

## QUASI-ISOMETRICALLY EMBEDDED SUBGROUPS OF BRAID AND DIFFEOMORPHISM GROUPS

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ABSTRACT. We show that a large class of right-angled Artin groups (in particular, those with planar complementary defining graph) can be embedded quasi-isometrically in pure braid groups and in the group  $\text{Diff}(D^2, \partial D^2, \text{vol})$  of area preserving diffeomorphisms of the disk fixing the boundary (with respect to the  $L^2$ -norm metric); this extends results of Benaim and Gambaudo who gave quasi-isometric embeddings of  $F_n$  and  $\mathbb{Z}^n$  for all  $n > 0$ . As a consequence we are also able to embed a variety of Gromov hyperbolic groups quasi-isometrically in pure braid groups and in the group  $\text{Diff}(D^2, \partial D^2, \text{vol})$ . Examples include hyperbolic surface groups, some HNN-extensions of these along cyclic subgroups and the fundamental group of a certain closed hyperbolic 3-manifold.

### 1. INTRODUCTION

Let  $(X, d)$  and  $(X', d')$  denote metric spaces. A (not necessarily continuous) function  $f : X \rightarrow X'$  is called a *quasi-isometric embedding* if, for some uniform constants  $\lambda \geq 1$  and  $C \geq 0$ , we have

$$\frac{1}{\lambda}d(x, y) - C \leq d'(f(x), f(y)) \leq \lambda d(x, y) + C, \quad \text{for all } x, y \in X.$$

The map  $f$  is called a *quasi-isometry* if, moreover, every point of  $X'$  is a distance at most  $C$  from a point of  $f(X)$ .

Let  $\Gamma, \Gamma'$  denote groups which are equipped with left-invariant metrics  $d, d'$  respectively. Note that if  $\phi : \Gamma \rightarrow \Gamma'$  is a homomorphism, then the quasi-isometric inequalities may be written as

$$\frac{1}{\lambda}|g| - C \leq |\phi(g)| \leq \lambda|g| + C, \quad \text{for all } g \in \Gamma,$$

where  $|g|$  denotes the distance from  $g$  to the identity. In this paper we shall only consider homomorphisms  $\phi : \Gamma \rightarrow \Gamma'$  where  $\Gamma$  is a finitely generated group. Such groups shall always be assumed to be equipped with the word metric  $d_S$  with respect to some finite set  $S$  of generators, and we recall that the quasi-isometry class of  $(\Gamma, d_S)$  is independent of the choice of finite generating set. Finally, we note that when  $S$  is a finite generating set for  $\Gamma$  the inequality  $|\phi(g)| \leq \lambda|g|_S + C$  is automatically satisfied for any choice of  $\lambda \geq \max\{|\phi(s)| : s \in S\}$  and  $C \geq 0$ .

We recall that a right-angled Artin group (also called a partially commutative group in the literature) is a group which can be described by a presentation with

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a finite number of generators, and a finite list of relations, each of which states that some pair of generators commutes. Thus free groups and free abelian groups are examples of right-angled Artin groups. To any simplicial graph  $\Delta$  we can associate a right-angled Artin group  $G(\Delta)$  by having generators corresponding to the vertices of  $\Delta$ , and a commutation relation between two generators if and only if the corresponding vertices of  $\Gamma$  are connected by an edge. Thus, if  $\Delta$  has vertex set  $\{1, 2, \dots, n\}$ , then

$$G(\Delta) = \langle a_1, a_2, \dots, a_n \mid a_i a_j = a_j a_i \text{ for each edge } \{i, j\} \text{ of } \Delta \rangle.$$

The main result of this paper is that each member of a large class of right-angled Artin groups, which we shall call *planar type* right-angled Artin groups, admits injective homomorphisms which are quasi-isometric embeddings into pure braid groups  $PB_m$ , for sufficiently large  $m$ , as well as into the group  $\text{Diff}(D^2, \partial D^2, \text{vol})$  of area-preserving diffeomorphisms of the unit disk, equipped with the so-called “hydrodynamical” or  $L^2$ -norm metric. Quite independently one may observe that many interesting groups admit quasi-isometric embeddings as a subgroup of some right-angled Artin group. As a corollary, we obtain that all surface groups, with the exception of the three simplest non-orientable surfaces, as well as at least one hyperbolic 3-manifold group, embed quasi-isometrically in  $PB_m$ , for some  $m$ , and in  $\text{Diff}(D^2, \partial D^2, \text{vol})$ .

In [4] the authors showed that each right-angled Artin group of planar type embeds as a subgroup of a pure braid group  $PB_m$ , for some  $m$  depending on the defining graph  $\Delta$ . However, it was not clear whether these homomorphisms are actually quasi-isometric embeddings. In the present paper we modify the construction in order to achieve this. Our techniques also yield quasi-isometric embeddings of arbitrary right-angled Artin groups into closed surface mapping class groups. (See Theorem 2 and Corollary 3.)

The initial motivation for the present paper, however, came from the work [2] of Benaim and Gambaudo, who introduce the hydrodynamical metric  $d_{\text{hydr}}$  on the group  $\text{Diff}(D^2, \partial D^2, \text{vol})$  of volume preserving diffeomorphisms of the closed disk  $D^2$  (see Section 3 for details). Observing that the metric  $d_{\text{hydr}}$  is unbounded they proposed a study of the large-scale properties of  $\text{Diff}(D^2, \partial D^2, \text{vol})$  with respect to this metric. Their main result in this direction states that, for any  $n$ , the free abelian and free groups,  $\mathbb{Z}^n$  and  $F_n$ , embed quasi-isometrically in  $\text{Diff}(D^2, \partial D^2, \text{vol})$ . Adapting their techniques to the case of right-angled Artin groups we are able to show that all of the examples referred to above (planar type right-angled Artin groups, surface groups, and other hyperbolic group examples) may also be embedded quasi-isometrically in  $\text{Diff}(D^2, \partial D^2, \text{vol})$  with respect to  $d_{\text{hydr}}$ .

We note that the method of [2] can only be applied to a *homomorphism* into the diffeomorphism group (because the length estimates are derived only for the norm  $|g|$  of an element  $g$  rather than the distance between pairs of elements). At first sight it seems to be a very difficult problem to choose surface diffeomorphisms which satisfy a given relation. In this regard, however, right-angled Artin groups are particularly well-adapted because they are defined using only commuting relations and one can easily find diffeomorphisms which commute by choosing their supports to be disjoint.

It should also be stressed that we do not actually construct a quasi-isometric embedding of  $PB_m$  in  $\text{Diff}(D^2, \partial D^2, \text{vol})$ . In fact, it is unknown whether  $PB_m$  can be embedded as a subgroup of  $\text{Diff}(D^2, \partial D^2, \text{vol})$  or  $\text{Homeo}(D^2, \partial D^2, \text{vol})$ , and it

is rather unlikely that this is possible in any natural way. For instance, it follows from the work of Morita [10] that there is no group-theoretic section of the natural homomorphism  $\mathcal{P}_m \rightarrow PB_m$  where  $\mathcal{P}_m < \text{Diff}(D^2, \partial D^2, \text{vol})$  denotes the subgroup of area-preserving diffeomorphisms fixing a given set of  $m$  disjoint closed disks in the interior of  $D^2$ .

The plan of this paper is as follows. In section 2 we define planar right-angled Artin groups and prove that each such group admits a quasi-isometric embedding (which is also a monomorphism) into a pure braid group  $PB_m$  for large enough  $m$ . This proof is the heart of the paper. In section 3 we review the technique developed by Benaim and Gambaudo in [2] and conclude that the maps described in section 2 are induced by quasi-isometric embeddings of the planar right-angled Artin groups into  $\text{Diff}(D^2, \partial D^2, \text{vol})$ . Finally, in section 4 we build on the work in [4] in order find many interesting quasi-isometrically embedded subgroups of  $PB_m$  and  $\text{Diff}(D^2, \partial D^2, \text{vol})$ .

## 2. PLANAR RIGHT-ANGLED ARTIN GROUPS IN PURE BRAID GROUPS

In this section we define the notion of “planarity” for right-angled Artin groups and show that any right-angled Artin group of planar type  $G(\Delta)$  may be embedded quasi-isometrically in an  $m$ -strand pure braid group  $PB_m$  (where  $m$  depends on the defining graph  $\Delta$ ); see Corollary 3 (2). More generally, our techniques show that any right-angled Artin group (not necessarily of planar type) may be embedded quasi-isometrically in the mapping class group  $\text{Mod}(S)$  of an orientable punctured surface, a surface with boundary or a closed surface (with implicit restrictions on the genera in each case); see Theorem 2 and Corollary 3 (1). In achieving these results we do not overly concern ourselves with the problem of minimising the number  $m$  of strands in the target pure braid group or the genera, or the number of punctures or boundary components, of the surface  $S$  when the target is the mapping class group of  $S$ . It would nevertheless be interesting to have a better understanding of exactly which braid and mapping class groups admit (quasi-isometric) embeddings into a given right-angled Artin group.

