HOLOMORPHIC GERMS AND THE PROBLEM OF SMOOTH CONJUGACY IN A PUNCTURED NEIGHBORHOOD OF THE ORIGIN

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ABSTRACT. We consider germs of conformal mappings tangent to the identity at the origin in $\mathbf{C}$. We construct a germ of a homeomorphism which is a C^∞ diffeomorphism except at the origin conjugating these holomorphic germs with the time-one map of the vector field $V(z) = z^m \frac{\partial}{\partial z}$. We then show that, in the case $m = 2$, for a germ of a homeomorphism which is real-analytic in a punctured neighborhood of the origin, with real-analytic inverse, conjugating these germs with the time-one map of the vector field exists if and only if a germ of a biholomorphism exists.

1. INTRODUCTION

We consider germs of conformal mappings at the origin which are tangent to the identity. Such germs are represented by power series of the form

$$(1.1) \quad F(z) = z + a_m z^m + O(z^{m+1}),$$

where $a_m \neq 0$. We say that two germs f and $\tilde{f}$ of the form (1.1) are equivalent (or conjugate) to one another if we can find a mapping H so that

$$(1.2) \quad (H^{-1} \circ f \circ H)(z) = \tilde{f}(z).$$

We can talk about formal, topological, smooth, or holomorphic equivalence depending on the map H . The thrust of this paper is as follows: given any two maps f and $\tilde{f}$ of the form (1.1), we can find a local homeomorphism H which is C^∞ off the origin conjugating f with $\tilde{f}$. However, in the case that $m = 2$ and $\tilde{f}(z) = \frac{z}{1-z}$, real-analytic equivalence in a punctured neighborhood of the origin will imply holomorphic equivalence in a *full* neighborhood of the origin. Precise statements will follow, but first, we look at some past results.

It is known that all such mappings (1.1) are topologically equivalent to the mapping

$$(1.3) \quad F_0 = g_V^1,$$

the time-one map of the vector field $V(z) = z^m \frac{\partial}{\partial z}$; this was proven by Shcherbakov (see [5]). Moreover, Shcherbakov shows that the conjugating mapping h taking f to g_V^1 can be written in the form $h(z) = z + \hat{h}(z)$, where $\hat{h}$ satisfies certain additional estimates (which are stated in our own theorem, below). Returning to the mapping

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f itself, in the case that $m = 2$, it is trivial to show that a necessary condition for formal (and hence holomorphic) equivalence of (1.1) with (1.3) is that

$$\frac{a_3}{a_2^2} = 1.$$

Thus, the continuous and holomorphic classifications for these germs do not agree. In fact, as Voronin shows in [6], the holomorphic classification admits functional moduli (the so-called Ecalle-Voronin moduli), and as a consequence, those germs (1.1) which are holomorphically equivalent to (1.3) are quite rare indeed. However, on sector regions of the form

$$S_j = \{z : |z| < r, |\arg z - \frac{\pi j}{m-1}| < \frac{\pi}{2(m-1)} + \alpha\}, \quad j = 1, \dots, 2(m-1),$$

where $r > 0$ is sufficiently small and $0 < \alpha < \frac{\pi}{2(m-1)}$, the classifications do agree (see [2]).

Later, Martinet and Ramis (in [3]) showed that even a C^1 equivalence between (1.1) and (1.3) will imply either a holomorphic or antiholomorphic equivalence (although the statement and proof which they give is incomplete, it can be corrected easily). In the case that $f(z) = z + z^2$, Ahern and Rosay [1] have proven that any germ of an entire function which is C^6 conjugate to f must in fact be holomorphically equivalent to f . However, there remains some space between the result of Shcherbakov and that of Martinet and Ramis. Furthermore, as Ahern and Rosay mention in [1], the Ecalle-Voronin moduli are so difficult to compute as to be almost useless for precise examples.

The goal of this paper is two-fold. First, we shall generalize the theorem by Shcherbakov, mentioned above, as follows.

Theorem 1.1. *Suppose that f is a germ of a holomorphic map at $0 \in \mathbf{C}$ of the form (1.1), and suppose that F_0 is the time-one map of the vector field $v(z) = z^m \frac{\partial}{\partial z}$. Let $\varepsilon > 0$ and $\delta > 0$ be given. Then, there exists a homeomorphism*

$$(1.4) \quad H = p + h,$$

defined in a small neighborhood of the origin, where p is a holomorphic polynomial of degree at most $m-1$, and furthermore where H is a C^∞ diffeomorphism except at the origin, satisfying the following:

i)

$$(1.5) \quad H \circ F_0 = f \circ H;$$

ii)

$$(1.6) \quad |h(z)| \leq |z|^{m-\varepsilon};$$

iii)

$$(1.7) \quad |h(z_2) - h(z_1)| \leq \delta |z_2 - z_1|.$$

In particular, the mapping which we construct satisfies all the estimates present in Shcherbakov's work, but can actually be constructed smoothly to obtain a C^∞ diffeomorphism off of the origin, and not merely a homeomorphism.

However, if we require the mapping to be real-analytic away from the origin with real-analytic inverse, the situation changes drastically. In particular, we show an equivalent condition for the existence of holomorphic conjugacy between germs of a special form as follows.

Theorem 1.2. *Suppose that the germ f has the special form*

$$(1.8) \quad f(z) = z + a_2 z^2 + O(z^3),$$

i.e., suppose that $m = 2$ in (1.1). Suppose there exists a homeomorphism R defined in a small neighborhood of the origin which is real-analytic except at the origin with real-analytic inverse, conjugating f with F_0 . Then, there exists a biholomorphism in a full neighborhood of the origin conjugating f with F_0 .

Hence, the C_*^∞ classification agrees with the topological one (where C_*^∞ means what we expect), but the real-analytic classification differs considerably. It is worth noting that Theorem 1.1 has been proved, in the case C^1 and $m = 2$, in the dissertation of Jérôme Rey [4]. However, his methods differ considerably from the ones presented here. Also, with regards to Theorem 1.2, it is important that f is equivalent to F_0 . It is easy to see that this theorem will fail for arbitrary $\tilde{f}$, that is, it is easy to find f and $\tilde{f}$ which are real-analytically equivalent but holomorphically inequivalent. For example, let $c \in \mathbf{C} - \mathbf{R}$ and consider

$$f(z) = z + z^2 + cz^3$$

and let

$$\tilde{f}(z) = z + z^2 + \bar{c}z^3.$$

Then, f is equivalent to $\tilde{f}$ via the antiholomorphic map $z \rightarrow \bar{z}$, but the two functions cannot be holomorphically equivalent, via the formal theory, since $c \neq \bar{c}$. In light of this, it is interesting that with the hypotheses of Theorem 1.2, the existence of a real-analytic map does indeed guarantee a *holomorphic* map, and rules out the antiholomorphic case.

