. \end{aligned}$$

Combining the above we find that $T_{0,\alpha+1}^{(2)}(n, r) \equiv 1 \pmod{2}$ if and only if n is a power of 2. This concludes the proof. $\square$

The following result plays a major role in our proof of Theorem 1.8.

Theorem 4.1. *Let $\alpha, c, d, e \in \mathbb{N}$ with $d < 2^e$. Then*

$$(4.1) \quad T_{l,\alpha+1}^{(2)}(2^\alpha(2^e + d), 2^\alpha c) \equiv \delta_{l,d} \pmod{2} \text{ for all } l = 0, \dots, d.$$

Proof. By Lemma 4.2, if $n \in \mathbb{Z}^+$, $r \in \mathbb{Z}$ and $n \equiv r \equiv 0 \pmod{2^\beta}$ with $\beta = 1$, then $T_{0,\beta+1}^{(2)}(n, r) \equiv T_{0,\beta}^{(2)}(n/2, r/2) \pmod{2}$. Thus, by modifying the proof of Theorem 1.5 (just the third case) slightly, we get a modified version of (1.7) with $p = 2$ and α replaced by $\beta = 1$. This, together with Theorem 1.5, shows that if $l \in \mathbb{N}$ and $\alpha > 0$, then

$$\begin{aligned} T_{l,\alpha+1}^{(2)}(2^\alpha(2^e + d), 2^\alpha c) &\equiv T_{l,\alpha}^{(2)}(2^{\alpha-1}(2^e + d), 2^{\alpha-1}c) \\ &\equiv \dots \equiv T_{l,1}^{(2)}(2^e + d, c) \pmod{2}. \end{aligned}$$

So, it suffices to show the following claim:

Claim. *If $l \in \mathbb{N}$, $n \in \mathbb{Z}^+$ and $d_n := n - 2^{\lfloor \log_2 n \rfloor} \geq l$, then*

$$T_l(n, r) := T_{l,1}^{(2)}(n, r) \equiv \delta_{l,d_n} \pmod{2} \text{ for all } r \in \mathbb{Z}.$$

We use induction on l to show the claim.

As $n \in \mathbb{Z}^+$ is a power of two if and only if $d_n = 0$, in the case $l = 0$ the claim follows from Lemma 4.2.

Now let $l \in \mathbb{Z}^+$ and assume the claim for $l-1$. Let $n \in \mathbb{Z}^+$ with $d_n \geq l$. Clearly $d_1 = 0 < l$ and hence $n > 1$. By Lemma 2.2,

$$T_l(n, r) = -rT_{l-1}(n, r+2) - T_{l-1}(n-1, r+1) \quad \text{for any } r \in \mathbb{Z}.$$

Since $d_n > d_{n-1} = d_n - 1 \geq l-1$, by the induction hypothesis we have

$$T_{l-1}(n, r+2) \equiv \delta_{l-1, d_n} \pmod{2} \quad \text{and} \quad T_{l-1}(n-1, r+1) \equiv \delta_{l-1, d_n-1} \pmod{2}.$$

Therefore

$$T_l(n, r) \equiv -r\delta_{l-1, d_n} - \delta_{l-1, d_n-1} \equiv \delta_{l, d_n} \pmod{2}.$$

This concludes the induction step, and we are done. $\square$

Proof of Theorem 1.8. Write $n = 2^\alpha(2^e + d) + c$ with $c, d, e \in \mathbb{N}$, $c < 2^\alpha$ and $d < 2^e$. Clearly $n_0 := 2^e + d = \lfloor n/2^\alpha \rfloor$. Since $2^e \leq n/2^\alpha < n_0 + 1 \leq 2^{e+1}$, we also have $e = \lfloor \log_2(n/2^\alpha) \rfloor$. By Vandermonde's identity,

$$\begin{aligned} & \sum_{k \equiv 0 \pmod{2^\alpha}} \binom{n}{k} (-1)^k \left(\frac{k}{2^\alpha} \right)^l \\ &= \sum_{k \equiv 0 \pmod{2^\alpha}} \sum_{j=0}^c \binom{c}{j} \binom{2^\alpha n_0}{k-j} (-1)^k \left(\frac{k}{2^\alpha} \right)^l \\ &= \sum_{j=0}^c \binom{c}{j} (-1)^j \sum_{k \equiv -j \pmod{2^\alpha}} \binom{2^\alpha n_0}{k} (-1)^k \left(\frac{k+j}{2^\alpha} \right)^l. \end{aligned}$$