**Notation** ( $\text{Mod}(S)$ ,  $P\text{Mod}(S)$ ). Let  $S$  denote a (not necessarily connected) *finitely punctured compact orientable surface*:  $S$  is homeomorphic to  $S_0 \setminus P$  where  $S_0$  is a compact orientable surface with boundary  $\partial S_0$ , and  $P$  is a finite set of points in the interior of  $S_0$ . We note that the boundary of  $S$  is just that of  $S_0$ , namely  $\partial S = \partial S_0$ .

We shall write simply  $\text{Mod}(S)$  for the orientable mapping class group of  $S$  relative to the boundary. That is, the diffeomorphisms of  $S$  are supposed to fix the boundary pointwise, and they are equivalent if they are related by a diffeotopy that is the identity on the boundary. We shall also write  $P\text{Mod}(S)$  for the subgroup of  $\text{Mod}(S)$  generated by Dehn twists. This is simply the finite index normal subgroup of  $\text{Mod}(S)$  whose elements leave invariant the components of  $S$  and do not permute the elements of the puncture set  $P$  [5].

Note that if  $S$  is a disjoint union of surfaces  $S_1$  and  $S_2$ , then we simply have  $\text{Mod}(S) \cong \text{Mod}(S_1) \times \text{Mod}(S_2)$ .

**2.1. Planarity, circle diagrams, and the basic representation.** Any finite collection  $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$  of connected compact subsets of  $\mathbb{R}^2$  (or  $D^2$  or  $S^2$ ) determines a *non-incidence* graph  $\Delta_{\mathcal{C}}$ , defined to be the simplicial graph with vertex set  $\mathcal{C}$  and edges  $\{C_i, C_j\}$  whenever  $C_i \cap C_j = \emptyset$ .

Note that any collection of plane diffeomorphisms  $\phi_1, \dots, \phi_n$  with  $\phi_i$  trivial outside a regular neighbourhood of the set  $C_i$ , for each  $i = 1, \dots, n$ , will generate a homomorphic image of the right-angled Artin group  $G(\Delta_{\mathcal{C}})$  associated to the graph  $\Delta_{\mathcal{C}}$ .

**Definition** (Circle diagram; planar type). We say that the right angled Artin group  $G(\Delta)$  is of *planar type* if  $\Delta$  is isomorphic to the non-incidence graph  $\Delta_{\mathcal{C}}$  where  $\mathcal{C}$  denotes a finite collection of smooth simple closed curves in general position in the interior of the disk  $D^2$ . We call  $\mathcal{C}$  a (*planar*) *circle diagram* for  $G(\Delta)$ .

More generally, we may define a *circle diagram* for  $G(\Delta)$  to be a collection  $\mathcal{C}$  of simple closed curves in some orientable surface whose non-incidence graph  $\Delta_{\mathcal{C}}$  is isomorphic to  $\Delta$ . Note that one may easily find a circle diagram, in this larger sense, for an arbitrary right-angled Artin group (cf. [4]).

If  $\Delta$  is a simplicial graph, then we define its *complementary* (or *opposite*) graph  $\Delta^{\text{op}}$  to be the simplicial graph with the same vertex set as  $\Delta$  and which has an edge between two vertices if and only if  $\Delta$  does not. We recall from [4] that if the complementary defining graph  $\Delta^{\text{op}}$  is planar, then  $G(\Delta)$  is of planar type. We also recall the idea of the proof. An embedding of the graph  $\Delta^{\text{op}}$  in the plane  $\mathbb{R}^2$ , with vertex set  $\mathcal{V} \subset \mathbb{R}^2$ , gives rise to a collection of simple closed curves  $\mathcal{C} = \{C_v : v \in \mathcal{V}\}$  in  $\mathbb{R}^2$ , where  $C_v$  is defined as the boundary of a regular neighbourhood of the union of  $v$  and one-half of each edge adjacent to  $v$ .

We stress that the planarity of  $\Delta^{\text{op}}$  is a sufficient, but by no means necessary condition for  $G(\Delta)$  to be of planar type. This point is nicely illustrated by the example described in section 4.4. Figure 6 shows a planar circle diagram for a graph (the 1-skeleton of an icosahedron) whose complementary graph is non-planar.

**Definition 1** (Surface associated to a circle diagram). Given a (smooth) circle diagram  $\mathcal{C} = \{C_1, \dots, C_n\}$  (in an arbitrary orientable surface) we define a compact surface with boundary  $S_{\mathcal{C}}$  associated to  $\mathcal{C}$ , as follows.

Let  $S'$  denote a regular closed neighbourhood of the circle diagram. Thus  $S'$  is a union of annuli  $A_i$  (with  $A_i$  a regular neighbourhood of  $C_i$ ). Moreover, each intersection point of the curves  $C_i$  and  $C_j$  gives rise to one square in the surface  $S'$ , which is just one path component of  $A_i \cap A_j$ . The whole surface  $S'$  is a compact orientable surface with boundary.

In each annulus  $A_i$  we introduce a pair of distinguished points which do not lie in the intersection with another annulus. The two points must be on opposite sides of the curve  $C_i$ , as indicated in Figure 1. We denote by  $P$  the union of all these distinguished points. Finally, we define  $S_{\mathcal{C}}$  to be the surface

$$S_{\mathcal{C}} := S' \setminus N(P),$$

where  $N(P)$  denotes a regular open neighbourhood of the finite set  $P$ .

We remark that in the preceding construction two annuli  $A_i$  and  $A_j$  are disjoint if and only if the corresponding generators  $a_i$  and  $a_j$  of the right-angled Artin group  $G(\Delta_{\mathcal{C}})$  commute.

**Definition** (Basic representation  $G(\Delta_{\mathcal{C}}) \rightarrow \text{Mod}(S_{\mathcal{C}})$ ). To a circle diagram  $\mathcal{C} = \{C_1, \dots, C_n\}$  in an orientable surface we can associate a representation  $G(\Delta_{\mathcal{C}}) \rightarrow \text{Mod}(S_{\mathcal{C}})$  (from the right-angled Artin group whose defining graph is the non-adjacency graph of  $\mathcal{C}$  to the mapping class group of the surface  $S_{\mathcal{C}}$ ) as follows.

In each annulus  $A_i$  of  $S$  we draw smooth simple closed curves  $B_i, C_i$  and  $D_i$ , as indicated in Figure 1, and define the following diffeomorphism for each  $i = 1, \dots, n$ :

$$f_i = \tau_{B_i} \circ \tau_{D_i}^2 \circ \tau_{C_i}^{-2} \circ \tau_{B_i} \in \text{Diff}(S_C, \partial S_C),$$

where  $\tau_C$  denotes a smooth Dehn twist along a curve  $C$ . We remark that the diffeomorphism  $\tau_{D_i}^{-2} \circ \tau_{C_i}^2$  may be thought of as induced by the pure braid of the set  $P$  of marked points on  $S'$  which is given by moving the puncture enclosed by the curves  $C_i$  and  $D_i$  twice in an anticlockwise sense around the annulus.

The diffeomorphisms  $f_i$  just described define the homomorphism

$$f: G(\Delta_C) \rightarrow \text{Diff}(S_C, \partial S_C)$$

by setting  $f(a_i) = f_i$ .

Whenever  $S_C$  is viewed as a subsurface of any other (not necessarily compact) surface  $\widehat{S}$  the homomorphism  $f$  extends naturally to a homomorphism

$$\widehat{f}: G(\Delta_C) \rightarrow \text{Diff}(\widehat{S}, \partial \widehat{S}),$$

where every element of the image acts by the identity on  $\widehat{S} \setminus S$ .

We shall denote by  $\varphi: G(\Delta_C) \rightarrow \text{Mod}(S_C)$  and  $\widehat{\varphi}: G(\Delta_C) \rightarrow \text{Mod}(\widehat{S})$  the homomorphisms induced by  $f$  and  $\widehat{f}$  respectively. Clearly  $\widehat{\varphi}$  is obtained from  $\varphi$  by composing with the map  $\text{Mod}(S_C) \rightarrow \text{Mod}(\widehat{S})$  induced by the inclusion.

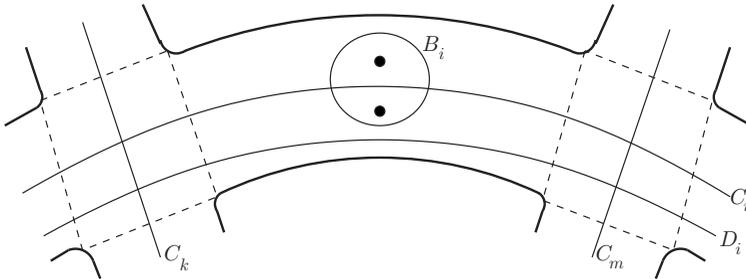


FIGURE 1. A segment of the  $i$ th annulus  $A_i$ , and its intersection with two other annuli,  $A_k$  and  $A_m$ . The curves  $B_i, C_i, D_i$  are indicated with solid lines, and the boundaries of the squares  $A_i \cap A_k$  and  $A_i \cap A_m$  with dashed lines.