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2. THE C^∞ CASE

In this section, we prove Theorem 1.1. Let f be a germ as in (1.1). We begin by assuming that $m = 2$. This is not necessary, as the case for arbitrary m does not depend on this base case. However, the reader may find that the proofs are more easily seen in this case. The general case will be proven, in a similar fashion, at the end of the section.

By applying the change of variable

$$z \mapsto \frac{z}{a_2},$$

we can assume that the germ f has the form

$$f(z) = z + z^2 + O(z^3).$$

Let us assume that f is defined in a disc centered at 0 of radius r . In order to construct the homeomorphism, it is convenient to work in a neighborhood of ∞ . Consider the change of variable

$$(2.1) \quad A(z) = -\frac{1}{z}.$$

Note that this change of variable converts our time-one map F_0 into

$$\tilde{F}_0(\eta) = (A \circ F_0 \circ A^{-1})(\eta) = \eta + 1.$$

Letting $R = \frac{1}{r}$, we may then assume that our map

$$\tilde{f}(\eta) = (A \circ f \circ A^{-1})(\eta)$$

is defined outside of a box about zero, that is, $\tilde{f}$ is defined in a neighborhood of the form

$$\Omega = \mathbf{C} - (\{|\operatorname{Re}\eta| \leq R\} \cap \{|\operatorname{Im}\eta| \leq R\}),$$

where $R \geq \frac{1}{r}$. Furthermore, $\tilde{f}$ has the form

$$\tilde{f}(\eta) = \frac{-1}{f(-\frac{1}{\eta})} = \eta + 1 + G^+(\eta)$$

where

$$(2.2) \quad |G^+(\eta)| < \frac{c}{|\eta|}, \quad |G^{+'}(\eta)| < \frac{c}{|\eta|^2}.$$

Similarly

$$\tilde{f}^{-1}(\eta) = \eta - 1 + G^-(\eta)$$

where

$$(2.3) \quad |G^-(\eta)| = \frac{c}{|\eta|}, \quad |G^{-'}(\eta)| = \frac{c}{|\eta|^2}.$$

Let us define the following sets:

$$\begin{aligned} A_1 &= \{\eta : \operatorname{Re}\eta \leq -R\}, & A_2 &= \{\eta : \operatorname{Re}\eta \geq R\}, \\ B_1 &= \{\eta : \operatorname{Im}\eta \leq -R\}, & B_2 &= \{\eta : \operatorname{Im}\eta \geq R\}, \\ \omega_1 &= \{\eta : -(R+1) \leq \operatorname{Re}\eta \leq -R\}, & \omega_2 &= \{\eta : R \leq \operatorname{Re}\eta \leq R+1\}. \end{aligned}$$

We denote

$$(2.4) \quad \tilde{H} = Id + \hat{h}.$$

Our goal is clear: we wish to construct $\hat{h}$ on the set Ω so that

$$(2.5) \quad \tilde{f} \circ \tilde{H} = \tilde{H} \circ (Id + 1).$$

In order to accomplish this, we will construct our mapping in two parts. Note that (2.5) yields the equation

$$(2.6) \quad \hat{h}(\eta + 1) = \hat{h}(\eta) + G^+(\eta + \hat{h}(\eta)).$$

Equivalently, the equation

$$(2.7) \quad \hat{h}(\eta - 1) = \hat{h}(\eta) + G^-(\eta + \hat{h}(\eta))$$

holds for f^{-1} and hence (2.6) and (2.7) yield the two functional equations

$$(2.8) \quad \hat{h}(\eta + n) = \hat{h}(\eta) + \sum_{j=0}^{n-1} G^+(\eta + j + \hat{h}(\eta + j)),$$

$$(2.9) \quad \hat{h}(\eta - n) = \hat{h}(\eta) + \sum_{j=0}^{n-1} G^-(\eta - j + \hat{h}(\eta - j)).$$

To begin, let us choose a function $\chi_1 \in C^\infty(\omega_1)$ with the following properties (let $\eta = \xi + i\zeta$):

- i) $\chi_1(\eta) = \chi_1(\operatorname{Re}\eta)$;
- ii) $0 \leq \chi_1 \leq 1$, $0 \leq \frac{\partial \chi_1}{\partial \xi} \leq 3$;

- iii) $\chi_1(\eta) = 0$ if $-(1+R) \leq \operatorname{Re}\eta \leq -(\frac{3}{4}+R)$;
- iv) $\chi_1(\eta) = 1$ if $-(\frac{1}{4}+R) \leq \operatorname{Re}\eta \leq -R$.

In the set ω_1 , we define

$$(2.10) \quad \hat{h}_1(\eta) = \chi_1(\eta)G^+(\eta-1).$$

We have that $\hat{h}_1(\eta) = 0$ if $\operatorname{Re}\eta = -(1+R)$, and thus it is clear that $\hat{h}_1$ satisfies the functional equation (2.6). Thus, we may use equations (2.8) and (2.9) to extend $\hat{h}_1$ to a function defined on the set $\Omega_1 = A_1 \cup B_1 \cup B_2$, and this extension is C^∞ on this set; to see this, we only need to check on lines of the form $\{\operatorname{Re}\eta = C\}$ where C is a constant of the form $-R+n$ for $n \in \mathbf{Z}$. Furthermore, by applying an inductive argument, it suffices to prove that the function is smooth in a small Ω -neighborhood of the two lines $\{\operatorname{Re}\eta = -R\}$ and $\{\operatorname{Re}\eta = -R-1\}$. It is easy to check one of these; note that if $\operatorname{Re}\eta = -R-\delta$ for δ sufficiently small, then $\hat{h}_1(\eta) = G^+(\eta-1)$. Also, (2.8) tells us that if $\operatorname{Re}\eta = -R+\delta$, then

$$\hat{h}_1(\eta) = \hat{h}_1(\eta-1) + G^+(\eta-1 + \hat{h}_1(\eta-1)),$$

and (2.10) tells us that the function is smooth over this line. To check the line $\{\operatorname{Re}\eta = -R-1\}$, we note that, on the right side of this line, $\hat{h}_1(\eta) = 0$, while on the left side, we have, by (2.9),