If $0 < j \leq c < 2^\alpha$, then

$$\begin{aligned} & \text{ord}_2 \left(\sum_{k \equiv -j \pmod{2^\alpha}} \binom{2^\alpha n_0}{k} (-1)^k \left(\frac{k+j}{2^\alpha} \right)^l \right) \\ & \geq \text{ord}_2(n_0!) + \tau_2(2^\alpha - j, j) > \text{ord}_2(n_0!) \end{aligned}$$

by the inequality (1.2). So, we need to show that

$$\sum_{k \equiv 0 \pmod{2^\alpha}} \binom{2^\alpha n_0}{k} (-1)^k \left(\frac{k}{2^\alpha} \right)^l = \sum_{k \in \mathbb{N}} \binom{2^\alpha n_0}{2^\alpha k} (-1)^{2^\alpha k} k^l$$

has the same 2-adic order as $n_0!$. If k is even, then $\text{ord}_2(k^l) \geq l \geq n_0 > \text{ord}_2(n_0!)$. Thus, it remains to prove that $\text{ord}_2(\Sigma) = \text{ord}_2(n_0!)$, where

$$\Sigma = \sum_{2 \nmid k} \binom{2^\alpha n_0}{2^\alpha k} k^l = \sum_{j \in \mathbb{N}} \binom{2^\alpha n_0}{2^\alpha(2j+1)} (2j+1)^l.$$

Observe that

$$(2j+1)^l = \sum_{s=0}^l \binom{l}{s} 2^s j^s = \sum_{s=0}^l \binom{l}{s} 2^s \sum_{t=0}^s S(s, t) t! \binom{j}{t},$$

and hence

$$\begin{aligned} \frac{\Sigma}{n_0!} &= \sum_{0 \leq t \leq s \leq l} \binom{l}{s} 2^s S(s, t) \frac{t!}{n_0!} \sum_{j \in \mathbb{N}} \binom{2^\alpha n_0}{2^{\alpha+1}j + 2^\alpha} \binom{j}{t} \\ &= \sum_{0 \leq t \leq s \leq l} \binom{l}{s} 2^{s-t} S(s, t) (-1)^{2^\alpha} T_{t, \alpha+1}^{(2)}(2^\alpha n_0, 2^\alpha) \\ &\equiv \sum_{t=0}^l \binom{l}{t} T_{t, \alpha+1}^{(2)}(2^\alpha n_0, 2^\alpha) \pmod{2}. \end{aligned}$$

Now we analyze the parity of $\binom{l}{t} T_{t, \alpha+1}^{(2)}(2^\alpha n_0, 2^\alpha)$ for each $0 \leq t \leq l$. As $l \geq n_0$ and $l \equiv n_0 \pmod{2^e}$, we can write $l = n_0 + 2^e q = 2^e(q+1) + d$ with $q \in \mathbb{N}$. Note that

$$\binom{l}{d} = \prod_{0 < r \leq d} \left(1 + \frac{2^e}{r}(q+1)\right) \equiv 1 \pmod{2}.$$

Also, if $0 \leq t \leq d$, then $T_{t, \alpha+1}^{(2)}(2^\alpha n_0, 2^\alpha) \equiv \delta_{t,d} \pmod{2}$ by Theorem 4.1. When $d < t \leq l$ and $T_{t, \alpha+1}^{(2)}(2^\alpha n_0, 2^\alpha) \neq 0$, we have $2^\alpha n_0 \geq 2^{\alpha+1}t + 2^\alpha$; hence $d < t < n_0/2 < 2^e$ and thus

$$\begin{aligned} \text{ord}_2 \binom{l}{t} &= \sum_{i=1}^{\infty} \left(\left\lfloor \frac{l}{2^i} \right\rfloor - \left\lfloor \frac{t}{2^i} \right\rfloor - \left\lfloor \frac{l-t}{2^i} \right\rfloor \right) \\ &\geq \left\lfloor \frac{l}{2^e} \right\rfloor - \left\lfloor \frac{t}{2^e} \right\rfloor - \left\lfloor \frac{l-t}{2^e} \right\rfloor = (q+1) - 0 - q > 0. \end{aligned}$$

Combining the above we find that

$$\frac{\Sigma}{n_0!} \equiv \sum_{t=0}^l \binom{l}{t} T_{t, \alpha+1}^{(2)}(2^\alpha n_0, 2^\alpha) \equiv 1 \pmod{2}.$$

So $\text{ord}_2(\Sigma) = \text{ord}_2(n_0!)$ as required. $\square$

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