We remark that in the above construction, the homeomorphism  $f_i$ , when restricted to the punctured annulus  $A_i$ , is pseudo-Anosov. This idea, which is essential to our proof that the homomorphism is a quasi-isometric embedding, is inspired by [6].

Note also that throughout the whole of the above discussion we did not need to suppose that any of the surfaces are connected. If  $S$  is a disjoint union of surfaces  $S_1$  and  $S_2$ , then we understand  $\text{Mod}(S) \cong \text{Mod}(S_1) \times \text{Mod}(S_2)$ .

**2.2. Quasi-isometric embeddings.** Our next aim is to prove that the basic representation  $G(\Delta_C) \rightarrow \text{Mod}(S_C)$  is a quasi-isometric embedding. In fact, in our case this implies that the homomorphism into  $\text{Mod}(S_C)$  is injective. This is simply because of the fact that any quasi-isometric embedding which is a homomorphism

must have a finite kernel, while the group  $G(\Delta_C)$  which we are considering is known to be torsion free.

The following is the main technical result of this paper:

**Theorem 2.** *Let  $\mathcal{C}$  be a circle diagram,  $\Delta_C$  the associated non-incidence graph, and  $S_C$  the surface of Definition 1. Suppose that  $S_C$  embeds as a subsurface of an orientable finitely punctured compact surface  $\widehat{S}$  and that the embedding  $S_C \hookrightarrow \widehat{S}$  is  $\pi_1$ -injective on each component of  $S_C$ . Then the homomorphism*

$$\widehat{\varphi}: G(\Delta_C) \rightarrow PMod(\widehat{S})$$

*is an injective quasi-isometric embedding (with respect to the word metrics).*

Before passing on to the proof of this theorem, we mention some easy consequences. Firstly, since it is a finite index subgroup, the inclusion of  $PMod(\widehat{S})$  into  $Mod(\widehat{S})$  is a quasi-isometric embedding. It then follows from the statement of Theorem 2 that we also obtain a quasi-isometric embedding of  $G(\Delta_C)$  into the slightly larger group  $Mod(\widehat{S})$ .

**Corollary 3.** *Let  $G = G(\Delta)$  denote a right-angled Artin group.*

- (1) *The group  $G$  embeds quasi-isometrically as a subgroup of  $Mod(S)$  for some connected closed orientable surface  $S$  (of genus depending on  $\Delta$ ).*
- (2) *If  $G$  is of planar type, then it embeds quasi-isometrically as a subgroup of the pure braid group  $PB_m$  (for a sufficiently large  $m$  depending on  $\Delta$ ).*

*Proof.* For an arbitrary right-angled Artin group  $G(\Delta)$  we may always find a circle diagram  $\mathcal{C}$  on some orientable surface such that  $\Delta = \Delta_C$ . Suppose that  $S_C$  has  $b$  boundary components. Then we define a  $\pi_1$ -injective inclusion of  $S_C$  into a closed connected surface  $\widehat{S}$  by gluing  $S_C$  along its boundary to an orientable genus zero surface with  $b$  boundary components. Statement (1) now follows by applying Theorem 2.

When  $G(\Delta)$  is of planar type we may find a planar circle diagram  $\mathcal{C}$  with  $\Delta = \Delta_C$ . The corresponding surface  $S_C$  is then also planar and may be viewed as a subsurface of  $D^2$ . Removing a single point from each disk component of  $D^2 \setminus S_C$  yields an  $m$ -punctured closed disk  $\widehat{S} \cong D^2 \setminus \{m \text{ points}\}$  (such that  $\partial\widehat{S} \cong S^1$ ). The inclusion  $S_C \hookrightarrow \widehat{S}$  is, by construction,  $\pi_1$ -injective on each connected component of  $S_C$ . We also recall the fact that  $Mod(\widehat{S})$  is naturally isomorphic to the  $m$ -string braid group  $B_m$ , for some  $m$ , and  $PMod(\widehat{S}) \cong PB_m$ , the pure braid group. Statement (2) now follows from Theorem 2. □

**2.3. Proof of Theorem 2.** We suppose throughout that  $\Delta = \Delta_C$  where  $\mathcal{C} = \{C_1, \dots, C_n\}$  is a smooth circle diagram in some orientable surface, and  $S = S_C$  the compact orientable surface of Definition 1. For simplicity (and without any loss of generality) we shall view  $\mathcal{C}$  as being a circle diagram in the surface  $S$ . We assume that an inclusion  $S \rightarrow \widehat{S}$  is given, and that the maps  $f, \widehat{f}, \varphi, \widehat{\varphi}$  are as described in the preceding definitions (subsection 2.1).

We shall first investigate the action of  $G(\Delta)$  on the surface  $S = S_C$  via the homomorphism  $\varphi: G(\Delta) \rightarrow Mod(S)$ , and more precisely the action of this group on the set of isotopy classes of simple closed curves in  $S$ . We let  $a_1, \dots, a_n$  denote the standard generators of  $G(\Delta)$ . Recall that  $S$  is the union of annuli  $A_i, i = 1, \dots, n$ , with a pair of open disks removed from each annulus; see subsection 2.1.

We define an  $(i, k)$ -intersection square to be one connected component of  $A_i \cap A_k$ , the intersection of the  $i$ th and the  $k$ th annulus,  $i \neq k$ . Note that an  $(i, k)$ -intersection square  $Q$  intersects  $\partial S$  in four corner points, and that  $\partial Q$  is a union of four interval segments, or sides, which are properly embedded in  $S$  and intersect in pairs at the corner points. We denote by  $\mathcal{Q}$  the collection of all sides of intersection squares in  $S$ .

The isotopy class of a simple closed curve  $C$  on a surface shall be denoted  $[C]$ . A simple closed curve in  $S$  shall be called a reduced representative of its isotopy class if it is reduced with respect to the set  $\mathcal{Q}$ , meaning that there is no bigon enclosed between the curve and any side of an intersection square. This is equivalent to asking that the number of intersections between the simple closed curve and the interiors of intersection squares be minimal in the isotopy class of the curve.

Standard arguments show that any two reduced representatives for the same isotopy class of simple closed curves are isotopic through a family of reduced representatives. In particular, the number and type of intersections that a reduced representative  $C$  has with each intersection square are invariants of the isotopy class  $[C]$ .

*Remark 4.* We observe that if a simple closed curve  $C$  is expressed as the union of open subsegments  $\gamma_i$  in such a way that each component of the overlap  $\gamma_i \cap \gamma_j$  between any two subsegments has at least one point of transverse intersection with  $\bigcup \mathcal{Q}$ , then  $C$  is reduced with respect to  $\mathcal{Q}$  if and only if each  $\gamma_i$  is. This is because any innermost arc of  $C$  which forms a bigon with some element of  $\mathcal{Q}$  must be contained in at least one of the given subsegments.

**Definition.** Let  $Q$  denote an  $(i, k)$ -intersection square, and  $C$  a simple closed curve. We say that the isotopy class  $[C]$  traverses the square  $Q$  a total of  $N$  times in the  $A_i$ -direction (resp. in the  $A_k$ -direction) if, for a reduced representative  $C_0$  of  $[C]$ , the following condition is satisfied: among the connected components of  $Q \cap C_0$  there are precisely  $N$  which connect opposite sides of the square without leaving the interior of  $A_i$  (resp.  $A_k$ ). See Figure 2. By the preceding discussion it is clear that this notion is well defined for isotopy classes.

For each  $i = 1, \dots, n$ , we define  $c_i([C])$  to be the largest number  $N$  such that  $[C]$  traverses every  $(i, k)$ -square on the annulus  $A_i$  at least  $N$  times in the  $A_i$ -direction.

**Lemma 5.** Let  $C$  be a simple closed curve in  $S$ ,  $p$  a nonzero integer and  $i \neq j \in \{1, \dots, n\}$ .

- (i) If  $A_i \cap A_j = \emptyset$ , then  $c_j(\varphi(a_i)^p[C]) = c_j([C])$ .
- (ii) If  $A_i \cap A_j \neq \emptyset$ , then  $c_i(\varphi(a_i)^p[C]) \geq 2^{|p|} c_j([C])$ .

*Proof.* We first recall that  $\varphi(a_i)$  is defined by the diffeomorphism

$$f_i = \tau_{B_i} \circ \tau_{D_i}^2 \circ \tau_{C_i}^{-2} \circ \tau_{B_i} \in \text{Diff}(S, \partial S),$$

which is the identity outside the annulus  $A_i$ . We may suppose that  $C$  is a reduced representative for  $[C]$ . It follows by the observation made in Remark 4 that a reduced representative  $C'$  for  $\varphi(a_i)^p[C]$  may be obtained from  $C$  by simply replacing each component  $\beta$  of  $C \cap A_i$  by an arc which is isotopic relative to its endpoints to  $f_i^p(\beta)$  and which is also reduced with respect to the intersection squares in  $A_i$ . It follows that application of  $\varphi(a_i)^p$  does not change the number and type of crossings (in either direction) at any  $(j, k)$ -square for  $j, k \neq i$ . In particular, if  $A_j \cap A_i = \emptyset$ , we have  $c_j([C']) = c_j([C])$ , establishing statement (i) of the lemma.