$$\hat{h}_1(\eta) = \hat{h}_1(\eta+1) + G^-(\eta+1 + \hat{h}_1(\eta+1)).$$

In order for this to be 0, we must have

$$\hat{h}_1(\eta+1) = -G^-(\eta+1 + \hat{h}_1(\eta+1)),$$

and by (2.10), this implies that

$$G^+(\eta) = -G^-(\eta+1 + G^+(\eta)).$$

But this follows immediately from the fact that $\tilde{f}^{-1} \circ \tilde{f} = Id$. Hence, $\hat{h}_1$ is smooth on the set Ω_1 . Thus, if we define

$$\tilde{H}_1(\eta) = \eta + \hat{h}_1(\eta),$$

then it is clear that, on Ω_1 , $\tilde{H}_1$ satisfies (2.5) (we note here that the mapping which Shcherbakov constructs in his paper fails to be C^∞ in the region Ω_1).

Similarly, we choose $\chi_2 \in C^\infty$ as above:

- i) $\chi_2(\eta) = \chi_2(\operatorname{Re}\eta)$;
- ii) $0 \leq \chi_2 \leq 1$, $-3 \leq \frac{\partial \chi_2}{\partial \xi} \leq 0$;
- iii) $\chi_2(\eta) = 0$ if $\frac{3}{4}+R \leq \operatorname{Re}\eta \leq 1+R$;
- iv) $\chi_2(\eta) = 1$ if $R \leq \operatorname{Re}\eta \leq \frac{1}{4}+R$.

Defining

$$(2.11) \quad \hat{h}_2(\eta) = \chi_2(\eta)G^-(\eta+1),$$

we see that this function can be extended into $\Omega_2 = A_2 \cup B_1 \cup B_2$ via (2.8) and (2.9). Writing

$$\tilde{H}_2(\eta) = \eta + \hat{h}_2(\eta),$$

we see also that $\tilde{H}_1$ satisfies (2.5). However, we must show that both $\tilde{H}_1$ and $\tilde{H}_2$ are actually invertible functions for suitably large R .

We now prove three estimates for $\hat{h}_1$ on Ω_1 (analogous estimates will hold for h_2 on Ω_2). In particular, we will show that, given any $\delta > 0$, we can find an R so that $\hat{h}_i$ is Lipschitz with constant δ . Thus, for $0 < \delta < 1$, this will then show that $\tilde{H}_i$ is one-to-one ($i = 1, 2$). Note that all of the results given here require a suitably large $\Omega = \Omega(R)$; that is, we will freely increase the size of R . The statements and proofs of these estimates are so similar to Shcherbakov's [5] that their proofs are presented merely for the convenience of the reader.

Proposition 2.1. *Define $F = \{\eta : |\arg \eta - \frac{\pi}{2}| \leq \frac{\pi}{3}\}$. Let R be suitably large. Then, for $\eta \in \omega_1 \cap B_2$ and any $n \in \mathbf{N}$ so that $\eta + n \in F$, we have*

$$(2.12) \quad |\hat{h}_1(\eta + n)| < 9c.$$

Recall that c is the number given in (2.3); $|G^+(\eta)| \leq \frac{c}{|\eta|}$.

Proof. We prove this via induction. For $n = 0$, we actually have $|\hat{h}_1(\eta)| < c$; this follows immediately from (2.3) and (2.10). Thus, suppose that $n > 0$ and that (2.12) holds for all j with $0 \leq j \leq n-1$. We will show that it holds also for $j = n$. We note the following:

$$n < 2|\eta|, \quad |\eta + j| > \frac{|\eta|}{2} \quad (0 \leq j \leq n-1)$$

(both of these statements are easily determined by basic geometry). Via the induction hypothesis, for R sufficiently large we have

$$\begin{aligned} |\eta + j + \hat{h}_1(\eta + j)| &> |\eta + j| - |\hat{h}_1(\eta + j)| \\ &> |\eta + j| - 9c \\ &> \frac{|\eta + j|}{2}. \end{aligned}$$

Putting this together, we get

$$\begin{aligned} |\hat{h}_1(\eta + n)| &= |\hat{h}_1(\eta) + \sum_{j=0}^{n-1} G^+(\eta + j + \hat{h}_1(\eta + j))| \\ &\leq c + \sum_{j=0}^{n-1} \frac{c}{|\eta + j + \hat{h}_1(\eta + j)|} \\ &\leq c + \frac{4cn}{|\eta|} \\ &< c + 8c = 9c. \end{aligned}$$

□

It is clear that Proposition 2.1 is true for $n < 0$ and for the region $-F$ by analogous reasoning. Furthermore, we note that the proof of Proposition 2.1 implies the following:

Corollary 2.2. *Let $\beta > 0$. Then, we can find R and $\alpha > 0$ so that, in the region*

$$F_\alpha = \{\eta \in \mathbf{C} : |\arg \eta - \frac{\pi}{2}| < \alpha\},$$

we have for $\eta \in F_\alpha$ that $\hat{h}(\eta) \leq \beta c$.

Hence, for α small enough, we can make $\hat{h}$ as small as we desire on the angular region F_α .

Proposition 2.3. *Let $0 < \varepsilon < 1$ be given. Suppose also that $\eta \in F \cup (-F) \cup \omega_1$ and that $\eta + 1 \in \Omega$, but $\eta + 1 \notin F \cup (-F)$. Then, for R large enough (depending only on ε), we have*

$$(2.13) \quad |\hat{h}_1(\eta + n)| < |\eta + n|^\varepsilon.$$

Proposition 2.3 tells us that the image of the set Ω under the map $\tilde{H}$ will include the complement of a large disc in the plane.