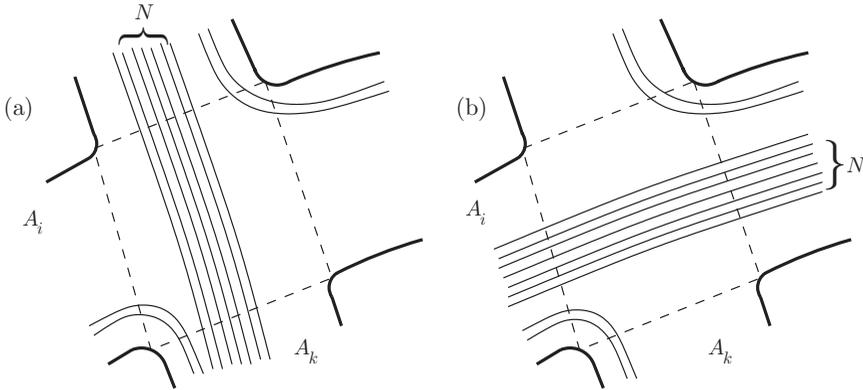


FIGURE 2. A reduced curve  $C$  traverses an  $(i, k)$ -intersection square  $N$  times in the  $A_k$ -direction (a), respectively  $N$  times in the  $A_i$  direction (b).

Let  $\alpha$  denote some subsegment of the curve  $C$  which crosses the annulus  $A_i$  as shown in Figure 3 (a). Then we claim that, in the reduced representative  $C'$ , the segment  $\alpha$  will be replaced with a segment  $\alpha'$  which traverses every intersection square on  $A_i$  at least  $2^{|p|}$  times in the  $A_i$ -direction. This establishes statement (ii) of the lemma, since if  $A_i \cap A_j \neq \emptyset$ , then  $C$  contains at least  $c_j([C])$  distinct parallel copies of the subsegment  $\alpha$ .

To prove the claim, we first consider the special case  $p = 1$ , that is, the action of  $\varphi(a_i)$  on the segment  $\alpha$  (the case  $p = -1$  is of course similar). Parts (b) and (c) of Figure 3 illustrate the different stages of the action of the diffeomorphism  $f_i$  on  $\alpha$  (up to isotopy relative to endpoints). Note that each segment shown in the figure is in reduced position with respect to  $\mathcal{Q}$ . In particular, under the action of  $\varphi(a_i)$ , the segment  $\alpha$  is replaced by the reduced segment illustrated in Figure 3 (c), and this new segment clearly traverses each intersection square at least twice in the  $A_i$ -direction, as required.

Now consider the more general case  $p \in \mathbb{N}$  (and again, the case  $-p \in \mathbb{N}$  is similar). Consider the traintrack with integer weights  $U, V, W \in \mathbb{N}$ , which is illustrated in Figure 4. This gives a convenient way of representing a line segment  $\beta(U, V, W)$  which is embedded in the annulus  $A_i$ . The segment  $f_i(\alpha)$  of Figure 3 (c) may thus be expressed as  $\beta(2, 2, 4)$  (up to isotopy relative to endpoints). Note also that, when placed in reduced position with respect to the intersection squares, the segment  $\beta(U, V, W)$  traverses every intersection square on the annulus  $A_i$  at least  $U + V - 2$  times in the  $A_i$ -direction.

Observe that the image of  $\beta(U, V, W)$  under any of the Dehn twists  $\tau_{B_i}, \tau_{C_i}^{-1}$ , or  $\tau_{D_i}$ , may easily be expressed using the same traintrack but with modified weights. Thus

$$\begin{aligned} \tau_{B_i}: \beta(U, V, W) &\mapsto \beta(U, V, W + 2U), \\ \tau_{C_i}^{-1}: \beta(U, V, W) &\mapsto \beta(U + 2W + 1, V, W), \\ \tau_{D_i}: \beta(U, V, W) &\mapsto \beta(U, V + 2, W). \end{aligned}$$

An easy calculation now shows that  $f_i(\beta(U, V, W))$  is isotopic to the segment

$$\beta(9U + 4W + 2, V + 4, 20U + 9W + 4).$$

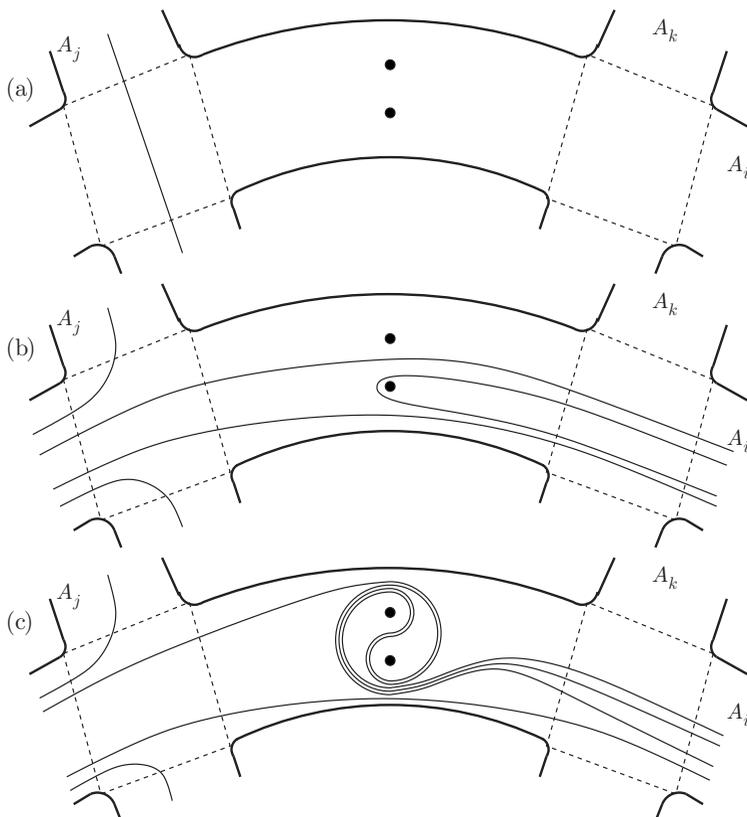


FIGURE 3. If there are  $N$  arcs traversing a square of  $A_i \cap A_j$  in the  $A_j$ -direction, then after the action of  $\varphi(a_i)$  there are  $2N$  arcs traversing any square of intersection involving  $A_i$  in the  $A_i$ -direction. (Part (b) shows the result of the action by  $\tau_{D_i}^2 \circ \tau_{C_i}^{-2} \circ \tau_{B_i}$ .)

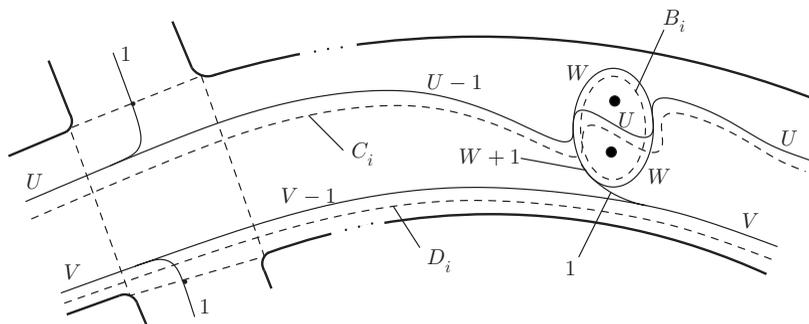


FIGURE 4. The traintrack  $\beta(U, V, W)$  for positive integers  $U, V, W$ .

By induction we conclude that, for  $p > 0$ , the segment  $f_i^p(\alpha)$  is isotopic to  $\beta(U', V', W')$  where  $V' \geq 2$  and  $U' \geq 2^p$ , and the desired result follows.  $\square$

We shall say that a word in the letters  $a_1^{\pm 1}, \dots, a_n^{\pm 1}$  is *reduced* if there is no shorter word in those letters representing the same element of the right-angled Artin group  $G(\Delta)$ . It is well known that any two reduced words differ by a finite sequence of “shuffles”: exchanges of adjacent letters  $a_i^{\pm}, a_j^{\pm}$  where  $i, j$  span an edge in  $\Delta$  (so that  $a_i, a_j$  commute). This seems to be first due to Baudisch [1]. A much more recent proof can be found in [4] (Proposition 9(i)). It follows from this result that the number of occurrences of a given letter in any reduced word is an invariant of the group element represented. Given  $w \in G(\Delta)$  and  $i \in \{1, \dots, n\}$ , we may therefore define  $\ell_i(w)$  to be the number of occurrences of the letters  $a_i, a_i^{-1}$  in some, or any, reduced word representing  $w$ . We also write  $\ell(w)$  for the wordlength of  $w$  over the standard generating set; thus  $\ell(w) = \sum_{i=1}^n \ell_i(w)$ . We shall say that  $w$  has *reduced form*  $w_1 w_2 \dots w_m$  whenever  $w_1, \dots, w_m \in G(\Delta)$  such that  $w = w_1 w_2 \dots w_m$  and  $\ell(w) = \ell(w_1) + \ell(w_2) + \dots + \ell(w_m)$ .