Proof. Again, we use induction. For $n = 0$ and R sufficiently large, this follows from Proposition 2.1. Suppose then that $n > 0$ (and hence that $\operatorname{Re} \eta > 0$), and suppose that (2.13) holds for all j with $0 \leq j \leq n$. Write $\eta' = \eta + n$. Since $\eta' + 1 \notin F$ and $\operatorname{Re} \eta' > 0$, we have that $|\eta' + 1| > |\eta'| + \frac{1}{2}$ (the key here is that $\operatorname{Re}(\eta') \geq \sqrt{3}\operatorname{Im}(\eta') - 1$). This yields

$$|\eta' + 1|^\varepsilon - |\hat{h}_1(\eta' + 1)| > |\eta' + 1|^\varepsilon - |\hat{h}_1(\eta')| - \left| G^+(\eta' + \hat{h}_1(\eta')) \right|.$$

Applying the induction hypothesis, we see that

$$|\hat{h}_1(\eta')| < |\eta'|^\varepsilon$$

and

$$|\eta' + \hat{h}_1(\eta')| > \frac{|\eta'|}{2}.$$

Therefore

$$\begin{aligned} |\eta' + 1|^\varepsilon - |\hat{h}_1(\eta' + 1)| &> \left(|\eta'| + \frac{1}{2} \right)^\varepsilon - |\eta'|^\varepsilon - \frac{2c}{|\eta'|} \\ &> \frac{\varepsilon}{2} |\eta'|^{\varepsilon-1} - \left(\frac{\varepsilon(1-\varepsilon)}{8} |\eta'|^{\varepsilon-2} + \frac{2c}{|\eta'|} \right) \\ &> 0, \end{aligned}$$

again, if R is chosen large enough. $\square$

Proposition 2.4. *Suppose that $\eta \in K \subset \omega_1$, where K is a disc of radius $\frac{1}{2}$, and suppose that $K + n \subset \Omega_1$, where $n \in \mathbf{Z}$. Then $\hat{h}_1$ satisfies a Lipschitz condition in the disc $K + n$ with Lipschitz constant $\delta = \delta(R)$, where $\delta(R) \rightarrow 0$ as $R \rightarrow \infty$.*

Proof. We use induction. For $n = 0$ we have

$$\begin{aligned} &|\hat{h}_1(\eta_2) - \hat{h}_1(\eta_1)| \\ &= |\chi_1(\eta_2)G^+(\eta_2 - 1) - \chi_1(\eta_1)G^+(\eta_1 - 1)| \\ &\leq |\chi_1(\eta_2)| |G^+(\eta_2 - 1) - G^+(\eta_1 - 1)| + |G^+(\eta_1 - 1)| |\chi_1(\eta_2) - \chi_1(\eta_1)| \\ &\leq |G^+(\eta_2 - 1) - G^+(\eta_1 - 1)| + 3|G^+(\eta_1 - 1)| |\eta_2 - \eta_1| \\ &\leq \left(\frac{c}{R^2} + \frac{3c}{R} \right) |\eta_2 - \eta_1|. \end{aligned}$$

Suppose now that $n > 0$. Furthermore, assume by the induction hypothesis that $\hat{h}_1$ satisfies a Lipschitz condition in $K + n$ with constant $\delta_n < 1$. Write $\eta'_j = \eta_j + n$,

$j = 1, 2$. We have

$$\begin{aligned} |\hat{h}_1(\eta'_2 + 1) - \hat{h}_1(\eta'_1 + 1)| &< \delta_n |\eta'_2 - \eta'_1| + \frac{2c}{|\eta'_1|^2} |\eta'_2 - \eta'_1 + \hat{h}_1(\eta'_2) - \hat{h}_1(\eta'_1)| \\ &< |\eta'_2 - \eta'_1| \left(\delta_n + \frac{2c(1 + \delta_n)}{|\eta'_1|^2} \right). \end{aligned}$$

We note the following fact:

$$|\eta_1 + n| < \frac{R + n}{4}$$

(this follows via elementary geometry). Thus

$$\delta_{n+1} < \delta_n + \frac{64c}{|R + n|^2},$$

and to complete the proof, we choose R large enough so that

$$64 \left(\sum_{n=1}^{\infty} |R + n|^{-2} \right) + \frac{c}{R^2} + \frac{3c}{R} < 1.$$

□

As mentioned above, Proposition 2.4 guarantees that the mappings $\tilde{H}_j$, $j = 1, 2$, are one-to-one in the domains

$$(2.14) \quad \Omega_j = A_j \cup B_1 \cup B_2,$$

provided that R is large enough. Let us define regions of the form

$$(2.15) \quad \tilde{S}_j = \{\eta \in \Omega_j : |\arg \eta - \pi j| < \frac{\pi}{2} + \alpha\},$$

where $0 < \alpha < \frac{\pi}{3}$. If we write $\tilde{H}_j^{-1} = Id + \tilde{h}_j$, Proposition 2.4 also guarantees that $\tilde{h}_j$ will satisfy a Lipschitz condition with constant $\delta' = \frac{\delta}{1-\delta}$. Finally, Propositions 2.1 and 2.3 yield that the maps $\tilde{H}_j$ will have domains and ranges containing and contained in sets of the form (2.15), again if R is large enough.

To summarize, we have found two mappings

$$\tilde{H}_j = Id + \hat{h}_j$$

in the regions Ω_j so that

$$(2.16) \quad \tilde{f} \circ \tilde{H}_j = \tilde{H}_j \circ (Id + 1).$$

We now will patch these maps together to form a global map $\tilde{H}$ in a neighborhood of ∞ which satisfies (2.16). It is here that our argument diverges most sharply from Shcherbakov's. In particular, in [5], Shcherbakov constructs a single mapping defined on the region Ω_1 satisfying Propositions 2.1, 2.3, and 2.4. In order to then construct a mapping defined in a full neighborhood of ∞ satisfying (2.16), he presents a linear interpolation argument. Unfortunately, that argument will not produce a C^∞ mapping. In order to counter this problem, we construct the two mappings h_1 and h_2 . We then use a well-chosen partition of unity to patch these mappings together in a smooth way in a neighborhood of ∞ ; this is the argument presented now. First, choose $\alpha > 0$ and R so that we have $\beta < \frac{1}{1000000c}$ and $\delta < \frac{1}{1000000}$. Let us define the regions $\tilde{S}_1$ and $\tilde{S}_2$ as in (2.15), taking α as chosen above. We define a mapping

$$(2.17) \quad \Phi_1(\eta) = \tilde{H}_2^{-1} \circ \tilde{H}_1$$

on the set

$$(2.18) \quad \tilde{S}_1 \cap \tilde{S}_2 \cap \{\operatorname{Im} \eta > D > 0\}.$$

Note that, for D large enough, the corollary to Proposition 2.1 insures that (2.17) is well defined. We note the following important property of Φ_1 :

$$(2.19) \quad \Phi_1(\eta + 1) = \Phi_1(\eta) + 1.$$

Hence, we may write

$$(2.20) \quad \Phi_1(\eta) = \eta + p(\eta),$$

where p is smooth and 1-periodic. Note also that p satisfies a Lipschitz condition with constant $\delta'' = \frac{2\delta}{1-\delta}$; this follows from Proposition 2.4. Hence, if we extend the mapping p to a mapping $\tilde{p}$ which is defined in an upper half plane, then the mapping

$$(2.21) \quad \tilde{\Phi}_1(\eta) = \eta + \tilde{p}(\eta)$$

will be one-to-one as well. Similarly, we can define a mapping on the set

$$(2.22) \quad \tilde{S}_1 \cap \tilde{S}_2 \cap \{\operatorname{Im} \eta < -D < 0\}$$

given by

$$(2.23) \quad \Phi_2 = \tilde{H}_1^{-1} \circ \tilde{H}_2.$$

Writing

$$(2.24) \quad \Phi_2(\eta) = \eta + q(\eta),$$

we may subsequently extend q to a mapping $\tilde{q}$ defined on a lower half plane, still satisfying a Lipschitz condition, and hence the mapping

$$(2.25) \quad \tilde{\Phi}_2(\eta) = \eta + \tilde{q}(\eta)$$

will also be one-to-one. Now, we patch our maps. Choose a map $\psi_1 \in C^\infty(\mathbf{C})$ with the following properties (again, let $\eta = \xi + i\zeta$):

- i) $\psi_1(\eta) = \psi_1(\operatorname{Im} \eta)$,
- ii) $0 \leq \psi_1 \leq 1$, $0 \leq \frac{\partial \psi}{\partial \zeta} \leq 3$,
- iii) $\psi_1(\eta) = 0$ if $\operatorname{Im} \eta \leq D$,
- iv) $\psi_1(\eta) = 1$ if $\operatorname{Im} \eta \geq D + 1$.

Define the function

$$(2.26) \quad \begin{aligned} K(\eta) &= (1 - \psi_1(\eta))\eta + \psi_1(\eta)(\eta + \tilde{p}(\eta)) \\ &= \eta + \psi_1(\eta)\tilde{p}(\eta). \end{aligned}$$

Note that K is smooth on $\mathbf{C}$ (with the obvious adjustments), one-to-one, and that K commutes with the unit shift. In similar fashion, choose $\psi_2 \in C^\infty(\mathbf{C})$ so that

- i) $\psi_2(\eta) = \psi_2(\operatorname{Im} \eta)$,
- ii) $0 \leq \psi_2 \leq 1$, $-3 \leq \frac{\partial \psi}{\partial \zeta} \leq 0$,
- iii) $\psi_2(\eta) = 0$ if $\operatorname{Im} \eta \geq -D$,
- iv) $\psi_2(\eta) = 1$ if $\operatorname{Im} \eta \leq -D - 1$,

and define

$$(2.27) \quad \begin{aligned} L(\eta) &= (1 - \psi_2(\eta)) \eta + \psi_2(\eta) (\eta + \tilde{q}(\eta)) \\ &= \eta + \psi_2(\eta) \tilde{q}(\eta). \end{aligned}$$

As with K , we see that L is smooth on $\mathbf{C}$, one-to-one, and of course L commutes with the unit shift. We use these two maps to patch together $\tilde{H}_1$ and $\tilde{H}_2$:

$$(2.28) \quad \tilde{H}(\eta) = \begin{cases} \tilde{H}_1(\eta) & \text{if } \operatorname{Re} \eta < -D, \operatorname{Im} \eta > 0, \\ (\tilde{H}_2 \circ K)(\eta) & \text{if } 0 \leq \arg \eta \leq \frac{\pi}{2} + \gamma, |\eta| > D, \\ \tilde{H}_2(\eta) & \text{if } \operatorname{Re} \eta > D, \operatorname{Im} \eta < 0, \\ (\tilde{H}_1 \circ L)(\eta) & \text{if } \pi \leq \arg \eta \leq \frac{3\pi}{2} + \gamma, |\eta| > D \end{cases}$$

(here, $\gamma < \frac{\pi}{C}$ for C very large). It is worth noting that the map $\tilde{H}$ is well defined. To check this, note that if $\operatorname{Im} \eta = 0$, then both $K = Id$ and $L = Id$. On the other hand, if $\operatorname{Re} \eta < -D$ and $\arg \eta < \frac{\pi}{2} + \gamma$, then $K = \tilde{H}_2^{-1} \circ \tilde{H}_1$, while if $\operatorname{Re} \eta > D$ and $\arg \eta < \frac{3\pi}{2} + \gamma$, then $L = \tilde{H}_1^{-1} \circ \tilde{H}_2$, and hence the function agrees in the overlaps of the domains. Also, $\tilde{H}$ is smooth in a neighborhood of ∞ (but NOT at $\infty!$); this can be checked easily from the definitions (2.26) and (2.27). Finally, the mapping $\tilde{H}$ does indeed conjugate $\tilde{f}$ with the unit shift. We check only one case, as the others follow analogously (or trivially):

$$\begin{aligned} (\tilde{H}_1 \circ K)^{-1} \circ \tilde{f} \circ (\tilde{H}_1 \circ K) &= K^{-1} \circ \tilde{H}_1^{-1} \circ \tilde{f} \circ \tilde{H}_1 \circ K \\ &= K^{-1} \circ (Id + 1) \circ K \\ &= K^{-1} \circ K \circ (Id + 1) \\ &= Id + 1. \end{aligned}$$

If we write $\tilde{H}(\eta) = \eta + \hat{h}(\eta)$, then Proposition 2.4 together with (2.26) and (2.27) guarantees that $\hat{h}$ satisfies a Lipschitz condition with constant δ as small as we like; taking $\delta < 1$ insures that $\tilde{H}$ possesses an inverse globally. Thus, we have constructed the mapping with the required properties in a neighborhood of ∞ , and thus the mapping

$$(2.29) \quad H(z) = -\frac{1}{\tilde{H}(-\frac{1}{z})} = \frac{z}{1 - z\hat{h}(-\frac{1}{z})}$$

clearly satisfies (1.5) and (1.6), and (1.7) may be checked easily by using the right-hand side of (2.29). Thus, the proof is complete for the case $m = 2$.