For the purposes of the following lemma we recall that the curves of the original circle diagram  $\mathcal{C} = \{C_1, \dots, C_n\}$  may be viewed as simple closed curves in the surface  $S$  (for each  $i = 1, \dots, n$ , the curve  $C_i$  runs along the centreline of the annulus  $A_i$  separating the two punctures).

**Lemma 6.** *Let  $w \in G(\Delta)$  with  $\ell(w) > 0$  and fix  $i \in \{1, \dots, n\}$ . Then there exist  $j \in \{1, \dots, n\}$  and  $C \in \mathcal{C}$  such that  $w$  has reduced form  $a_j^p v$  with  $|p| > 0$  and  $c_j(\varphi(w)[C]) \geq 2^{\ell_i(w)}$ .*

*Remark.* The induction hypothesis needed in the following proof is a little stronger than the statement of the lemma. As a result, we actually also prove that if  $w$  happens to have reduced form  $a_i^p v$  with  $|p| > 0$ , then one may take  $j = i$  in the conclusion of the lemma (namely  $c_i(\varphi(w)[C]) \geq 2^{\ell_i(w)}$ ).

*Proof.* Fix  $i \in \{1, \dots, n\}$ . We shall prove the following statement (for all  $w$  and  $k$ ) by induction on  $\ell(w)$ :

- Let  $w \in G(\Delta)$  and  $k \in \{1, \dots, n\}$ , and suppose that the element  $w$  has reduced form  $w = a_k^p v$  with  $p \neq 0$ . Then either:
  - (A) there exists a curve  $C \in \mathcal{C}$  such that  $c_k(\varphi(w)[C]) \geq 2^{\ell_i(w)}$ , or
  - (B)  $k \neq i$  and the conclusion of the lemma holds with  $j \neq k$ .

As a basis for the induction, we first consider the case where  $w = a_k^p$  for some  $k$  and some  $p \neq 0$ . In this case, taking  $C$  to be any curve in  $\mathcal{C}$  which crosses the annulus  $A_k$  transversely (i.e.,  $C = C_j$  for  $j \neq k$  and  $A_j \cap A_k \neq \emptyset$ ), it follows from Lemma 5 (ii) that  $c_k(\varphi(w)[C]) \geq 2^{|p|} \geq 2^{\ell_i(w)}$ .

Suppose now that  $w$  has reduced form  $a_k^p v$  with  $p \neq 0$  and  $\ell(v) > 0$ . By choosing  $|p|$  to be as large as possible we may further assume that whenever  $v$  has reduced form  $a_j^{\pm 1} u$  we have  $j \neq k$ .

By induction, we may now choose  $j \neq k$  and  $C \in \mathcal{C}$  such that  $v$  has reduced form  $v = a_j^q u$  with  $q \neq 0$  and  $c_j(\varphi(v)[C]) \geq 2^{\ell_i(v)}$ .

If  $A_j \cap A_k \neq \emptyset$ , then it follows from Lemma 5(ii) that

$$c_k(\varphi(w)[C]) \geq 2^{|p|} \cdot c_j(\varphi(v)[C]) \geq 2^{|p|+\ell_i(v)} \geq 2^{\ell_i(w)},$$

and we have (A).

We may henceforth suppose that  $A_j \cap A_k = \emptyset$ . Then  $a_j$  and  $a_k$  commute and, setting  $v' = a_k^p u$ , we have

$$w = a_k^p v = a_k^p a_j^q u = a_j^q a_k^p u = a_j^q v'$$

all in reduced form. By Lemma 5(i) we have

$$(1) \quad c_j(\varphi(w)[C]) = c_j(\varphi(v)[C]) \geq 2^{\ell_i(v)}$$

and

$$(2) \quad c_k(\varphi(w)[C']) = c_k(\varphi(v')[C'])$$

for all  $C' \in \mathcal{C}$ .

If  $k = i$ , then  $\ell_i(v') = \ell_i(w)$  and the induction hypothesis (A) applied to  $v'$ , together with Equation (2), implies that  $c_k(\varphi(w)[C']) \geq 2^{\ell_i(w)}$ , for some  $C' \in \mathcal{C}$  (possibly different from  $C$ ), giving (A). Otherwise  $k \neq i$ , in which case  $\ell_i(v) = \ell_i(w)$ , and Equation (1) gives the original statement of the lemma, and we have (B).  $\square$

We now turn to study the homomorphism  $\widehat{\varphi}: G(\Delta) \rightarrow PMod(\widehat{S})$ . Our first observation is the following:

**Proposition 7.** *The homomorphism  $\widehat{\varphi}: G(\Delta) \rightarrow PMod(\widehat{S})$  is faithful.*

*Proof.* Suppose that  $w \in G(\Delta)$  is some nontrivial element. By Lemma 6 we have  $c_i(\varphi(w)[C_j]) \geq 2$  for some  $i, j \in \{1, \dots, n\}$ . On the other hand, for  $i, j \in \{1, \dots, n\}$ , we always have  $c_i([C_j]) = 1$  or  $0$  depending on whether or not  $i = j$ . Thus the mapping class  $\varphi(w)$  is nontrivial, and so the homomorphism  $\varphi: G(\Delta) \rightarrow Mod(S)$  is injective.

Now, since the inclusion  $S \hookrightarrow \widehat{S}$  is  $\pi_1$ -injective on each component, the numbers  $c_i([C])$  are actually invariants of the isotopy class of a simple closed curve  $C$  in  $\widehat{S}$ , as well as in  $S$ . It follows, by the same argument, that the homomorphism  $\widehat{\varphi}$  is injective.  $\square$

We note, as previously remarked, that injectivity of  $\widehat{\varphi}$  is also a consequence of the statement that we are about to prove, namely that  $\widehat{\varphi}$  satisfies a quasi-isometric inequality.

**Definition** (Intersection numbers). If  $C, C'$  are simple closed curves on a surface, then we define the (geometric) intersection number  $|[C] \cap [C']|$  to be the minimal number of intersection points between representatives of the two isotopy classes which are in transverse position with respect to one another.

*Remark 8.* (i) Since the inclusion  $S \rightarrow \widehat{S}$  is  $\pi_1$ -injective on each component, the intersection number  $|[C] \cap [C']|$  between two simple closed curves  $C$  and  $C'$  in  $S$  will be the same whether measured in  $S$  or in  $\widehat{S}$ . We therefore make no distinction in our notation.

(ii) If  $A_i \cap A_j \neq \emptyset$ , for  $i \neq j$ , then, for any simple closed curve  $C$  in  $S$ , the value  $c_i([C])$  is a lower bound for the intersection number  $|[C] \cap [C_j]|$ .

**Definition** (Curve diagrams). By a curve diagram on a surface we shall mean a finite collection of isotopy classes of simple closed curves on the surface, no two of which are homotopy equivalent. We define the intersection number between a pair of curve diagrams  $D, D'$  to be the number

$$|D \cap D'| = \sum_{[C] \in D, [C'] \in D'} |[C] \cap [C']|.$$

Note that we do not assume that different curves of the same curve diagram are mutually disjoint. That is, the self-intersection number  $|D \cap D|$  of a diagram  $D$  need not be zero.

A basic example of a curve diagram is given by the collection  $\mathcal{C} = \{C_1, \dots, C_n\}$  of simple closed curves in  $S$ . Since no two of these are homotopic they represent a curve diagram in  $S$ , which we shall denote  $E$ , and shall think of as the *trivial* curve diagram in  $S$ .

Let  $\widehat{\mathcal{C}}$  denote a finite collection of essential simple closed curves in  $\widehat{S}$  which include the set  $\mathcal{C} = \{C_i : i = 1, \dots, n\}$  and such that the Dehn twists along these curves generate  $PMod(\widehat{S})$ . Observe that since the inclusion  $S \rightarrow \widehat{S}$  is  $\pi_1$ -injective on each component, no two curves of  $\mathcal{C}$  are homotopic in  $\widehat{S}$  (note also that no curve  $C_i$  can ever be parallel to a boundary component of  $S$ ). We may therefore choose  $\widehat{\mathcal{C}}$  to be minimal in the sense that no two curves are homotopic. It follows that the collection  $\widehat{\mathcal{C}}$  represents a curve diagram  $\widehat{E}$  in  $\widehat{S}$ . We note that  $E \subset \widehat{E}$  may be viewed as a subdiagram, and we shall think of  $\widehat{E}$  as the (extended) trivial curve diagram in  $\widehat{S}$ .