Now, let us suppose that m is arbitrary. Though the proof given here is very similar to the case $m = 2$, it is worthwhile to look at the cases separately. First, by the formal classification theorem (e.g. see [2]), we can apply a (holomorphic) polynomial change of coordinate p with degree no more than $m - 1$ so that

$$(2.30) \quad (p^{-1} \circ f \circ p)(z) = z + z^m + \mu z^{2m-1} + O(z^{2m}),$$

and hence we assume that f has this form. The change of variable which conjugates g_V^1 with the unit shift in a neighborhood of ∞ now becomes

$$(2.31) \quad A_m(z) = -\frac{1}{(m-1)z^{m-1}},$$

which unfortunately has a multi-valued inverse

$$(2.32) \quad A_m^{-1}(z) = \sqrt[m-1]{\frac{-1}{(m-1)z}}.$$

To remedy the situation, we look at the Riemann surface for the map $\sqrt[m-1]{z}$; let $0 < r \ll 1$ and consider sectors of the form

$$(2.33) \quad S_j = \{z \in \mathbf{C} : |z| < r, |\arg z - \frac{\pi j}{m-1}| < \alpha\},$$

where $\frac{\pi}{2(m-1)} < \alpha < \frac{\pi}{m-1}$, $j = 1, \dots, 2(m-1)$. Note that these sectors cover a disc of radius r with center at the origin. Let us look at the image of these sectors under the map (2.31) in the plane. When j is odd, S_j is mapped to

$$(2.34) \quad \tilde{S}_j = \{\eta \in \mathbf{C} : |\eta| > R, |\arg \eta - \pi| < \frac{\pi}{2} + \tilde{\alpha}\}$$

while for j even, S_j is mapped to

$$(2.35) \quad \tilde{S}_j = \{\eta \in \mathbf{C} : |\eta| > R, |\arg \eta| < \frac{\pi}{2} + \tilde{\alpha}\},$$

where $0 < \tilde{\alpha} < \frac{\pi}{2}$. It is important to keep in mind that, for any given j , the sector $\tilde{S}_j$ will only touch those sectors $\tilde{S}_{j-1}$ and $\tilde{S}_{j+1}$ (the exceptions being $\tilde{S}_1$ and $\tilde{S}_{2(m-1)}$, which overlap). We may think of these sectors as living in copies of $\mathbf{C}$, that is,

$$\tilde{S}_j \subseteq \mathbf{C} \times \{j\}.$$

Given $\eta \in \tilde{S}_j$, we say that $\eta \in \tilde{S}_j \cap \tilde{S}_k$ if we can find $\tilde{\eta} \in S_k$ so that

$$A^{-1}(\eta) = A^{-1}(\tilde{\eta}).$$

It is easy to see that $\tilde{S}_1 \cap \tilde{S}_2$ lies in an upper half plane, $\tilde{S}_2 \cap \tilde{S}_3$ lies in a lower half plane, etc., until we reach $\tilde{S}_{2(m-1)} \cap \tilde{S}_1$, which lies in a lower half plane. This takes care of the multi-valuedness of A^{-1} .

We note that we can write

$$\begin{aligned} \tilde{f}(\eta) &= (A_m \circ f \circ A_m^{-1})(\eta) \\ &= \frac{\eta}{(1 - \frac{1}{(m-1)\eta} + \frac{1}{(m-1)^2\eta^2} + \dots)^{m-1}}. \end{aligned}$$

Computing the Taylor series for the function $\left(\frac{1}{1-z}\right)^{m-1}$, we see that we can write

$$(2.36) \quad \tilde{f}(\eta) = \eta + 1 + G^+(\eta)$$

and similarly

$$(2.37) \quad \tilde{f}^{-1}(\eta) = \eta - 1 + G^-(\eta)$$

where G^+ and G^- satisfy the estimates (2.2). We are now basically in the situation of the case $m = 2$; by following the constructions given in that case, we may construct maps $\tilde{H}_j$ defined in the sectors $\tilde{S}_j$ (for $j = 1, 2, \dots, 2m-2$) satisfying Propositions 2.1, 2.3 and 2.4 and of course satisfying

$$(2.38) \quad \tilde{f} \circ \tilde{H}_j = \tilde{H}_j \circ (Id + 1).$$

We now may look at the transition functions

$$\Phi_j = \tilde{H}_{j+1}^{-1} \circ \tilde{H}_j$$

for $j = 1, 2, \dots, 2m-3$ and

$$\Phi_{2m-2} = \tilde{H}_1^{-1} \circ \tilde{H}_{2m-2}.$$

As in the case $m = 2$, these functions may be written as

$$\Phi_j(\eta) = \eta + p_j(\eta)$$

where p_j is smooth and 1-periodic, for all $j = 1, \dots, 2m-2$. We may then extend this mapping to $\tilde{p}_j$, defined in an upper half plane $\{\text{Im}\eta > D\}$ in $\mathbf{C} \times \{j\}$ for j odd, and a lower half plane $\{\text{Im}\eta < -D\}$ in $\mathbf{C} \times \{j\}$ for j even. Furthermore, the functions

$$(2.39) \quad \tilde{\Phi}_j(\eta) = \eta + \tilde{p}_j(\eta)$$

will be one-to-one in these upper or lower half planes. Now, choose functions $\psi_j \in C^\infty(\mathbf{C})$ which satisfy

- i) $\psi_j(\eta) = \psi_j(\text{Im}\eta)$,
- ii) $0 \leq \psi_j \leq 1$, $0 \leq \frac{\partial \psi_j}{\partial(\text{Im}\eta)} \leq 3$,
- iii) $\psi_{2n-1}(\eta) = 0$ if $\text{Im}\eta \leq D$ and $\psi_{2n-1}(\eta) = 1$ if $\text{Im}\eta \geq D+1$,
- iv) $\psi_{2n}(\eta) = 0$ if $\text{Im}\eta \geq -D$ and $\psi_{2n}(\eta) = 1$ if $\text{Im}\eta \leq -(D+1)$

(where $n = 1, 2, \dots, m-1$). Using these maps, we can define smooth maps K_j , defined on $\mathbf{C} \times \{j\}$ as follows:

$$(2.40) \quad \begin{aligned} K_j(\eta) &= (1 - \psi_j(\eta))\eta + \psi_j(\eta)(\eta + \tilde{p}_j(\eta)) \\ &= \eta + \psi_j(\eta)\tilde{p}_j(\eta). \end{aligned}$$

Note that K_j is one-to-one on $\mathbf{C} \times \{j\}$ and commutes with the unit shift for all $j = 1, \dots, 2(m-1)$.

Finally, we can patch our mappings $\tilde{H}_j$ together to form

$$(2.41) \quad \tilde{H}(\eta) = \begin{cases} (\tilde{H}_{2n-1})(\eta) & \text{if } \eta \in \tilde{S}_{2n-1}, 0 \leq \arg \eta \leq \frac{\pi}{2} + \gamma, \\ (\tilde{H}_{2n} \circ K_{2n-1})(\eta) & \text{if } \eta \in \tilde{S}_{2n}, \text{Re}\eta < -D, \text{Im}\eta > 0, \\ (\tilde{H}_{2n})(\eta) & \text{if } \eta \in \tilde{S}_{2n}, \pi \leq \arg \eta \leq \frac{3\pi}{2} + \gamma, \\ (\tilde{H}_{2n+1} \circ K_{2n})(\eta) & \text{if } \eta \in \tilde{S}_{2n+1}, \text{Re}\eta > D, \text{Im}\eta < 0, \end{cases}$$

where $n = 1, \dots, m-1$ (here, we define $\tilde{H}_{2m-1} = \tilde{H}_1$). As in the case $m = 2$, we choose γ so that the mapping is well defined. This mapping satisfies a Lipschitz condition with constant as small as we like (via estimates analogous to Proposition 2.4). Now, we take this back into a neighborhood of the origin. We thus consider the mapping

$$(2.42) \quad \begin{aligned} H(z) &= (A_m^{-1} \circ \tilde{H} \circ A_m)(z) \\ &= \sqrt[m-1]{\frac{-1}{(m-1) \left(\tilde{H} \left(\frac{-1}{(m-1)z^{m-1}} \right) \right)}} \\ &= \sqrt[m-1]{\frac{-1}{(m-1) \left(\frac{-1}{(m-1)z^{m-1}} + \hat{h} \left(\frac{-1}{(m-1)z^{m-1}} \right) \right)}} \\ &= z \left(\sqrt[m-1]{\frac{1}{1 - (m-1)z^{m-1}\hat{h} \left(\frac{-1}{(m-1)z^{m-1}} \right)}} \right). \end{aligned}$$

Though this formula is quite ugly, by computing the Taylor series for the function ${}^{m-1}\sqrt{\frac{1}{1-z}}$, we can rewrite the final line of (2.42) as

$$(2.43) \quad H(z) = z + z^m \hat{h} \left(\frac{-1}{(m-1)z^{m-1}} \right) + o(z^{m-\varepsilon}).$$

Now, it is clear that if we write $H(z) = z + h(z)$, then h satisfies (1.6), and (1.7) may be checked in a fashion similar to the case $m = 2$. Hence, the theorem is proved for arbitrary m .

3. THE REAL-ANALYTIC CASE

In the previous section, we showed that given any two germs of the form

$$(3.1) \quad f(z) = z + a_2 z^2 + \dots$$

where $a \neq 0$, these two germs were equivalent via a local homeomorphism H which is a diffeomorphism off of the origin. In this section, we prove Theorem 1.2, which says that given $F(z) = \frac{z}{1-z}$, any local homeomorphism R which is real-analytic away from the origin, with real-analytic inverse, conjugating a germ f of the form (3.1) with the germ F must necessarily give rise to a biholomorphic map H in a full neighborhood of the origin conjugating f with F . In particular, by the formal theory, this will put restrictions on the formal invariant a_3/a_2^2 of the function f (as mentioned in the introduction).

The key is the following “basic lemma” of Voronin [6]:

Lemma 3.1. *Suppose that f is a germ as in (3.1), defined in a small disc $D(0, \rho)$, and define the set*

$$\Omega_\rho = \{Re\eta > \frac{1}{\rho}\} \cup \{|Im\eta| > \frac{1}{\rho}\}.$$

Suppose that G is a quasi-conformal homeomorphism of Ω_ρ into D so that

$$G^{-1} \circ f \circ G = Id + 1.$$

Define $G(\Omega_\rho) = \Omega$. Then, there exists an injective holomorphic mapping $H : \Omega \rightarrow \mathbf{C}$ so that, if we define $\tilde{\Omega} = H(\Omega)$, then

$$(3.2) \quad H \circ f \circ H^{-1} = Id + 1$$

on $\tilde{\Omega}$. The set $\tilde{\Omega}$ will contain a set of the form Ω_ε for $\varepsilon > 0$ sufficiently small. Moreover, the map H is unique in the following sense: given another map $\tilde{H}$ satisfying the conclusions above, there exists $c \in \mathbf{C}$ and ε' so that $\tilde{H} = H + c$ on $\Omega_{\varepsilon'} \subseteq \Omega$.

Recall that a homeomorphism $G : \Omega \rightarrow \mathbf{C}$ is called quasi-conformal if G has locally integrable generalized derivatives satisfying the estimate

$$\left| \frac{\partial G}{\partial \bar{z}} \right| \leq k \left| \frac{\partial G}{\partial z} \right|.$$

Another characterization of quasi-conformal mappings is the following: define the quantity

$$(3.3) \quad K_G(z_0) = \overline{\lim}_{r \rightarrow 0} \left(\frac{\sup_{|z-z_0|=r} |f(z) - f(z_0)|}{\inf_{|z-z_0|=r} |f(z) - f(z_0)|} \right);$$

this quantity is called the quasi-conformal deviation of G at z_0 . Then, the mapping G is quasi-conformal if and only if $K_G(z_0) < \infty$ for all $z_0 \in \Omega$ and $K = |K_G|_{L^\infty} < \infty$.