Observe that the group  $Mod(\widehat{S})$  acts naturally (on the left) on the set of curve diagrams in  $\widehat{S}$ . We now use this fact to define a notion of complexity for a pure mapping class in terms of its action on the curve diagram  $\widehat{E}$ .

**Definition** (Complexity of an element of  $PMod(\widehat{S})$ ). Suppose that the generating curves  $\widehat{\mathcal{C}}$  are chosen as above, so that  $\widehat{E}$  is a curve diagram in  $\widehat{S}$ . We define the *complexity* of a curve diagram  $D$  in  $\widehat{S}$  to be

$$complexity(D) = \log_2(|D \cap \widehat{E}|) - \log_2(|\widehat{E} \cap \widehat{E}|).$$

The complexity of an element  $\phi \in Mod(\widehat{S})$  is defined by

$$complexity(\phi) = complexity(\phi(\widehat{E})).$$

Note that the definition is normalized so that  $complexity(\widehat{E}) = 0$ .

**Proof of Theorem 2.** At this point we fix the set of Dehn twists  $\{\tau_C : C \in \widehat{\mathcal{C}}\}$  as our choice of generating set for  $PMod(\widehat{S})$ , and we write  $d(\cdot, \cdot)$  to denote the word metric in  $PMod(\widehat{S})$  with respect to these generators. Theorem 2 is now a consequence of the faithfulness of  $\widehat{\varphi}$  established in Proposition 7, and the following two propositions. Together, Propositions 9 and 10 imply that for an element  $a \in G(\Delta)$  of wordlength  $\ell(a)$  we have

$$\ell(a) \leq K_1 \cdot complexity(\widehat{\varphi}(a)) + K_0 \leq K_1 K_2 \cdot d(\widehat{\varphi}(a), 1) + K_0,$$

from which it follows that the homomorphism  $\widehat{\varphi}$  is a quasi-isometric embedding.

**Proposition 9.** *Let  $\ell(a)$  denote the length of a shortest representative word of an element  $a \in G(\Delta)$ . Then the complexity of  $\widehat{\varphi}(a)$  grows at least linearly with  $\ell(a)$ . In other words,*

$$\ell(a) \leq K_1 \cdot complexity(\widehat{\varphi}(a)) + K_0,$$

where  $K_0$  and  $K_1$  are some positive constants (e.g.,  $K_1$  equal to the number of generators in  $G(\Delta)$ , and  $K_0 = K_1 \cdot \log_2(|\widehat{E} \cap \widehat{E}|)$  suffice).

*Proof.* Let  $D = \varphi(a)(E)$ ,  $\widehat{D} = \widehat{\varphi}(a)(\widehat{E})$ , and suppose without loss of generality that  $a \neq 1$ . Note that  $D$  is a subdiagram of  $\widehat{D}$  and, by Remark 8(i), we clearly have  $|D \cap E| \leq |\widehat{D} \cap \widehat{E}|$ . Now, by Lemma 6, for every  $i \in \{1, \dots, n\}$  there exist

$j, k \in \{1, \dots, n\}$  such that  $2^{\ell_i(a)} \leq c_j(\varphi(a)[C_k])$ . Using Remark 8(ii), it follows that, for each  $i = 1, \dots, n$ ,

$$2^{\ell_i(a)} \leq |D \cap E| \leq |\widehat{D} \cap \widehat{E}|.$$

Thus

$$2^{\ell(a)/n} \leq \max\{2^{\ell_i(a)} : i = 1, \dots, n\} \leq |\widehat{D} \cap \widehat{E}|,$$

and the proposition follows with  $K_1 = n$  and  $K_0 = K_1 \cdot \log_2(|\widehat{E} \cap \widehat{E}|)$ . □

**Proposition 10.** *The complexity of the curve diagram of an element  $\phi$  of  $PMod(\widehat{S})$  grows at most linearly with the distance of  $\phi$  from the neutral element in the Cayley graph of  $PMod(\widehat{S})$ ; that is, we have*

$$\text{complexity}(\phi) \leq K_2 \cdot d(\phi, 1),$$

where  $K_2$  is a positive constant (equal to the base 2 logarithm of the number of curves in  $\widehat{C}$ ).

*Proof.* We take the Dehn twists along the curves in  $\widehat{C}$  as our finite generating set for  $PMod(\widehat{S})$ . If  $\tau_C$  is a given generator (the Dehn twist along the curve  $C \in \widehat{C}$ ) and  $\widehat{D}$  is a curve diagram of known complexity, then we can easily estimate the complexity of  $\tau_C(\widehat{D})$ . Namely, for each point of intersection of  $\widehat{D}$  with  $C$ , application of  $\tau_C$  may introduce at most  $r$  new points of intersection with curves in  $\widehat{E}$ , where

$$r = \#\{\text{curves of } \widehat{C} \text{ which intersect } C\} \leq N - 1,$$

where  $N = |\widehat{C}|$ . Since  $|[C] \cap \widehat{D}| \leq |\widehat{E} \cap \widehat{D}|$ , we then have

$$|\widehat{E} \cap \tau_C(\widehat{D})| \leq |\widehat{E} \cap \widehat{D}| + (N - 1) \cdot |[C] \cap \widehat{D}| \leq N \cdot |\widehat{E} \cap \widehat{D}|.$$

Thus  $\text{complexity}(\tau_C(\widehat{D})) \leq \text{complexity}(\widehat{D}) + \log_2(N)$ . By a straightforward induction on the wordlength of  $\phi$ , we then obtain

$$\text{complexity}(\phi) \leq K_2 \cdot d(\phi, 1),$$

where  $K_2 = \log_2(N)$ . □

### 3. FROM PURE BRAID GROUPS TO $Diff(D^2, \partial D^2, vol)$

In the previous section we proved that there exists a homomorphic quasi-isometric embedding of any right-angled Artin group of planar type  $G$  into the pure braid group of a certain number of points  $P_1, \dots, P_m$  in a disk  $D^2$ . The number of points needed depends, of course, on the group's circle diagram. We shall assume that the disk is the disk with radius 1 and centre  $(0, 0)$  in the plane, and that the distinguished points are  $P_i = (\frac{i}{m-1} - \frac{1}{2}, 0)$  (with  $i = 0, \dots, m - 1$ ).

We also recall that the homomorphism constructed in the last section factors through a certain subgroup of  $Diff(D^2, \partial D^2)$ , namely the group of diffeomorphisms of the disk which fix pointwise the distinguished points.

In the current section we shall point out that there is, in fact, an embedding in the group  $\mathcal{P}_m$  of volume-preserving diffeomorphisms of the disk which, moreover, fix pointwise not only the distinguished points and the boundary of the disk, but even disks of radius  $r$  centered on each of the distinguished points, as well as a regular neighbourhood of the boundary; here  $r$  is a sufficiently small positive real number. This yields an embedding of  $G$  in the group  $\mathcal{P}_m$ , which is itself a subgroup

of the group  $\text{Diff}(D^2, \partial D^2, \text{vol})$  of volume-preserving diffeomorphisms of the disk  $D^2$  which are the identity on a neighbourhood of  $\partial D^2$ .

The aim of the current section is to prove that, if we equip  $\text{Diff}(D^2, \partial D^2, \text{vol})$  with the *hydrodynamical metric*, then this homomorphism  $G \rightarrow \text{Diff}(D^2, \partial D^2, \text{vol})$  is itself a quasi-isometric embedding. (This is stronger than saying that the composition  $G \rightarrow \mathcal{P}_m \rightarrow PB(D^2, P_1 \cup \dots \cup P_m)$  is a quasi-isometric embedding.)

We recall from [2] the definition of the hydrodynamical metric: if  $\{\phi_t\}_{t \in [0,1]}$  is a path in  $\text{Diff}(D^2, \partial D^2, \text{vol})$ , then the length of this path is

$$\int_0^1 \sqrt{\int_{D^2} \left\| \frac{d\phi_t}{dt}(x) \right\|^2 dx} dt,$$

where the symbol  $\|\cdot\|$  denotes the Euclidean norm of a tangent vector to the disk. The hydrodynamical length  $l_{\text{hydr}}(\phi)$  of an element  $\phi$  of  $\text{Diff}(D^2, \partial D^2, \text{vol})$  is then the infimum length of a path from the identity map to  $\phi$ . This defines a left-invariant metric  $d_{\text{hydr}}$  by setting  $d_{\text{hydr}}(\phi, \psi) = l_{\text{hydr}}(\phi^{-1}\psi)$ .