∞ . Using this condition, it is clear that the homeomorphism H constructed in section 2 is quasi-conformal as long as the Lipschitz constant $\delta < 1$, for then we have

$$(3.4) \quad |K_H(z_0)| \leq \frac{1+\delta}{1-\delta}.$$

Hence, we may assume that, given a holomorphic mapping f defined on some disc $D(0, \rho)$, we have a local homeomorphism R defined on D , which is real-analytic with real-analytic inverse away from 0, so that

$$(3.5) \quad R \circ f \circ R^{-1}(z) = \frac{z}{1-z}.$$

Also, by Lemma 3.1 (the “basic lemma”), we can find a holomorphic map H , defined on some open subset U of D , so that

$$(3.6) \quad H \circ f \circ H^{-1}(\eta) = \eta + 1,$$

and satisfying that $\Omega_\varepsilon \subseteq H(U)$ for some $\varepsilon > 0$ sufficiently small. Again, let $A(z) = -\frac{1}{z}$, so that we have

$$(3.7) \quad A \circ R \circ f \circ R^{-1} \circ A^{-1} = Id + 1.$$

Then, (3.6) and (3.7) yield that, on a set of the form $\Omega_{\varepsilon'}$, $0 < \varepsilon' < \varepsilon$ we have

$$(3.8) \quad A \circ R \circ H^{-1}(\eta + 1) = A \circ R \circ H^{-1}(\eta) + 1,$$

so that we may write

$$(3.9) \quad A \circ R \circ H^{-1}(\eta) = \eta + p(\eta),$$

where p is 1-periodic and real-analytic on $\Omega_{\varepsilon'}$. Extending p to $\tilde{p}$ defined on the complex plane, we write

$$(3.10) \quad B(\eta) = \eta + \tilde{p}(\eta).$$

It is clear that B is injective; to see this, suppose that

$$(3.11) \quad B(\eta_2) = B(\eta_1).$$

Choose a single $n \in \mathbf{Z}$ and $\eta'_i \in \Omega_{\varepsilon'}$, so that

$$(3.12) \quad \eta_i = \eta'_i + n,$$

for $i = 1, 2$. Now, we have

$$B(\eta'_1 + n) = B(\eta'_2 + n),$$

which implies that $B(\eta'_1) = B(\eta'_2)$. Since B is certainly injective in the region $\Omega_{\varepsilon'}$, we must have $\eta'_1 = \eta'_2$, which then shows that $\eta_1 = \eta_2$.

It is important to note that

$$(3.13) \quad |B(\eta)| \rightarrow \infty \text{ as } |\eta| \rightarrow \infty.$$

In order to prove (3.13), we need the following lemma:

Lemma 3.2. *Suppose that f is holomorphic on the disc $D(0, \delta)$, and let H be the map given by the “basic lemma” (i.e. Lemma 3.1) with domain Ω (and recall that $H(\Omega) \supseteq \Omega_\rho$ for some $\rho > 0$). Let $0 < \delta' < \delta$ be given. Then, we can find ρ' so that*

$$H(D(0, \delta') \cap \Omega) \supseteq \Omega_{\rho'}.$$

Proof. This follows from the uniqueness property of Lemma 3.1. To be more precise, the mapping f is defined on the disc $D(0, \delta')$, and hence Lemma 3.1 tells us that we can find a map $\tilde{H}$, defined on a domain $\tilde{\Omega}$ with $\tilde{H}(\tilde{\Omega}) \supseteq \Omega_{\rho''}$, with $\rho'' > 0$, which conjugates f with the unit shift. By uniqueness, on the set $\tilde{\Omega}$, we can write

$$H = \tilde{H} + c$$

where $c \in \mathbf{C}$. Thus, if we write $L_c(z) = z + c$, then $H(\tilde{\Omega})$ contains the set $L_c\Omega_{\rho''} \supseteq \Omega_{\rho'}$, completing the proof of the lemma. $\square$

We now prove (3.13). Let $C > 0$. By Lemma 3.2, we can choose ε' so that, on the set $\Omega_{\varepsilon'}$, we have $|B| > C$. Now, consider the set

$$(3.14) \quad \mathbf{C} - \Omega_{\varepsilon'} = \{\eta \in \mathbf{C} : \operatorname{Re} \eta \leq \frac{1}{\varepsilon'} \text{ and } |\operatorname{Im} \eta| \leq \frac{1}{\varepsilon'}\}.$$

On this set, the values of $\tilde{p}$ are completely defined by its values in a closed rectangle of base 1 and height $\frac{2}{\varepsilon'}$; in particular, $\tilde{p}$ is *bounded* on the set $\mathbf{C} - \Omega_{\varepsilon'}$. Thus, we can choose $\varepsilon'' > 0$ so that, if $\eta \in \mathbf{C} - \Omega_{\varepsilon'}$ and $\operatorname{Re} \eta < -\varepsilon''$, then $|B| > C$. Thus, $|B| > C$ on a full neighborhood of ∞ , and this proves (3.13).

Using (3.13), we see that the map $R^{-1} \circ A^{-1} \circ B$ is real-analytic in a neighborhood U of ∞ and agrees with the map H^{-1} on some open subset $U' \subseteq U$. Thus, this map defines a real-analytic, and hence holomorphic, extension of H^{-1} to U . Hence, the map $\tilde{H} = R^{-1} \circ A^{-1} \circ B \circ A$ is defined and holomorphic in a punctured neighborhood of the origin, and is bounded in a full neighborhood of the origin, by (3.13). By the theorem on removable singularities, $\tilde{H}$ may be extended to a *full* neighborhood of the origin, with $\tilde{H}(0) = 0$ (again, by (3.13)). Furthermore, $\tilde{H}$ has a holomorphic inverse by virtue of injectivity. Finally, it is clear that

$$(3.15) \quad (\tilde{H}^{-1} \circ f \circ \tilde{H})(z) = \frac{z}{1-z}$$

(for example, one could apply the identity theorem), which completes the proof.

From here, a question comes to mind. As mentioned in the introduction, Theorem 1.2 is obviously false in general. Thus, we can ask the following:

Question 1. Given that f and $\tilde{f}$ are conjugate via a mapping satisfying the hypotheses of Theorem 1.2, under what conditions on $\tilde{f}$ will the conclusion of Theorem 1.2 hold?

The proof of Theorem 1.2 given here depends largely on the fact that the holomorphic mapping which takes $F_0(z) = \frac{z}{1-z}$ to $\tilde{F}_0(\eta) = \eta + 1$ in a neighborhood of ∞ is globally defined in a punctured neighborhood of the origin; $A(z) = -\frac{1}{z}$. This is *not* the case in general. For example, consider the time-one map g_μ^1 of the vector field $V(z) = \frac{z^2}{1+\mu z} \frac{\partial}{\partial z}$ for $\mu \neq 0$. Then, the map $A_\mu(z) = -\frac{1}{z} + \mu \log z$ will take g_μ^1 to F_0 , but cannot be globally defined in a punctured neighborhood of the origin. Also, the proof depends on the fact that $m = 2$; if we instead assume that f is conjugate to the time-one map of the vector field $V(z) = z^m \frac{\partial}{\partial z}$ for $m \geq 3$, this proof will fail. It would be nice to see precise conditions under which we have such a characterization of holomorphic conjugacy.

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