Now we recall a technical result of Benaim and Gambaudo (Lemma 4 in [2]). There exists a constant  $K > 0$  and a function  $C: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\lim_{r \rightarrow 0} C(r) = 0$  with the following property. Suppose that  $\phi$  is an element of  $\mathcal{P}_m$  (fixing disks of radius  $r$  around each of the distinguished points), and which represents an element of the pure braid group whose shortest expression as a product of Artin’s standard generators  $\sigma_1^{\pm 1}, \dots, \sigma_{m-1}^{\pm 1}$  has length  $l_{\text{Artin}}$ . Then

$$l_{\text{hydr}}(\phi) \geq \frac{1}{K} \cdot l_{\text{Artin}} \cdot (1 - C(r)) \cdot (\text{area } D_0)^2.$$

Here  $D_0$  denotes a disk of radius  $r$ .

This technical result implies immediately the following theorem:

**Theorem 11** (Benaim-Gambaudo [2]). *Suppose that  $f$  is a homomorphism from a group  $G$  to  $\mathcal{P}_m$  for some choice of  $m$  fixed disks around the puncture points, and suppose that the induced homomorphism  $\varphi: G \rightarrow PB_m$  is a quasi-isometric embedding. Then so is  $f$ .*

We are now ready to prove the second main result of this paper.

**Theorem 12.** *If  $G$  is a planar right-angled Artin group, then there exists an injective homomorphism of  $G$  into  $\text{Diff}(D^2, \partial D^2, \text{vol})$  that is a quasi-isometric embedding.*

*Proof.* Corollary 3(ii) gives a homomorphism  $\widehat{\varphi}: G(\Delta) \rightarrow PB_m$  which, by the construction given in Section 2.1, factors through a homomorphism  $\widehat{f}: G(\Delta) \rightarrow \text{Diff}(S, \partial S)$  where  $S$  is a compact subsurface of the  $m$ -punctured disk. We may suppose, in fact, that  $S$  is just the closed disk with  $m$  open disks removed, and is contained in the exterior of a suitably chosen collection of  $m$  open disks of some constant radius  $r > 0$ . We claim that the diffeomorphisms  $\widehat{f}(a_i)$  which generate the image of  $\widehat{f}$  may be chosen to be area preserving. It then follows that the image of  $\widehat{f}$  lies in the subgroup  $\mathcal{P}_m < \text{Diff}(D^2, \partial D^2, \text{vol})$  (defined by the collection of disks of radius  $r$  just mentioned). Since the induced map  $\widehat{\varphi}: G(\Delta) \rightarrow PB_m$  is already shown to be a quasi-isometric embedding (see Corollary 3(ii)), the result now follows by Theorem 11.

To justify the claim, it suffices to observe that any Dehn twist about a smooth curve  $C$  in  $D^2$  may be realised by a volume preserving diffeomorphism with support in an arbitrarily small neighbourhood of the curve  $C$ . This follows from the following two observations:

- (i) a smooth embedding  $c: S^1 \rightarrow D^2$  can be extended, for a sufficiently small  $\epsilon > 0$ , to a smooth area preserving embedding  $S^1 \times [-\epsilon, \epsilon] \rightarrow D^2$ ;
- (ii) if we choose a smooth function  $h: [-\epsilon, \epsilon] \rightarrow [0, 2\pi]$  such that  $h(x) = 0$  for  $x < -\frac{\epsilon}{2}$ , and  $h(x) = 2\pi$  for  $x > \frac{\epsilon}{2}$ , then

$$T_h: S^1 \times [-\epsilon, \epsilon] \rightarrow S^1 \times [-\epsilon, \epsilon] \quad \text{such that} \quad T_h(t, s) = (t + h(s), s)$$

defines a smooth area preserving diffeomorphism which is the identity outside  $S^1 \times [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ . □

#### 4. QUASI-ISOMETRICALLY EMBEDDED HYPERBOLIC SUBGROUPS

In this section we consider examples of quasi-isometric embeddings of groups in right-angled Artin groups of planar type.

**4.1. Hyperbolic surface groups.** As proved in [4] every closed hyperbolic surface group with  $\chi(S) \leq -2$  embeds quasi-isometrically as a subgroup of some right-angled Artin group of planar type. Therefore each of these surface groups admits a homomorphic quasi-isometric embedding into both  $PB_m$  for some  $m$  and  $Diff(D^2, \partial D^2, vol)$ .

The case of an orientable surface  $F$  of genus 2 is illustrated in Figure 5. The figure describes a homomorphism  $\pi_1(F) \rightarrow G(C_5)$  where  $C_5$  denotes the 5-cycle graph with generators  $a, b, c, d, e$ . The homomorphism is determined by fixing a basepoint  $\star$  and, for each loop  $\gamma$  based at  $\star$  in  $F$ , reading the sequence of crossings with sign that  $\gamma$  makes with the transversely labelled curves on the surface  $F$ . We remark that this homomorphism actually projects to an embedding into the corresponding right-angled Coxeter group  $W(C_5)$ . (Note that  $W(C_5)$  acts by isometries of the hyperbolic plane and that the surface subgroup obtained here has finite index in the Coxeter group).

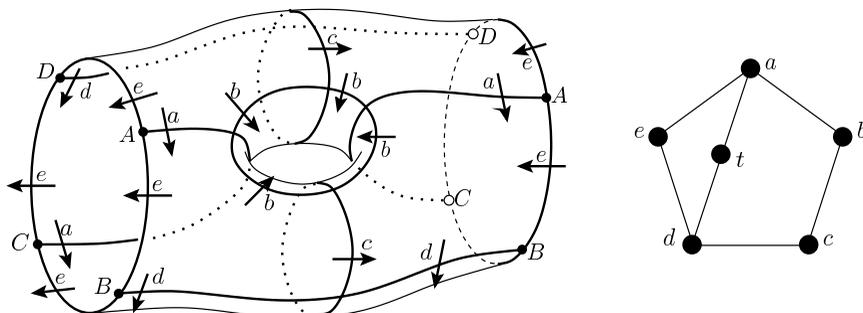


FIGURE 5. We obtain the surface  $F$  by identifying the two boundary circles of the surface on the left according to the labels  $ABCD$ . The fundamental group of the complex  $X$  embeds in  $G(\Delta)$ , where  $\Delta$  is the graph on the right.

Embeddings of arbitrary higher genus orientable surface groups may be obtained by simply restricting to finite index subgroups of  $\pi_1(F)$ . The treatment of the non-orientable case is similar but more complicated to describe (see [4], Section 4).

The fact that these homomorphisms are all quasi-isometric embeddings is a consequence of the method used in [4] to prove injectivity: namely, each embedding is realised as the homomorphism induced on the fundamental groups by a locally isometric embedding of CAT(0) cubical complexes. We refer the reader to the final sentence in the statement of Theorem 1 of [4].

**4.2. Some HNN extensions.** Other hyperbolic groups can be obtained by taking HNN extensions of surfaces and applying the Bestvina-Feighn combination theorem. These are two-dimensional but have boundary not homeomorphic to the circle. An explicit construction of such an example is given below. These examples necessarily have local cut points in the boundary. It is known from work of M. Kapovich and B. Kleiner [8] that if the boundary of a one-ended Gromov hyperbolic group is 1-dimensional and has no local cut points, then it is homeomorphic to either the Sierpinski carpet or the Menger curve (see [9], section 8, for a further discussion). This raises the following question, which we are so far unable to resolve:

*Question 13.* Do the groups  $\text{Diff}(D^2, \partial D^2, \text{vol})$ ,  $PB_m$ , or  $\text{Mod}(S)$ , for  $S$  a closed orientable surface, admit quasi-isometric embeddings of 2-dimensional hyperbolic groups with boundary homeomorphic to either a Sierpinski carpet or a Menger curve?

**Construction of HNN extension examples.** Consider the genus 2 closed orientable surface  $F$  shown in Figure 5. A *dissection* of  $F$ , as defined in [4], consists of a system of simple closed curves in  $F$  which are transversely oriented and labelled by generators of some right-angled Artin group in such a way that two curves intersect only if they are labelled by commuting generators. The curves of a dissection labelled by a generator  $x$  shall be referred to as  $x$ -curves. The dissection of the surface  $F$  which is illustrated in Figure 5 has labels in the group  $G(C_5)$ , where  $C_5$  denotes the five cycle graph. It defines a homomorphism which is a quasi-isometric embedding of the group  $\pi_1(F)$  into this right-angled Artin group  $G(C_5)$ .

We form a double cover  $F_2$  of  $F$  by cutting along the  $e$ -curve and gluing together two copies of the subsurface shown in Figure 5. We further modify the dissection of  $F_2$  by doubling each of the two  $e$ -curves (replacing it with a parallel pair of  $e$ -curves having the same transverse orientation). Each of the parallel pairs of  $e$ -curves thus formed bounds an annulus. We choose a core curve for each of these two annuli, that is, a simple closed curve in the interior of the annulus which is parallel to the bounding  $e$ -curves and in minimal position with respect to the dissection. Denote these core curves  $\gamma$  and  $\gamma'$  respectively. We now construct a complex  $X$  by attaching an annulus  $A$  to our surface, joining the two curves  $\gamma$  and  $\gamma'$ . We extend the dissecting curves over the new annulus  $A$ , by drawing four segments which connect the two boundary components of  $A$ , and which are transversely labelled  $a$  and  $d$  in the obvious way. We also introduce a new dissecting curve, namely a core curve of the annulus  $A$ , which shall be transversely labelled  $t$ .

We next modify the right-angled Artin group by adding a generator  $t$  to  $G$  which commutes with  $a$  and  $d$ , but not with  $e$ ,  $b$ , or  $c$ . Thus  $G = G(\Delta)$  for a new graph  $\Delta$  containing  $C_5$ . It is easy to see that  $G(\Delta)$  is of planar type (in fact  $\Delta^{\text{op}}$  is a planar graph).

Now, the dissection of the surface, extended to the complex  $X$ , defines a homomorphism  $\pi_1(X) \rightarrow G(\Delta)$ . Moreover, by the technique of [4] this homomorphism can easily be seen to be an injective quasi-isometric embedding: the complex  $X$  admits a CAT(0) squaring  $X_Q$  dual to the dissection, and the obvious labelling on the edges of this squaring determines a locally isometric embedding of  $X_Q$  into the standard cubical complex associated to  $G(\Delta)$  (see [4] for more details).

Finally we observe that  $\pi_1(X)$  is an HNN-extension. Choosing a basepoint in  $F_Q^2$  and paths in  $F_Q^2$  out to the endpoints of a  $t$ -edge  $E$  of the annulus  $A$  we define elements  $g, g'$  and  $t$  in  $\pi_1(X)$  corresponding to the loops  $\gamma, \gamma'$  and the path  $E$ , respectively. We then have

$$\pi_1(X) = \pi_1(F_Q^2) \star_{t: \langle g \rangle \rightarrow \langle g' \rangle, t(g)=g'} = \pi_1(F_Q^2) \star \langle t \rangle / \langle tgt^{-1} = g' \rangle.$$

It follows from the Bestvina-Feighn Combination Theorem [3] that  $\pi_1(X)$  is a word hyperbolic group. (Note that since the curves  $\gamma$  and  $\gamma'$  are non-parallel geodesics in  $F_Q^2$ , the hypotheses of [3, Corollary 2.3] are readily satisfied; namely, no powers of  $g$  and  $g'$  are conjugate in  $\pi_1(F_Q^2)$ , and  $g, g'$  are not proper powers of other elements.)

**4.3. The commutator subgroup of a right-angled Coxeter group.** In this subsection we present a natural quasi-isometric embedding of the commutator subgroup of a right-angled Coxeter group into the corresponding Artin group. We note that many Coxeter groups are Gromov hyperbolic groups. In particular, Januszkiewicz and Świątkowski [7] show that there exist Gromov hyperbolic right-angled Coxeter groups of virtual cohomological dimension  $n$ , for all  $n \geq 1$ .

Fix an arbitrary simplicial graph  $\Delta$  and let  $G(\Delta)$  be the associated right-angled Artin group, with standard generators  $a_1, \dots, a_n$ . One may define the corresponding right-angled Coxeter group by adding the further relations that each generator has order 2:

$$W(\Delta) = G(\Delta) / \langle a_i^2 : i = 1, \dots, n \rangle.$$

In the following we shall simply write  $W = W(\Delta)$  and  $G = G(\Delta)$ . Observe that  $W$  is a group with 2-torsion. However, its commutator subgroup  $[W, W]$  has index  $2^n$  and is torsion free (as a consequence, for instance, of the following lemma). An element  $w \in W$  lies in  $[W, W]$  precisely when each letter  $a_i$  appears an even number of times in any word representing  $w$ .

**Lemma 14.** *The group  $[W, W]$  embeds quasi-isometrically as a subgroup of  $G$ .*

*Proof.* We define a map  $\phi: [W, W] \rightarrow G$  as follows. Let  $w = b_1 b_2 \dots b_r$  be any word in the usual generators for an element  $w \in [W, W]$ . Thus  $b_i \in \{a_1, \dots, a_n\}$ . For each  $i \in \{1, \dots, r\}$  we define  $\epsilon_i \in \{\pm 1\}$  by  $\epsilon_i = (-1)^{d+1}$  if  $b_i$  is the  $d$ th occurrence of that particular letter in the word  $w$ . Then we set  $\phi(w) = b_1^{\epsilon_1} b_2^{\epsilon_2} \dots b_r^{\epsilon_r} \in G$ . This gives a well-defined function since the element  $\phi(w) \in G$  is invariant under modification of  $w$  by trivial insertion or deletion of a square  $a_i^2$  and substitutions  $a_i a_j \leftrightarrow a_j a_i$  for  $i, j$  adjacent in  $\Delta$ . Moreover, since each letter  $a_j$  appears an even number of times in any word for an element  $w \in [W, W]$ , the above map defines a homomorphism  $\phi: [W, W] \rightarrow G$ . It is clear, since minimal length (or reduced) words for elements in  $[W, W]$  are mapped to reduced words for elements in  $G$ , that  $\phi$  is an injective homomorphism and a quasi-isometric embedding. □

**4.4. A closed hyperbolic 3-manifold group.** Let  $\Upsilon$  denote a regular dodecahedron in  $\mathbb{H}^3$  with dihedral angles  $\pi/2$ , and define  $W_\Upsilon$  to be the right-angled Coxeter group generated by reflections in the faces of  $\Upsilon$ . The defining graph for  $W_\Upsilon$  as a right-angled Coxeter group shall be denoted  $\Delta_\Upsilon$  and is isomorphic to the 1-skeleton of the icosahedron. We observe that  $\Gamma_\Upsilon := [W_\Upsilon, W_\Upsilon]$  is the fundamental group of a closed orientable hyperbolic 3-manifold.

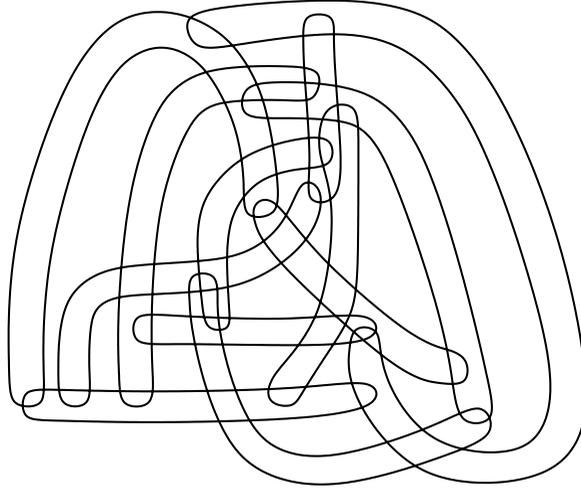


FIGURE 6. A planar circle diagram for the icosahedral graph  $\Delta_\Upsilon$ .

It follows from Lemma 14 that the group  $\Gamma_\Upsilon = [W_\Upsilon, W_\Upsilon]$  may be embedded quasi-isometrically as a subgroup of a right-angled Artin group, namely in  $G(\Delta_\Upsilon)$ . Moreover, Figure 6 shows that the defining graph  $\Delta_\Upsilon$  is in fact of planar type. (This would be far from obvious at first sight. In fact, while the icosahedral graph  $\Delta_\Upsilon$  is planar, its complementary graph is not. It is obtained from  $\Delta_\Upsilon$  by adding an extra edge joining each antipodal pair of vertices, and this graph contains a  $K_{3,3}$  graph embedded as a subgraph.) Observe that in Figure 6 there are 12 circles, and each is disjoint from exactly 5 others whose incidence graph is in each case a 5-cycle. Thus we have here a circle diagram for a regular graph on 12 vertices with vertex valence 5, in which the link of every vertex spans a 5-cycle (the 5-cycle being self-dual). In other words, we have a circle diagram for the icosahedral graph  $\Delta_\Upsilon$ . Thus we have the following:

**Theorem 15.** *The closed hyperbolic 3-manifold group  $\Gamma_\Upsilon = [W_\Upsilon, W_\Upsilon]$  may be quasi-isometrically embedded as a subgroup of a pure braid group and as a subgroup of the group  $\text{Diff}(D^2, \partial D^2, \text{vol})$ .*

By Theorem 2 of [7], a Gromov hyperbolic right-angled Coxeter group can be a virtual  $n$ -manifold group only if  $n \leq 4$ . An example in dimension 4, similar to the 3-manifold group discussed above, would be provided by considering the group of reflections in the faces of a hyperbolic hyperdodecahedron (120-cell) with dihedral angles  $\frac{\pi}{2}$ . The commutator subgroup of this reflection group is a torsion free subgroup of index  $2^{120}$ , and the fundamental group of a closed hyperbolic 4-manifold. While this group embeds in a closed surface mapping class group (by

Corollary 3(i) and Lemma 14), we do not know whether it can be embedded in a braid group. To obtain such an embedding, it would be sufficient to find a planar circle diagram for the corresponding right-angled Artin group on 120 generators.

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